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An explicit formula for the determinants of tridiagonal 2-Toeplitz and 3-Toeplitz matrices

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We give an explicit formula for the determinants of tridiagonal 2-Toeplitz and 3-Toeplitz matrices defined over $\mathbb{C}$ without of use the Chebyshev polynomials.

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1. Introduction

The tridiagonal matrix is any matrix of the form

$$\begin{bmatrix} d_1 & a_1 & & & & \\ b_1 & d_2 & a_2 & & & \\ & b_2 & d_3 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{n-2} & d_{n-1} & a_{n-1} \\ & & & & b_{n-1} & d_n \end{bmatrix}. \tag{1}$$

(The empty places symbolize the zero entries.)

The tridiagonal matrices appear in many various applications, e.g. interpolation problems [5, 4], parallel computing [10] and some engineering problems [18]. They also appear in the context of difference and differential equations [17, 20]. Thus, their determinants, eigenvalues, eigenvectors and inverses are of great interest. However, there are not many explicit formulas for these quantities. It is well-known that D_n – the determinants of (1) satisfy the recurrence relation

$$D_n = d_n D_{n-1} - a_{n-1} b_{n-1} D_{n-2}. \tag{2}$$

The above recurrence is quite easy to solve in the case when we deal with a Toeplitz matrix. This formula is given for example in [21]. However, the general case is far more troublesome and to find D_n one usually uses various algorithms [7, 8]. Some of them work for matrices with some specific properties, like for instance being positive definite [6] what simplifies recurrence (2) and improves the algorithm. Extending the problem of calculating the determinants to more complex matrices, one can find it even more difficult. Also in this case, one usually uses algorithms [1, 16]. More explicit results are known for some more specific cases; for example, when a matrix has only the main, the k-th and $-k$-th diagonal distinct from 0 (like in [13, 2]) or it is skew-symmetric and pentadiagonal [9]. However, in [2, 9] the explicit results are known only for Toeplitz matrices. (Note that in the case of Toeplitz matrices very useful way of finding the determinant is using the Day formula (see [3, 15]).)

In this paper, we are going to derive an explicit formulas for determinants of some particular tridiagonal matrices over $\mathbb{C}$. First, we will study tridiagonal 2-Toeplitz matrices, i.e. those satisfying conditions $d_{2k+1} = d_1$, $d_{2k} = d_2$, $a_{2k+1} = a_1$, $a_{2k} = a_2$, $b_{2k+1} = b_1$, $b_{2k} = b_2$. Next, we will move to tridiagonal 3-Toeplitz matrices, i.e. $d_{3k+1} = d_1$, $d_{3k+2} = d_2$, $d_{3k} = d_3$, analogously for a_i, b_i. Such matrices were considered in [19, 11, 12]. The results obtained in these articles are given in terms of the Chebyshev polynomials. In this work, we would like to present somewhat different approach.

Before we present the results, let us introduce some notation. The following is used in [14]: by $[S]$ we mean a number that is equal to 1 when the sentence S is true and 0 if it is false. For example

$$A(n) = [n < 5] \cdot 1 + [n = 5] \cdot 3^{12} + [n > 5] \cdot n^5$$

reads as follows:

$$A(n) = \begin{cases} 1 \quad \text{if } n < 5 \\ 3^{12} \quad \text{if } n = 5 \\ n^5 \quad \text{if } n > 5. \end{cases}$$

Moreover, it should be noted that writing $\sqrt{\alpha}$ means the primitive square root of α.

The two following theorems hold.

Theorem 1.1. *Denote by D_n the determinant of a tridiagonal 2-Toeplitz $n \times n$ complex matrix. Let us put $A = a_1b_1a_2b_2$, $B = a_1b_1 + a_2b_2 - d_1d_2$.*

1. *If n is even, then*

$$D_n = \begin{cases} [n=0]\cdot 1 + [n=2]\cdot a_2b_2 \quad \text{if } A=B=0 \\ [n=0]\cdot 1 + [n\geq 2]\cdot (d_1d_2 - a_1b_1)(-B)^{\frac{n}{2}-1} \quad \text{if } A=0,\ B\neq 0 \\ \left(-\frac{2A}{B}\right)^{\frac{n}{2}} \frac{4\left(\frac{n}{2}+1\right)A - na_2b_2B}{B^2} \quad \text{if } A\neq 0,\ B^2=4A \\ -\frac{A^2}{\sqrt{B^2-4A}}\left[\frac{((-B+\sqrt{B^2-4A})a_2b_2+1)(2A)^{\frac{n}{2}+1}}{(-B+\sqrt{B^2-4A})^{\frac{n}{2}+1}} \right. \\ \quad \left. + \frac{((-B-\sqrt{B^2-4A})a_2b_2+1)(2A)^{\frac{n}{2}+1}}{(-B-\sqrt{B^2-4A})^{\frac{n}{2}+1}}\right] \\ \quad \text{if } A\neq 0,\ B^2\neq 4A. \end{cases}$$

