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Weak convergence of two-step inertial iteration for countable family of quasi-nonexpansive mappings*

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In this paper, we propose and study an iterative method with two-step inertial extrapolation to find a common fixed point of countable family of certain quasi-nonexpansive mappings in real Hilbert spaces. We obtain weak convergence analysis of the proposed method under some standard conditions. Our results unify and extend several inertial-type methods already appeared in the literature.

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1. Introduction

Suppose H is a real Hilbert space with scalar product $\langle ., . \rangle$ and induced norm $\| \cdot \|$. Given a nonlinear continuous mapping $T : H \to H$, let us denote the set of fixed points of T by

$$F(T) := \{x \in H \mid Tx = x\}.$$

Several optimization algorithms in the literature can be converted to fixed point methods for solving fixed point problems for which the underline fixed point operators are averaged quasi-nonexpansive mappings. Prominent examples for averaged quasi-nonexpansive mappings in Hilbert spaces are the projection mapping, the proximal point mapping, the resolvent operator, and several composite maps which involve at least one of these two mappings, see, e.g., [6] for more details. Several fixed point iterations have also been discussed in the literature cf. [8, 12, 13] and references therein for some relevant results in this direction.

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When T is averaged quasi-nonexpansive mapping, the fixed point iteration

$$x^{k+1} = Tx^k \tag{1}$$

has been shown to converge weakly to a fixed point of T (see, e.g., [6, Proposition 5.15]). Inertial version of fixed point iteration (1), which is a discrete version of a second order dissipative dynamical system has been studied for example, in [2, 7, 9, 10, 16, 17, 20, 22, 23, 24, 26, 31, 32] and this inertial-type iteration has can be regarded as a procedure of speeding up convergence properties of fixed point iteration (1).

In [24], Maingé studied the following method to find common fixed points of infinitely many averaged quasi-nonexpansive mappings $\{T_k\}$: $x^0, x^1 \in H$,

$$x^{k+1} = T_k(x^k + \theta_k(x^k - x^{k-1})), \tag{2}$$

where $\theta_k \in [0, 1)$. Under suitable conditions on θ_k and the operators $\{T_k\}$, Maingé [24] proved that the method (2) generates a sequence which converges weakly to an element of $S \subset F(T_k)$. In [31], fixed point iteration (2) with relaxation parameter was proposed and weak convergence results obtained in real Hilbert spaces. The results in [31] are unifications of several versions of fixed point iteration (1) for averaged quasi-nonexpansive mappings.

Advantages of two-step inertial extrapolation. Poon and Liang [28, 29] gave some limitations of fixed point iteration (2) in the case when $T_k = T := \frac{1}{2}\Big(I + (2P_{H_1} - I)(2P_{H_2} - I)\Big)$, where $P_{H_i}, i = 1, 2$ is a projection onto H_i. Let consider the following feasibility problem in $\mathbb{R}^2$.

Example 1.1. *Let $H_1, H_2 \subset \mathbb{R}^2$ be two subspaces such that $H_1 \cap H_2 \neq \emptyset$. Find $x \in \mathbb{R}^2$ such that $x \in H_1 \cap H_2$.*

It was shown in [29, Section 4] that two-step inertial fixed point iteration

$$x^{k+1} = T(x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}))$$

converges faster in terms of number of iterations and CPU time than the one-step inertial fixed point iteration (2):

$$x^{k+1} = T(x^k + \theta(x^k - x^{k-1}))$$

for Example 1.1. Furthermore, it was shown using Example 1.1 that one-step inertial fixed point iteration (2) converges slower than the fixed point iteration (1). This example therefore shows that one-step inertial fixed point iteration (2) may fail to provide acceleration in the case when T is the Douglas-Rachford splitting operator. Therefore, for certain cases, the use of inertia of more than two points could be of numerical advantage. It was remark in [21, Chapter 4] that the use of more than two points x^k, x^{k-1} could provide acceleration. For example, the following two-step inertial extrapolation

$$w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \tag{3}$$

with $\theta > 0$ and $\delta < 0$ can provide acceleration. The failure of one-step inertial acceleration of ADMM was also discussed in [28, Section 3] and adaptive acceleration for ADMM was proposed instead. Polyak [27] also discussed that the multi-step inertial methods can boost the speed of optimization methods even though neither the convergence nor the rate result of such multi-step inertial methods is established in [27]. Some results on multi-step inertial methods have recently been studied in [15, 18].

Our Contributions. Motivated by the results of Maingé [24], Shehu et al. [31] and Example 1.1, our contributions in this paper are summarily given as:

- we propose a two-step inertial fixed point iteration for countable families of average quasi-nonexpansive mappings and obtain weak convergence result in real Hilbert spaces. Our results extend the results obtained in [24, 31] from one-step inertial extrapolation to two-step extrapolation step and extend the results in [1, 10, 14, 16, 17, 20, 33] to a more general average quasi-nonexpansive mappings setting;
- we give applications of our results to solve composite convex and nonconvex optimization problems and variational inequalities.

Organization of the Paper. The paper is therefore organized as follows: We give some basic results we will need later in the sequel in Section 2. We propose our method in Section 3 and give its weak convergence analysis. Section 4 gives some applications of our results to composite convex optimization problems and variational inequalities. We conclude with final remarks in Section 5.

2. Preliminaries

Here, we give some definitions and lemmas that would be used in the next sections.

