Annals of Mathematical Sciences and Applications
Volume 8, Number 1, 21–76, 2023

On a degenerate mixed-type boundary value problem for the two-dimensional self-similar Euler equations

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This paper is concerned with the semi-hyperbolic structures originated from the study of the two-dimensional Riemann problem for the compressible Euler equations in gas dynamics. Given two piece of smooth curves in the self-similar plane such that one is a sonic curve and the other is a characteristic curve, we establish the existence of classical supersonic solutions in the angular region near the corner point. The main difficulty arises from the coupling of nonlinearity and degeneracy at the corner. With the help of the characteristic decomposition technique, the problem is solved by transforming the self-similar Euler equations into a new degenerate hyperbolic system with explicitly singularity-regularity structures. Based on the solution in the partial hodograph plane, we construct a smooth sonic-supersonic solution of the original degenerate mixed-type boundary value problem in the self-similar plane.

AMS 2000 subject classifications: 35Q31, 35Q35, 35M33, 35A09, 76H05.
Keywords and phrases: Self-similar Euler equations, two-dimensional Riemann problem, sonic curve, sonic-supersonic solution, partial hodograph transformation.

1. Introduction

Consider the two-dimensional (2-D) isentropic compressible Euler equations

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1}$$

where ρ is the density, (u, v) is the velocity and p is the pressure given by the polytropic gas equation $p(\rho) = A\rho^\gamma$, $A > 0$ is a constant can be scaled to be one, $\gamma > 1$ is the adiabatic gas constant [5].

We are interested in the Riemann problem of (1), which is a kind of Cauchy problem with special initial data that is constant along each ray from the origin. Based on these special initial data, the flow is pseudo-steady for which a solution depends on the self-similar variables $(\xi, \eta) = (x/t, y/t)$. The 2-D Riemann problem of (1) with four piecewise constants was initiated by Zhang and Zheng [37]. By using the generalized characteristic analysis method, they conjectured the configurations of the global solutions. The solution configurations were completed and confirmed afterward by numerical simulations [24, 40]. Many very important and interesting phenomena, such as shock reflection and dam collapse, are included in the framework of the 2-D Riemann problem (see the survey [22]). The rigorous proof of the numerical simulations are considerably difficult due to the fact that each solution configuration typically contains transonic and small-scale structures. Many efforts have been made to understand these configurations for more specific initial data. A representative example is the expansion problem of a semi-infinite wedge of gas into vacuum, which is often interpreted as the dam collapse problem in hydraulics [20, 33]. In the context of 2-D Riemann problem, it is that of the interaction of two 2-D planar rarefaction waves. In [21], Li established the first global existence result for the interaction of rarefaction waves in the hodograph plane. This solution was converted into the physical plane by Li and Zheng [26, 27] by applying the characteristic decomposition technique which is a powerful tool for studying the degenerate hyperbolic problems developed in [6, 25]. Subsequently, the existence of global solutions to the interaction of two arbitrary planar rarefaction waves was solved directly in the physical plane [3, 14, 23]. The interaction of a centered simple wave and a planar rarefaction wave was discussed by Sheng et al. [18, 18, 29]. The results of shock reflection and shock diffraction problems can be found, among others, in [1, 2, 7, 41].

The numerical simulations in [8] show that shock waves may formate near sonic curves even in the interaction of four-rarefaction waves, which illustrate that the properties of supersonic solutions near sonic curves are indeed extremely complicated. In [32], Song and Zheng proposed the concept of semi-hyperbolic region which is not parabolic or hyperbolic in the classical sense. A semi-hyperbolic region is a small region in which a family of characteristics starts on a sonic curve and ends on either a sonic curve or a transonic shock wave. This type of region may also appear in many other situations, such as in the transonic flow over an airfoil [4, 17] and in Guderley shock reflection [34, 35]. It is worth noting that the study of semi-hyperbolic solutions may provide us the important information about sonic curves. A semi-hyperbolic patch as shown in Figure 1 was first extracted theoretically

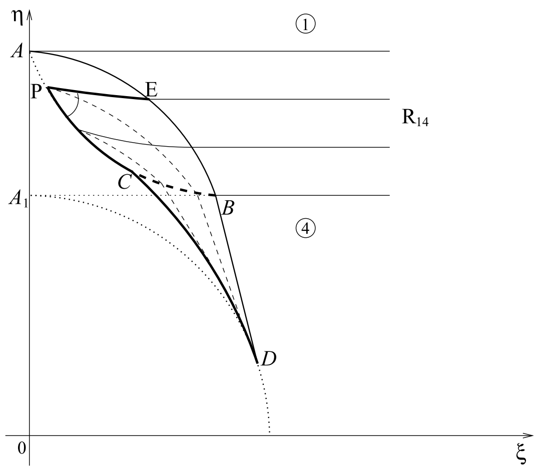

Figure 1: A semi-hyperbolic patch produced by a planar rarefaction wave $R_{14}(\eta) : \eta = v + \sqrt{p'(\rho)},\ v = v_4 + \int_{\rho_4}^{\rho} \sqrt{p'(s)}/s \ \mathrm{d}s,\ u = u_1 = u_4 = 0$ with $v_4 + \sqrt{p'(\rho_4)} \le \eta \le v_1 + \sqrt{p'(\rho_1)}$, $0 < \rho_4 \le \rho \le \rho_1$.

in [32] for the 2-D pressure gradient system and then extended to the isentropic and isothermal Euler equations in [28, 14]. A similar semi-hyperbolic problem to the 2-D nonlinear wave system with Chaplygin gases was considered in [16]. In [30], a semi-hyperbolic patch arising from a transonic shock in simple waves interaction was constructed for the pressure gradient system. The regularity of the semi-hyperbolic problems for the pressure-gradient and Euler systems were discussed in [15, 31, 36]. Hu and Li [12] established a global supersonic-sonic solution in a region surrounded by a streamline and a characteristic curve for the steady full Euler equations with a special relation of entropy and vorticity, also see Hu [9] and Hu and Li [13] for the related results.

The framework for studying the semi-hyperbolic problem as described in Figure 1 in previous papers [32, 28, 14, 16, 15, 31, 36] is to give the negative characteristic curve $\widehat{BC}$ and then solve two degenerate Goursat problems in the regions ABC and BCD with the possible sonic boundary $\widehat{AC}$ and envelope curve $\widehat{CD}$ respectively. Particularly, the flow in the region BCD is a simple wave. Unlike the previous framework, we plan to study the semi-hyperbolic structure by giving the sonic curve $\widehat{AC}$ and then solving the problem in the region ABC to determine the negative characteristic $\widehat{BC}$. In order to realize this programme, it is necessary to study first the existence of classical sonic-supersonic solutions of the degenerate boundary value and degenerate mixed-type boundary value problems. The existence of classical sonic-supersonic solutions of the degenerate boundary value problem to the Euler equations were investigated in [38, 39, 11]. In [10], Hu and Chen constructed a classical sonic-supersonic solution of a degenerate mixed-type boundary value problem for the steady full Euler equations. In the present paper, we consider a degenerate mixed-type boundary value problem for the 2-D self-similar Euler equations and establish the existence of classical sonic-supersonic solutions in an angular region bounded by a sonic curve and a characteristic curve. Specifically, we consider the degenerate mixed-type problem as follows.

Problem 1.1. *Let $\widehat{PC}$ and $\widehat{PE}$ be two piece of smooth curves in the self-similar plane, see Figure 1. We assign the boundary data on $\widehat{PC}$ and $\widehat{PE}$ such that $\widehat{PC}$ is a sonic curve and $\widehat{PE}$ is a negative characteristic curve. We look for a classical self-similar supersonic solution for* (1) *in the region bounded by $\widehat{PC}$ and $\widehat{PE}$ near point P.*

The degenerate mixed-type boundary value problem 1.1 is also called the degenerate Cauchy-Goursat problem. The main difficulty of this problem is to deal with the coupling of nonlinearity and degeneracy near the corner point. We adopt the pseudo-Mach angle and pseudo-velocity potential as the auxiliary coordinate system to transform the pseudo-steady Euler equations into a new nonlinear system with clear singularity-regularity structures. An iterative sequence generated by a nonlinear integral system is formed and then shown to be uniformly convergent. To overcome the influence of nonlinearity on the convergence of iterative sequence, the difference of iterative fluctuations need to be analyzed carefully and the difference of boundary values on the characteristic need to be estimated accurately. Based on the solution in the partial hodograph plane, we construct a classical sonic-supersonic solution to the original degenerate mixed-type boundary value problem for the 2-D self-similar isentropic irrotational Euler equations.

The rest of the paper is organized as follows. In Section 2, we introduce the angle variables and derive their characteristic decompositions to formulate the problem and state the main result of the paper. In Section 3, we transform the problem into a new degenerate mixed-type boundary value problem in a partial hodograph plane and use the iteration method to solve this new problem under some higher-order compatibility conditions at the corner point. In Section 4, with the aid of the solution in terms of partial hodograph variables, we establish the local existence of classical supersonic solutions to the original problem in the self-similar plane and then complete the proof of the main theorem.

2. Formulation of the problem and the main result

In this section, we introduce a set of dependent variables and derive their characteristic decompositions to formulate the degenerate mixed-type boundary value problem and state the main result of the paper.

2.1. Preliminary characteristic decompositions

In terms of self-similar variables (ξ, η), system (1) can be written as

$$\begin{cases} U\rho_\xi + V\rho_\eta + \rho(u_\xi + v_\eta) = 0, \\ Uu_\xi + Vu_\eta + \left(\dfrac{c^2}{\gamma - 1}\right)_\xi = 0, \\ Uv_\xi + Vv_\eta + \left(\dfrac{c^2}{\gamma - 1}\right)_\eta = 0, \end{cases} \tag{2}$$

where $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity and $c = \sqrt{p'(\rho)}$ is the sound speed. We further assume that the flow is irrotational, that is, $u_y = v_x$ in the (x, y) plane or equivalent $u_\eta = v_\xi$ in the (ξ, η) plane. Then system (2) reduces to

$$\begin{cases} (c^2 - U^2)u_\xi - UV(u_\eta + v_\xi) + (c^2 - V^2)v_\eta = 0, \\ u_\eta - v_\xi = 0, \end{cases} \tag{3}$$

which is supplemented with the pseudo-Bernoulli law

$$\frac{c^2}{\gamma - 1} + \frac{U^2 + V^2}{2} = -\phi, \quad \phi_\xi = U, \quad \phi_\eta = V, \tag{4}$$

where ϕ is the pseudo-velocity potential.