2. *If n is odd, then*

$$D_n = \begin{cases} [n=1]\cdot d_1 \quad \text{if } A=B=0 \\ d_1(-B)^{\frac{n-1}{2}} \quad \text{if } A=0,\ B\neq 0 \\ \frac{d_1(n+1)}{B}\left(-\frac{2A}{B}\right)^{\frac{n+1}{2}} \quad \text{if } A\neq 0,\ B^2=4A \\ -\frac{d_1}{\sqrt{B^2-4A}}\left[\left(\frac{2A}{-B+\sqrt{B^2-4A}}\right)^{\frac{n+1}{2}} - \left(\frac{2A}{-B-\sqrt{B^2-4A}}\right)^{\frac{n+1}{2}}\right] \\ \quad \text{if } A\neq 0,\ B^2\neq 4A. \end{cases}$$

Theorem 1.2. *Denote by D_n the determinant of a tridiagonal 3-Toeplitz $n\times n$ complex matrix. Let $A = a_1b_1a_2b_2a_3b_3$, $B = d_1a_2b_2 + d_2a_3b_3$, $C = d_3a_1b_1 - d_1d_2d_3 + 1$.*

1. *If $n = 0 \bmod 3$, then*

$$D_n = \begin{cases} [n=0]\cdot 1 + [n=3]\cdot d_2a_3b_3 \quad \text{if } A=B=0 \\ [n=0]\cdot \frac{1}{C} + [n\geq 3]\frac{B-d_2a_3b_3C}{BC}\left(-\frac{B}{C}\right)^{\frac{n}{3}} \quad \text{if } A=0, B\neq 0 \\ \frac{Bd_2a_3b_3\left[2B-4A\left(\frac{n}{3}+1\right)\right]+8A^2\left(\frac{n}{3}+1\right)}{B^3}\left(-\frac{2A}{B}\right)^{\frac{n}{3}} \quad \text{if } A\neq 0, B^2=4AC \\ \frac{(-B+\sqrt{B^2-4AC})d_2a_3b_3+2A}{2B\sqrt{B^2-4AC}}\left(\frac{2A}{-B+\sqrt{B^2-4AC}}\right)^{\frac{n}{3}} \\ +\frac{(-B-\sqrt{B^2-4AC})d_2a_3b_3+2A}{2B\sqrt{B^2-4AC}}\left(\frac{2A}{-B-\sqrt{B^2-4AC}}\right)^{\frac{n}{3}} \\ \quad \text{if } A\neq 0, B^2\neq 4AC \end{cases}$$

2. *If* $n = 1 \bmod 3$, *then*

$$D_n = \begin{cases} [n=1]\cdot\frac{d_1}{C} + [n=4]\cdot a_1b_1a_3b_3 \quad \textit{if } A=B=0 \\ [n=1]\frac{d_1}{C} + [n\geq 4]\frac{d_1B - Ca_1b_1a_3b_3}{BC}\left(-\frac{B}{C}\right)^{\frac{n-1}{3}} \quad \textit{if } A=0, B\neq 0 \\ \frac{Ba_1b_1a_3b_3[2B-4A\frac{n+2}{3}]+8A^2d_1\frac{n+2}{3}}{B^3}\left(-\frac{2A}{B}\right)^{\frac{n-1}{3}} \quad \textit{if } A\neq 0, B^2=4AC \\ \frac{(-B+\sqrt{B^2-4AC})a_1b_1a_3b_3+2Ad_1}{2A\sqrt{B^2-4AC}}\left(\frac{2A}{-B+\sqrt{B^2-4AC}}\right)^{\frac{n+2}{3}} \\ +\frac{(-B-\sqrt{B^2-4AC})a_1b_1a_3b_3+2Ad_1}{2A\sqrt{B^2-4AC}}\left(\frac{2A}{-B-\sqrt{B^2-4AC}}\right)^{\frac{n+2}{3}} \\ \qquad \textit{if } A\neq 0, B^2\neq 4AC \end{cases}$$

3. *If* $n = 2 \bmod 3$, *then*

$$D_n = \begin{cases} [n=2](d_1d_2 - a_1b_1) \quad \textit{if } A=B=0 \\ \frac{d_1d_2-a_1b_1}{C}\left(-\frac{B}{C}\right)^{\frac{n-2}{3}} \quad \textit{if } A=0, B\neq 0 \\ \frac{4A\frac{n+1}{3}(a_1b_1-d_1d_2)}{B^2}\left(-\frac{2A}{B}\right)^{\frac{n-2}{3}} \quad \textit{if } A\neq 0, B^2=4AC \\ \frac{a_1b_1-d_1d_2}{\sqrt{B^2-4AC}}\left(\frac{2A}{-B+\sqrt{B^2-4AC}}\right)^{\frac{n+1}{3}} + \frac{d_1d_2-a_1b_1}{\sqrt{B^2-4AC}}\left(\frac{2A}{-B-\sqrt{B^2-4AC}}\right)^{\frac{n+1}{3}} \\ \quad \textit{if } A\neq 0, B^2\neq 4AC. \end{cases}$$

2. Proof

To prove the results given in the introduction we are going to solve recurrence (2) that for 2- and 3-tridiagonal matrices may take 2 or 3 different forms. Because of them we will define 2 (for 2-Toeplitz matrices) or 3 (for 3-Toeplitz) sequences that depend on each other. Then we will solve those recurrences by associating with each of them a formal Laurent series. This beautiful and quite simple technique is described in detail in [14].

