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Wigner rotation and its $SO(3)$ model: an active-frame approach

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As an important issue in special relativity, Wigner rotation is notoriously difficult for beginners for two major reasons: this physical phenomenon is highly unintuitive, and the mathematics behind it can be extremely challenging. To remove the first obstacle, we introduce a clear and easy toy model under the guidance of group theory. To overcome the second, a concise mathematical method is developed by the integration of geometric algebra and the active-frame formalism.

Keywords and phrases: Wigner rotation, special relativity, geometric algebra, Clifford algebra.

1. Introduction

First discovered by L. Silberstein and then rediscovered by L. Thomas [1, 2], the phenomenon that two successive non-parallel boosts (i.e., Lorentz transformations that contain neither rotation nor reflection) lead to a boost and a rotation is generally called Wigner rotation [3]. It has been studied by many authors for almost a century [4, 5, 6, 7, 8], the mysterious aura persists nevertheless. H. Goldstein, author of the classic work *Classical Mechanics,* compared it to the twin paradox as two famous counter-intuitive consequences of special relativity [9].

As for what is wrong with our intuition, we begin our discussion with the classical counterpart of boost transformation. In classical mechanics, for two inertial reference frames with coordinates (x_1, x_2, x_3, t) and (x'_1, x'_2, x'_3, t'), if their relative velocity is the constant three-dimensional vector $\vec{V}$, then the relation between the two set of coordinates is the so-called Galilean transformation

$$\begin{aligned} x'_i &= x_i - V_i t,\ \ i = 1, 2, 3; \\ t' &= t. \end{aligned}$$

The first equation of the above transformation is nothing but a time-dependent passive translation, and the second implies the two frames share the same time coordinate. Because the composition of two translations is another translation, there is no doubt that two successive Galilean transformations lead to another one. Being the relativistic version of Galilean transformation, boost is usually mistaken for possessing this closure property as well, i.e., two successive boosts lead to another boost. But the truth is, as mentioned earlier, as long as the directions of the two boosts are not parallel, their composition is not a single boost, but a boost along with a Wigner rotation.

Getting to the bottom of the matter, it is because boost can not be identified with or compared to any kind of translation. Consequently, analogizing boost to Galilean transformation is groundless and dangerous, at least in the current case. From the mathematical point of view, a boost is a kind of rotation in the four-dimensional spacetime, or more precisely a pseudo-rotation. It should come as no surprise that the composition law of boosts is more complicated than that of translations.

However, even if we understand a boost is essentially a pseudo-rotation in the four-dimensional spacetime, there is still a fact violating our common sense. When a problem involves two non-parallel boosts and three reference frames, say K_0, K_1, and K_2, with K_0 and K_1 being associated by the first boost, and K_1 and K_2 by the second, it makes sense that some of the spatial axes of K_0 and K_2 might be non-parallel since not all of the spatial axes of K_0 (K_1) are parallel to that of K_1 (K_2) from the four-dimensional point of view. What contradicts intuition is another frame K_3 can be obtained from K_2 by a boost so that K_3 and K_0 are at rest with respect to each other, but some spatial axes of K_3 and K_0 still differ by a rotation—the same Wigner rotation—even though there is no temporal dimension involved.

To comprehend this fact, it is better to use a geometric picture to replace the physical one, i.e., consider a series of frame pseudo-rotations ($K_0 \to K_1 \to K_2 \to K_3$) instead of a snap shot of these four frames. Even then, since the pseudo-rotation is quite different to the ordinary one, it is hard to build a clear picture in one's mind. The best policy is to find a toy model for this process which contains only ordinary rotations.

Although Wigner rotation emerges whenever the two boost velocities are not parallel, people usually let these velocities be perpendicular to each other to simplify the calculations. It will be called the simple Wigner rotation in this paper. Interpreting this simple case as a series of pseudo-rotations, we are able to build an $SO(3)$ toy model to mimic this process (Section 3). Being a model, it contains the essence of the original problem nonetheless. Thus we can use what we learn to study the simple Wigner rotation (Section 4) then generalize it to the general case (Section 5). This model may be considered as the first achievement of this paper.

As for the second achievement, by working directly on the active frame we show there is no need to consider the passive coordinate transformation at all. Allying with geometric algebra, the active-frame formalism enables us to derive all the important results of Wigner rotation and condense them into three neat geometric theorems.

2. Preliminaries

2.1. Active frame

In the two-dimensional Euclidean space $\mathbb{R}^2$, the position vector of a point is $x\hat{x}+y\hat{y} = (\hat{x}\,\hat{y})(x\,y)^\top$, where (x, y) are the coordinates of that point, $\top$ is the notation for matrix transpose, and $(\hat{x}, \hat{y})$ is the frame which may be taken as a set of orthonormal bases at the origin. Now consider a passive linear transformation in this space, if the transformation law for the coordinates is $(x'\,y')^\top = [T](x\,y)^\top$, where $[T]$ is the matrix representation of the transformation T, then the frame transformation must obey $(\hat{x}'\,\hat{y}') = (\hat{x}\,\hat{y})[T]^{-1}$ to balance the change made by $[T]$ and render the position vector intact.

When the transformation is a two-dimensional rotation, i.e., an element of the special orthogonal group $SO(2)$, we have $[T]^{-1} = [T]^\top$ so that the transformations for the coordinates and frames are formally the same, i.e.,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = [T]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

and

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \end{pmatrix} = [T]\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}. \tag{1}$$

For a positive ω, (1) represents a counterclockwise (i.e., positive-sense) rotation of the frame $(\hat{x}, \hat{y})$. We shall regard this rotation as taking place "along" the xy-plane around $(0, 0)$ instead of some axis, the reason will be clear soon.

If the coordinates (x, y) are replaced by (x, ct), where c is the light speed and t is the time coordinate, the space corresponding to the new coordinates is usually called the two-dimensional Minkowski space $\mathbb{R}^{1,1}$. It differs from the two-dimensional Euclidean space $\mathbb{R}^2$ in the following aspects.

1. Although the second coordinate is ct with the dimension of length, the corresponding basis is $\hat{t}$ which is a dimensionless quantity like $\hat{x}$ and $\hat{y}$.

2. There is no ordinary rotation in $\mathbb{R}^{1,1}$, instead we have the pseudo-rotation (or more precisely the hyperbolic rotation) defined by

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\Omega & -\sinh\Omega \\ -\sinh\Omega & \cosh\Omega \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \tag{2}$$

which leaves $x^2 - c^2t^2$ unchanged. We may regard it as taking place along the xt-plane just like ordinary rotations along a Euclidean plane.

3. The transformation matrix in (2) belongs to the group $SO^+(1,1)$ whose elements are those reflection-free pseudo-orthogonal transformations in $\mathbb{R}^{1,1}$. It is symmetric but not orthogonal unless $\Omega = 0$.

4. With regard to the coordinate transformation (2), the transformation law for the Minkowski frame is

$$\begin{pmatrix} \hat{x}' \\ \hat{t}' \end{pmatrix} = \begin{pmatrix} \cosh\Omega & \sinh\Omega \\ \sinh\Omega & \cosh\Omega \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{t} \end{pmatrix}. \tag{3}$$

It is obvious that, as long as $\Omega \neq 0$, the new bases will no longer have unit length and no longer be perpendicular to each other from the Euclidean point of view (Figure 1).

From the physical perspective, there are two more noteworthy points.

5. If the frame $(\hat{x}, \hat{t})$ is interpreted as a physical reference frame, then the spatial basis $\hat{x}$ is usually thought of as a rigid rule and the temporal basis $\hat{t}$ a set of clocks fixed on the rule. When talking about a moving frame, we mean the rule carries those clocks moving along the x-direction.

6. If we interpret the hyperbolic angle Ω as the rapidity of the relative speed u between the two frames, i.e., $u = c\tanh\Omega$, then (2) and (3) become the boost transformations of the spacetime coordinates and the frames respectively. Note that the rigid rules corresponding to the spatial bases $\hat{x}$ and $\hat{x}'$ are still parallel to each other from the one-dimensional point of view.