Definition 2.1. Suppose C is a nonempty, closed and convex subset of H. P_C is called the *metric projection* of H onto C if, for any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| \quad \forall y \in C.$$

It is well known that P_C is a nonexpansive mapping of H onto C (see, [6]). Furthermore,

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \quad \forall x, y \in H \tag{4}$$

and

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y \rangle \geq 0 \quad \forall y \in C. \tag{5}$$

This implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \forall x \in H, \forall y \in C. \tag{6}$$

Definition 2.2. A mapping $A : H \to H$ is called monotone on H if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in H$;

and L-Lipschitz continuous on H if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in H$.

Definition 2.3. Given an operator $A : H \to H$, we say that A is η-inverse strongly monotone(η-co-coercive), if there exists $\eta > 0$ such that

$$\langle A(x) - A(y), x - y \rangle \geq \eta \|A(x) - A(y)\|^2, \forall x, y \in H. \tag{7}$$

An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0 \quad \forall x, y \in D(A), \quad u \in Ax, v \in Ay,$$

and A is maximal monotone if its graph

$$G(A) := \{(x, y) : x \in D(A), y \in Ax\}$$

is not properly contained in the graph of any other monotone operator.

Suppose that $g : H \to \mathbb{R}$ is a proper, convex and lower semicontinuous functional. The subdifferential of g at x is

$$\partial g(x) := \big\{ s \mid g(y) \geq g(x) + \langle s, y - x \rangle \ \forall y \big\}.$$

The proximity operator prox_g of g, is defined by

$$\operatorname{prox}_g(x) := \operatorname{argmin}_y \big\{ g(y) + \frac{1}{2} \|y - x\|^2 \big\}.$$

From [6], $\operatorname{prox}_g(x) = (x + \partial g(x))^{-1}, x \in H$ and ∂g is maximal monotone.

Definition 2.4. For each $k \geq 1$, let $T_k : H \to H$ be a nonlinear mapping and I the identity mapping on H. We say that $I - T_k$ is demiclosed at the origin if $\forall \{\xi_k\} \subset H, \forall \xi \in H, \xi$ is a weak cluster point of $\{\xi_k\}$ and $\xi_k - T_k \xi_n \to 0$ strongly implies $\xi \in F(T_k)$.

Definition 2.5. For each $\beta \in (0, 1)$ and $n \geq 1$, T_k is β-averaged quasi-nonexpansive mappings on H if

$$\|T_k x - x^*\|^2 \leq \|x - x^*\|^2 - \frac{1 - \beta}{\beta} \|x - T_k x\|^2, \quad \forall (x, x^*) \in H \times F(T_k).$$

Lemma 2.6. *The following identities hold for all $u, v, w \in H$:*

$$2\langle u, v \rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2.$$

Lemma 2.7. *Let $x, y, z \in H$ and $a, b \in \mathbb{R}$. Then*

$$\begin{aligned} \|(1 + a)x - (a - b)y - bz\|^2 \quad = \quad & (1 + a)\|x\|^2 - (a - b)\|y\|^2 - b\|z\|^2 \\ & + (1 + a)(a - b)\|x - y\|^2 \\ & + b(1 + a)\|x - z\|^2 \\ & - b(a - b)\|y - z\|^2. \end{aligned}$$

3. Main results

We first give the following assumptions in order to obtain our convergence analysis.

Assumption 3.1. *(a) For each $k \geq 1$, $\{T_k\}_{k=1}^{\infty}$ is a countable family of β-averaged quasi-nonexpansive mappings on H;*

(b) For each $k \geq 1$, $S \subset F(T_k) \neq \emptyset$;

(c) *$\forall \{\xi_k\} \subset H, \forall \xi \in H, \xi$ is a weak cluster point of $\{\xi_k\}$ and $\xi_k - T_k\xi_k \to 0$ strongly implies $\xi \in S$.*

Algorithm 1 Two-Step Inertial Iteration

1: Choose $\theta \in [0,1), \delta \leq 0, x^{-1}, x^0, x^1 \in H$ arbitrarily and set $k = 1$.
2: Compute

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \\ x^{k+1} = T_k w^k \end{cases} \tag{8}$$

3: Set $k \leftarrow k + 1$ and **goto 2**.

Let us assume the following conditions on the inertial factors θ and δ.

Assumption 3.2. *(i) $0 \leq \theta < \min\{\frac{1}{2}, \frac{1-\beta}{1+\beta}\}$;*
(ii) $\delta \leq 0$ such that

$$\max\Big\{-\frac{(1-\beta-\theta-\beta\theta)}{1-\beta}, \frac{\beta\theta(1+\theta)-(1-\beta)(1-\theta)^2}{1+\theta}\Big\} < \delta;$$

and

$$\beta\theta(1+\theta) - (1-\beta)(1-\theta)^2 < (2\theta - \beta + 2)\delta + (1-2\beta)\delta^2.$$

Remark 3.3. We recall some proposed related methods in the literature which are special cases of our proposed Algorithm 1.