It is obtained by direct calculations that the two eigenvalues of (3) are

$$\Lambda_\pm = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2}, \tag{5}$$

from which one can clearly see that system (3) is of mixed-type: supersonic for $U^2 + V^2 > c^2$, subsonic for $U^2 + V^2 < c^2$ and sonic for $U^2 + V^2 = c^2$. A curve is called a sonic curve if each point (ξ, η) on it satisfies $U^2(\xi, \eta) + V^2(\xi, \eta) = c^2(\xi, \eta)$. We perform the standard manipulation to achieve that the left eigenvectors of (3) are $\ell_\pm = (1, \Lambda_\mp)$ and then the characteristic forms are

$$\begin{cases} \partial^+ u + \Lambda_- \partial^+ v = 0, \\ \partial^- u + \Lambda_+ \partial^- v = 0, \end{cases} \qquad \partial^\pm = \partial_\xi + \Lambda_\pm \partial_\eta. \tag{6}$$

Following the previous works [11, 26], it is convenient to handle the sonic degenerate problems of the compressible Euler equations in terms of the angle variables. Introduce the pseudo-flow angle θ and pseudo-Mach angle ω as follows

$$\tan\theta = \frac{V}{U}, \quad \sin\omega = \frac{c}{q}, \tag{7}$$

and denote

$$\alpha := \theta + \omega, \quad \beta := \theta - \omega. \tag{8}$$

According to the expression of $\Lambda_\pm$ in (5), one obtains that

$$\tan\alpha = \Lambda_+, \quad \tan\beta = \Lambda_-, \tag{9}$$

which mean that the angles α and β are the inclination angles of positive and negative characteristic curves, respectively. Moreover, we combine (4) and (7) to express the functions (c, u, v) in terms of ϕ, θ, ω as follows:

$$c = \sqrt{\frac{-2\phi\kappa \sin^2\omega}{\kappa + \sin^2\omega}}, \quad u = \xi - c\frac{\cos\theta}{\sin\omega}, \quad v = \eta - c\frac{\sin\theta}{\sin\omega}, \tag{10}$$

where $\kappa = (\gamma-1)/2 > 0$. And now the sonic curve is $\{(\xi,\eta) : \sin\omega(\xi,\eta) = 1\}$. In addition, we introduce the following normalized directional derivatives

$$\begin{aligned} &\bar\partial^+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta, \quad \bar\partial^- = \cos\beta\partial_\xi + \sin\beta\partial_\eta, \\ &\bar\partial^0 = \cos\theta\partial_\xi + \sin\theta\partial_\eta, \quad \bar\partial^\perp = \sin\theta\partial_\xi - \cos\theta\partial_\eta, \end{aligned} \tag{11}$$

from which one has

$$\begin{cases} \partial_\xi = \dfrac{\cos\theta\sin\omega(\bar\partial^+ + \bar\partial^-) - \sin\theta\cos\omega(\bar\partial^+ - \bar\partial^-)}{\sin(2\omega)}, \\ \partial_\eta = \dfrac{\sin\theta\sin\omega(\bar\partial^+ + \bar\partial^-) + \cos\theta\cos\omega(\bar\partial^+ - \bar\partial^-)}{\sin(2\omega)}, \end{cases} \quad \begin{cases} \bar\partial^0 = \dfrac{\bar\partial^+ + \bar\partial^-}{2\cos\omega}, \\ \bar\partial^\perp = \dfrac{\bar\partial^- - \bar\partial^+}{2\sin\omega}. \end{cases} \tag{12}$$

Combining with (6), (10) and (11) gives a new system in terms of the variables (θ, ω)

$$
\begin{cases}
\bar{\partial}^+\theta + \dfrac{\cos\omega}{\kappa+\varpi^2}\bar{\partial}^+\varpi = \dfrac{\varpi^2}{c}\cdot\dfrac{\kappa-1+2\varpi^2}{\kappa+\varpi^2}, \\
\bar{\partial}^-\theta - \dfrac{\cos\omega}{\kappa+\varpi^2}\bar{\partial}^-\varpi = -\dfrac{\varpi^2}{c}\cdot\dfrac{\kappa-1+2\varpi^2}{\kappa+\varpi^2}.
\end{cases} \tag{13}
$$

Here and below, we use the mixed variables ω and $\varpi := \sin\omega$ in a system for convenience. Set

$$
R = \frac{\bar{\partial}^+ c}{c}, \quad S = \frac{\bar{\partial}^- c}{c}. \tag{14}
$$

Then we can obtain the relations between ϖ and (R, S) by the pseudo-Bernoulli law (4)

$$
\bar{\partial}^+\varpi = \frac{\varpi(\kappa+\varpi^2)}{\kappa}R - \frac{\cos\omega\varpi^2}{c}, \quad \bar{\partial}^-\varpi = \frac{\varpi(\kappa+\varpi^2)}{\kappa}S - \frac{\cos\omega\varpi^2}{c}, \tag{15}
$$

and the equations for (R, S) by the characteristic decomposition technique

$$
\begin{cases}
\bar{\partial}^- R = R\Big\{-\dfrac{2\cos\omega\varpi}{c} + \dfrac{(\kappa+1)(R+S)}{2\kappa\cos^2\omega} - \dfrac{\kappa+2\varpi^2}{\kappa}S\Big\}, \\
\bar{\partial}^+ S = S\Big\{-\dfrac{2\cos\omega\varpi}{c} + \dfrac{(\kappa+1)(R+S)}{2\kappa\cos^2\omega} - \dfrac{\kappa+2\varpi^2}{\kappa}R\Big\},
\end{cases} \tag{16}
$$

The detailed derivation of (16) can be found in [28]. We further introduce

$$
\bar{R} = \varpi\sqrt{\kappa+\varpi^2}R, \quad \bar{S} = -\varpi\sqrt{\kappa+\varpi^2}S, \tag{17}
$$

from which and (16) arrives at

$$
\begin{cases}
\bar{\partial}^-\bar{R} = \bar{R}\Big\{\dfrac{\kappa+1}{2\kappa\varpi\sqrt{\kappa+\varpi^2}}\cdot\dfrac{\bar{R}-\bar{S}}{\cos^2\omega} - \dfrac{\varpi(3\kappa+4\varpi^2)\cos\omega}{c(\kappa+\varpi^2)}\Big\}, \\
\bar{\partial}^+\bar{S} = \bar{S}\Big\{\dfrac{\kappa+1}{2\kappa\varpi\sqrt{\kappa+\varpi^2}}\cdot\dfrac{\bar{R}-\bar{S}}{\cos^2\omega} - \dfrac{\varpi(3\kappa+4\varpi^2)\cos\omega}{c(\kappa+\varpi^2)}\Big\}.
\end{cases} \tag{18}
$$

2.2. The problem and the main result

We now specify the boundary conditions and formulate the mixed-type boundary value problem 1.1 in terms of the angle variables. Given a smooth curve $\widehat{PC}: \eta = \varphi(\xi)(\xi \in [\xi_P, \xi_C])$, we suppose that the function φ and the boundary values $(c, \theta, \varpi)|_{\widehat{PC}} = (\hat{c}, \hat{\theta}, \hat{\varpi})(\xi)$ satisfy

$$\varphi(\xi) \in C^3([\xi_P, \xi_C]), \quad (\hat{c}, \hat{\theta})(\xi) \in C^3([\xi_P, \xi_C]), \quad \hat{\varpi}(\xi) = 1. \tag{19}$$

Thus $\widehat{PC}$ is a sonic curve. Let $\widehat{PE}: \eta = \psi(\xi)$ $(\xi \in [\xi_P, \xi_E])$ be a smooth curve satisfying $\varphi(\xi_P) = \psi(\xi_P)$. We assume that the function ψ and the boundary values $(c, \theta, \varpi)|_{\widehat{PE}} = (\tilde{c}, \tilde{\theta}, \tilde{\varpi})(\xi)$ satisfy

$$\begin{aligned} &\psi(\xi) \in C^1([\xi_P, \xi_E)) \cap C^4((\xi_P, \xi_E)), \\ &(\tilde{c}, \tilde{\theta}, \tilde{\varpi})(\xi) \in C^1([\xi_P, \xi_E)) \cap C^4((\xi_P, \xi_E)), \\ &\tilde{\theta}(\xi) = \arctan \psi'(\xi) + \arcsin \tilde{\varpi}(\xi). \end{aligned} \tag{20}$$

Then $\widehat{PE}$ is a negative characteristic curve. We further assume that the functions $(\hat{c}, \hat{\theta}, \hat{\varpi})(\xi)$ and $(\tilde{c}, \tilde{\theta}, \tilde{\varpi})(\xi)$ satisfy the following basic compatibility conditions at the corner P and the characteristic curve $\widehat{PE}$

$$\begin{gathered} \hat{c}(\xi_P) = \tilde{c}(\xi_P), \quad \hat{\theta}(\xi_P) = \tilde{\theta}(\xi_P), \quad \hat{\varpi}(\xi_P) = \tilde{\varpi}(\xi_P) = 1, \\ \tilde{\theta}' - \frac{\sqrt{1-\tilde{\varpi}^2}}{\kappa + \tilde{\varpi}^2}\tilde{\varpi}' = -\frac{\tilde{\varpi}^2\sqrt{(\psi')^2+1}}{\tilde{c}} \cdot \frac{\kappa - 1 + 2\tilde{\varpi}^2}{\kappa + \tilde{\varpi}^2} \quad \forall\, \xi \in [\xi_P, \xi_E). \end{gathered} \tag{21}$$

The last relation in (21) comes from the second governing equation in (13). In addition, we can express ξ as a function of $\cos\tilde{\omega}$ on the curve $\widehat{PE}$ to get $\xi = \tilde{\xi}(\cos\tilde{\omega})$. Set

$$\begin{aligned} &\tilde{b}_0(\cos\tilde{\omega}) \\ &\quad := \frac{\kappa}{\sqrt{1-\cos^2\tilde{\omega}}(\kappa + 1 - \cos^2\tilde{\omega})}\left(\frac{\tilde{\varpi}'(\tilde{\xi}(\cos\tilde{\omega}))}{\sqrt{(\psi')^2+1}} + \frac{\cos\tilde{\omega}(1-\cos^2\tilde{\omega})}{\tilde{c}(\tilde{\xi}(\cos\tilde{\omega}))}\right). \end{aligned} \tag{22}$$

Actually, $\tilde{b}_0$ is the boundary value of S on the curve $\widehat{PE}$. As the function of $\cos\tilde{\omega}$, we further require $\tilde{b}_0$ to satisfy the regularity

$$\tilde{b}_0(\cos\tilde{\omega}) \in C^3([0, \cos\tilde{\omega}(E))). \tag{23}$$

We comment that this requirement can be achievable by the degeneracy of the derivative of $\tilde{\xi}(\cos\tilde{\omega})$ at point P.