2.1. Tridiagonal 2-Toeplitz matrices

First, we consider 2-tridiagonal matrices:

$$\begin{bmatrix} d_1 & a_1 & & & & \\ b_1 & d_2 & a_2 & & & \\ & b_2 & d_1 & a_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_1 & d_2 & a_2 \\ & & & & b_2 & d_1 \end{bmatrix}_{n\times n\ (2\nmid n)}, \quad \begin{bmatrix} d_1 & a_1 & & & & \\ b_1 & d_2 & a_2 & & & \\ & b_2 & d_1 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_2 & d_1 & a_1 \\ & & & & b_1 & d_2 \end{bmatrix}_{n\times n\ (2\mid n)}.$$

One can see that initial conditions for (2) are

$$D_1 = d_1, \quad D_2 = d_1 d_2 - a_1 b_1,$$
$$D_3 = d_1 d_2 d_3 - d_3 a_1 b_1 - d_1 a_2 b_2 = d_1^2 d_2 - d_1 a_1 b_1 - d_1 a_2 b_2.$$

Additionally, we put $D_0 = 1$.

Clearly, the recurrence relation (2) reduces to

$$D_n = \begin{cases} d_2 D_{n-1} - a_1 b_1 D_{n-2} & \text{if } 2|n \\ d_1 D_{n-1} - a_2 b_2 D_{n-2} & \text{if } 2 \nmid n. \end{cases} \tag{3}$$

As we deal with two cases, we introduce two sequences $(U_n)_{n=0}^\infty$, $(V_n)_{n=0}^\infty$ defined by the rules

$$U_n = D_{2n}, \qquad V_n = D_{2n+1} \qquad \text{for all } n \geq 0.$$

From (3) it follows that these sequences satisfy the conditions

$$U_0 = 1, \quad U_1 = d_1 d_2 - a_1 b_1, \quad U_n = d_2 V_{n-1} - a_1 b_1 U_{n-1} \quad \text{for } n \geq 2, \tag{4}$$

and

$$V_0 = d_1, \quad V_1 = d_1^2 d_2 - d_1 a_1 b_1 - d_1 a_2 b_2, \quad V_n = d_1 U_n - a_2 b_2 V_{n-1} \quad \text{for } n \geq 2, \tag{5}$$

which form a system of difference equations. Clearly, its solution is a solution of our problem. In order to obtain it, we associate with $(U_n)_{n=0}^\infty$, $(V_n)_{n=0}^\infty$ the formal power series u and v[1]:

$$u(z) = \sum_{n=0}^\infty U_n z^n, \qquad v(z) = \sum_{n=0}^\infty V_n z^n.$$

Using them we will prove

Proposition 2.1. *Let $(U_n)_{n=0}^\infty$, $(V_n)_{n=0}^\infty$ be sequences given by (4), (5). Then*

$$U_n = \begin{cases} [n=0]\cdot 1 + [n=1]\cdot a_2 b_2 & \textit{if } A = B = 0 \\ [n=0]\cdot 1 + [n \geq 1]\cdot (d_1 d_2 - a_1 b_1)(-B)^{n-1} & \textit{if } A = 0,\ B \neq 0 \\ \left(-\frac{2A}{B}\right)^n \frac{4(n+1)A - 2n a_2 b_2 B}{B^2} & \textit{if } A \neq 0,\ B^2 = 4A \\ -\frac{A^2}{\sqrt{B^2-4A}} \left[\frac{((-B+\sqrt{B^2-4A})a_2 b_2 + 1)(2A)^{n+1}}{(-B+\sqrt{B^2-4A})^{n+1}} + \frac{((-B-\sqrt{B^2-4A})a_2 b_2 + 1)(2A)^{n+1}}{(-B-\sqrt{B^2-4A})^{n+1}} \right] & \\ \qquad \textit{if } A \neq 0,\ B^2 \neq 4A \end{cases}$$

[1]Note that in this case we do not have to assume that they are convergent.

$$V_n = \begin{cases} [n=0]\cdot d_1 \quad \textit{if } A=B=0 \\ d_1(-B)^n \quad \textit{if } A=0,\ B\neq 0 \\ \frac{2d_1(n+1)}{B}\left(-\frac{2A}{B}\right)^{n+1} \quad \textit{if } A\neq 0,\ B^2=4A \\ -\frac{d_1}{\sqrt{B^2-4A}}\left[\left(\frac{2A}{-B+\sqrt{B^2-4A}}\right)^{n+1}-\left(\frac{2A}{-B-\sqrt{B^2-4A}}\right)^{n+1}\right] \\ \quad \textit{if } A\neq 0,\ B^2\neq 4A \end{cases}$$

where $A = a_1b_1a_2b_2$, $B = a_1b_1 + a_2b_2 - d_1d_2$.