1. Suppose we take $\theta = 0, \beta = \frac{1}{2}$ in Algorithm 1 and $T_k = T, \forall k \geq 1$. Then Algorithm 1 reduces to the methods in [14, 16, 17, 20].
2. If $\alpha = 1$ and $\theta = 0$, then Algorithm 1 reduces to the method studied in [10]. Taking $\delta = 0$ and $\theta \in [0,1)$, then Algorithm 1 reduces to the method studied in [24].
3. Take $\theta = 0 = \delta$, then Algorithm 1 reduces to the method studied in [33].
4. If $T_k = T$, $\delta = 0, \theta \in [0,1)$ and $\beta = \frac{1}{2}$ in Algorithm 1, then we have the methods studied in [1].
5. Our Algorithm 1 reduces to [25, Algorithm 1] when $\beta = \frac{1}{2}$. $\Diamond$

Lemma 3.4. *Suppose Assumptions 3.1 and Assumptions 3.2 are fulfilled. Then the sequence $\{x^k\}$ generated by Algorithm 1 is bounded.*

Proof. Let $x^* \in F(T)$. Then

$$\begin{array}{rcl} w^k & = & x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x_{k-2}) - x^* \\ & = & (1+\theta)(x^k - x^*) - (\theta - \delta)(x^{k-1} - x^*) - \delta(x^{k-2} - x^*). \end{array}$$

Consequently, we have

$$\begin{array}{rcl} \|w^k - x^*\|^2 & = & \|(1+\theta)(x^k - x^*) - (\theta - \delta)(x^{k-1} - x^*) - \delta(x^{k-2} - x^*)\|^2 \\ & = & (1+\theta)\|x^k - x^*\|^2 - (\theta - \delta)\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2 \\ & & +(1+\theta)(\theta-\delta)\|x^k - x^{k-1}\|^2 + \delta(1-\theta)\|x^k - x^{k-2}\|^2 \\ & & -\delta(\theta - \delta)\|x^{k-1} - x^{k-2}\|^2. \end{array} \tag{9}$$

Observe that

$$\begin{array}{rcl} 2\theta\langle x^{k+1} - x^k, x^k - x^{k-1}\rangle & = & 2\langle \theta(x^{k+1} - x^k), x^k - x^{k-1}\rangle \\ & \leq & 2|\theta|\|x^{k+1} - x^k\|\|x^k - x^{k-1}\| \\ & = & 2\theta\|x^{k+1} - x^k\|\|x^k - x^{k-1}\| \end{array}$$

and so

$$-2\theta\langle x^{k+1} - x^k, x^k - x^{k-1}\rangle \geq -2\theta\|x^{k+1} - x^k\|\|x^k - x^{k-1}\|. \tag{10}$$

Also,

$$\begin{array}{rcl} 2\delta\langle x^{k+1} - x^k, x^{k-1} - x^{k-2}\rangle & = & 2\langle \delta(x^{k+1} - x^k), x^{k-1} - x^{k-2}\rangle \\ & \leq & 2|\delta|\|x^{k+1} - x^k\|\|x^{k-1} - x^{k-2}\| \end{array}$$

which implies that

$$-2\delta\langle x^{k+1} - x^k, x^{k-1} - x^{k-2}\rangle \geq -2|\delta|\|x^{k+1} - x^k\|\|x^{k-1} - x^{k-2}\|. \tag{11}$$

Similarly, we note that

$$\begin{array}{rcl} 2\delta\theta\langle x^{k-1} - x^k, x^{k-1} - x^{k-2}\rangle & = & 2\langle \delta\theta(x^{k-1} - x^k), x^{k-1} - x^{k-2}\rangle \\ & \leq & 2|\delta|\theta\|x^{k-1} - x^k\|\|x^{k-1} - x^{k-2}\| \\ & = & 2|\delta|\theta\|x^k - x^{k-1}\|\|x^{k-1} - x^{k-2}\| \end{array}$$

and thus,

$$\begin{array}{rcl} 2\delta\theta\langle x^k - x^{k-1}, x^{k-1} - x^{k-2}\rangle & = & -2\delta\theta\langle x^{k-1} - x^k, x^{k-1} - x^{k-2}\rangle \\ & \geq & -2|\delta|\theta\|x^k - x^{k-1}\|\|x^{k-1} - x^{k-2}\|. \end{array} \tag{12}$$

By (10), (11) and (12), we obtain

$$
\begin{aligned}
\|x^{k+1}-w^k\|^2 &= \|x^{k+1}-(x^k+\theta(x^k-x^{k-1})+\delta(x^{k-1}-x^{k-2}))\|^2\\
&= \|x^{k+1}-x^k-\theta(x^k-x^{k-1})-\delta(x^{k-1}-x^{k-2})\|^2\\
&= \|x^{k+1}-x^k\|^2-2\theta\langle x^{k+1}-x^k,x^k-x^{k-1}\rangle\\
&\quad -2\delta\langle x^{k+1}-x^k,x^{k-1}-x^{k-2}\rangle+\theta^2\|x^k-x^{k-1}\|^2\\
&\quad +2\delta\theta\langle x^k-x^{k-1},x^{k-1}-x^{k-2}\rangle+\delta^2\|x^{k-1}-x^{k-2}\|^2\\
&\geq \|x^{k+1}-x^k\|^2-2\theta\|x^{k+1}-x^k\|\|x^k-x^{k-1}\|\\
&\quad -2|\delta|\|x^{k+1}-x^k\|\|x^{k-1}-x^{k-2}\|+\theta^2\|x^k-x^{k-1}\|^2\\
&\quad -2|\delta|\theta\|x^k-x^{k-1}\|\|x^{k-1}-x^{k-2}\|\\
&\quad +\delta^2\|x^{k-1}-x^{k-2}\|^2\\
&\geq \|x^{k+1}-x^k\|^2-\theta\|x^{k+1}-x^k\|^2-\theta\|x^k-x^{k-1}\|^2\\
&\quad -|\delta|\|x^{k+1}-x^k\|^2-|\delta|\|x^{k-1}-x^{k-2}\|^2\\
&\quad +\theta^2\|x^k-x^{k-1}\|^2-|\delta|\theta\|x^k-x^{k-1}\|^2\\
&\quad -|\delta|\theta\|x^{k-1}-x^{k-2}\|^2+\delta^2\|x^{k-1}-x^{k-2}\|^2\\
&= (1-|\delta|-\theta)\|x^{k+1}-x^k\|^2\\
&\quad +(\theta^2-\theta-|\delta|\theta)\|x^k-x^{k-1}\|^2\\
&\quad +(\delta^2-|\delta|-|\delta|\theta)\|x^{k-1}-x^{k-2}\|^2. \qquad (13)
\end{aligned}
$$