Our main conclusion can be stated in the following theorem.

Theorem 2.1. *Let the boundary conditions* (19)–(20) *and* (23) *hold. Assume that the compatibility conditions* (21) *and a higher-order compatibility* *condition* $(\mathcal{C})$ *are satisfied at the corner* P *(the condition* $(\mathcal{C})$ *is given in* (58) *in Subsection 3.1). We further suppose that the following inequality conditions at point* P *hold*

$$\begin{aligned} &\hat{c}'(\xi_P) < 0, \quad \tilde{\varpi}'(\xi_P) < 0, \\ &(\hat{c}\hat{\theta}' - \varphi' \cos\hat{\theta} + \sin\hat{\theta})(\xi_P) > 0, \quad (\cos\hat{\theta} + \varphi' \sin\hat{\theta})(\xi_P) < 0. \end{aligned} \tag{24}$$

Then there exists a classical supersonic solution (c, θ, ϖ) *for system* (13) *with* $(c, \theta, \varpi)|_{\widehat{PC}} = (\hat{c}, \hat{\theta}, \hat{\varpi})(\xi)$ *and* $(c, \theta, \varpi)|_{\widehat{PE}} = (\tilde{c}, \tilde{\theta}, \tilde{\varpi})(\xi)$ *in the angular region around* P.

Remark 1. The inequality conditions in (24) are matched the study of semi-hyperbolic patch in the previous works [28, 14].

2.3. The boundary values of $(\bar{R}, \bar{S})$

The strategy of this paper is to construct the solution of system (13) by solving a problem corresponding to system (18) in a partial hodograph plane. Hence we need the information of $\bar{R}$ and $\bar{S}$ on the boundaries $\widehat{PC}$ and $\widehat{PE}$.

We first assert that system (13) is compatible at the corner P. The compatibility of the second equation of (13) follows from (21). For the first equation, we note by the fact $\cos\omega(P) = 0$ and (21) that

$$\begin{aligned} \bar{\partial}^+\theta(P) =& \cos(\theta(P) + \omega(P))\theta_\xi(P) + \sin(\theta(P) + \omega(P))\theta_\eta(P) \\ =& -\sin(\theta(P))\theta_\xi(P) + \cos(\theta(P))\theta_\eta(P) \\ =& -\cos(\theta(P) - \omega(P))\theta_\xi(P) - \sin(\theta(P) - \omega(P))\theta_\eta(P) \\ =& -\bar{\partial}^-\theta(P) = -\frac{1}{\sqrt{1 + (\psi')^2}}\tilde{\theta}'(P) = \frac{1}{\tilde{c}(P)}, \end{aligned}$$

which implies that the first equation of (13) holds at P.

Since $\widehat{PE}$ is a negative characteristic curve, we can get the boundary data of S by (15) and (22)

$$S|_{\widehat{PE}} = \left\{ \frac{\kappa}{\varpi(\kappa + \varpi^2)} \left(\bar{\partial}^-\varpi + \frac{\cos\omega\varpi^2}{c} \right) \right\}\Big|_{\widehat{PE}} = \tilde{b}_0(\cos\tilde{\omega}), \tag{25}$$

from which and (17) one gets

$$\begin{aligned} \bar{S}|_{\widehat{PE}} &= -(\varpi\sqrt{\kappa + \varpi^2} S)|_{\widehat{PE}} \\ &= -\sqrt{(1 - \cos^2\tilde{\omega})(\kappa + 1 - \cos^2\tilde{\omega})}\tilde{b}_0(\cos\tilde{\omega}) := \tilde{b}_1(\cos\tilde{\omega}). \end{aligned} \tag{26}$$

The boundary data of $\bar{R}$ on $\widehat{PE}$ will be obtained later by solving a singular ODE problem.

For the data of (R, S) on the sonic curve $\widehat{PC}$, we first note by (12) and (14) that

$$R + S = \frac{2\cos\omega\bar{\partial}^0 c}{c}, \tag{27}$$

which indicates by $\cos\hat{\omega} = 0$ that $R|_{\widehat{PC}} = -S|_{\widehat{PC}}$. On the other hand, adding the two equations of (13) and applying (15) yields

$$R - S = -\frac{2\kappa\bar{\partial}^0\theta}{\varpi}. \tag{28}$$

Thus it follows that

$$R|_{\widehat{PC}} = -S|_{\widehat{PC}} = -\kappa(\bar{\partial}^0\theta)|_{\widehat{PC}},$$

and then

$$\bar{R}|_{\widehat{PC}} = \bar{S}|_{\widehat{PC}} = -\kappa\sqrt{\kappa+1}(\bar{\partial}^0\theta)|_{\widehat{PC}}. \tag{29}$$

To obtain the boundary data of $\bar{\partial}^0\theta$, we subtract the two equations of (13) and notice the definition of $\bar{\partial}^\perp$ to acquire

$$\bar{\partial}^\perp\theta = \frac{\cos\omega(\bar{\partial}^+\varpi + \bar{\partial}^-\varpi)}{2\varpi(\kappa+\varpi^2)} - \frac{\varpi(\kappa - 1 + 2\varpi^2)}{c(\kappa+\varpi^2)}, \tag{30}$$

which along with the fact $\cos\hat{\omega} = 0$ leads to

$$\sin\hat{\theta}(\theta_\xi)|_{\widehat{PC}} - \cos\hat{\theta}(\theta_\eta)|_{\widehat{PC}} = -\frac{1}{\hat{c}},$$

which together with the boundary value $\theta(\xi, \varphi(\xi)) = \hat{\theta}(\xi)$ arrives at

$$(\theta_\xi)|_{\widehat{PC}} = \frac{\hat{c}\cos\hat{\theta}\hat{\theta}' - \varphi'}{\hat{c}(\cos\hat{\theta} + \varphi'\sin\hat{\theta})}, \quad (\theta_\eta)|_{\widehat{PC}} = \frac{\hat{c}\sin\hat{\theta}\hat{\theta}' + 1}{\hat{c}(\cos\hat{\theta} + \varphi'\sin\hat{\theta})},$$

from which we have

$$(\bar{\partial}^0\theta)|_{\widehat{PC}} = \cos\hat{\theta}(\theta_\xi)|_{\widehat{PC}} + \sin\hat{\theta}(\theta_\eta)|_{\widehat{PC}} = \frac{\hat{c}\hat{\theta}' - \varphi'\cos\hat{\theta} + \sin\hat{\theta}}{\hat{c}(\cos\hat{\theta} + \varphi'\sin\hat{\theta})}. \tag{31}$$

Combining with (29) and (31) gives

$$\bar{R}|_{\widehat{PC}} = \bar{S}|_{\widehat{PC}} = -\frac{\kappa\sqrt{\kappa+1}(\hat{c}\hat{\theta}' - \varphi'\cos\hat{\theta} + \sin\hat{\theta})}{\hat{c}(\cos\hat{\theta} + \varphi'\sin\hat{\theta})} := \hat{a}_0(\xi). \tag{32}$$

Moreover, for late use, we here derive the boundary data of $\bar{\partial}^0 c$ on $\widehat{PC}$. It suggests by (14) and (17) that

$$\bar{R} + \bar{S} = \frac{\varpi\sqrt{\kappa+\varpi^2}}{c}(\bar{\partial}^+ c - \bar{\partial}^- c) = -\frac{2\varpi^2\sqrt{\kappa+\varpi^2}}{c}\bar{\partial}^\perp c,$$

from which and (32) obtains

$$(\bar{\partial}^\perp c)|_{\widehat{PC}} = -\frac{\hat{c}\hat{a}_0}{\sqrt{\kappa+1}}. \tag{33}$$

which combined with the boundary value $c(\xi, \varphi(\xi)) = \hat{c}(\xi)$ achieves

$$(c_\xi)|_{\widehat{PC}} = \frac{\hat{c}'\cos\hat{\theta} - \frac{\hat{c}\hat{a}_0}{\sqrt{\kappa+1}}\varphi'}{\cos\hat{\theta} + \varphi'\sin\hat{\theta}}, \quad (c_\eta)|_{\widehat{PC}} = \frac{\hat{c}'\sin\hat{\theta} + \frac{\hat{c}\hat{a}_0}{\sqrt{\kappa+1}}}{\cos\hat{\theta} + \varphi'\sin\hat{\theta}}.$$

Hence one has

$$(\bar{\partial}^0 c)|_{\widehat{PC}} = \frac{\sqrt{\kappa+1}\hat{c}' + \hat{c}\hat{a}_0(\sin\hat{\theta} - \varphi'\cos\hat{\theta})}{\sqrt{\kappa+1}(\cos\hat{\theta} + \varphi'\sin\hat{\theta})}, \tag{34}$$

and then by (27) and (17)

$$\begin{aligned} \frac{\bar{R} - \bar{S}}{2\cos\omega}\bigg|_{\widehat{PC}} &= \frac{\varpi\sqrt{\kappa+\varpi^2}\bar{\partial}^0 c}{c}\bigg|_{\widehat{PC}} \\ &= \frac{\sqrt{\kappa+1}\hat{c}' + \hat{c}\hat{a}_0(\sin\hat{\theta} - \varphi'\cos\hat{\theta})}{\hat{c}(\cos\hat{\theta} + \varphi'\sin\hat{\theta})} := \hat{a}_1(\xi). \end{aligned} \tag{35}$$