2.2. $SO^+(2,1)$ and $SO(3)$ groups

Although physical spacetime is the four-dimensional Minkowski space $\mathbb{R}^{3,1}$, we work on its subspace in many cases without losing generality. For example, when discussing a boost transformation between two frames $(\hat{x}, \hat{y}, \hat{z}, \hat{t})$ and $(\hat{x}', \hat{y}', \hat{z}', \hat{t}')$, we may assume the relative velocity is along the x-direction and consider just the transformation between the frames $(\hat{x}, \hat{t})$ and $(\hat{x}', \hat{t}')$,

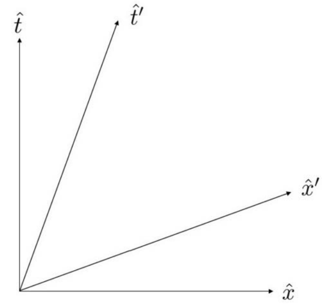

Figure 1: Hyperbolic rotation of the Minkowski frame $(\hat{x}, \hat{t})$.

which is depicted by (3). Similarly, since the problem of Wigner rotation involves only two relative velocities, it is legitimate to put them in the xy-plane so that none of the z-components shows up in the calculations. Therefore the space $\mathbb{R}^{2,1}$ and the transformation group $SO^+(2,1)$ are sufficient for us to derive all of the related results.

From the point of view of group theory, there are many similarities between $SO^+(2,1)$ and $SO(3)$. For example, neither of them contains any kind of reflection, and the invariants of these two groups are $x^2 + y^2 - c^2t^2$ and $x^2 + y^2 + z^2$ respectively. This allows us to model the Wigner rotation problem with a series of rotations in $\mathbb{R}^3$ which provides a concrete and clear picture.

2.3. Geometric algebra

Wigner rotation, like many other problems in special relativity, is usually studied by using vectors and matrices as the mathematical tools. However, there is an alternative choice named geometric algebra (also known as Clifford algebra) which might be more suitable. Putting it simply, geometric algebra is nothing but the traditional vector algebra plus a new operation, the so-called geometric product.

For the four-dimensional Minkowski space $\mathbb{R}^{3,1}$ with a set of orthonormal bases $(\hat{x}, \hat{y}, \hat{z}, \hat{t})$, the geometric product is defined as below.

1. $\hat{x}\hat{x} = \hat{y}\hat{y} = \hat{z}\hat{z} = 1$ and $\hat{t}\hat{t} = -1$. The first three correspond to the unit length of those bases and the fourth one reflects the "Minkowskiness" of $\mathbb{R}^{3,1}$.

2. The geometric product is associative, e.g., $(\hat{x}\hat{y})\hat{z} = \hat{x}(\hat{y}\hat{z}) = \hat{x}\hat{y}\hat{z}$, etc.

3. The geometric product of different bases obeys the anti-commutative relation, e.g., $\hat{x}\hat{y} = -\hat{y}\hat{x}$, $\hat{z}\hat{t} = -\hat{t}\hat{z}$, etc.

4. The dagger conjugation changes the ordering in the products, e.g., $(\hat{x}\hat{y})^\dagger = \hat{y}\hat{x}$, $(\hat{x}\hat{y}\hat{t})^\dagger = \hat{t}\hat{y}\hat{x}$, etc.

With the aid of geometric algebra, the three-dimensional version of (1) can be expressed in a neat form.

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} R^\dagger, \tag{4}$$

where

$$R = \exp\left(-\frac{\omega}{2}\hat{x}\hat{y}\right)$$

is called rotor. Note that $\hat{z}$ behaves like a constant because $(\hat{x}\hat{y})\hat{z} = \hat{z}(\hat{x}\hat{y})$.

Using the identity $(\hat{x}\hat{y})(\hat{x}\hat{y}) = -1$, R can be expanded as $\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\hat{x}\hat{y}$ and its action to the bases can be calculated easily. For example,

$$\hat{x}' = R\hat{x}R^\dagger = \left(\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\hat{x}\hat{y}\right)\hat{x}\left(\cos\frac{\omega}{2} + \sin\frac{\omega}{2}\hat{x}\hat{y}\right) = \cos\omega\hat{x} + \sin\omega\hat{y}.$$

If we replace ω with $\omega + 2\pi$, $R = \exp(-\frac{\omega}{2}\hat{x}\hat{y})$ becomes $-R$ but the rotation transformation (4) is not affected at all. Therefore we adopt the identification $R \equiv -R$.

The rotor R introduced above can be derived rigorously from the so-called Cartan-Dieudonné theorem [6, 10]. Here we start with the semi-finished result $R = C(\hat{x} + \hat{x}')\hat{x} = C(\hat{y} + \hat{y}')\hat{y}$, where $\hat{x}'$ and $\hat{y}'$ are given by (1) and C is the real normalization constant in order that $RR^\dagger = 1$. The term $(\hat{x} + \hat{x}')$ in $C(\hat{x} + \hat{x}')\hat{x}$ may be interpreted metaphorically as "halfway between $\hat{x}$ and $\hat{x}'$" and $\hat{x}$ on its right as "initial position". This combination may be taken as a general rule.

By using the explicit expression of $\hat{x}'$ and some trigonometric identities, it is straightforward to derive $C(\hat{x} + \hat{x}')\hat{x} = \cos\frac{\omega}{2} - \sin\frac{\omega}{2}\hat{x}\hat{y}$. Obviously $C(\hat{y} + \hat{y}')\hat{y}$ can generate the same result.

The rotors for the hyperbolic rotations in $\mathbb{R}^{2,1}$ are similar to those in $\mathbb{R}^3$. For example, when the rotation is along the xt-plane, the corresponding rotor B takes the form

$$B = \exp\left(-\frac{\Omega}{2}\hat{x}\hat{t}\right),$$

and the basis $\hat{x}$ is transformed as

$$B\hat{x}B^\dagger = \left(\cosh\frac{\Omega}{2} - \sinh\frac{\Omega}{2}\hat{x}\hat{t}\right)\hat{x}\left(\cosh\frac{\Omega}{2} + \sinh\frac{\Omega}{2}\hat{x}\hat{t}\right) = \cosh\Omega\hat{x} + \sinh\Omega\hat{t},$$

where the hyperbolic functions come from the identity $(\hat{x}\hat{t})(\hat{x}\hat{t}) = 1$.

Analogous to (4), when we apply B to the frame $(\hat{x}, \hat{y}, \hat{t})$, the result is

$$B \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} B^{\dagger} = \begin{pmatrix} \cosh\Omega & 0 & \sinh\Omega \\ 0 & 1 & 0 \\ \sinh\Omega & 0 & \cosh\Omega \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} =: \begin{pmatrix} \hat{x}^* \\ \hat{y}^* \\ \hat{t}^* \end{pmatrix}.$$

To derive the hyperbolic rotors, we need to use the Minkowski version of Cartan-Dieudonné theorem. Starting with $B = C'(\hat{x} + \hat{x}^*)\hat{x} = -C'(\hat{t} + \hat{t}^*)\hat{t}$ with the condition $BB^{\dagger} = 1$, the rest is similar to that of the rotors in $\mathbb{R}^3$.

2.4. Composition of velocities

In special relativity, a boost transformation takes place between two inertial frames, hence each boost is defined by a constant velocity which is the relative velocity between the frames. Since Wigner rotation involves two successive boosts, it is inherently related to the problem of adding two velocities relativistically. We give a brief review of this problem below.

In the four-dimensional Minkowski space $\mathbb{R}^{3,1}$, when a four-velocity undergoes a passive boost $B(\vec{V})$ with $\vec{V}$ being the relative velocity between the two frames, the transformation formula $W' = [B(\vec{V})]W$ is analogous to the boost transformation of spacetime coordinates, where W and W' are the four-velocities in the old and the new frames respectively, and $[B(\vec{V})]$ is the matrix representation of $B(\vec{V})$ which can be proved to be always symmetric. Conversely, the inverse boost transformation $W = [B(\vec{V})]^{-1}W' = [B(-\vec{V})]W'$ allows us to calculate the four-velocity in the old frame from that in the new one.