Since T is β-averaged quasi-nonexpansive, we obtain

$$
\begin{aligned}
\|x^{k+1}-x^*\|^2 &= \|Tw^k-x^*\|^2\\
&\leq \|w^k-x^*\|^2-\frac{(1-\beta)}{\beta}\|w^k-Tw^k\|^2. \qquad (14)
\end{aligned}
$$

Using (9) and (13) in (14), we get

$$
\begin{aligned}
\|x^{k+1}-x^*\|^2 &\leq (1+\theta)\|x^k-x^*\|^2-(\theta-\delta)\|x^{k-1}-x^*\|^2\\
&\quad -\delta\|x^{k-2}-x^*\|^2\\
&\quad +(1+\theta)(\theta-\delta)\|x^k-x^{k-1}\|^2+\delta(1+\theta)\|x^k-x^{k-2}\|^2\\
&\quad -\delta(\theta-\delta)\|x^{k-1}-x^{k-2}\|^2\\
&\quad -\frac{(1-\beta)}{\beta}(1-|\delta|-\theta)\|x^{k+1}-x^k\|^2\\
&\quad -\frac{(1-\beta)}{\beta}(\theta^2-\theta-|\delta|\theta)\|x^k-x^{k-1}\|^2\\
&\quad -\frac{(1-\beta)}{\beta}(\delta^2-|\delta|-|\delta|\theta)\|x^{k-1}-x^{k-2}\|^2
\end{aligned}
$$

$$\begin{aligned}
&= (1+\theta)\|x^k - x^*\|^2 - (\theta-\delta)\|x^{k-1} - x^*\|^2 \\
&\quad -\delta\|x^{k-2} - x^*\|^2 + (1+\theta)(\theta-\delta)\|x^k - x^{k-1}\|^2 \\
&\quad +\delta(1+\theta)\|x^k - x^{k-2}\|^2 \\
&\quad -\left(\frac{1-\beta}{\beta}\right)(\theta^2 - \theta - |\delta|\theta)\|x^k - x^{k-1}\|^2 \\
&\quad -\left(\frac{1-\beta}{\beta}\right)(1 - |\delta| - \theta)\|x^{k+1} - x^k\|^2 \\
&\quad -\left(\frac{1-\beta}{\beta}\right)(\delta^2 - |\delta| - |\delta|\theta)\|x^{k-1} - x^{k-2}\|^2 \\
&\quad -\delta(\theta-\delta)\|x^{k-1} - x^{k-2}\|^2 \\
&\leq (1+\theta)\|x^k - x^*\|^2 - (\theta-\delta)\|x^{k-1} - x^*\|^2 \\
&\quad -\delta\|x^{k-2} - x^*\|^2 - \left(\frac{1-\beta}{\beta}\right)(1 - |\delta| - \theta)\|x^{k+1} - x^k\|^2 \\
&\quad -\left(\frac{1-\beta}{\beta}\right)(\delta^2 - |\delta| - |\delta|\theta)\|x^{k-1} - x^{k-2}\|^2 \\
&\quad -\delta(\theta-\delta)\|x^{k-1} - x^{k-2}\|^2 \\
&\quad +(1+\theta)(\theta-\delta)\|x^k - x^{k-1}\|^2 \\
&\quad -\left(\frac{1-\beta}{\beta}\right)(\theta^2 - \theta - |\delta|\theta)\|x^k - x^{k-1}\|^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 - \theta\|x^k - x^*\|^2 - \delta\|x^{k-1} - x^*\|^2 \\
&\quad + \left(\frac{1-\beta}{\beta}\right)(1 - |\delta| - \theta)\|x^{k+1} - x^k\|^2 \\
&\leq \|x^k - x^*\|^2 - \theta\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2 \\
&\quad + \left(\frac{1-\beta}{\beta}\right)(1 - |\delta| - \theta)\|x^k - x^{k-1}\|^2 \\
&\quad +(\theta-\delta)(1+\theta)\|x^k - x^{k-1}\|^2 \\
&\quad -\left(\frac{1-\beta}{\beta}\right)(\theta^2 - 2\theta - |\delta|\theta - |\delta| + 1)\|x^k - x^{k-1}\|^2 \\
&\quad -\left[\delta(\theta-\delta) + \left(\frac{1-\beta}{\beta}\right)(\delta^2 - |\delta| - |\delta|\theta)\right]\|x^{k-1} - x^{k-2}\|^2. \qquad (15)
\end{aligned}$$