Finally, we discuss the properties of the boundary data of $(\bar{R}, \bar{S})$. We first note by (22), (26) and (32) that $\hat{a}_0(\xi_P) = \tilde{b}_1(0)$. According to the regularity assumptions in (19)–(20) and (23), we see that $\hat{a}_0, \hat{a}_1$ are C^2-continuous functions and $\tilde{b}_1$ is C^3-continuous function provided $\cos\hat{\theta} + \varphi'\sin\hat{\theta} \neq 0$. Recalling the inequality conditions in (24) and employing the expression of $\hat{a}_0$, we know by continuity that there exist two small positive constants ε_0 and δ_0 such that $\hat{c}'(\xi) \leq -\varepsilon_0$, $\tilde{\varpi}'(\xi) \leq -\varepsilon_0$, $(\hat{c}\hat{\theta}' - \varphi'\cos\hat{\theta} + \sin\hat{\theta})(\xi) \geq \varepsilon_0$,

$(\cos\hat{\theta}+\varphi'\sin\hat{\theta})(\xi)\le -\varepsilon_0$, $\hat{a}_0(\xi)\ge\varepsilon_0$ and $\tilde{b}_1(\xi)\ge\varepsilon_0$ for any $\xi\in[\xi_P,\xi_P+\delta_0]$. Without loss of generality, we may assume these inequalities hold on the boundary curve $\widehat{PC}$ and $\widehat{PE}$. Otherwise, we can use the points C_1 and E_1 instead of C and E respectively, to make them hold on $\widehat{PC_1}$ and $\widehat{PE_1}$. Therefore we have the following conditions for the boundary data of $(\bar{R},\bar{S})$

$$
\begin{aligned}
&(\hat{a}_0,\hat{a}_1)(\xi)\in C^2([\xi_P,\xi_C)),\ \tilde{b}_1(\cos\tilde{\omega})\in C^3([0,\cos\tilde{\omega}(E)),\\
&\hat{c}'(\xi)\le-\varepsilon_0,\ \hat{a}_0(\xi)\ge\varepsilon_0\quad\forall\,\xi\in[\xi_P,\xi_C),\\
&\tilde{\varpi}'(\xi)\le-\varepsilon_0,\ \tilde{b}_1(\xi)\ge\varepsilon_0\quad\forall\,\xi\in[\xi_P,\xi_E).
\end{aligned}
\tag{36}
$$

3. Solutions in a partial hodograph plane

In this section, we transform system (18) into a new nonlinear degenerate hyperbolic system with clear singularity-regularity structures by introducing a partial hodograph transformation. The new problem will be solved by the iteration method.

3.1. Reformulated problem in a partial hodograph plane

To characterize the singularity of system (18) caused by the sonic degeneracy, we consider the problem in a partial hodograph plane. Introduce

$$
t'=\cos^2\omega,\quad z'=-\phi. \tag{37}
$$

Applying (4), (12), (15) and (17) yields the Jacobian of the transformation (37)

$$
\begin{aligned}
J:=&\frac{\partial(z',t')}{\partial(\xi,\eta)}=2\sin\omega(U\varpi_\eta-V\varpi_\xi)\\
=&c\frac{\bar{\partial}^-\varpi-\bar{\partial}^+\varpi}{\sin\omega}=-\frac{c\sqrt{\kappa+1-t'}(\bar{R}+\bar{S})}{\kappa\sqrt{1-t'}}.
\end{aligned}
\tag{38}
$$

Thanks to (36), we see that $J<0$ near the boundary curve $\widehat{PC}$.

By direct calculations, the operators $\bar{\partial}^\pm$ can be transformed into

$$
\begin{aligned}
\bar{\partial}^+&=\left\{-\frac{2\sqrt{(\kappa+1-t')(1-t')}\bar{R}}{\kappa}+\frac{2\sqrt{t'}\sqrt{(1-t')^3}}{c}\right\}\partial_{t'}+\frac{c\sqrt{t'}}{\sqrt{1-t'}}\partial_{z'},\\
\bar{\partial}^-&=\left\{\frac{2\sqrt{(\kappa+1-t')(1-t')}\bar{S}}{\kappa t}+\frac{2\sqrt{t'}\sqrt{(1-t')^3}}{c}\right\}\partial_{t'}+\frac{c\sqrt{t'}}{\sqrt{1-t'}}\partial_{z'},
\end{aligned}
\tag{39}
$$

where $c = c(z', t') = \sqrt{2\kappa(1-t')z'/(\kappa+1-t')}$. Then the functions $(\bar{R}, \bar{S})$ in terms of (z', t') satisfy

$$
\begin{cases}
\bar{R}_{t'} + \dfrac{c\kappa\sqrt{t'}}{2(1-t')T_1'}\bar{R}_{z'} = \dfrac{(\kappa+1)\bar{R}}{(1-t')\sqrt{\kappa+1-t'}T_1'} \cdot \dfrac{\bar{R}-\bar{S}}{4t'} \\
\qquad\qquad - \dfrac{\kappa(3\kappa+4-4t')\bar{R}}{2c(\kappa+1-t')T_1'}\sqrt{t'}, \\
\bar{S}_{t'} - \dfrac{c\kappa\sqrt{t'}}{2(1-t')T_2'}\bar{S}_{z'} = \dfrac{(\kappa+1)\bar{S}}{(1-t')\sqrt{\kappa+1-t'}T_2'} \cdot \dfrac{\bar{S}-\bar{R}}{4t'} \\
\qquad\qquad + \dfrac{\kappa(3\kappa+4-4t')\bar{S}}{2c(\kappa+1-t')T_2'}\sqrt{t'},
\end{cases}
\tag{40}
$$

where

$$
T_1' = \sqrt{\kappa+1-t'}\bar{S} + \frac{\kappa(1-t')}{c(z',t')}\sqrt{t'},
$$

$$
T_2' = \sqrt{\kappa+1-t'}\bar{R} - \frac{\kappa(1-t')}{c(z',t')}\sqrt{t'}.
$$

We note that the term $(\bar{R}-\bar{S})/t'$ in (40) corresponds to the term $(\bar{R}-\bar{S})/\cos^2\omega$ in the (ξ,η) plane, which is still singular at the sonic curve. Then we further introduce

$$
t = \sqrt{t'}, \quad z = z'. \tag{41}
$$

It is clear that the transformation $(z', t') \mapsto (z, t)$ and its inverse transformation are one-to-one, despite the fact that the Jacobian of this transformation has singularities at $t' = 0$. In terms of (z, t), the operators $\bar{\partial}^{\pm}$ are

$$
\begin{aligned}
\bar{\partial}^{+} &= \left\{ -\frac{\sqrt{(\kappa+1-t^2)(1-t^2)}\bar{R}}{\kappa t} + \frac{\sqrt{(1-t^2)^3}}{c} \right\}\partial_t + \frac{ct}{\sqrt{1-t^2}}\partial_z, \\
\bar{\partial}^{-} &= \left\{ \frac{\sqrt{(\kappa+1-t^2)(1-t^2)}\bar{S}}{\kappa t} + \frac{\sqrt{(1-t^2)^3}}{c} \right\}\partial_t + \frac{ct}{\sqrt{1-t^2}}\partial_z,
\end{aligned}
\tag{42}
$$

where $c = c(z,t) = \sqrt{2\kappa(1-t^2)z/(\kappa+1-t^2)}$. Hence, we can obtain the system of $(\bar{R},\bar{S})(z,t)$

$$
\begin{cases}
\bar{R}_t+\dfrac{c\kappa t^2}{(1-t^2)T_1}\bar{R}_z=\dfrac{(\kappa+1)\bar{R}}{(1-t^2)\sqrt{\kappa+1-t^2}T_1}\cdot\dfrac{\bar{R}-\bar{S}}{2t}\\
\qquad\qquad -\dfrac{\kappa(3\kappa+4-4t^2)\bar{R}}{c(\kappa+1-t^2)T_1}t^2,\\
\bar{S}_t-\dfrac{c\kappa t^2}{(1-t^2)T_2}\bar{S}_z=\dfrac{(\kappa+1)\bar{S}}{(1-t^2)\sqrt{\kappa+1-t^2}T_2}\cdot\dfrac{\bar{S}-\bar{R}}{2t}\\
\qquad\qquad +\dfrac{\kappa(3\kappa+4-4t^2)\bar{S}}{c(\kappa+1-t^2)T_2}t^2,
\end{cases}
\tag{43}
$$

where

$$
T_1=\sqrt{\kappa+1-t^2}\bar{S}+\frac{\kappa(1-t^2)}{c(z,t)}t,\quad T_2=\sqrt{\kappa+1-t^2}\bar{R}-\frac{\kappa(1-t^2)}{c(z,t)}t.
$$

Set

$$
\tilde{R}=\frac{1}{\bar{R}},\quad \tilde{S}=\frac{1}{\bar{S}}.
\tag{44}
$$

Then system (43) can be rewritten as

$$
\begin{cases}
\tilde{R}_t+\dfrac{\kappa c\tilde{S}t^2}{(1-t^2)T_3}\tilde{R}_z=\dfrac{\kappa+1}{(1-t^2)\sqrt{\kappa+1-t^2}T_3}\cdot\dfrac{\tilde{R}-\tilde{S}}{2t}\\
\qquad\qquad +\dfrac{\kappa(3\kappa+4-4t^2)}{c(\kappa+1-t^2)T_3}\tilde{R}\tilde{S}t^2,\\
\tilde{S}_t-\dfrac{\kappa c\tilde{R}t^2}{(1-t^2)T_4}\tilde{S}_z=\dfrac{\kappa+1}{(1-t^2)\sqrt{\kappa+1-t^2}T_4}\cdot\dfrac{\tilde{S}-\tilde{R}}{2t}\\
\qquad\qquad -\dfrac{\kappa(3\kappa+4-4t^2)}{c(\kappa+1-t^2)T_4}\tilde{R}\tilde{S}t^2,
\end{cases}
\tag{45}
$$

where

$$
T_3=\sqrt{\kappa+1-t^2}+\frac{\kappa(1-t^2)}{c(z,t)}\tilde{S}t,
$$
$$
T_4=\sqrt{\kappa+1-t^2}-\frac{\kappa(1-t^2)}{c(z,t)}\tilde{R}t.
$$

We next consider the boundary conditions of $(\tilde{R},\tilde{S})$ on the the (z,t) coordinates. According to the assumption $\hat{c}'(\xi)\le-\varepsilon_0$ in (36), we find by (4) that the function

$$z = \left(\frac{1}{\gamma-1} + \frac{1}{2}\right)\hat{c}^2(\xi), \quad (\xi \in [\xi_P, \xi_C))$$

is strictly decreasing, which implies that there exists an inverse function, denoted by $\xi = \hat{\xi}(z)$ $(z \in (z_1, z_2])$, where

$$z_1 = \left(\frac{1}{\gamma-1} + \frac{1}{2}\right)\hat{c}^2(\xi_C), \quad z_2 = \left(\frac{1}{\gamma-1} + \frac{1}{2}\right)\hat{c}^2(\xi_P).$$