Proof. From (4) we have

$$\begin{aligned} u(z) = &\sum_{n=0}^{\infty} U_n z^n = 1 + (d_1d_2 - a_1b_1)z + \sum_{n=2}^{\infty}(d_2V_{n-1} - a_1b_1U_{n-1})z^n \\ = &\ 1 + (d_1d_2 - a_1b_1)z + d_2z\sum_{n=1}^{\infty} V_n z^n - a_1b_1z\sum_{n=1}^{\infty} U_n z^n \\ = &\ 1 + (d_1d_2 - a_1b_1)z + d_2z(v(z) - d_1) - a_1b_1z(u(z) - 1) \end{aligned}$$

This way we get

$$(1 + a_1b_1z)u(z) - d_2z \cdot v(z) = 1.$$

Analogously, from (5)

$$\begin{aligned} v(z) = &\sum_{n=0}^{\infty} V_n z^n \\ = &\ d_1 + (d_1^2d_2 - d_1a_1b_1 - d_1a_2b_2)z + \sum_{n=2}^{\infty}(d_1U_n - a_2b_2V_{n-1})z^n \\ = &\ d_1 + (d_1^2d_2 - d_1a_1b_1 - d_1a_2b_2)z + d_1\sum_{n=2}^{\infty} U_n z^n - a_2b_2z\sum_{n=1}^{\infty} V_n z^n \\ = &\ d_1 + (d_1^2d_2 - d_1a_1b_1 - d_1a_2b_2)z + d_1[u(z) - 1 - (d_1d_2 - a_1b_1)z] \\ &-a_2b_2z[v(z) - d_1] \end{aligned}$$

and this time we obtain

$$-d_1 \cdot u(z) + (1 + a_2b_2z)v(z) = 0.$$

The solution of the system

$$\begin{cases} (1 + a_1b_1z)u(z) - d_2z \cdot v(z) = 1 \\ -d_1 \cdot u(z) + (1 + a_2b_2z)v(z) = 0 \end{cases}$$

is the pair

$$\begin{aligned} u(z) &= \tfrac{1+a_2b_2z}{a_1b_1a_2b_2z^2+(a_1b_1+a_2b_2-d_1d_2)z+1}, \\ v(z) &= \tfrac{d_1}{a_1b_1a_2b_2z^2+(a_1b_1+a_2b_2-d_1d_2)z+1}. \end{aligned}$$

Now we have to discuss the Laurent expansions of u and v. Obviously, they depend on the degree of

$$P(z) = a_1b_1a_2b_2z^2 + (a_1b_1 + a_2b_2 - d_1d_2)z + 1.$$

Let's denote $a_1b_1a_2b_2$ by A and $a_1b_1 + a_2b_2 - d_1d_2$ by B.

Now we consider various cases.

1. $A = a_1b_1a_2b_2 = 0$
   In this case, the degree of P is at most 1. Again, consider two possibilities.
   (a) $B = a_1b_1 + a_2b_2 - d_1d_2 = 0$
   Then $u(z) = 1 + a_2b_2z$, $v(z) = d_1$. Hence

$$U_n = \begin{cases} 1 & \text{for } n = 0 \\ a_2b_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2, \end{cases} \qquad V_n = \begin{cases} d_1 & \text{for } n = 0 \\ 0 & \text{for } n \geq 1. \end{cases}$$

   (b) $B \neq 0$
   Then $\deg(P) = 1$ and u, v can be written as follows:

$$\begin{aligned} u(z) = & \ (1 + a_2b_2z) \cdot \tfrac{1}{1-(-Bz)} = (1 + a_2b_2z) \textstyle\sum_{n=0}^{\infty}(-B)^n z^n \\ = & \ 1 + \textstyle\sum_{n=1}^{\infty}[(-B)^n + a_2b_2(-B)^{n-1}]z^n, \end{aligned}$$

$$v(z) = d_1 \cdot \frac{1}{1 - (-Bz)} = d_1 \sum_{n=0}^{\infty}(-B)^n z^n,$$

   so

$$U_n = \begin{cases} 1 & \text{for } n = 0 \\ (-B)^{n-1}(d_1d_2 - a_1b_1) & \text{for } n \geq 1, \end{cases}$$

$$V_n = d_1(-B)^n \quad \text{for all } n \in \mathbb{N}_0.$$

2. $A \neq 0$
   Also this time we consider two cases.

(a) $B^2 = 4A$

Notice that in this case $B \neq 0$ because otherwise $B^2 = 4A$ would force $A = 0$ – a contradiction.

Here P has a double root equal to $-\frac{B}{2A}$ and the functions u, v take the forms

$$u(z) = \frac{1 + a_2 b_2 z}{A\left(z + \frac{B}{2A}\right)^2}, \qquad v(z) = \frac{d_1}{A\left(z + \frac{B}{2A}\right)^2}.$$

Their Laurent expansions are as follows:

$$\begin{aligned} u(z) = &\ \tfrac{2a_2b_2}{B} \cdot \tfrac{1}{1-\left(-\frac{2A}{B}z\right)} + \tfrac{2(2A-a_2b_2B)}{B^2} \cdot \tfrac{1}{\left(1+\frac{2A}{B}z\right)^2} \\ = &\ \tfrac{2a_2b_2}{B} \textstyle\sum_{n=0}^{\infty} \left(-\tfrac{2A}{B}\right)^n z^n + \tfrac{a_2b_2-2A}{AB} \left(\textstyle\sum_{n=0}^{\infty} \left(-\tfrac{2A}{B}\right)^n z^n\right)' \\ = &\ \textstyle\sum_{n=0}^{\infty} \left[\tfrac{2a_2b_2}{B} \left(-\tfrac{2A}{B}\right)^n + (n+1)\tfrac{a_2b_2-2A}{AB} \left(-\tfrac{2A}{B}\right)^{n+1}\right] z^n, \end{aligned}$$

$$\begin{aligned} v(z) = &\ \tfrac{d_1}{\left(z+\frac{B}{2A}\right)^2} = -\tfrac{d_1}{A} \left(\tfrac{1}{z+\frac{B}{2A}}\right)' \\ = &\ \textstyle\sum_{n=0}^{\infty} \left(-\tfrac{2d_1}{B}\right) \left(-\tfrac{2A}{B}\right)^{n+1} (n+1) z^n. \end{aligned}$$