Now consider an object resting in the new frame, since its three-velocity relative to the old frame equals the relative velocity between the two frames, transforming its four-velocity in the new frame back to that in the old one reveals the information of the boost velocity.

$$W = [B(\vec{V})]^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix} = \gamma(\vec{V}) \begin{pmatrix} V_x \\ V_y \\ V_z \\ c \end{pmatrix}, \text{ where } \gamma(\vec{V}) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}. \quad (5)$$

The above result has an intriguing geometric representation in the active-frame formalism. Given a boost transformation $B(\vec{V})$, the temporal basis $\hat{t}'$

of the new frame is proportional to the four-velocity of the boost velocity.

$$(6)\qquad \hat{t}' = \begin{pmatrix}\hat{x}' & \hat{y}' & \hat{z}' & \hat{t}'\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\hat{x} & \hat{y} & \hat{z} & \hat{t}\end{pmatrix}[B(\vec{V})]^{-1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
$$= \frac{\gamma(\vec{V})}{c}(\vec{V} + c\hat{t}).$$

When the problem involves three inertial frames and two successive boosts, say first $B(\vec{V}_1)$ then $B(\vec{V}_2)$, the four-velocity of a rest object in the third frame can be transformed to that in the first frame by

$$W = \Big([B(\vec{V}_2)][B(\vec{V}_1)]\Big)^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix} = [B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}.$$

The three-velocity contained in this four-velocity is the composition of the two boost velocities in that order and is usually denoted by $\vec{V}_1 \oplus \vec{V}_2$, therefore the above formula is equivalent to

$$(7)\qquad [B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix} = [B(\vec{V}_1 \oplus \vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}$$
$$= \gamma(\vec{V}_1 \oplus \vec{V}_2)\begin{pmatrix}(\vec{V}_1 \oplus \vec{V}_2)_x\\(\vec{V}_1 \oplus \vec{V}_2)_y\\(\vec{V}_1 \oplus \vec{V}_2)_z\\c\end{pmatrix}.$$

The explicit expression of $\vec{V}_1 \oplus \vec{V}_2$ can always be extracted from (7). For example, when $\vec{V}_1 = c\tanh\Omega_1\hat{x}$ and $\vec{V}_2 = c\tanh\Omega_2\hat{x}$, (7) becomes

$$\begin{pmatrix}\cosh\Omega_1 & 0 & 0 & \sinh\Omega_1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\\sinh\Omega_1 & 0 & 0 & \cosh\Omega_1\end{pmatrix}\begin{pmatrix}\cosh\Omega_2 & 0 & 0 & \sinh\Omega_2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\\sinh\Omega_2 & 0 & 0 & \cosh\Omega_2\end{pmatrix}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}$$

$$= \cosh(\Omega_1 + \Omega_2) \begin{pmatrix} c \tanh(\Omega_1 + \Omega_2) \\ 0 \\ 0 \\ c \end{pmatrix},$$

which yields the result

$$\vec{V}_1 \oplus \vec{V}_2 = c \tanh(\Omega_1 + \Omega_2)\hat{x} = \frac{V_1 + V_2}{1 + V_1 V_2/c^2}\hat{x},$$

where $V_1 V_2/c^2$ in the denominator is the relativistic correction term. If $\vec{V}_1$ and $\vec{V}_2$ are not parallel, the expression of $\vec{V}_1 \oplus \vec{V}_2$ is rather complicated [11], but we still have $\vec{V}_1 \oplus \vec{V}_2 = \vec{V}_1 + \vec{V}_2$ in the classical limit $c \to \infty$.

Substituting $\vec{V}_1 \oplus \vec{V}_2$ for $\vec{V}$, we can rewrite (6) as

$$\text{(8)} \qquad \hat{t}'' = \frac{\gamma(\vec{V}_1 \oplus \vec{V}_2)}{c}[\vec{V}_1 \oplus \vec{V}_2 + c\hat{t}\,], \text{ or } c\hat{t}'' = \gamma(\vec{V}_1 \oplus \vec{V}_2)[\vec{V}_1 \oplus \vec{V}_2 + c\hat{t}\,].$$

In short, $c\hat{t}''$ equals the four-velocity which corresponds to the three-velocity $\vec{V}_1 \oplus \vec{V}_2$ that defines the composite boost.

If the order of the two boosts is exchanged, then (7) changes to

$$[B(\vec{V}_2)]^{-1}[B(\vec{V}_1)]^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix} = [B(\vec{V}_2 \oplus \vec{V}_1)]^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}$$

$$= \gamma(\vec{V}_2 \oplus \vec{V}_1) \begin{pmatrix} (\vec{V}_2 \oplus \vec{V}_1)_x \\ (\vec{V}_2 \oplus \vec{V}_1)_y \\ (\vec{V}_2 \oplus \vec{V}_1)_z \\ c \end{pmatrix}.$$

An important identity $\gamma(\vec{V}_1 \oplus \vec{V}_2) = \gamma(\vec{V}_2 \oplus \vec{V}_1)$ can be proved by using the following two facts: (i) $[B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}$ and $[B(\vec{V}_2)]^{-1}[B(\vec{V}_1)]^{-1}$ share the same diagonal elements since boost matrices are all symmetric, and (ii) $\gamma(\vec{V}_1 \oplus \vec{V}_2)$ equals the $(4,4)$ element of $[B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}$ and $\gamma(\vec{V}_2 \oplus \vec{V}_1)$ equals that of $[B(\vec{V}_2)]^{-1}[B(\vec{V}_1)]^{-1}$.

3. The toy model

3.1. *process a* and Theorem 1

To construct the $SO(3)$ toy model of the simple Wigner rotation, first we define a series of rotations of a Euclidean frame $(\hat{x}, \hat{y}, \hat{z})$ as below and name it *process a*.

Step 1. Rotate the frame along the zx-plane by an angle θ, and call the new frame $(\hat{x}'_a, \hat{y}'_a, \hat{z}'_a)$, where $\hat{y}'_a = \hat{y}$.

Step 2. Rotate the new frame along the new yz-plane by an angle ϕ, and call the newer frame $(\hat{x}''_a, \hat{y}''_a, \hat{z}''_a)$, where $\hat{x}''_a = \hat{x}'_a$.

Step 3. Rotate the newer frame along the plane spanned by $\hat{z}$ and $\hat{z}''_a$ to the extent that the final $\hat{z}'''_a$ coincides with the original $\hat{z}$.

Using the formulation of geometric algebra, these three rotations can be expressed as follows:

$$a1. \begin{pmatrix} \hat{x}'_a \\ \hat{y}'_a \\ \hat{z}'_a \end{pmatrix} = \exp(-\frac{\theta}{2}\hat{z}\hat{x}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \exp(-\frac{\theta}{2}\hat{z}\hat{x})^{\dagger}.$$

$$\begin{aligned} a2. \begin{pmatrix} \hat{x}''_a \\ \hat{y}''_a \\ \hat{z}''_a \end{pmatrix} &= \exp(-\frac{\phi}{2}\hat{y}'_a\hat{z}'_a) \begin{pmatrix} \hat{x}'_a \\ \hat{y}'_a \\ \hat{z}'_a \end{pmatrix} \exp(-\frac{\phi}{2}\hat{y}'_a\hat{z}'_a)^{\dagger} \\ &= \exp(-\frac{\theta}{2}\hat{z}\hat{x}) \exp(-\frac{\phi}{2}\hat{y}\hat{z}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \exp(-\frac{\phi}{2}\hat{y}\hat{z})^{\dagger} \exp(-\frac{\theta}{2}\hat{z}\hat{x})^{\dagger} \\ &= XY \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} Y^{\dagger}X^{\dagger}, \end{aligned}$$

where $X = \exp(-\frac{\theta}{2}\hat{z}\hat{x})$ and $Y = \exp(-\frac{\phi}{2}\hat{y}\hat{z})$, and the following equalities have been used.

$$\exp(-\frac{\theta}{2}\hat{z}\hat{x})^{\dagger} \exp(-\frac{\theta}{2}\hat{z}\hat{x}) = \exp(+\frac{\theta}{2}\hat{z}\hat{x}) \exp(-\frac{\theta}{2}\hat{z}\hat{x}) = 1;$$

$$\hat{y}'_a\hat{z}'_a = \exp(-\frac{\theta}{2}\hat{z}\hat{x})\hat{y}\hat{z} \exp(-\frac{\theta}{2}\hat{z}\hat{x})^{\dagger};$$

$$\exp(-\frac{\phi}{2}\hat{y}'_a\hat{z}'_a) = \exp(-\frac{\theta}{2}\hat{z}\hat{x}) \exp(-\frac{\phi}{2}\hat{y}\hat{z}) \exp(-\frac{\theta}{2}\hat{z}\hat{x})^{\dagger}.$$

$$a3.\ \begin{pmatrix}\hat{x}_a'''\\ \hat{y}_a'''\\ \hat{z}_a'''\end{pmatrix} = M \begin{pmatrix}\hat{x}_a''\\ \hat{y}_a''\\ \hat{z}_a''\end{pmatrix} M^\dagger = MXY \begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix} Y^\dagger X^\dagger M^\dagger,$$

where M is constructed by employing the Cartan-Dieudonné theorem,

$$M = C(\hat{z}_a'' + \hat{z})\hat{z}_a'' = C(1 + \hat{z}\hat{z}_a'') = C(1 + X^\dagger Y^{\dagger 2} X^\dagger) \tag{9}$$

with the condition $MM^\dagger = 1$. Note that

$$\hat{z}XY = \hat{z}\exp(-\frac{\theta}{2}\hat{z}\hat{x})\exp(-\frac{\phi}{2}\hat{y}\hat{z}) = \exp(+\frac{\theta}{2}\hat{z}\hat{x})\exp(+\frac{\phi}{2}\hat{y}\hat{z})\hat{z} = X^\dagger Y^\dagger \hat{z}$$

has been used to obtain (9).