It is clear that the sonic boundary $\widehat{PC}$ on the (ξ,η)-plane is transformed to a segment $\widehat{P'C'}$ on $t=0$ with $z \in (z_1, z_2]$ on the (z,t)-plane. On the segment $\widehat{P'C'}$, we have

$$(\tilde{R}, \tilde{S})(z, 0) = (\hat{a}_0, \hat{a}_0)(z) \quad \forall\ z \in (z_1, z_2], \tag{46}$$

where $\hat{a}_0(z) = 1/\hat{a}_0(\hat{\xi}(z))$. In addition, if system (45) admits a smooth solution $(\tilde{R}, \tilde{S})$, then by the exact form of (45), the solution should be satisfied

$$\tilde{R}_t|_{t=0} = \left.\frac{\tilde{R}-\tilde{S}}{2t}\right|_{t=0}, \quad \tilde{S}_t|_{t=0} = \left.\frac{\tilde{S}-\tilde{R}}{2t}\right|_{t=0}. \tag{47}$$

Therefore, we also have by (35)

$$\tilde{R}_t(z, 0) = \hat{a}_1(z), \quad \tilde{S}_t(0, z) = -\hat{a}_1(z) \quad \forall\ z \in (z_1, z_2], \tag{48}$$

where $\hat{a}_1(z) = -\hat{a}_1(\hat{\xi}(z))/\hat{a}_0^2(\hat{\xi}(z))$.

Now we discuss the image of boundary $\widehat{PE}$ on the (z,t)-plane. Applying (22) gives

$$\begin{aligned}&\left.\left(\frac{\kappa+\varpi^2}{\kappa\varpi^2}S - \frac{\bar{\partial}^-\varpi}{\varpi^3}\right)\right|_{\widehat{PE}}\\ =&\frac{1}{\tilde{\varpi}^3}\left(\frac{\tilde{\varpi}(\kappa+\tilde{\varpi}^2)}{\kappa}\tilde{b}_0 - \frac{\tilde{\varpi}'}{\sqrt{1+(\psi')^2}}\right) = \frac{\cos\tilde{\omega}}{\tilde{\varpi}\tilde{c}} > 0,\end{aligned}$$

for $\xi > \xi_P$, which indicates by (4) that the function

$$z = \left(\frac{\tilde{c}^2}{\gamma-1} + \frac{\tilde{c}^2}{2\tilde{\varpi}^2}\right)(\xi), \quad (\xi \in [\xi_P, \xi_E))$$

is strictly increasing. Thus there exists an inverse function $\xi = \tilde{\xi}(z)$ on $z \in [z_2, z_3)$, where

$$z_3 = \left(\frac{\tilde{c}^2}{\gamma - 1} + \frac{\tilde{c}^2}{2\tilde{\varpi}^2}\right)(\xi_E).$$

Let us use $\widehat{P'E'}: z = \tilde{z}(t)$ $(z \in [z_2, z_3))$ to denote the curve $\{(t, z)|\ t = \sqrt{1 - \tilde{\varpi}^2(\tilde{\xi}(z))}, z \in [z_2, z_3)\}$. Due to the assumption $\tilde{\varpi}'(\xi) \leq -\varepsilon_0$ in (36), we know that $z = \tilde{z}(t)$ is a strictly increasing function on $t \in [0, t_0)$, where $t_0 = \cos\tilde{\omega}(\xi_E)$. By (26), we obtain that the boundary value of $\tilde{S}$ on $\widehat{P'E'}$ is

$$\tilde{S}(\tilde{z}(t), t) = \tilde{b}_2(t) \quad \forall\, t \in [0, t_0), \tag{49}$$

where $\tilde{b}_2(t) = 1/\tilde{b}_1(t)$. Moreover, it is not difficult to check that the curve $\widehat{P'E'}$ is a positive characteristic of system (45) passing through point $(0, z_2)$ and the expression of $\tilde{z}(t)$ is

$$\tilde{z}(t) = z_2 + \int_0^t \frac{\kappa c(\tilde{z}(s), s)\tilde{b}_2(s)s^2}{(1 - s^2)[\sqrt{\kappa + 1 - s^2} + \frac{\kappa(1-s^2)}{c(\tilde{z}(s),s)}\tilde{b}_2(s)s]}\, \mathrm{d}s, \quad t \in [0, t_0). \tag{50}$$

For the boundary data of $\tilde{R}$ on $\widehat{P'E'}$, we consider the following ODE problem

$$\begin{cases} \tilde{d}_0'(t) = \dfrac{\kappa + 1}{(1 - t^2)\sqrt{\kappa + 1 - t^2}\tilde{T}_3} \cdot \dfrac{\tilde{d}_0(t) - \tilde{b}_2(t)}{2t} \\ \qquad\qquad + \dfrac{\kappa(3\kappa + 4 - 4t^2)\tilde{d}_0(t)\tilde{b}_2(t)t^2}{c(\tilde{z}(t), t)(\kappa + 1 - t^2)\tilde{T}_3}, \\ \tilde{d}_0(0) = \hat{a}_0(z_2), \end{cases} \tag{51}$$

where

$$\tilde{T}_3 = \sqrt{\kappa + 1 - t^2} + \frac{\kappa(1 - t^2)}{c(\tilde{z}(t), t)}\tilde{b}_2(t)t.$$

The solvability for the ODE problem (51) will be shown in Lemma 3.1 in Subsection 3.2.1. Hence the boundary data of $(\tilde{R}, \tilde{S})$ on $\widehat{P'E'}$ are

$$(\tilde{R}, \tilde{S})(\tilde{z}(t), t) = (\tilde{b}_2, \tilde{d}_0)(t), \quad \forall\, t \in [0, t_0). \tag{52}$$

We combine (46), (48) and (52) to obtain the mixed-type boundary conditions of system (45) as follows

(53)
$$
\begin{aligned}
(\tilde{R},\tilde{S})(z,0)=(\hat{a}_0,\hat{a}_0)(z),\ (\tilde{R}_t,\tilde{S}_t)(z,0)=(\hat{a}_1,-\hat{a}_1)(z),\quad \forall\ z\in(z_1,z_2],\\
(\tilde{R},\tilde{S})(\tilde{z}(t),t)=(\tilde{d}_0,\tilde{b}_2)(t),\quad \forall\ t\in[0,t_0).
\end{aligned}
$$

Moreover, it follows by (36) that the functions $\hat{a}_0,\hat{a}_1$ and $\tilde{b}_2$ satisfy

$$
\begin{gathered}
(\hat{a}_0,\hat{a}_1)\in C^2((z_1,z_2]),\ \tilde{b}_2\in C^3([0,t_0)),\\
\hat{a}_0(z)\geq\varepsilon_0\quad\forall z\in(z_1,z_2],\\
\tilde{b}_2(0)=\hat{a}_0(z_2),\ \tilde{b}_2'(0)=\hat{a}_1(z_2).
\end{gathered}
\tag{54}
$$

The compatibility conditions in (54) come from the compatibility conditions (21) and the definitions of functions $\hat{a}_0(z),\hat{a}_1(z)$ and $\tilde{b}_2(t)$).

To obtain a classical solution for the degenerate problem (45), (53), we need more information about the derivative of $\tilde{S}$ at the corner $P'(z_2,0)$. Denote

$$
\begin{aligned}
\tilde{\chi}_0(t)=&\frac{\kappa+1}{(1-t^2)\sqrt{\kappa+1-t^2}\tilde{T}_4}\cdot\frac{\tilde{b}_2(t)-\tilde{d}_0(t)}{2t}\\
&-\frac{\kappa(3\kappa+4-4t^2)}{c(\tilde{z}(t),t)(\kappa+1-t^2)\tilde{T}_4}\tilde{d}_0(t)\tilde{b}_2(t)t^2,
\end{aligned}
\tag{55}
$$

where

$$
\tilde{T}_4=\sqrt{\kappa+1-t^2}-\frac{\kappa(1-t^2)}{c(\tilde{z}(t),t)}\tilde{d}_0(t)t.
$$

Actually, $\tilde{\chi}_0(t)$ is the value of the right-hand term of the equation for $\tilde{S}$ in (45) on the boundary $\widehat{P'E'}$. In view of the boundary value $\tilde{S}(\tilde{z}(t),t)=\tilde{b}_2(t)$, we find by (50) that

$$
\left\{
\begin{aligned}
&\tilde{S}_t(\tilde{z}(t),t)-\frac{\kappa c(\tilde{z}(t),t)\tilde{d}_0(t)t^2}{(1-t^2)\tilde{T}_4}\tilde{S}_z(\tilde{z}(t),t)=\tilde{\chi}_0(t),\\
&\tilde{S}_t(\tilde{z}(t),t)+\frac{\kappa c(\tilde{z}(t),t)\tilde{b}_2(t)t^2}{(1-t^2)\tilde{T}_3}\tilde{S}_z(\tilde{z}(t),t)=\tilde{b}_2'(t),
\end{aligned}
\right.
\tag{56}
$$

from which one gets

$$
\tilde{S}_z(\tilde{z}(t),t)=\frac{(1-t^2)\tilde{T}_3\tilde{T}_4}{\kappa c(\tilde{z}(t),t)(\tilde{T}_4\tilde{b}_2(t)+\tilde{T}_3\tilde{d}_0(t))}\cdot\frac{\tilde{b}_2'(t)-\tilde{\chi}_0(t)}{t^2}:=\tilde{\chi}_1(t).
\tag{57}
$$

We further assume that the boundary data in (19) and (20) satisfy some appropriate conditions such that the following compatibility condition $(\mathcal{C})$ at $P'(z_2, 0)$ holds:

$$(\mathcal{C}): \ \tilde{\chi}_1(0) = \hat{a}_0'(z_2), \quad \tilde{\chi}_1'(0) = -\hat{a}_1'(z_2). \tag{58}$$

Remark 2. The conditions in $(\mathcal{C})$ ensure that $\tilde{S}_z$ and the derivative of $\tilde{S}_z$ with respect to t are continuous at point P'. This higher-order compatibility condition plays an important role in dealing with the singularities at the initial line $t = 0$ in the current paper.