For each $k \geq 1$, define

$$\Gamma_k := \|x^k - x^*\|^2 - \theta\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2$$

$$+\left(\frac{1-\beta}{\beta}\right)(1-|\delta|-\theta)\|x^k-x^{k-1}\|^2. \tag{16}$$

We first show that $\Gamma_k \geq 0, \ \forall k \geq 1$. Note that

$$\|x^{k-1}-x^*\|^2 \leq 2\|x^k-x^{k-1}\|^2+2\|x^k-x^*\|^2.$$

Hence,

$$\begin{aligned}
\Gamma_k &= \|x^k-x^*\|^2-\theta\|x^{k-1}-x^*\|^2-\delta\|x^{k-2}-x^*\|^2\\
&\quad+\left(\frac{1-\beta}{\beta}\right)(1-|\delta|-\theta)\|x^k-x^{k-1}\|^2\\
&\geq \|x^k-x^*\|^2-2\theta\|x^k-x^{k-1}\|^2-2\theta\|x^k-x^*\|^2\\
&\quad-\delta\|x^{k-2}-x^*\|^2+\left(\frac{1-\beta}{\beta}\right)(1-|\delta|-\theta)\|x^k-x^{k-1}\|^2\\
&= (1-2\theta)\|x^k-x^*\|^2-\delta\|x^{k-2}-x^*\|^2\\
&\quad+\left[\left(\frac{1-\beta}{\beta}\right)(1-|\delta|-\theta)-2\theta\right]\|x^k-x^{k-1}\|^2. \qquad (17)
\end{aligned}$$

By Condition 3.2 (i), (ii) and (iii), we obtain

$$\begin{aligned}
|\delta| &< 1-\theta-\frac{2\theta}{\left(\frac{1-\beta}{\beta}\right)}\\
&= \frac{1-\beta-\theta-\beta\theta}{1-\beta}. \qquad (18)
\end{aligned}$$

We then obtain from (17) and (18) that $\Gamma_k \geq 0, \quad \forall k \geq 1$. Consequently, we obtain from (15) that

$$\begin{aligned}
\Gamma_{k+1} &\leq \Gamma_k+c_1\left(\|x^{k-1}-x^{k-2}\|^2-\|x^k-x^{k-1}\|^2\right)\\
&\quad -c_2\|x^{k-1}-x^{k-2}\|^2, \qquad (19)
\end{aligned}$$

where

$$c_1 := -\left((\theta-\delta)(1+\theta)-\left(\frac{1-\beta}{\beta}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right)$$

and

$$c_2 := -\Big((\theta-\delta)(1+\theta)-\left(\frac{1-\beta}{\beta}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)$$

$$-\delta(\theta-\delta)-\Big(\frac{1-\beta}{\beta}\Big)(\delta^2-|\delta|-|\delta|\theta)\Big).$$

Noting that $|\delta| = -\delta$ since $\delta \leq 0$, we then have that

$$c_1 = -\left((\theta-\delta)(1+\theta)-\left(\frac{1-\beta}{\beta}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right) > 0$$

which is equivalent to

$$\frac{\theta(1+\theta)-\left(\frac{1-\beta}{\beta}\right)(1-\theta)^2}{(1+\theta)\left(1+\frac{1-\beta}{\beta}\right)} < \delta. \tag{20}$$

By Condition 3.2 (iii), we see that (20) holds and thus $c_1 > 0$. Also,

$$\begin{aligned} c_2 := -\Big((\theta-\delta)(1+\theta)-\left(\frac{1-\beta}{\beta}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1) \\ -\delta(\theta-\delta)-\Big(\frac{1-\beta}{\beta}\Big)(\delta^2-|\delta|-|\delta|\theta)\Big) > 0 \end{aligned} \tag{21}$$

implies that

$$\begin{aligned} \theta(1+\theta)-\left(\frac{1-\beta}{\beta}\right)(1-\theta)^2 < \left(1+\frac{1-\beta}{\beta}\right)\delta(1+\theta) \\ +\delta(\theta-\delta)+\left(\frac{1-\beta}{\beta}\right)(\delta^2+\delta(1+\theta)). \end{aligned} \tag{22}$$

By Condition 3.2 (ii), we have that the inequality (22) is satisfied. Therefore, $c_2 > 0$ from (21). From (19), we then obtain

$$\begin{aligned} \Gamma_{k+1}+c_1\|x^k-x^{k-1}\|^2 \quad \leq \quad & \Gamma_k+c_1\|x^{k-1}-x^{k-2}\|^2 \\ & -c_2\|x^{k-1}-x^{k-2}\|^2. \end{aligned} \tag{23}$$

Letting $\bar{\Gamma}_k := \Gamma_k + c_1\|x^{k-1}-x^{k-2}\|^2$, we obtain from (23) that

$$\bar{\Gamma}_{k+1} \leq \bar{\Gamma}_k. \tag{24}$$

This implies from (24) that the sequence $\{\bar{\Gamma}_k\}$ is decreasing and thus $\lim\limits_{k\to\infty}\bar{\Gamma}_k$ exists. Consequently, we have from (23) that

$$\lim_{k\to\infty} c_2\|x^{k-1}-x^{k-2}\|^2 = 0. \tag{25}$$

Hence,

$$\lim_{k\to\infty} \|x^{k-1} - x^{k-2}\| = 0. \tag{26}$$

Using (26) and existence of limit of $\{\bar{\Gamma}_k\}$, we have that

$$\begin{aligned} \lim_{k\to\infty} \Gamma_k &:= \lim_{k\to\infty} \Big[\|x^k - x^*\|^2 - \theta\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2 \\ &\quad + \left(\frac{1-\beta}{\beta}\right)(1 - |\delta| - \theta)\|x^k - x^{k-1}\|^2 \Big] \end{aligned} \tag{27}$$

exists. Also,

$$\begin{aligned} \|x^{k+1} - w^k\| &= \|x^{k+1} - x^k - \theta(x^k - x^{k-1}) - \delta(x^{k-1} - x^{k-2})\| \\ &\leq \|x^{k+1} - x^k\| + \theta\|x^k - x^{k-1}\| + |\delta|\|x^{k-1} - x^{k-2}\|. \end{aligned}$$