Thus

$$U_n = \frac{4(n+1)A - 2na_2b_2B}{B^2} \left(-\frac{2A}{B}\right)^n,$$

$$V_n = -\frac{2d_1(n+1)}{B} \left(-\frac{2A}{B}\right)^{n+1}.$$

(b) $B^2 < 4A$

This time we have two single roots:

$$r_1 = \frac{-B + \sqrt{B^2 - 4A}}{2A}, \qquad r_2 = \frac{-B - \sqrt{B^2 - 4A}}{2A}$$

and u, v are as follows:

$$\begin{aligned} u(z) &= \tfrac{r_1 a_2 b_2 + 1}{A(r_1 - r_2)} \cdot \tfrac{1}{z - r_1} - \tfrac{r_2 a_2 b_2 + 1}{A r_2 (r_1 - r_2)} \cdot \tfrac{1}{z - r_2}, \\ v(z) &= \tfrac{d_1}{A(r_1 - r_2)} \cdot \tfrac{1}{z - r_1} - \tfrac{d_1}{A(r_1 - r_2)} \cdot \tfrac{1}{z - r_2}. \end{aligned}$$

Therefore

$$\begin{aligned} u(z) &= \tfrac{r_1 a_2 b_2 + 1}{A(r_2 - r_1)} \textstyle\sum_{n=0}^{\infty} \tfrac{z^n}{r_1^n} - \tfrac{r_2 a_2 b_2 + 1}{A r_2 (r_2 - r_1)} \textstyle\sum_{n=0}^{\infty} \tfrac{z^n}{r_2^n}, \\ v(z) &= \tfrac{d_1}{A r_1 (r_2 - r_1)} \textstyle\sum_{n=0}^{\infty} \tfrac{z^n}{r_1^n} - \tfrac{d_1}{A r_2 (r_2 - r_1)} \textstyle\sum_{n=0}^{\infty} \tfrac{z^n}{r_2^n}. \end{aligned}$$

Finally, we get

$$\begin{aligned} U_n &= \tfrac{r_1 a_2 b_2+1}{A r_1^{n+1}(r_2-r_1)} - \tfrac{r_2 a_2 b_2+1}{A r_2^{n+1}(r_2-r_1)} \\ &= -\tfrac{A^2}{\sqrt{B^2-4A}}\Big[\tfrac{((-B+\sqrt{B^2-4A})a_2b_2+1)(2A)^{n+1}}{(-B+\sqrt{B^2-4A})^{n+1}} \\ &\quad + \tfrac{((-B-\sqrt{B^2-4A})a_2b_2+1)(2A)^{n+1}}{(-B-\sqrt{B^2-4A})^{n+1}}\Big] \end{aligned}$$

$$\begin{aligned} V_n &= \tfrac{d_1}{A(r_2-r_1)}\left(\tfrac{1}{r_1^{n+1}} - \tfrac{1}{r_2^{n+1}}\right) \\ &= -\tfrac{d_1}{\sqrt{B^2-4A}}\left[\left(\tfrac{2A}{-B+\sqrt{B^2-4A}}\right)^{n+1} - \left(\tfrac{2A}{-B-\sqrt{B^2-4A}}\right)^{n+1}\right]. \quad \square \end{aligned}$$

Now Theorem 1.1 is simply a corollary from Proposition 2.1.

2.2. Tridiagonal 3-Toeplitz matrices

We will work similarly as in the previous section.

This time we have three sequences $(U_n)_{n=0}^{\infty}$, $(V_n)_{n=0}^{\infty}$, $(W_n)_{n=0}^{\infty}$:

$$U_n = D_{3n}, \quad V_n = D_{3n+1}, \quad W_n = D_{3n+2} \qquad \text{for } n \geq 0$$

that correspond to the series

$$U(z) = \sum_{n=0}^{\infty} U_n z^n, \quad V(z) = \sum_{n=0}^{\infty} V_n z^n, \quad W(z) = \sum_{n=0}^{\infty} W_n z^n.$$

Those sequences satisfy

$$\begin{aligned} &U_n = d_3 W_{n-1} - a_2 b_2 V_{n-1}, \quad V_n = d_1 U_n - a_3 b_3 W_{n-1}, \\ &W_n = d_2 V_n - a_1 b_1 U_n \qquad \text{for } n \geq 2. \end{aligned} \tag{6}$$

The first values of D_n are

$$\begin{aligned} &U_0 = D_0 = 1, \quad V_0 = D_1 = d_1, \quad W_0 = D_2 = d_1 d_2 - a_1 b_1, \\ &U_1 = D_3 = d_1 d_2 d_3 - d_3 a_1 b_1 - d_1 a_2 b_2, \\ &V_1 = D_4 = d_1^2 d_2 d_3 - d_1 d_3 a_1 b_1 - d_1^2 a_2 b_2 - d_1 d_2 a_3 b_3 + a_1 b_1 a_3 b_3, \\ &W_1 = D_5 \\ &= d_1^2 d_2^2 d_3 - 2 d_1 d_2 d_3 a_1 b_1 - d_1^2 d_2 a_2 b_2 - d_1 d_2^2 a_3 b_3 \\ &\quad + d_2 a_1 b_1 a_3 b_3 + d_3 a_1^2 b_1^2 + d_1 a_1 b_1 a_2 b_2. \end{aligned} \tag{7}$$