In terms of the partial hodograph variables, we have the the following existence theorem.

Theorem 3.1. *Let (54) and (58) be satisfied. The degenerate mixed-type boundary value problem (45), (53) admits a unique classical solution around point $P'(z_2, 0)$.*

3.2. The proof of Theorem 3.1

This subsection is devoted to solving the degenerate mixed-type boundary value problem (45), (53) in the partial hodograph plane. We divide the process into four steps. In the first step, we homogenize the boundary conditions and define the admissible functions. In the second step, we construct an iterative sequence by the integral system relative to (45). In the third step, we establish several key lemmas for the iterative sequence. Finally, we complete the proof of Theorem 3.1 in the fourth step.

3.2.1. The homogeneous problem. In order to deal with the degenerate boundary conditions conveniently, we homogenize the boundary values (53) by introducing the higher-order error terms for the variables $(\tilde{R}, \tilde{S})$ as follows:

$$\begin{cases} W(z,t) = \tilde{R}(z,t) - a_0(z) - ta_1(z), \\ V(z,t) = \tilde{S}(z,t) - a_0(z) + ta_1(z), \end{cases} \tag{59}$$

where

$$a_0(z) = \hat{a}_0(z + z_2), \quad a_1(z) = \hat{a}_1(z + z_2).$$

Here we moved the point $P'(z_2, 0)$ to the origin. Corresponding to (53), we acquire that the boundary conditions for the variables (W, V) are

$$\begin{aligned} &(W, V, W_t, V_t)(z, 0) = 0 \qquad \forall\ z \in (z_1 - z_2, 0], \\ &(W, V)(\bar{z}(t), t) = (d, b)(t) \quad \forall\ t \in [0, t_0), \end{aligned} \tag{60}$$

where $\bar{z}(t) = \tilde{z}(t) - z_2$ and

$$
\begin{aligned}
d(t) &= \tilde{d}_0(t) - a_0(\bar{z}(t)) - ta_1(\bar{z}(t)),\\
b(t) &= \tilde{b}_2(t) - a_0(\bar{z}(t)) + ta_1(\bar{z}(t)).
\end{aligned}
\tag{61}
$$

It follows from (54) that

$$
\begin{gathered}
(a_0, a_1)(z) \in C^2((z_1 - z_2, 0]),\ \ b(t) \in C^2([0, t_0)),\\
a_0(z) \geq \varepsilon_0 \ \ \forall z \in (z_1 - z_2, 0],\\
b(0) = b'(0) = 0.
\end{gathered}
\tag{62}
$$

Combining with (57) and (59) leads to

$$
V_z(\bar{z}(t), t) = \tilde{\chi}_1(t) - a_0(\bar{z}(t)) + ta_1(\bar{z}(t)) := \chi(t). \tag{63}
$$

By the definition of $\chi(t)$ and the conditions in (54), we employ the compatibility condition (58) to find that there exists a positive constant $\widehat{K}$ such that

$$
|\chi(t)| \leq \widehat{K} t^2. \tag{64}
$$

From (45), we obtain the equations for (W, V)

$$
\begin{cases}
W_t + \dfrac{c\kappa(V+f)t^2}{(1-t^2)T_5} W_z = \dfrac{W - V}{2t} + A_1(V, z, t)W + A_2(V, z, t)V\\
\qquad\qquad\qquad\qquad\qquad + A_3(V, z, t)t^2 + F(z, t)t,\\
V_t - \dfrac{c\kappa(W+g)t^2}{(1-t^2)T_6} V_z = \dfrac{V - W}{2t} + B_1(W, z, t)W + B_2(W, z, t)V\\
\qquad\qquad\qquad\qquad\qquad + B_3(W, z, t)t^2 + F(z, t)t,
\end{cases}
\tag{65}
$$

where

$$
\begin{aligned}
& f = a_0 - ta_1, \quad g = a_0 + ta_1, \quad F(z, t) = -\tfrac{a_0 a_1 \kappa}{c\sqrt{\kappa + 1 - t^2}},\\
& T_5(V) = \sqrt{\kappa + 1 - t^2} + \kappa \tfrac{1 - t^2}{c}(V + f)t,\\
& T_6(W) = \sqrt{\kappa + 1 - t^2} - \kappa \tfrac{1 - t^2}{c}(W + g)t,
\end{aligned}
$$

and

$$
\begin{aligned}
A_1(V,z,t) &= \frac{t(\kappa+2-t^2)+\kappa\sqrt{\kappa+1-t^2}\,\frac{1-t^2}{c}(V+f)(t^2-1)}{2(1-t^2)\sqrt{\kappa+1-t^2}T_5} + \frac{\kappa(3\kappa+4-4t^2)(V+f)t^2}{c(\kappa+1-t^2)T_5},\\
A_2(V,z,t) &= -\frac{t(\kappa+2-t^2)+\kappa\sqrt{\kappa+1-t^2}\,\frac{1-t^2}{c}(V+f)(t^2-1)}{2(1-t^2)\sqrt{\kappa+1-t^2}T_5} - \frac{ta_1\kappa(1-t^2)}{cT_5},\\
A_3(V,z,t) &= \frac{\kappa(3\kappa+4-4t^2)g(V+f)}{c(\kappa+1-t^2)T_5} - \frac{(a_0'+ta_1')(V+f)c\kappa}{(1-t^2)T_5} + \frac{a_1(\kappa+2-t^2)}{\sqrt{\kappa+1-t^2}(1-t^2)T_5}\\
&\quad + \frac{fta_1\kappa}{cT_5} + \frac{a_1^2\kappa}{cT_5} + \frac{\kappa^2(1-t^2)(V+f)a_0a_1}{c^2\sqrt{\kappa+1-t^2}T_5},
\end{aligned}
$$

and

$$
\begin{aligned}
B_1(W,z,t) &= -\frac{t(\kappa+2-t^2)+\kappa\sqrt{\kappa+1-t^2}\,\frac{1-t^2}{c}(W+g)(1-t^2)}{2(1-t^2)\sqrt{\kappa+1-t^2}T_6} - \frac{ta_1\kappa(1-t^2)}{cT_6},\\
B_2(W,z,t) &= \frac{t(\kappa+2-t^2)+\kappa\sqrt{\kappa+1-t^2}\,\frac{1-t^2}{c}(W+g)(1-t^2)}{2(1-t^2)\sqrt{\kappa+1-t^2}T_6} - \frac{\kappa(3\kappa+4-4t^2)(W+g)t^2}{c(\kappa+1-t^2)T_6},\\
B_3(W,z,t) &= -\frac{\kappa(3\kappa+4-4t^2)(W+g)f}{c(\kappa+1-t^2)T_6} + \frac{(a_0'-ta_1')(W+g)c\kappa}{(1-t^2)T_6} - \frac{a_1(\kappa+2-t^2)}{\sqrt{\kappa+1-t^2}(1-t^2)T_6}\\
&\quad + \frac{gta_1\kappa}{cT_6} - \frac{a_1^2\kappa}{cT_6} - \frac{\kappa^2(1-t^2)(W+g)a_0a_1}{c^2\sqrt{\kappa+1-t^2}T_6}.
\end{aligned}
$$

It is clear that the positive/negative eigenvalues of system (65) are

$$
\lambda_+(V,z,t) = \frac{c\kappa(V+f)t^2}{(1-t^2)T_5(V)}, \quad \lambda_-(W,z,t) = -\frac{c\kappa(W+g)t^2}{(1-t^2)T_6(W)}. \tag{66}
$$

and the positive/negative characteristics passing through point (ζ,τ) are defined by

$$
\begin{cases}
\dfrac{\mathrm{d}z_\pm(t;\zeta,\tau)}{\mathrm{d}t} = \lambda_\pm(z_\pm(t;\zeta,\tau),t),\\
z_\pm(\tau;\zeta,\tau) = \zeta.
\end{cases} \tag{67}
$$

We now verify that the ODE problem (51) is solvable, which follows directly from the next lemma by (59).

Lemma 3.1. *Assume that* (62) *holds. Then the following ODE problem*

$$
\begin{cases}
d'(t) = \dfrac{d(t)-b(t)}{2t} + A_1(b(t),\bar z(t),t)d(t) + A_2(b(t),\bar z(t),t)b(t)\\
\qquad\qquad + A_3(b(t),\bar z(t),t)t^2 + F(\bar z(t),t)t,\\
d(0) = d'(0) = 0.
\end{cases} \tag{68}
$$

admits a unique C^2-solution on $t \in [0,\delta_1]$ for a small positive constant $\delta_1 < t_0$.

Proof. For convenience, we use K_0 in this paper to denote a positive constant which depends only on the C^2-norms of $\hat{a}_0, \hat{a}_1$, the C^3-norm of $\tilde{b}_1$ and the constants κ, ε_0. The value of K_0 may change from one expression to another.