So, we obtain

$$\lim_{k\to\infty} \|w^k - T_k w^k\| = 0. \tag{28}$$

Again, Note that

$$\|w^k - x^k\| \leq \theta\|x^k - x^{k-1}\| + |\delta|\|x^{k-1} - x^{k-2}\| \to 0, \ k \to \infty.$$

Since $\lim_{k\to\infty} \Gamma_k$ exists and $\lim_{k\to\infty} \|x^k - x^{k-1}\| = 0$, we have from (17) that $\{x^k\}$ is bounded. ☐

Theorem 3.1. *Suppose Assumptions 3.1 and Assumptions 3.2 are fulfilled. Then the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to a point in S.*

Proof. Using (26) in (27), we have that

$$\lim_{k\to\infty} \left[\|x^k - x^*\|^2 - \theta\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2 \right] \tag{29}$$

exists. By Lemma 3.4, we have that $\{x^k\}$ is bounded. We first show that any weak cluster point of $\{x^k\}$ is in S. Suppose $\{x^{k_n}\} \subset \{x^k\}$ such that $x^{k_n} \rightharpoonup v^* \in H$. Since $\|w^k - x^k\| \to 0, \ k \to \infty$, we have $w^{k_n} \rightharpoonup v^* \in H$. Invoking Assumption 3.1 (c) and (28), we conclude that v^* belongs to S.

We now show that $x^k \rightharpoonup x^* \in S$. Let us assume that there exist $\{x^{k_n}\} \subset \{x^k\}$ and $\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_n} \rightharpoonup v^*, n \to \infty$ and $x^{k_j} \rightharpoonup x^*, j \to \infty$. We show that $v^* = x^*$.

Observe that

$$2\langle x^k, x^* - v^*\rangle = \|x^k - v^*\|^2 - \|x^k - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2, \tag{30}$$

$$\begin{aligned} 2\langle -\theta x^{k-1}, x^* - v^*\rangle &= -\theta\|x^{k-1} - v^*\|^2 + \theta\|x^{k-1} - x^*\|^2 \\ &\quad +\theta\|v^*\|^2 - \theta\|x^*\|^2 \end{aligned} \tag{31}$$

and

$$\begin{aligned} 2\langle -\delta x^{k-2}, x^* - v^*\rangle &= -\delta\|x^{k-2} - v^*\|^2 + \delta\|x^{k-2} - x^*\|^2 \\ &\quad +\delta\|v^*\|^2 - \delta\|x^*\|^2. \end{aligned} \tag{32}$$

Addition of (30), (31) and (32) gives

$$\begin{aligned} 2\langle x^k - \theta x^{k-1} - \delta x^{k-2}, x^* - v^*\rangle &= \Big(\|x^k - v^*\|^2 - \theta\|x^{k-1} - v^*\|^2 \\ &\quad -\delta\|x^{k-2} - v^*\|^2\Big) - \Big(\|x^k - x^*\|^2 \\ &\quad -\theta\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2\Big) \\ &\quad +(1-\theta-\delta)(\|x^*\|^2 - \|v^*\|^2). \end{aligned}$$

According to (27), we have

$$\lim_{k\to\infty}\Big[\|x^k - x^*\|^2 - \theta\|x^{k-1} - x^*\|^2 - \delta\|x^{k-2} - x^*\|^2\Big]$$

exists and

$$\lim_{k\to\infty}\Big[\|x^k - v^*\|^2 - \theta\|x^{k-1} - v^*\|^2 - \delta\|x^{k-2} - v^*\|^2\Big]$$

exists. This implies that

$$\lim_{k\to\infty}\langle x^k - \theta x^{k-1} - \delta x^{k-2}, x^* - v^*\rangle$$

exists. Now,

$$\begin{aligned} \langle v^* - \theta v^* - \delta v^*, x^* - v^*\rangle &= \lim_{n\to\infty}\langle x^{k_n} - \theta x^{k_n-1} - \delta x^{k_n-2}, x^* - v^*\rangle \\ &= \lim_{n\to\infty}\langle x^k - \theta x^{k-1} - \delta x^{k-2}, x^* - v^*\rangle \\ &= \lim_{j\to\infty}\langle x^{k_j} - \theta x^{k_j-1} - \delta x^{k_j-2}, x^* - v^*\rangle \end{aligned}$$

$$= \langle x^* - \theta x^* - \delta x^*, x^* - v^* \rangle,$$

and this yields

$$(1 - \theta - \delta)\|x^* - v^*\|^2 = 0.$$

Since $\delta \leq 0 < 1 - \theta$, we obtain that $x^* = v^*$. Hence, $\{x^k\}$ converges weakly to a point in S. $\square$