In the light of (4), the expressions of the first two steps can be easily transformed to matrix forms. In contrast, the elegance of (9) will be lost if we use matrix formulation to replace geometric algebra.

Now we are equipped to prove that the result of *process a* is generated by a rotor along the original xy-plane.

Theorem 1. $MXY = \exp(\frac{\epsilon}{2}\hat{x}\hat{y}) =: R_{\mathbf{w}}$ *generates the toy Wigner rotation, where the toy Wigner angle* ϵ *is defined by*

$$\tan\frac{\epsilon}{2} = \tan\frac{\theta}{2}\tan\frac{\phi}{2} \text{ with } \epsilon \in (-\pi, \pi].$$

The proof is provided by some simple calculations with three notes as follows:

$$\begin{aligned} MXY &= C(1 + X^\dagger Y^{\dagger 2} X^\dagger)XY = C(XY + X^\dagger Y^\dagger) \\ &= C\left[\exp(-\frac{\theta}{2}\hat{z}\hat{x})\exp(-\frac{\phi}{2}\hat{y}\hat{z}) + \exp(\frac{\theta}{2}\hat{z}\hat{x})\exp(\frac{\phi}{2}\hat{y}\hat{z})\right] \\ &= 2C\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\hat{x}\hat{y}\right) \\ &\propto \cos\frac{\epsilon}{2} + \sin\frac{\epsilon}{2}\hat{x}\hat{y} \\ &= \exp\left(\frac{\epsilon}{2}\hat{x}\hat{y}\right). \end{aligned}$$

Note 1. A normalized M implies the product MXY is also normalized. Hence we may assert the coefficient of $\exp(\frac{\epsilon}{2}\hat{x}\hat{y})$ equals unity without doing practical calculation.

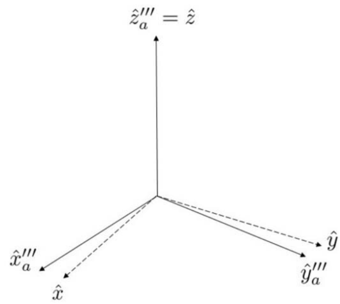

Figure 2: The frame rotates clockwise along the xy-plane by a toy Wigner angle at the end of *process a*. The result of *process b* is the same except the rotation is counterclockwise.

Note 2. Since $R_{\mathbf{w}} = \exp(-\frac{-\epsilon}{2}\hat{x}\hat{y})$, a positive ϵ corresponds to a clockwise (i.e., negative-sense) rotation according to (4). If the range of θ and ϕ is taken to be $(-\pi, \pi]$, then $\epsilon > 0$ if and only if $\theta\phi > 0$. The maximum of the toy Wigner angle corresponds to $\theta = \pi$ and $\phi \neq 0$, or $\phi = \pi$ and $\theta \neq 0$.

Note 3. This theorem tells us although the basis $\hat{z}$ comes back to its original orientation at the end of the process, the other two bases deviate from the original $(\hat{x}, \hat{y})$ by a toy Wigner rotation (Figure 2).

$$MXY \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} Y^\dagger X^\dagger M^\dagger = R_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} R_{\mathbf{w}}^\dagger = \begin{pmatrix} R_{\mathbf{w}}\hat{x}R_{\mathbf{w}}^\dagger \\ R_{\mathbf{w}}\hat{y}R_{\mathbf{w}}^\dagger \\ \hat{z} \end{pmatrix}.$$

3.2. *process b* and Theorem 2 & 3

To fully model the problem of simple Wigner rotation, we have to create another process which will be named *process b*. The main difference between these two processes is the order of the first two rotations. In *process b*, we let the frame rotate along the yz-plane by an angle ϕ first and then let the new frame rotate along the new zx-plane by an angle θ. The result of this process can be expressed as

$$\begin{pmatrix} \hat{x}_b''' \\ \hat{y}_b''' \\ \hat{z}_b''' \end{pmatrix} = N \begin{pmatrix} \hat{x}_b'' \\ \hat{y}_b'' \\ \hat{z}_b'' \end{pmatrix} N^\dagger = NYX \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} X^\dagger Y^\dagger N^\dagger,$$

where X and Y are the same as those in *process a*, and N as the counterpart of M can be constructed in a similar way,

(10) $$N = C'(\hat{z}_b'' + \hat{z})\hat{z}_b'' = C'(1 + \hat{z}\hat{z}_b'') = C'(1 + Y^\dagger X^{\dagger 2} Y^\dagger)$$

with the condition $NN^\dagger = 1$.

It is easy to prove NYX is also a rotor along the xy-plane. Moreover, it is the inverse of the rotor MXY.

Theorem 2. $NYX = (MXY)^\dagger = R_{\mathbf{w}}^\dagger = R_{\mathbf{w}}^{-1}$.

The proof is also made of a few simple calculations:

$$\begin{aligned} NYX &= C'(1 + Y^\dagger X^{\dagger 2} Y^\dagger)YX = C'(YX + Y^\dagger X^\dagger) \\ &= C' Y^\dagger X^\dagger (1 + XY^2X) = \frac{C'}{C} Y^\dagger X^\dagger M^\dagger = (MXY)^\dagger, \end{aligned}$$

where $C = C'$ comes from both MXY and NYX are normalized.

This theorem implies that, except for the sense of the rotation, the result of *process b* is the same as that of *process a*.

$$NYX \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} X^\dagger Y^\dagger N^\dagger = R_{\mathbf{w}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} R_{\mathbf{w}} = \begin{pmatrix} R_{\mathbf{w}}^\dagger \hat{x} R_{\mathbf{w}} \\ R_{\mathbf{w}}^\dagger \hat{y} R_{\mathbf{w}} \\ \hat{z} \end{pmatrix}.$$

In addition to Theorem 2, there is another interesting relation between the rotors N and M.

Theorem 3. $R_{\mathbf{w}} N R_{\mathbf{w}}^\dagger = M$.

The trick of the proof is substituting $X^\dagger Y^\dagger N^\dagger$ for $R_{\mathbf{w}}$ and NYX for $R_{\mathbf{w}}^\dagger$.

An important relation $R_{\mathbf{w}} \hat{z}_b'' R_{\mathbf{w}}^\dagger = \hat{z}_a''$ can be derived from Theorem 3 when M is expressed by $C(1+\hat{z}\hat{z}_a'')$ and N by $C(1+\hat{z}\hat{z}_b'')$. It implies the two rotations generated by M and N perform along two different planes and the N-plane can be transformed to the M-plane by the toy Wigner rotation (Figure 3).

In summary, each theorem of this toy model has a precise geometric meaning.

Theorem 1. *process a* brings about a toy Wigner rotation of the frame along the xy-plane.

Theorem 2. The only difference between the results of *process a* and *process b* is the sense of the toy Wigner rotation (clockwise vs. counter-clockwise).

Theorem 3. The third rotation planes of the two processes differ from each other by a toy Wigner rotation, hence the angle between them equals the toy Wigner angle.