According to the conditions of $b(t)$ in (62), we know that

$$|b(t)| \leq K_0 t^2, \quad |b'(t)| \leq K_0 t. \tag{69}$$

Thus we find that there exists a small positive constant $\bar{\delta}_1$ such that

$$T_5(b(t)) = \sqrt{\kappa + 1 - t^2} + \kappa \frac{1 - t^2}{c(\bar{z}(t), t)}(b(t) + f)t \geq \frac{\sqrt{\kappa}}{2}$$

for $t \leq \bar{\delta}_1$, from which and the exact expressions of A_i $(i = 1, 2, 3)$ and F, one obtains that

$$\begin{aligned} M_1 = \max_{t \in [0, \bar{\delta}_1]} \Big\{ & K_0, |A_1(b(t), \bar{z}(t), t)|, |A_2(b(t), \bar{z}(t), t)|, \\ & |A_3(b(t), \bar{z}(t), t)|, |F(\bar{z}(t), t)| \Big\}, \end{aligned} \tag{70}$$

is uniformly bounded. We next construct the iterative sequence. Denote $d^{(0)}(t) = 0$ and then define quantities $d^{(k)}(t)$ $(k \geq 1)$ by the following relation

$$\begin{aligned} d^{(k)}(t) = \int_0^t \Big\{ & \frac{d^{(k-1)}(s) - b(s)}{2s} + A_1(b(s), \bar{z}(s), s) d^{(k-1)} \\ & + A_2(b(s), \bar{z}(s), s) b(s) + A_3(b(s), \bar{z}(s), s) s^2 + F(\bar{z}(s), s) s \Big\} \, \mathrm{d}s. \end{aligned} \tag{71}$$

Let $\delta_1 = \min\{\bar{\delta}_1, 1/(4M_1)\}$. Then for $t \in [0, \delta_1]$, we can get by a standard argument of induction that for all $k \geq 1$

$$\left| d^{(k)}(t) \right| \leq M_1 t^2 \sum_{i=0}^{k} \left(\frac{2}{3} \right)^j, \quad \left| d^{(k+1)}(t) - d^{(k)}(t) \right| \leq M_1 t^2 \left(\frac{2}{3} \right)^k, \tag{72}$$

which indicates that the sequence $d^{(k)}(t)$ converges uniformly to a continuous function $d(t)$. Moreover, one can see by (72) that the function $d(t)$ satisfies

$$|d(t)| \leq 3M_1 t^2 \quad \forall \, t \in [0, \delta_1], \tag{73}$$

which together with (71) leads to the smoothness result for $d(t)$. The proof of the lemma is completed. □

We next define the admissible functions and strong determinate domain to system (65). Let $\overline{D}_{\delta_1}$ be a closed domain in the (z,t)-plane defining as follows

$$\overline{D}_{\delta_1}=\{(z,t)|\ t\in[0,\delta_1],\ \frac{z_1-z_2}{2}\le z\le\bar{z}(t)\},$$

Denote $\mathcal{S}(\overline{D}_{\delta_1})$ a function class incorporating all vectors $\mathbf{F}=(f_1,f_2)^T$: $\overline{D}_{\delta_1}\to\mathbb{R}^2$ that satisfy the following properties:

$$\begin{aligned}
&(\mathrm{P}_1):\ f_1,f_2 \text{ are continuous on } \overline{D}_{\delta_1};\\
&(\mathrm{P}_2):(f_1,f_2)^T(z,0)=(0,0)\ \ \forall\ z\in[\tfrac{z_1-z_2}{2},0];\\
&(\mathrm{P}_3):(f_1,f_2)^T(\bar{z}(t),t)=(d,b)^T(t)\ \ \forall\ t\in[0,\delta_1];\\
&(\mathrm{P}_4):\max_{(z,t)\in\overline{D}_{\delta_1}}\{|f_1(z,t)|,|f_2(z,t)|\}\le\widetilde{M}t^2,\\
&\qquad\text{where } \widetilde{M}\ (\ge 3M_1) \text{ is a fixed constant.}
\end{aligned}\tag{74}$$

Thanks to Lemma 3.1, (73) and (69), we see that $(d,b)^T(t)$ belongs to $\mathcal{S}(\overline{D}_{\delta_1})$. Thus $\mathcal{S}(\overline{D}_{\delta_1})$ is not empty. Let $(f_1,f_2)^T\in\mathcal{S}(\overline{D}_{\delta_1})$ be any element in $\mathcal{S}(\overline{D}_{\delta_1})$. We apply the property (P_4) in (74) and the exact expressions of f,g and T_5,T_6 to find by (62) and (66) that there exists a small positive constant $\delta_2<\min\{\delta_1,1/\widetilde{M}\}$ such that

$$\left.\begin{aligned}
&f,g\ge\frac{\varepsilon_0}{2},\ T_5(f_2),T_6(f_1)\ge\frac{\sqrt{\kappa}}{2},\\
&0<\underline{k}\le\frac{\lambda_+(f_2,z,t)}{t^2},\frac{-\lambda_-(f_1,z,t)}{t^2}\le\overline{K},
\end{aligned}\right.\quad\forall\ (z,t)\in\overline{D}_{\delta_1}\cap\{t\le\delta_2\},\tag{75}$$

and

$$\underline{k}\ge 2\overline{K}\delta_2,\qquad \frac{z_1-z_2}{2}\le\bar{z}(\delta_2)-\overline{K}\delta_2^3:=z_*\tag{76}$$

for some constants $\underline{k}$ and $\overline{K}$. From the relation $\bar{z}(t)=\tilde{z}(t)-z_2$ and (50), one has

$$\begin{aligned}
z_*&=\int_0^{\delta_2}\frac{\kappa c(\tilde{z}(s),s)\tilde{b}_2(s)s^2}{(1-s^2)[\sqrt{\kappa+1-s^2}+\frac{\kappa(1-s^2)}{c(\tilde{z}(s),s)}\tilde{b}_2(s)s]}\,\mathrm{d}s-\overline{K}\delta_2^3\\
&\le\int_0^{\delta_2}\overline{K}s^2\,\mathrm{d}s-\overline{K}\delta_2^3=-\frac{2}{3}\overline{K}\delta_2^3,
\end{aligned}\tag{77}$$

which means that the number z_* is negative. Set $\hat{z}(t)=z_*+\overline{K}t^3$ $(t\in[0,\delta_2])$. Then it suggests that

$$\hat{z}(t)<\bar{z}(t)\ \ \forall\ t\in[0,\delta_2),$$

and $\hat{z}(\delta_2) = \bar{z}(\delta_2)$. We now denote

$$\overline{D}_{\delta_2} = \{(z,t) |\ t \in [0, \delta_2],\ \hat{z}(t) \le z \le \bar{z}(t)\}, \tag{78}$$

and $\mathcal{S}(\overline{D}_{\delta_2})$ the corresponding function class defined on $\overline{D}_{\delta_2}$. For any vector function $(f_1, f_2)^T \in \mathcal{S}(\overline{D}_{\delta_2})$ and for any point $(\zeta, \tau) \in \overline{D}_{\delta_2}$, it is not hard to check that the positive/negative characteristic curves $z_\pm(t; \zeta, \tau)$, defined in (67) but with $(f_1, f_2)^T$ replacing $(W, V)^T$ in $\lambda_\pm$, stay insider $\overline{D}_{\delta_2}$ until the intersection with the boundary curves $z = \bar{z}(t)$ or $t = 0$. Thus the domain $\overline{D}_{\delta_2}$ is a strong determinate domain for system (65).

For later use, we here derive $|z_+(t; \zeta, \tau) - z_-(t; \zeta, \tau)|$ by (75)

$$\big|z_+(t; \zeta, \tau) - z_-(t; \zeta, \tau)\big| \le \int_0^\tau 2\overline{K} t^2 \ \mathrm{d}t \le \overline{K}\tau^3 \tag{79}$$

for $t \in [\tau_-, \tau]$. Here and below τ_- is the intersection time of the negative characteristic $z = z_-(t; \zeta, \tau)$ and the boundary of $\overline{D}_{\delta_2}$. Furthermore, one also has by the expression of F in (65)

$$\begin{aligned} &\big|F(z_+(t; \zeta, \tau), t) - F(z_-(t; \zeta, \tau), t)\big| \\ \le& K_0 \big|z_+(t; \zeta, \tau) - z_-(t; \zeta, \tau)\big| \le K_0 \tau^3. \end{aligned} \tag{80}$$

3.2.2. The construction of iterative sequence. We are now based on the differential equations (65) to construct an iterative sequence. Let (ζ, τ) be any point in $\overline{D}_{\delta_2}$. Integrating the system (65) along the characteristic curves $z = z_\pm(t)$ and employing the boundary conditions in (60) gives

$$\begin{cases} W(\zeta, \tau) = \displaystyle\int_0^\tau \Big\{ \frac{W - V}{2t} + A_1(V) W + A_2(V) V \\ \qquad\qquad + A_3(V) t^2 + F(z,t) t \Big\} (z_+(t), t) \ \mathrm{d}t, \\ V(\zeta, \tau) = b(\tau_-) + \displaystyle\int_0^\tau \Big\{ \frac{V - W}{2t} + B_1(W) W + B_2(W) V \\ \qquad\qquad + B_3(W) t^2 + F(z,t) t \Big\} (z_-(t), t) \ \mathrm{d}t. \end{cases} \tag{81}$$

Here we used the fact that the positive characteristic curve $z = z_+(t)$ only intersects the line $t = 0$.

Set $W^{(0)}(z,t)=d(t)$ and $V^{(0)}(z,t)=b(t)$. We solve the following ODE problem

$$\frac{\mathrm{d}z_-^{(0)}(t)}{\mathrm{d}t}=\lambda_-(W^{(0)})(z_-^{(0)}(t),t),\quad z_-^{(0)}(0)=0, \tag{82}$$

and denote the solution as $z=\check{z}_-^{(0)}(t)$ $(t\in[0,\delta_2])$. The solvability of problem (82) follows from (75). It is clear that the curve $z=\check{z}_-^{(0)}(t)$ divides the domain $\overline{D}_{\delta_2}$ into two disjoint subdomains

$$\overline{D}_{\delta_2}=D_{\delta_2}^{(01)}\cup D_{\delta_2}^{(02)},$$

where $D_{\delta_2}^{(01)}=\{(z,t)|\ z\le\check{z}_-^{(0)}(t)\}\cap\overline{D}_{\delta_2}$ and $D_{\delta_2}^{(02)}=\{(z,t)|\ z>\check{z}_-^{(0)}(t)\}\cap\overline{D}_{\delta_2}$. For any point $(\zeta,\tau)\in\overline{D}_{\delta_2}$, the characteristic curves $z=z_\pm^{(0)}(t)=: z_\pm^{(0)}(t;\zeta,\tau)$ are defined as

$$\begin{cases}\dfrac{\mathrm{d}z_\pm^{(0)}(t;\zeta,\tau)}{\mathrm{d}t}=\lambda_\pm(W^{(0)},V^{(0)},z,t)(t,z_\pm^{(0)}(t;\xi,\eta)),\\ z_\pm^{(0)}(\tau;\zeta,\tau)=\zeta.\end{cases} \tag{83}$$