In this case, we have

Proposition 2.2. *Let $(U_n)_{n=0}^{\infty}$, $(V_n)_{n=0}^{\infty}$, $(W_n)_{n=0}^{\infty}$ be given by (6), (7). Then*

$$U_n = \begin{cases} [n=0]\cdot 1 + [n=1]\cdot d_2a_3b_3 \quad \textit{if } A=B=0 \\ [n=0]\cdot \frac{1}{C} + [n\geq 1]\frac{B-d_2a_3b_3C}{BC}\left(-\frac{B}{C}\right)^n \quad \textit{if } A=0, B\neq 0 \\ \frac{Bd_2a_3b_3[2B-4A(n+1)]+8A^2(n+1)}{B^3}\left(-\frac{2A}{B}\right)^n \quad \textit{if } A\neq 0, B^2=4AC \\ \frac{(-B+\sqrt{B^2-4AC})d_2a_3b_3+2A}{2B\sqrt{B^2-4AC}}\left(\frac{2A}{-B+\sqrt{B^2-4AC}}\right)^n \\ +\frac{(-B-\sqrt{B^2-4AC})d_2a_3b_3+2A}{2B\sqrt{B^2-4AC}}\left(\frac{2A}{-B-\sqrt{B^2-4AC}}\right)^n \\ \quad \textit{if } A\neq 0, B^2\neq 4AC, \end{cases}$$

$$V_n = \begin{cases} [n=0]\cdot 1 + [n=1]\cdot a_1b_1a_3b_3 \quad \textit{if } A=B=0 \\ [n=0]\frac{d_1}{C} + [n\geq 1]\frac{d_1B-Ca_1b_1a_3b_3}{BC}\left(-\frac{B}{C}\right)^n \quad \textit{if } A=0, B\neq 0 \\ \frac{Ba_1b_1a_3b_3[2B-4A(n+1)]+8A^2d_1(n+1)}{B^3}\left(-\frac{2A}{B}\right)^n \quad \textit{if } A\neq 0, B^2=4AC \\ \frac{(-B+\sqrt{B^2-4AC})a_1b_1a_3b_3+2Ad_1}{2A\sqrt{B^2-4AC}}\left(\frac{2A}{-B+\sqrt{B^2-4AC}}\right)^{n+1} \\ +\frac{(-B-\sqrt{B^2-4AC})a_1b_1a_3b_3+2Ad_1}{2A\sqrt{B^2-4AC}}\left(\frac{2A}{-B-\sqrt{B^2-4AC}}\right)^{n+1} \\ \textit{if } A\neq 0, B^2\neq 4AC, \end{cases}$$

$$W_n = \begin{cases} [n=0](d_1d_2-a_1b_1) \quad \textit{if } A=B=0 \\ \frac{d_1d_2-a_1b_1}{C}\left(-\frac{B}{C}\right)^n \quad \textit{if } A=0, B\neq 0 \\ \frac{4A(n+1)(a_1b_1-d_1d_2)}{B^2}\left(-\frac{2A}{B}\right)^n \quad \textit{if } A\neq 0, B^2=4AC \\ \frac{a_1b_1-d_1d_2}{\sqrt{B^2-4AC}}\left(\frac{2A}{-B+\sqrt{B^2-4AC}}\right)^{n+1} + \frac{d_1d_2-a_1b_1}{\sqrt{B^2-4AC}}\left(\frac{2A}{-B-\sqrt{B^2-4AC}}\right)^{n+1} \\ \textit{if } A\neq 0, B^2\neq 4AC. \end{cases}$$

Proof. From the recurrence relations defining $(U_n)_{n=0}^{\infty}$, $(V_n)_{n=0}^{\infty}$, $(W_n)_{n=0}^{\infty}$ we obtain

$$\begin{aligned} u(z) = &\quad U_0 + U_1z + \sum_{n=2}^{\infty}(d_3W_{n-1} - a_2b_2V_{n-1})z^n \\ = &\quad U_0 + (U_1 - d_3W_0 + a_2b_2V_0)z + d_3z\cdot W(z) - a_2b_2z\cdot V(z) \\ = &\quad 1 + d_3z\cdot W(z) - a_2b_2z\cdot V(z), \end{aligned}$$

$$\begin{aligned} v(z) = &\quad V_0 + V_1z + \sum_{n=2}^{\infty}(d_1U_n - a_3b_3W_{n-1})z^n \\ = &\quad V_0 - d_1U_0 + (V_1 - d_1U_1 + a_3b_3W_0)z + d_1\cdot u(z) - a_3b_3z\cdot w(z) \\ = &\quad d_1\cdot u(z) - a_3b_3z\cdot w(z) \end{aligned}$$

$$\begin{aligned} w(z) = & \quad W_0 + W_1 z + \sum_{n=2}^{\infty}(d_2V_n - a_1b_1U_n)z^n \\ = & \quad W_0 - d_2V_0 + (W_1 - d_2V_1 + a_1b_1U_1)z + d_2 \cdot v(z) - a_1b_1 \cdot u(z) \\ = & \quad d_2 \cdot v(z) - a_1b_1 \cdot u(z) \end{aligned}$$