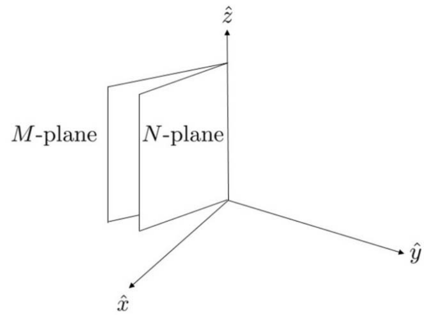

Figure 3: The third rotation plane of *process b* (N-plane) can be transformed to that of *process a* (M-plane) by the toy Wigner rotation.

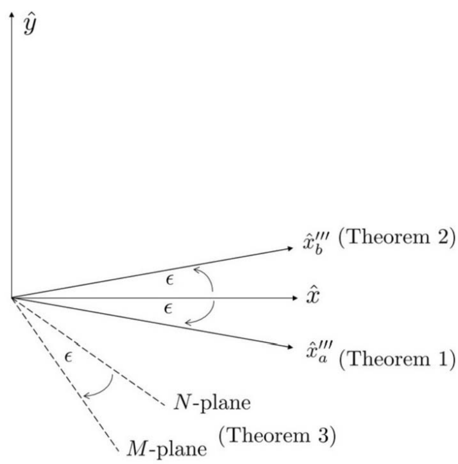

Figure 4: Schematic diagram for Theorem 1∼3 of the toy Wigner rotation.

These statements may be illustrated in one schematic diagram as shown in Figure 4.

4. Simple Wigner rotation

4.1. *process c* and Theorem I

After familiarizing ourselves with the $SO(3)$ toy model, we are ready to study the processes that yield the simple Wigner rotation. First we define *process c* which contains three hyperbolic rotations of a Minkowski frame $(\hat{x}, \hat{y}, \hat{t})$.

Step 1. Rotate the frame along the xt-plane by a hyperbolic angle Θ, and call the new frame $(\hat{x}'_c, \hat{y}'_c, \hat{t}'_c)$, where $\hat{y}'_c = \hat{y}$.

Step 2. Rotate the new frame along the new yt-plane by a hyperbolic angle Φ, and call the newer frame $(\hat{x}''_c, \hat{y}''_c, \hat{t}''_c)$, where $\hat{x}''_c = \hat{x}'_c$.

Step 3. Rotate the newer frame hyperbolically along the plane spanned by $\hat{t}$ and $\hat{t}''_c$ to the extent that the final $\hat{t}'''_c$ coincides with the original $\hat{t}$.

The expressions of these three hyperbolic rotations are similar to those of *process a*.

$$c1.\ \begin{pmatrix} \hat{x}'_c \\ \hat{y}'_c \\ \hat{t}'_c \end{pmatrix} = \exp(-\frac{\Theta}{2}\hat{x}\hat{t}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger,$$

$$\begin{aligned} c2.\ \begin{pmatrix} \hat{x}''_c \\ \hat{y}''_c \\ \hat{t}''_c \end{pmatrix} &= \exp(-\frac{\Phi}{2}\hat{y}'_c\hat{t}'_c) \begin{pmatrix} \hat{x}'_c \\ \hat{y}'_c \\ \hat{t}'_c \end{pmatrix} \exp(-\frac{\Phi}{2}\hat{y}'_c\hat{t}'_c)^\dagger \\ &= \exp(-\frac{\Theta}{2}\hat{x}\hat{t}) \exp(-\frac{\Phi}{2}\hat{y}\hat{t}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \exp(-\frac{\Phi}{2}\hat{y}\hat{t})^\dagger \exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger \\ &= \mathcal{X}\mathcal{Y} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{Y}^\dagger\mathcal{X}^\dagger, \end{aligned}$$

$$c3.\ \begin{pmatrix} \hat{x}'''_c \\ \hat{y}'''_c \\ \hat{t}'''_c \end{pmatrix} = \mathcal{M} \begin{pmatrix} \hat{x}''_c \\ \hat{y}''_c \\ \hat{t}''_c \end{pmatrix} \mathcal{M}^\dagger = \mathcal{M}\mathcal{X}\mathcal{Y} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{Y}^\dagger\mathcal{X}^\dagger\mathcal{M}^\dagger,$$

where $\mathcal{X} = \exp(-\frac{\Theta}{2}\hat{x}\hat{t}), \mathcal{Y} = \exp(-\frac{\Phi}{2}\hat{y}\hat{t}),$ and $\mathcal{M} = \mathcal{C}(1 + \mathcal{X}^\dagger\mathcal{Y}^{\dagger 2}\mathcal{X}^\dagger)$ are the analogues of X, Y, and M of *process a* respectively. The construction of $\mathcal{M}$ is analogous to (9),

$$\mathcal{M} = -\mathcal{C}(\hat{t}''_c + \hat{t})\hat{t}''_c = \mathcal{C}(1 - \hat{t}\hat{t}''_c) = \mathcal{C}(1 + \mathcal{X}^\dagger\mathcal{Y}^{\dagger 2}\mathcal{X}^\dagger) \tag{11}$$

with the condition $\mathcal{M}\mathcal{M}^\dagger = 1$.

With regard to this process, we can deduce a theorem which is analogous to Theorem 1 of the toy model.

Theorem I. *$\mathcal{MXY} = \exp(\frac{\varepsilon}{2}\hat{x}\hat{y}) =: \mathcal{R}_{\mathbf{w}}$ is the rotor of the simple Wigner rotation, where the simple Wigner angle ε is defined by*

$$\tan\frac{\varepsilon}{2} = \tanh\frac{\Theta}{2}\tanh\frac{\Phi}{2} \ \textit{with}\ \varepsilon \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

The proof is omitted owing to its resemblance to that of Theorem 1.

Similar to the toy Wigner angle ϵ, the simple Wigner angle ε is positive if and only if $\Theta\Phi > 0$. However, the range of the latter is half of that of the former because the range of $\tanh x$ is $(-1, 1)$ while that of $\tan x$ is $(-\infty, \infty)$.

4.2. *process d* and Theorem II & III

Imitating the procedure for constructing the toy model, we now exchange the first two rotations in *process c*, i.e., let the frame rotate along the yt-plane by a hyperbolic angle Φ first and then let the new frame rotate along the new xt-plane by a hyperbolic angle Θ. This new process will be named *process d* and its result can be expressed as

$$\begin{pmatrix} \hat{x}'''_d \\ \hat{y}'''_d \\ \hat{t}'''_d \end{pmatrix} = \mathcal{N}\begin{pmatrix} \hat{x}''_d \\ \hat{y}''_d \\ \hat{t}''_d \end{pmatrix}\mathcal{N}^\dagger = \mathcal{NYX}\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix}\mathcal{X}^\dagger\mathcal{Y}^\dagger\mathcal{N}^\dagger,$$

where $\mathcal{N}$ is the counterpart of $\mathcal{M}$,

$$\mathcal{N} = -\mathcal{C}'(\hat{t}''_d + \hat{t})\hat{t}''_d = \mathcal{C}'(1 - \hat{t}\hat{t}''_d) = \mathcal{C}'(1 + \mathcal{Y}^\dagger\mathcal{X}^{\dagger 2}\mathcal{Y}^\dagger) \tag{12}$$

with the condition $\mathcal{NN}^\dagger = 1$.

Thanks to the isomorphisms between (9) and (11), and between (10) and (12), we acquire the following two theorems by change of notations.

Theorem II. *$\mathcal{NYX} = (\mathcal{MXY})^\dagger = \mathcal{R}_{\mathbf{w}}^{-1}$.*

Theorem III. *$\mathcal{R}_{\mathbf{w}}\mathcal{N}\mathcal{R}^\dagger{}_{\mathbf{w}} = \mathcal{M}$.*

4.3. The physical meanings

To discuss the physical meanings of the processes and theorems introduced in this section, we begin with identifying the rotors $\mathcal{X}$ and $\mathcal{Y}$ with the boosts defined by the velocities $\vec{u} = c\tanh\Theta\hat{x}$ and $\vec{v} = c\tanh\Phi\hat{y}$ respectively. Under these identifications, the (2+1)-dimensional version of (8) gives us the following results.

$$c\hat{t}''_c = \gamma(\vec{u}\oplus\vec{v})[\vec{u}\oplus\vec{v} + c\hat{t}\,], \text{ and } c\hat{t}''_d = \gamma(\vec{v}\oplus\vec{u})[\vec{v}\oplus\vec{u} + c\hat{t}\,], \tag{13}$$

where $\gamma(\vec{u} \oplus \vec{v}) = \gamma(\vec{v} \oplus \vec{u})$ as proved at the end of Section 2.