Corollary 3.5. *Let $\{T_i\}_{i\geq 0} \subset F_\beta$, where $\beta \in (0,1)$ and $\cap_{i\geq 0} F(T_i) \neq \emptyset$. Assume that T_i is demi-closed for each i. Let the sequence $\{x^k\}$ in H be generated by choosing $x^{-1}, x^0, x^1 \in H$ and using the recursion*

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \\ x^{k+1} = \sum_{i=0}^\infty \sigma_{i,k} T_i w^k, \quad k \geq 1, \end{cases}$$

where $\sigma_{i,k} \in [0,\infty)$ such that $\sum_{i=0}^\infty \sigma_{i,k} = 1$; for all $i \geq 0, \{\sigma_{i,k}\}$ is bounded away from zero for k large enough (i.e., $\forall i \geq 0, \exists N_i \in \mathbb{N}$ and $\exists \sigma_i > 0$ such that $\forall k \geq N_i, \sigma_{i,k} \geq \sigma_i$). Suppose θ and δ satisfy Assumption 3.2. Then $\{x^k\}$ weakly converges to a point in $\cap_{i\geq 0} F(T_i)$.

Proof. Following the same line of arguments as in and of [24, Lemma 4.1, Theorem 4.2], one can show that Assumptions 3.1 hold with $T_k = \sum_{i=0}^\infty \sigma_{i,k} T_i$. The rest of the proof follows from Theorem 3.1. $\square$

Corollary 3.6. *Let $\{T_i\}_{i\geq 0} \subset F_\beta$, where $\beta \in (0,1)$ and $\cap_{i\geq 0} F(T_i) \neq \emptyset$. Assume that T_i is demi-closed for each i. Let the sequence $\{x^k\}$ in H be generated by choosing $x^{-1}, x^0, x^1 \in H$ and using the recursion*

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \\ x^{k+1} = \left(\sum_{j=1}^k \gamma_j\right)^{-1} \sum_{i=1}^k \gamma_i T_i w^k, \quad k \geq 1, \end{cases}$$

where $\gamma_k \in (0,\infty)$ with $\sum_{j=1}^\infty \gamma_j < \infty$ with θ and δ satisfy Assumption 3.2. Then $\{x^k\}$ weakly converges to a point in $\cap_{i\geq 0} F(T_i)$.

Proof. Take $\sigma_{i,k} = \frac{\gamma_i}{\sum_{j=1}^k \gamma_j}$ for $1 \leq j \leq k$ and $\sigma_{i,k} = 0$ for $i \geq k+1$ in Corollary 3.5. Then $\sum_{i=1}^\infty \sigma_{i,k} = 1$. Furthermore, since $\sum_{j=1}^\infty \gamma_j < \infty$, we have for all $i \geq 0$ and large enough k, $\sigma_{i,k} \geq \frac{\gamma_i}{\sum_{j=1}^\infty \gamma_j} > 0$. By Corollary 3.5, we have the desired conclusion. $\square$

Corollary 3.7. *Let $\{T_i\}_{i\geq 0}$ be an infinite countable family of β-averaged mappings defined on H, where $\beta \in (0,1)$ and $\cap_{i\geq 0} F(T_i) \neq \emptyset$. Let the sequence $\{x^k\}$ in H be generated by choosing $x^{-1}, x^0, x^1 \in H$ and using the* *recursion*

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \\ x^{k+1} = \left(\sum_{j=1}^k \gamma_j\right)^{-1} \sum_{i=1}^k \gamma_i T_i w^k, \quad k \geq 1, \end{cases}$$

where $\gamma_k \in (0, \infty)$ *with* $\sum_{j=1}^\infty \gamma_j < \infty$ *with* θ *and* δ *satisfying Assumption 3.2. Then* $\{x^k\}$ *weakly converges to a point in* $\cap_{i\geq 0} F(T_i)$.

Proof. If T_i is β-averaged on H, then $\|T_i x - T_i y\|^2 \leq \|x - y\|^2 - \frac{1-\beta}{\beta}\|x - y - (T_i x - T_i y)\|^2, \forall x, y \in H$. Consequently,

$$\|T_i x - x^*\|^2 \leq \|x - x^*\|^2 - \frac{1-\beta}{\beta}\|x - T_i x\|^2, \forall x \in H, x^* \in \cap_{i\geq 0} F(T_i). \tag{33}$$

Thus, we get that $T_i \in F_\beta$. Since T_i is nonexpansive for each $i \geq 0$, we have that each T_i is demi-closed. Hence, by invoking Corollary 3.6, we obtain the desired conclusion. □

Remark 3.8. Our results reduce to the results given in [2, 3, 4, 10, 24, 30] when $\delta = 0$.

4. Some applications

We give some applications of our proposed method in Section 3 to composite optimization problem and variational inequalities.

4.1. Application to optimization problems

In this subsection, we apply our results in Section 3 to solve both composite convex optimization problems in Hilbert spaces. This we do by converting the composite optimization problem to equivalent fixed point problem of averaged quasi-nonexpansive mappings and consequently apply the results in Section 3.

Consider the composite convex optimization problem:

$$\min_{x\in H} F(x) = f(x) + g(x), \tag{34}$$

where f, g are proper, lower-semicontinuous convex functions taking values in $(-\infty, \infty]$ with f being L-smooth (i.e., the gradient ∇f of f is L-Lipschitz continuous). The proximal-gradient algorithm

$$x^{k+1} = \text{prox}_{\lambda_k g}(x^k - \lambda_k \nabla f(x^k)), \tag{35}$$

where $\lambda \in (0, \frac{2}{L})$.