what yields the following system

$$\begin{cases} u(z) + a_2b_2z \cdot v(z) - d_3z \cdot w(z) = 1 \\ d_1 \cdot u(z) - v(z) - a_3b_3z \cdot w(z) = 0 \\ a_1b_1 \cdot u(z) - d_2 \cdot v(z) + w(z) = 0 \end{cases} \tag{8}$$

whose solution is the triple

(9)

$$u(z) = \frac{d_2a_3b_3z + 1}{Az^2 + Bz + C}, \quad v(z) = \frac{a_1b_1a_3b_3z + d_1}{Az^2 + Bz + C}, \quad w(z) = \frac{d_1d_2 - a_1b_1}{Az^2 + Bz + C},$$

where $A = a_1b_1a_2b_2a_3b_3$, $B = d_1a_2b_2 + d_2a_3b_3$, $C = d_3a_1b_1 - d_1d_2d_3 + 1$. Note that since the sequences $(U_n)_{n=0}^{\infty}$, $(V_n)_{n=0}^{\infty}$, $(W_n)_{n=0}^{\infty}$ are defined by (6) and system (8) follows from this definition, Eq.(9) is a solution of (8), so we can not have $A = B = C = 0$.

Now we discuss all the cases.

1. $A = 0$
   (a) $B = 0$
   This case is particularly easy, i.e.

$$U_n = \begin{cases} \frac{1}{C} & \text{if } n = 0 \\ \frac{d_2a_3b_3}{C} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \qquad V_n = \begin{cases} \frac{d_1}{C} & \text{if } n = 0 \\ \frac{a_1b_1a_3b_3}{C} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

$$W_n = \begin{cases} \frac{d_1d_2 - a_1b_1}{C} & \text{if } n = 0 \\ 0 & \text{if } n \geq 1. \end{cases}$$

   (b) $B \neq 0$
   This time

$$u(z) = \frac{d_2a_3b_3z + 1}{Bz + C} = \frac{d_2a_3b_3}{B} + \frac{B - d_2a_3b_3C}{BC} \sum_{n=0}^{\infty} \left(-\frac{B}{C}\right)^n z^n,$$

$$\begin{aligned} v(z) = & \quad \frac{a_1b_1a_3b_3z + d_1}{Bz + C} \\ = & \quad \frac{a_1b_1a_3b_3}{B} + \frac{d_1B - Ca_1b_1a_3b_3}{BC} \sum_{n=0}^{\infty} \left(-\frac{B}{C}\right)^n z^n, \end{aligned}$$

$$w(z) = \frac{d_1d_2 - a_1b_1}{Bz + C} = \frac{d_1d_2 - a_1b_1}{C} \sum_{n=0}^{\infty} \left(-\frac{B}{C}\right)^n z^n.$$

Thus

$$U_n = \begin{cases} \frac{1}{C} & \text{if } n = 0 \\ \frac{B - d_2a_3b_3C}{BC} \left(-\frac{B}{C}\right)^n & \text{if } n \geq 1, \end{cases}$$

$$V_n = \begin{cases} \frac{d_1}{C} & \text{if } n = 0 \\ \frac{d_1B - Ca_1b_1a_3b_3}{BC} \left(-\frac{B}{C}\right)^n & \text{if } n \geq 1, \end{cases}$$

$$W_n = \frac{d_1d_2 - a_1b_1}{C} \left(-\frac{B}{C}\right)^n.$$

2. $A \neq 0$

   (a) $B^2 = 4AC$

   In this case $Az^2 + Bz + C = A\left(z + \frac{B}{2A}\right)^2$ and

$$\begin{aligned} u(z) = & \frac{2d_2a_3b_3}{B} \cdot \frac{1}{1+\frac{2A}{B}z} + \frac{2(Bd_2a_3b_3 - 2A)}{B^2} \cdot \left(\frac{1}{1+\frac{2A}{B}z}\right)' \\ = & \frac{2d_2a_ab_3}{B} \sum_{n=0}^{\infty} \left(-\frac{2A}{B}\right)^n \\ & + \frac{2(Bd_2a_3b_3 - 2A)}{B^2} \sum_{n=0}^{\infty} \left(-\frac{2A}{B}\right)^{n+1} (n+1)z^n, \end{aligned}$$

$$\begin{aligned} v(z) = & \frac{a_1b_1a_3b_3z + d_1}{A\left(z+\frac{B}{2A}\right)^2} \\ = & \frac{2a_1b_1a_3b_3}{B} \cdot \frac{1}{1+\frac{2A}{B}z} + \frac{2(a_1b_1a_3b_3B - 2Ad_1)}{B^2} \left(\frac{1}{1+\frac{2A}{B}z}\right)' \\ = & \frac{2a_1b_1a_3b_3}{B} \sum_{n=0}^{\infty} \left(-\frac{2A}{B}\right)^n z^n \\ & + \frac{2(a_1b_1a_3b_3B - 2Ad_1)}{B^2} \sum_{n=0}^{\infty} \left(-\frac{2A}{B}\right)^{n+1} (n+1)z^n, \end{aligned}$$

$$\begin{aligned} w(z) = & \frac{d_1d_2 - a_1b_1}{A\left(z+\frac{B}{2A}\right)^2} = \frac{4A(a_1b_1 - d_1d_2)}{B^2} \left(\frac{1}{1+\frac{2A}{B}z}\right)' \\ = & \frac{4A(a_1b_1 - d_1d_2)}{B^2} \sum_{n=0}^{\infty} \left(-\frac{2A}{B}\right)^{n+1} (n+1)z^n. \end{aligned}$$