Since $\hat{t}''_c = \mathcal{M}^\dagger \hat{t}'''_c \mathcal{M} = \mathcal{M}^\dagger \hat{t} \mathcal{M}$ according to Step 3 of *process c*, it implies that $\mathcal{M}^\dagger = \mathcal{M}^{-1}$ as a boost is defined by the velocity $\vec{u} \oplus \vec{v}$. Therefore the boost $\mathcal{M}$ is defined by $-(\vec{u} \oplus \vec{v})$, and for the same reason $\mathcal{N}^\dagger$ is defined by $\vec{v} \oplus \vec{u}$ and $\mathcal{N}$ by $-(\vec{v} \oplus \vec{u})$.

Now we are ready to use physical language to rephrase the two processes in this section (omitting the subscripts c and d).

process c:

Step 1. A physical reference frame $(\hat{x}', \hat{y}', \hat{t}')$ is found which moves with velocity $\vec{u}$ relative to the original frame $(\hat{x}, \hat{y}, \hat{t})$.

Step 2. Another frame $(\hat{x}'', \hat{y}'', \hat{t}'')$ is found which moves with velocity $\vec{v}$ relative to $(\hat{x}', \hat{y}', \hat{t}')$.

Step 3. The third frame $(\hat{x}''', \hat{y}''', \hat{t}''')$ is found which moves with velocity $-(\vec{u} \oplus \vec{v})$ relative to $(\hat{x}'', \hat{y}'', \hat{t}'')$.

Since $\hat{t}''' = \hat{t}$, we know from (6) there is no relative velocity between the third and the original unprimed frames. The spatial bases of these two frames differ by a simple Wigner rotation according to Theorem I.

$$\begin{pmatrix} \hat{x}''' \\ \hat{y}''' \\ \hat{t}''' \end{pmatrix} = \mathcal{M}\mathcal{X}\mathcal{Y} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{Y}^\dagger \mathcal{X}^\dagger \mathcal{M}^\dagger = \mathcal{R}_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{R}_{\mathbf{w}}^\dagger = \begin{pmatrix} \mathcal{R}_{\mathbf{w}} \hat{x} \mathcal{R}_{\mathbf{w}}^\dagger \\ \mathcal{R}_{\mathbf{w}} \hat{y} \mathcal{R}_{\mathbf{w}}^\dagger \\ \hat{t} \end{pmatrix}.$$

Integrating with Theorem III, the above relation can be expressed in the following form.

$$\mathcal{X}\mathcal{Y} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{Y}^\dagger \mathcal{X}^\dagger = \mathcal{M}^\dagger \mathcal{R}_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{R}_{\mathbf{w}}^\dagger \mathcal{M} = \mathcal{R}_{\mathbf{w}} \mathcal{N}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{N} \mathcal{R}_{\mathbf{w}}^\dagger, \tag{14}$$

which means the action of the first two boosts to the original frame is equivalent to that of a boost of velocity $\vec{u} \oplus \vec{v}$ preceded by a simple Wigner rotation, or a boost of velocity $\vec{v} \oplus \vec{u}$ followed by the same rotation.

process d:

Step 1. A physical reference frame $(\hat{x}', \hat{y}', \hat{t}')$ is found which moves with velocity $\hat{v}$ relative to the original frame $(\hat{x}, \hat{y}, \hat{t})$.

Step 2. Another frame $(\hat{x}'', \hat{y}'', \hat{t}'')$ is found which moves with velocity $\vec{u}$ relative to $(\hat{x}', \hat{y}', \hat{t}')$.

Step 3. The third frame $(\hat{x}''', \hat{y}''', \hat{t}''')$ is found which moves with velocity $-(\vec{v} \oplus \vec{u})$ relative to $(\hat{x}'', \hat{y}'', \hat{t}'')$.

Similarly, according to Theorem II and Theorem III, we can conclude that (i) the spatial bases of the third and the original frames differ by an inverse simple Wigner rotation,

$$\begin{pmatrix} \hat{x}''' \\ \hat{y}''' \\ \hat{t}''' \end{pmatrix} = \mathcal{N}\mathcal{Y}\mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger \mathcal{Y}^\dagger \mathcal{N}^\dagger = \mathcal{R}_{\mathbf{w}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{R}_{\mathbf{w}} = \begin{pmatrix} \mathcal{R}_{\mathbf{w}}^\dagger \hat{x} \mathcal{R}_{\mathbf{w}} \\ \mathcal{R}_{\mathbf{w}}^\dagger \hat{y} \mathcal{R}_{\mathbf{w}} \\ \hat{t} \end{pmatrix},$$

and (ii) the action of the first two boosts to the original frame is equivalent to that of a boost of velocity $\vec{v} \oplus \vec{u}$ preceded by an inverse simple Wigner rotation, or a boost of velocity $\vec{u} \oplus \vec{v}$ followed by the same rotation,

$$\mathcal{Y}\mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger \mathcal{Y}^\dagger = \mathcal{N}^\dagger \mathcal{R}_{\mathbf{w}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{R}_{\mathbf{w}} \mathcal{N} = \mathcal{R}_{\mathbf{w}}^\dagger \mathcal{M}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{M} \mathcal{R}_{\mathbf{w}}. \tag{15}$$

In addition to the above results, Theorem III leads to a relation $\mathcal{R}_{\mathbf{w}} \hat{t}_d'' \mathcal{R}_{\mathbf{w}}^\dagger = \hat{t}_c''$ which is analogous to $R_{\mathbf{w}} \hat{z}_b'' R_{\mathbf{w}}^\dagger = \hat{z}_a''$ of the toy model. We can use (13) to convert this relation to $\mathcal{R}_{\mathbf{w}}(\vec{v} \oplus \vec{u})\mathcal{R}_{\mathbf{w}}^\dagger = \vec{u} \oplus \vec{v}$ which implies the angle between these two composite velocities equals the simple Wigner angle.

In summary, each of the three theorems in this section has a precise physical meaning.

Theorem I. *process c* brings about a simple Wigner rotation of the frame along the xy-plane.

Theorem II. The only difference between the results of *process c* and *process d* is the sense of the simple Wigner rotation.

Theorem III. The composite velocities of the two processes differ from each other by a simple Wigner rotation, implying the angle between them equals the simple Wigner angle.

These statements are illustrated in one schematic diagram as shown in Figure 5.

5. General Wigner rotation

5.1. *process e* and Theorem I′

Now we release the 90° constraint on the two boost velocities, allowing the angle between them to be arbitrary. Without loss of generality, we still put the velocity $\vec{u}$ along the x-direction, while the velocity $\vec{v}$ deviates from the y-direction clockwise by an angle $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The basis $\hat{y}$ can be rotated to

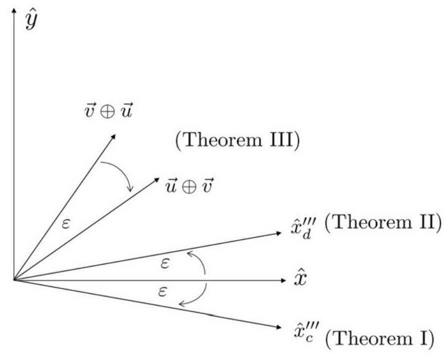

Figure 5: Schematic diagram for Theorem I∼III of the simple Wigner rotation.

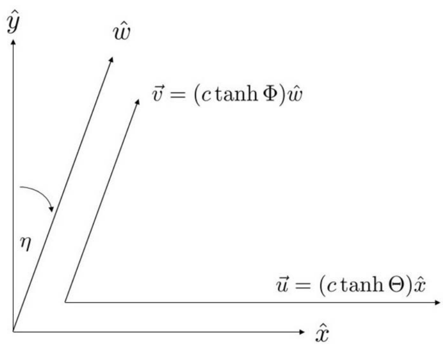

Figure 6: The orientations of velocities $\vec{u}$ and $\vec{v}$ in the problem of general Wigner rotation.

the direction of $\vec{v}$ by a rotor and the new basis will be called $\hat{w}$ (Figure 6), i.e.,

$$\hat{w} = \exp(\frac{\eta}{2}\hat{x}\hat{y})\hat{y}\exp(\frac{\eta}{2}\hat{x}\hat{y})^\dagger = \sin\eta\hat{x} + \cos\eta\hat{y}.$$

Using this new basis $\hat{w}$, we generalize *process c* to the following one which will be named *process e*.