Remark 4.1. Observe that ∇f is L-Lipschitz continuous implies that ∇f is $\frac{1}{L}$-ism [5]. This further implies that $\lambda \nabla f$ is $\frac{1}{\lambda L}$-ism. So by [33, Proposition 3.4(iii)], $I - \lambda \nabla f$ is $\frac{\lambda L}{2}$-averaged. Now since $\text{Prox}_{\lambda g}$ is $\frac{1}{2}$-averaged, we see from [33, Proposition 3.2(iv)] that $T := \text{prox}_{\lambda g}(I - \lambda \nabla f)$ is $\frac{2+\lambda L}{4}$-averaged.

Using Remark 4.1 and Algorithm 1, we have the following two-step inertial proximal-gradient method to solve problem (34):

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \\ x^{k+1} = \text{prox}_{\lambda_k g}(w^k - \lambda_k \nabla f(w^k)), \end{cases} \tag{36}$$

where θ and δ satisfy Assumption 3.2, $0 < \liminf_{k\to\infty} \lambda_k \le \limsup_{k\to\infty} \lambda_k < \frac{2}{L}$. Consequently, we have the following weak convergence result for problem (34).

Theorem 4.1. Consider the composite optimization problem (34), where f, g are proper, lower-semicontinuous convex functions with f L-smooth. Suppose the set of solutions of (34), denoted by S, is nonempty. Suppose θ and δ satisfy Assumption 3.2, $0 < a \le \liminf_{k\to\infty} \lambda_k \le \limsup_{k\to\infty} \lambda_k \le b < \frac{2}{L}$. Then the sequence $\{x^k\}$ generated by Algorithm (36) converges weakly to a point in S.

Proof. Define

$$T = \text{prox}_{\lambda g}(I - \lambda \nabla f).$$

Then T is $\frac{2+\lambda L}{4}$-averaged mapping and hence averaged quasi-nonexpansive mapping. Furthermore, $S = F(T)$ and Algorithm (1) reduces to Algorithm (36). Then by Theorem 3.1 we have that $\{x^k\}$ converges to a point in S. ☐

The give the following remark regarding our method (36) and some related methods.

Remark 4.2. Suppose $\delta = 0$, $\theta = 0$ and g is an indicator function in Algorithm (36), then Algorithm (36) reduces to [33, (2),(3)]. ◊

4.2. Application to variational inequalities

Suppose C is a nonempty, closed and convex subset of H and $A : H \to H$ is a given continuous operator. A variational inequality is defined as: find

$x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{37}$$

We denote by SOL, the solution set of (37). One of the projection-type methods for solving (37) is the projection and contraction method [11, 19], which has been studied by several authors in the literature.

Example 4.3 ([31, Example 2.6]). *Let $A : H \to H$ be a monotone and L-Lipschitz operator on a nonempty closed and convex subset C and λ be a positive number. For all $x \in H$, $\gamma \in (0,2)$ and $\lambda_k > 0, \forall n \geq 1$, define $y^k := P_C(x - \lambda_k Ax)$, $d^k := (x - y^k) - \lambda_k(Ax - Ay^k)$ and $T_k x := x - \gamma\rho_k d^k$ where*

$$\rho_k := \begin{cases} \frac{\langle x - y_n, d^k \rangle}{\|d^k\|^2}, & d^k \neq 0 \\ 0, & d^k = 0 \end{cases}$$

Then

(i) T_k is β averaged quasi-nonexpansive mapping with $\beta = \frac{\gamma}{2}$. Thus, we have

$$\|T_k x - x^*\|^2 \leq \|x - x^*\|^2 - \frac{1-\beta}{\beta}\|x - T_k x\|^2, \forall x \in H, x^* \in F(T_k), k \geq 1.$$

(ii) Suppose that $0 < a \leq \lambda_k \leq b < \frac{1}{L}$. Then, for each $n \geq 1$, $I - T_k$ is demiclosed at the origin.

In the view of Example 4.3, we propose the following algorithm for solving variational inequality problem (37).

$$\begin{cases} w^k = x^k + \theta(x^k - x^{k-1}) + \delta(x^{k-1} - x^{k-2}) \\ x^{k+1} = Tw^k, \end{cases} \tag{38}$$

where T is as given in Example 4.3, θ and δ satisfy Assumption 3.2, and $\lambda_k \in (0, \frac{1}{L})$. Consequently, we have the following weak convergence result for variational inequality problem (37).

Theorem 4.2. Let $A : H \to H$ be a monotone and L-Lipschitz operator on a nonempty closed and convex subset C. Suppose SOL is nonempty with θ and δ satisfy Assumption 3.2, and $\lambda_k \in (0, \frac{1}{L})$. Then $\{x^k\}$ generated by (38) converges weakly to a point in SOL.

5. Conclusion

This paper studies a two-step inertial fixed point iteration for finding a common fixed point of a countable family of averaged quasi-nonexpansive mappings in real Hilbert spaces. As far as we know, this is the first time a fixed point iteration with two-step inertial extrapolation is proposed and studied without assuming on-line rule on the inertial parameters as done in [18, 21]. Many already existing methods in the literature are recovered as special cases of our method. We obtain weak convergence results under some conditions and applications to composite convex optimization problems and variational inequalities are given.

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