   Hence

$$U_n = \frac{Bd_2a_3b_3[2B - 4A(n+1)] + 8A^2(n+1)}{B^3} \left(-\frac{2A}{B}\right)^n,$$

$$V_n = \frac{Ba_1b_1a_3b_3[2B - 4A(n+1)] + 8A^2d_1(n+1)}{B^3} \left(-\frac{2A}{B}\right)^n,$$

$$W_n = \frac{4A(n+1)(a_1b_1 - d_1d_2)}{B^2} \left(-\frac{2A}{B}\right)^n.$$

(b) $B^2 \neq 4AC$

Let

$$r_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \qquad r_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

Then

$$\begin{aligned} u(z) = &\ \tfrac{d_2 a_3 b_3 z + 1}{A(z-r_1)(z-r_2)} = -\tfrac{r_1 d_2 a_3 b_3 + 1}{A r_1 (r_1 - r_2)} \cdot \tfrac{1}{1-\frac{z}{r_1}} + \tfrac{r_2 d_2 a_3 b_3 + 1}{A r_2 (r_1 - r_2)} \cdot \tfrac{1}{1-\frac{z}{r_2}} \\ = &\ -\tfrac{r_1 d_2 a_3 b_3 + 1}{A r_1 (r_1 - r_2)} \sum_{n=0}^{\infty} \tfrac{1}{r_1^n} z^n + \tfrac{r_2 d_2 a_3 b_3 + 1}{A r_2 (r_1 - r_2)} \sum_{n=0}^{\infty} \tfrac{1}{r_2^n} z^n, \end{aligned}$$

$$\begin{aligned} v(z) = &\ \tfrac{a_1 b_1 a_3 b_3 z + d_1}{A(z-r_1)(z-r_2)} \\ = &\ -\tfrac{r_1 a_1 b_1 a_3 b_3 + d_1}{A r_1 (r_1 - r_2)} \cdot \tfrac{1}{1-\frac{z}{r_1}} + \tfrac{r_2 a_1 b_1 a_3 b_3 + d_1}{A r_2 (r_1 - r_2)} \cdot \tfrac{1}{1-\frac{z}{r_2}} \\ = &\ -\tfrac{r_1 a_1 b_1 a_3 b_3 + d_1}{A r_1 (r_1 - r_2)} \sum_{n=0}^{\infty} \tfrac{1}{r_1^n} z^n + \tfrac{r_2 a_1 b_1 a_3 b_3 + d_1}{A r_2 (r_1 - r_2)} \sum_{n=0}^{\infty} \tfrac{1}{r_2^n} z^n, \end{aligned}$$

$$\begin{aligned} w(z) = &\ \tfrac{d_1 d_2 - a_1 b_1}{A(z-r_1)(z-r_2)} = \tfrac{a_1 b_1 - d_1 d_2}{A r_1 (r_1 - r_2)} \cdot \tfrac{1}{1-\frac{z}{r_1}} + \tfrac{d_1 d_2 - a_1 b_1}{A r_2 (r_1 - r_2)} \cdot \tfrac{1}{1-\frac{z}{r_2}} \\ = &\ \tfrac{a_1 b_1 - d_1 d_2}{A r_1 (r_1 - r_2)} \sum_{n=0}^{\infty} \tfrac{1}{r_1^n} z^n + \tfrac{d_1 d_2 - a_1 b_1}{A r_2 (r_1 - r_2)} \sum_{n=0}^{\infty} \tfrac{1}{r_2^n} z^n. \end{aligned}$$

Therefore

$$U_n = -\frac{r_1 d_2 a_3 b_3 + 1}{A(r_1 - r_2)} \cdot \frac{1}{r_1^n} + \frac{r_2 d_2 a_3 b_3 + 1}{A(r_1 - r_2)} \cdot \frac{1}{r_2^n},$$

$$V_n = -\frac{r_1 a_1 b_1 a_3 b_3 + d_1}{A(r_1 - r_2)} \cdot \frac{1}{r_1^{n+1}} + \frac{r_2 a_1 b_1 a_3 b_3 + d_1}{A(r_1 - r_2)} \cdot \frac{1}{r_2^{n+1}},$$

$$W_n = \frac{a_1 b_1 - d_1 d_2}{A(r_1 - r_2)} \cdot \frac{1}{r_1^n} + \frac{d_1 d_2 - a_1 b_1}{A(r_1 - r_2)} \cdot \frac{1}{r_2^n}. \qquad \square$$

Obviously, from Proposition 2.2 we immediately get Theorem 1.2.

2.3. Closing comments

At the end of the paper, let us notice that using the presented method it is also possible that the formula for any $n \times n$ tridiagonal k-Toeplitz matrix determinant (where $k < n$). However, one can observe, that this requires even more complicated computations than in the cases $k = 2, 3$. Moreover, this approach has sense only in the case when the ratio $\frac{k}{n}$ is small.

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