Step 1. Rotate the frame along the xt-plane by a hyperbolic angle Θ, and call the new frame $(\hat{x}_e', \hat{y}_e', \hat{t}_e')$, where $\hat{y}_e' = \hat{y}$. Accordingly $\hat{w}$ is transformed to $\hat{w}_e'$.

Step 2. Rotate the new frame along the new wt-plane by a hyperbolic angle Φ, and call the newer frame $(\hat{x}_e'', \hat{y}_e'', \hat{t}_e'')$. Note that $\hat{x}_e'' \neq \hat{x}_e'$.

Step 3. Rotate the newer frame hyperbolically along the plane spanned by $\hat{t}$ and $\hat{t}''_e$ to the extent that the final $\hat{t}'''_e$ coincides with the original $\hat{t}$.

The expressions for this process are as follows:

$$e1.\ \begin{pmatrix} \hat{x}'_e \\ \hat{y}'_e \\ \hat{t}'_e \\ \hat{w}'_e \end{pmatrix} = \exp(-\frac{\Theta}{2}\hat{x}\hat{t}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \\ \hat{w} \end{pmatrix} \exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger,$$

$$\begin{aligned} e2.\ \begin{pmatrix} \hat{x}''_e \\ \hat{y}''_e \\ \hat{t}''_e \end{pmatrix} &= \exp(-\frac{\Phi}{2}\hat{w}'_e\hat{t}'_e) \begin{pmatrix} \hat{x}'_e \\ \hat{y}'_e \\ \hat{t}'_e \end{pmatrix} \exp(-\frac{\Phi}{2}\hat{w}'_e\hat{t}'_e)^\dagger \\ &= \exp(-\frac{\Theta}{2}\hat{x}\hat{t}) \exp(-\frac{\Phi}{2}\hat{w}\hat{t}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \exp(-\frac{\Phi}{2}\hat{w}\hat{t})^\dagger \exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger \\ &= \mathcal{X}\mathcal{W} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{W}^\dagger \mathcal{X}^\dagger, \end{aligned}$$

$$e3.\ \begin{pmatrix} \hat{x}'''_e \\ \hat{y}'''_e \\ \hat{t}'''_e \end{pmatrix} = \boldsymbol{\mathcal{M}} \begin{pmatrix} \hat{x}''_e \\ \hat{y}''_e \\ \hat{t}''_e \end{pmatrix} \boldsymbol{\mathcal{M}}^\dagger = \boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{W}^\dagger \mathcal{X}^\dagger \boldsymbol{\mathcal{M}}^\dagger,$$

where $\mathcal{X} = \exp(-\frac{\Theta}{2}\hat{x}\hat{t})$, $\mathcal{W} = \exp(-\frac{\Phi}{2}\hat{w}\hat{t})$, and

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{C}}(1 - \hat{t}\hat{t}''_e) = \boldsymbol{\mathcal{C}}(1 + \mathcal{X}^\dagger \mathcal{W}^{\dagger 2} \mathcal{X}^\dagger) \text{ with } \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}^\dagger = 1.$$

Now we can generalize Theorem I of the simple Wigner rotation to the general case.

Theorem I′.

$$\boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} = \exp\left(\frac{\boldsymbol{\varepsilon}}{2}\hat{x}\hat{y}\right) =: \boldsymbol{\mathcal{R}}_{\mathbf{w}},$$

$$\textit{where } \tan\frac{\boldsymbol{\varepsilon}}{2} = \frac{\cos\eta}{\coth\frac{\Theta}{2}\coth\frac{\Phi}{2} + \sin\eta} \quad \textit{with} \quad \boldsymbol{\varepsilon} \in (-\frac{\pi}{2} - \eta, \frac{\pi}{2} - \eta).$$

The proof is contained in the following calculations:

$$\boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} = \boldsymbol{\mathcal{C}}(\mathcal{X}\mathcal{W} + \mathcal{X}^\dagger\mathcal{W}^\dagger)$$

$$
\begin{aligned}
&= \boldsymbol{\mathcal{C}}\left[\exp(-\frac{\Theta}{2}\hat{x}\hat{t})\exp(-\frac{\Phi}{2}\hat{w}\hat{t}) + \exp(\frac{\Theta}{2}\hat{x}\hat{t})\exp(\frac{\Phi}{2}\hat{w}\hat{t})\right] \\
&= 2\boldsymbol{\mathcal{C}}\left(\cosh\frac{\Theta}{2}\cosh\frac{\Phi}{2} + \sinh\frac{\Theta}{2}\sinh\frac{\Phi}{2}\hat{x}\hat{w}\right) \\
&= 2\boldsymbol{\mathcal{C}}\left[\left(\cosh\frac{\Theta}{2}\cosh\frac{\Phi}{2} + \sin\eta\sinh\frac{\Theta}{2}\sinh\frac{\Phi}{2}\right) + \cos\eta\sinh\frac{\Theta}{2}\sinh\frac{\Phi}{2}\hat{x}\hat{y}\right] \\
&= \cos\frac{\varepsilon}{2} + \sin\frac{\varepsilon}{2}\hat{x}\hat{y}.
\end{aligned}
$$

5.2. *process f* and Theorem II′ & III′

Just like *process e* is a generalization of *process c*, *process f* is generalized from *process d*.

$$
\begin{pmatrix} \hat{x}_f''' \\ \hat{y}_f''' \\ \hat{t}_f''' \end{pmatrix} = \boldsymbol{\mathcal{N}} \begin{pmatrix} \hat{x}_f'' \\ \hat{y}_f'' \\ \hat{t}_f'' \end{pmatrix} \boldsymbol{\mathcal{N}}^\dagger = \boldsymbol{\mathcal{N}}\mathcal{W}\mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger\mathcal{W}^\dagger\boldsymbol{\mathcal{N}}^\dagger,
$$

where $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{C}}'(1 - \hat{t}\hat{t}_f'') = \boldsymbol{\mathcal{C}}'(1 + \mathcal{W}^\dagger\mathcal{X}^{\dagger 2}\mathcal{W}^\dagger)$ with $\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}^\dagger = 1$.

Because the expressions of the rotors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ are respectively isomorphic to those of $\mathcal{M}$ and $\mathcal{N}$ of the simple Wigner rotation, Theorem II and Theorem III are generalized to the following.

Theorem II′. $\boldsymbol{\mathcal{N}}\mathcal{W}\mathcal{X} = (\boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W})^\dagger = \boldsymbol{\mathcal{R}}_{\mathbf{w}}^{-1}$.

Theorem III′. $\boldsymbol{\mathcal{R}}_{\mathbf{w}}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger = \boldsymbol{\mathcal{M}}$.

In summary, except for the fact that the formula for the Wigner angle is more complicated, the related theorems and their physical meanings have no significant change for the general Wigner rotation. For example, the analogue of (14) is

$$
\mathcal{X}\mathcal{W} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{W}^\dagger\mathcal{X}^\dagger = \boldsymbol{\mathcal{M}}^\dagger\boldsymbol{\mathcal{R}}_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{R}}_{\mathbf{w}}\boldsymbol{\mathcal{N}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger, \tag{16}
$$

and that of (15) is

$$
\mathcal{W}\mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger\mathcal{W}^\dagger = \boldsymbol{\mathcal{N}}^\dagger\boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{R}}_{\mathbf{w}}\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger\boldsymbol{\mathcal{M}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{R}}_{\mathbf{w}}. \tag{17}
$$

We can take either (16) or (17) as the mathematical manifestation of the claim "two successive non-parallel boosts lead to a boost and a rotation" in Section 1.

6. Conclusion

By the joint effort of group theory, geometric algebra, and the active-frame formalism, the geometry of Wigner rotation problem is clarified and the mathematics is simplified. Among other things, the $SO(3)$ toy model provides an easy way to comprehend the essence of this problem.

Appendix: comparison with matrix-coordinate formalism

The results in this paper are mainly obtained and expressed by geometric algebra. In order to make comparisons with those in other literature, there is a need to translate these results into matrix formulation.

First, we work on an example from *process c* as a demonstration,

$$\begin{aligned}
&\exp(-\frac{\Theta}{2}\hat{x}\hat{t})\exp(-\frac{\Phi}{2}\hat{y}\hat{t})\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\exp(-\frac{\Phi}{2}\hat{y}\hat{t})^{\dagger}\exp(-\frac{\Theta}{2}\hat{x}\hat{t})^{\dagger}\\
&=\exp(-\frac{\Theta}{2}\hat{x}\hat{t})\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\Phi & \sinh\Phi\\ 0 & \sinh\Phi & \cosh\Phi\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\exp(-\frac{\Theta}{2}\hat{x}\hat{t})^{\dagger},\\
&=\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\Phi & \sinh\Phi\\ 0 & \sinh\Phi & \cosh\Phi\end{pmatrix}\begin{pmatrix}\cosh\Theta & 0 & \sinh\Theta\\ 0 & 1 & 0\\ \sinh\Theta & 0 & \cosh\Theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}.
\end{aligned}$$

Using the aforementioned notations $\mathcal{X}$ and $\mathcal{Y}$ for the rotors, and denoting the corresponding matrices by $[\mathcal{X}]$ and $[\mathcal{Y}]$, the above equality becomes

$$\mathcal{X}\mathcal{Y}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{Y}^{\dagger}\mathcal{X}^{\dagger}=[\mathcal{Y}][\mathcal{X}]\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}.$$

The correspondence between these two formulations may be put formally as $\mathcal{X}\mathcal{Y} \iff [\mathcal{Y}][\mathcal{X}]$. However, in order to make those matrices act on the coordinates (x, y, ct) instead of the frame $(\hat{x}, \hat{y}, \hat{t})$, we need to go further to take the transpose-inverse of $[\mathcal{Y}][\mathcal{X}]$. Hence in this example the correspondence between the rotor-frame formalism and matrix-coordinate formalism should be $\mathcal{X}\mathcal{Y} \iff ([\mathcal{Y}]^{\top})^{-1}([\mathcal{X}]^{\top})^{-1}$.

Now we are ready to apply this rule to the results of general Wigner rotation discussed in Section 5. Starting with Theorem I′ and bearing in mind the boost matrix is symmetric while the rotation matrix orthogonal, we have

$$\boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} = \boldsymbol{\mathcal{R}}_{\mathbf{w}} \iff [\mathcal{W}]^{-1}[\mathcal{X}]^{-1}[\boldsymbol{\mathcal{M}}]^{-1} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}].$$

It is better to introduce the notation $B(\vec{V})$ from Section 2 to unify those boost matrices. Since $B(\vec{V})$ is a passive transformation, we have $[\mathcal{W}]^{-1} = [B(\vec{v})]$, $[\mathcal{X}]^{-1} = [B(\vec{u})]$, and $[\boldsymbol{\mathcal{M}}]^{-1} = [B(-(\vec{u}\oplus\vec{v}))]$, and the above matrix equality becomes

$$[B(\vec{v})][B(\vec{u})][B(-(\vec{u}\oplus\vec{v}))] = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}],$$

or

$$[B(\vec{v})][B(\vec{u})] = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}][B(\vec{u}\oplus\vec{v})]. \tag{18}$$

Applying the same rule to Theorem II′ gives us a similar result,

$$[B(\vec{u})][B(\vec{v})] = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})].$$

As for Theorem III′, the rotor-matrix correspondence is

$$\boldsymbol{\mathcal{R}}_{\mathbf{w}}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{R}}_{\mathbf{w}}^{\dagger} = \boldsymbol{\mathcal{M}} \iff$$
$$[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[\boldsymbol{\mathcal{N}}]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}] = [\boldsymbol{\mathcal{M}}]^{-1},$$

or

$$[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(-(\vec{v}\oplus\vec{u}))][\boldsymbol{\mathcal{R}}_{\mathbf{w}}] = [B(-(\vec{u}\oplus\vec{v}))],$$

or

$$[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})][\boldsymbol{\mathcal{R}}_{\mathbf{w}}] = [B(\vec{u}\oplus\vec{v})]. \tag{19}$$

Integrating (18) with (19), we obtain the correspondent of (16),

$$[B(\vec{v})][B(\vec{u})] = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}][B(\vec{u}\oplus\vec{v})] = [B(\vec{v}\oplus\vec{u})][\boldsymbol{\mathcal{R}}_{\mathbf{w}}]. \tag{20}$$

The transpose of (20) gives us the correspondent of (17),

$$[B(\vec{u})][B(\vec{v})] = [B(\vec{u}\oplus\vec{v})][\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})]. \tag{21}$$

Needless to say, (20) and (21) can also be obtained by applying the correspondence rule to the rotor-frame formulas (16) and (17).

Lastly, the explicit forms of these boost matrices are provided below for reference.

$$[B(\vec{u})] = \begin{pmatrix} \cosh\Theta & 0 & -\sinh\Theta \\ 0 & 1 & 0 \\ -\sinh\Theta & 0 & \cosh\Theta \end{pmatrix}, \ [B(\vec{u})]^{-1} = \begin{pmatrix} \cosh\Theta & 0 & \sinh\Theta \\ 0 & 1 & 0 \\ \sinh\Theta & 0 & \cosh\Theta \end{pmatrix}.$$

$$[B(\vec{v})] = \begin{pmatrix} \cos^2\eta + \sin^2\eta\cosh\Phi & \sin\eta\cos\eta(\cosh\Phi - 1) & -\sin\eta\sinh\Phi \\ \sin\eta\cos\eta(\cosh\Phi - 1) & \sin^2\eta + \cos^2\eta\cosh\Phi & -\cos\eta\sinh\Phi \\ -\sin\eta\sinh\Phi & -\cos\eta\sinh\Phi & \cosh\Phi \end{pmatrix},$$

$$[B(\vec{v})]^{-1} = \begin{pmatrix} \cos^2\eta + \sin^2\eta\cosh\Phi & \sin\eta\cos\eta(\cosh\Phi - 1) & \sin\eta\sinh\Phi \\ \sin\eta\cos\eta(\cosh\Phi - 1) & \sin^2\eta + \cos^2\eta\cosh\Phi & \cos\eta\sinh\Phi \\ \sin\eta\sinh\Phi & \cos\eta\sinh\Phi & \cosh\Phi \end{pmatrix}.$$

$$[B(\vec{u}\oplus\vec{v})] = \frac{1}{P}\begin{pmatrix} P+Q^2 & QR & -PQ \\ QR & P+R^2 & -PR \\ -PQ & -PR & P^2-P \end{pmatrix},$$

$$[B(\vec{u}\oplus\vec{v})]^{-1} = \frac{1}{P}\begin{pmatrix} P+Q^2 & QR & PQ \\ QR & P+R^2 & PR \\ PQ & PR & P^2-P \end{pmatrix},$$

$$\begin{aligned} \text{where } P &= 1 + \cosh\Theta\cosh\Phi + \sin\eta\sinh\Theta\sinh\Phi, \\ Q &= \sinh\Theta\cosh\Phi + \sin\eta\cosh\Theta\sinh\Phi, \\ R &= \cos\eta\sinh\Phi. \end{aligned}$$

According to (5), $\gamma(\vec{u}\oplus\vec{v})$ and $\vec{u}\oplus\vec{v}$ can be read out from the last column of $[B(\vec{u}\oplus\vec{v})]^{-1}$, i.e., $\gamma(\vec{u}\oplus\vec{v}) = P-1$ and $\vec{u}\oplus\vec{v} = \frac{c}{P-1}[Q,R]$.

With the following modifications for the expressions of Q and R, the two matrices above can represent $[B(\vec{v}\oplus\vec{u})]$ and $[B(\vec{v}\oplus\vec{u})]^{-1}$ as well, and $\vec{v}\oplus\vec{u}$ can be extracted by using the same rule.

$$\begin{aligned} Q &= \cos^2\eta\sinh\Theta + \sin\eta\cosh\Theta\sinh\Phi + \sin^2\eta\sinh\Theta\cosh\Phi, \\ R &= \cos\eta\cosh\Theta\sinh\Phi + \sin\eta\cos\eta\sinh\Theta(\cosh\Phi - 1). \end{aligned}$$

Supplementary material

An animation of the six processes discussed in this paper is available at https://www.youtube.com/watch?v=HyVouwd7X2o

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