Denote $\tau_-^{(0)}$ the intersection time of the negative characteristic $z=z_-^{(0)}(t;\zeta,\tau)$ and the boundary of $\overline{D}_{\delta_2}$. Obviously, $\tau_-^{(0)}=0$ if $(\zeta,\tau)\in D_{\delta_2}^{(01)}$, while $\tau_-^{(0)}>0$ if $(\zeta,\tau)\in D_{\delta_2}^{(02)}$. Then we construct the functions $(W^{(1)},V^{(1)})(\zeta,\tau)$ by (81) as follows

$$\begin{cases}W^{(1)}(\zeta,\tau)=\displaystyle\int_0^\tau\Big\{\frac{W^{(0)}-V^{(0)}}{2t}+A_1(V^{(0)})W^{(0)}+A_2(V^{(0)})V^{(0)}\\ \qquad\qquad +A_3(V^{(0)})t^2+Ft\Big\}(z_+^{(0)}(t),t)\ \mathrm{d}t,\\ V^{(1)}(\zeta,\tau)=b(\tau_-^{(0)})+\displaystyle\int_{\tau_-^{(0)}}^\tau\Big\{\frac{V^{(0)}-W^{(0)}}{2t}+B_1(W^{(0)})W^{(0)}\\ \qquad\qquad +B_2(W^{(0)})V^{(0)}+B_3(W^{(0)})t^2+Ft\Big\}(z_-^{(0)}(t),t)\ \mathrm{d}t.\end{cases} \tag{84}$$

After defining the functions $(W^{(k)},V^{(k)})(z,t)$ $(k\ge1)$, we solve the following ODE problem

$$\frac{\mathrm{d}z_-^{(k)}(t)}{\mathrm{d}t} = \lambda_-(W^{(k)})(z_-^{(k)}(t), t), \quad z_-^{(0)}(0) = 0, \tag{85}$$

and denote the solution as $z = \check{z}_-^{(k)}(t)$ $(t \in [0, \delta_2])$. We will prove that $(W^{(k)}, V^{(k)})$ is in $\mathcal{S}(\overline{D}_{\delta_2})$, which together with (75) leads to the solvability of problem (85). Similarly, the curve $z = \check{z}_-^{(k)}(t)$ divides the domain $\overline{D}_{\delta_2}$ into two disjoint subdomains

$$\overline{D}_{\delta_2} = D_{\delta_2}^{(k1)} \cup D_{\delta_2}^{(k2)},$$

where $D_{\delta_2}^{(k1)} = \{(z,t) |\ z \le \check{z}_-^{(k)}(t)\} \cap \overline{D}_{\delta_2}$ and $D_{\delta_2}^{(k2)} = \{(z,t) |\ z > \check{z}_-^{(k)}(t)\} \cap \overline{D}_{\delta_2}$. For any point $(\zeta, \tau) \in \overline{D}_{\delta_2}$, we define the characteristic curves $z = z_\pm^{(k)}(t) =: z_\pm^{(k)}(t; \zeta, \tau)$ as

$$\begin{cases} \dfrac{\mathrm{d}z_\pm^{(k)}(t;\zeta,\tau)}{\mathrm{d}t} = \lambda_\pm(W^{(k)}, V^{(k)}, z, t)(t, z_\pm^{(k)}(t; \xi, \eta)), \\ z_\pm^{(k)}(\tau; \zeta, \tau) = \zeta, \end{cases} \tag{86}$$

and then denote the intersection time of the negative characteristic $z = z_-^{(k)}(t; \zeta, \tau)$ and the boundary of $\overline{D}_{\delta_2}$ by $\tau_-^{(k)}$. By the construction, we see that $\tau_-^{(k)} = 0$ if $(\zeta, \tau) \in D_{\delta_2}^{(k1)}$, while $\tau_-^{(k)} > 0$ if $(\zeta, \tau) \in D_{\delta_2}^{(k2)}$. We then construct the functions $(W^{(k+1)}, V^{(k+1)})(\zeta, \tau)$

$$\begin{cases} W^{(k+1)}(\zeta,\tau) = \displaystyle\int_0^\tau \Big\{ \frac{W^{(k)} - V^{(k)}}{2t} + A_1(V^{(k)})W^{(k)} \\ \qquad\qquad + A_2(V^{(k)})V^{(k)} + A_3(V^{(k)})t^2 + Ft \Big\} (z_+^{(k)}(t), t)\ \mathrm{d}t, \\ V^{(k+1)}(\zeta,\tau) = b(\tau_-^{(k)}) + \displaystyle\int_{\tau_-^{(k)}}^\tau \Big\{ \frac{V^{(k)} - W^{(k)}}{2t} + B_1(W^{(k)})W^{(k)} \\ \qquad\qquad + B_2(W^{(k)})V^{(k)} + B_3(W^{(k)})t^2 + Ft \Big\} (z_-^{(k)}(t), t)\ \mathrm{d}t. \end{cases} \tag{87}$$

We shall show that there exist two positive constants $\delta < \delta_2$ and $\widetilde{M}$ such that the sequences $(W^{(k)}, V^{(k)})$ $(k \ge 0)$ converge uniformly in the function class $\mathcal{S}(\overline{D}_\delta)$.

3.2.3. Several lemmas. We now establish several key lemmas for the iterative sequences $(W^{(k)}, V^{(k)})$ $(k \geq 0)$. According to the exact expressions of A_i, B_i $(i = 1, 2, 3)$ and F in (65), if $(W, V)^T \in \mathcal{S}(\overline{D}_{\delta_2})$, one has

$$\begin{aligned} &|A_i(V, z, t)|; |B_i(W, z, t)|; |F(z, t)|; |F_z(z, t)|; |F_{zz}(z, t)| \leq \overline{M}, \\ &|A_{iz}(V, z, t)|; |A_{iV}(V, z, t)|; |B_{iz}(W, z, t)|; |B_{iW}(W, z, t)| \leq \overline{M} \end{aligned} \tag{88}$$

for some positive constant $\overline{M}$.

Let M, δ and $\widetilde{M}$ be three positive constants satisfying

$$M = \max\{1, 4K_0, 4\widehat{K}, 4M_1, \overline{M}\}, \quad \delta \leq \left\{\delta_2, \frac{1}{100M}, \frac{\underline{k}}{2M}\right\}, \quad \widetilde{M} \geq 3M, \tag{89}$$

where $\widehat{K}, M_1$ and $\underline{k}$ are given in (64), (70) and (75), respectively. The choices of M and δ in (89) ensure that the following inequalities hold

$$\left(\frac{1}{2} + 13M\delta\right) \exp(2M\delta^3) \leq \frac{2}{3}, \ 4M\delta + \frac{M\delta}{\underline{k}} < \frac{2}{3}. \tag{90}$$

For the number δ, we define a closed domain $\overline{D}_\delta$ as follows

$$\overline{D}_\delta = \{(z, t) |\ 0 \leq t \leq \delta,\ \bar{z}(\delta) - \overline{K}\delta^3 + \overline{K}t^3 \leq z \leq \bar{z}(t)\}.$$

Clearly, one has $\overline{D}_\delta \subset \overline{D}_{\delta_2}$.

We now have

Lemma 3.2. *For any $(\zeta, \tau) \in \overline{D}_\delta$ and for all $k \geq 1$, the following inequalities hold*

$$\begin{aligned} &\left|W^{(k)}(\zeta, \tau)\right|;\ \left|V^{(k)}(\zeta, \tau)\right| \leq M\tau^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j, \\ &\left|W^{(k)}(\zeta, \tau) - V^{(k)}(\zeta, \tau)\right| \leq M\tau^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j. \end{aligned} \tag{91}$$

Proof. The proof is based on the standard argument of induction. We first check (91) for $n = 1$ and then assume all the inequalities are true for $n = k$ to derive (91) for $n = k + 1$.

Due to $(W^{(0)}, V^{(0)})^T = (d, b)^T \in \mathcal{S}(\overline{D}_{\delta_2})$, we apply (88) and (89) to acquire

$$|A_i(V^{(0)}, z, t)|; |B_i(W^{(0)}, z, t)|; |F(z, t)|; |F_z(z, t)| \leq M \ (i = 1, 2, 3) \tag{92}$$

for any $(z,t)\in\overline{D}_\delta$. It follows from (69), (84) and (92) that

$$
\begin{aligned}
\big|V^{(1)}(\zeta,\tau)\big| &= \big|b(\tau_-^{(0)})\big| + \int_{\tau_-^{(0)}}^{\tau}\left\{\frac{\big|V^{(0)}-W^{(0)}\big|}{2t} + \big|B_1(W^{(0)})\big|\cdot\big|W^{(0)}\big|\right.\\
&\qquad \left. + \big|B_2(W^{(0)})\big|\cdot\big|V^{(0)}\big| + \big|B_3(W^{(0)})\big|t^2 + |F|t\right\}(z_-^{(0)}(t),t)\ \mathrm{d}t\\
&\le K_0\tau^2 + \int_0^{\tau}\left\{\frac{K_0+3M_1}{2}t + M\cdot Mt^2 + M\cdot Mt^2 + Mt^2 + Mt\right\}\mathrm{d}t\\
&\le \frac{M}{4}\tau^2 + \left\{\frac{M}{4}\tau^2 + M^2\tau^3 + \frac{M}{2}\tau^2\right\} = M\tau^2(1+M\delta)\\
&\le M\tau^2\sum_{j=0}^{1}\left(\frac{2}{3}\right)^j.
\end{aligned}
\tag{93}
$$

A similar argument obtains the inequality for $W^{(1)}$. To estimate $|W^{(1)}(\zeta,\tau)-V^{(1)}(\zeta,\tau)|$, we use (84) again to achieve

$$
\big|W^{(1)}(\zeta,\tau)-V^{(1)}(\zeta,\tau)\big| \le \big|b(\tau_-^{(0)})\big| + I_1 + I_2,
\tag{94}
$$

where

$$
\begin{aligned}
I_1 = \int_0^{\tau_-^{(0)}} &\left\{\frac{\big|W^{(0)}-V^{(0)}\big|}{2t} + \big|A_1(V^{(0)})\big|\cdot\big|W^{(0)}\big|\right.\\
&\left. + \big|A_2(V^{(0)})\big|\cdot\big|V^{(0)}\big| + \big|A_3(V^{(0)})\big|t^2 + |F|t\right\}(z_+^{(0)}(t),t)\ \mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = \int_{\tau_-^{(0)}}^{\tau} &\left\{2\frac{\big|W^{(0)}-V^{(0)}\big|}{2t} + \big|A_1(V^{(0)})W^{(0)}\big| + \big|B_1(W^{(0)})W^{(0)}\big|\right.\\
&+ \big|A_2(V^{(0)})V^{(0)}\big| + \big|B_2(W^{(0)})V^{(0)}\big| + \big|A_3(V^{(0)})\big|t^2\\
&\left. + \big|B_3(W^{(0)})\big|t^2 + \big|F(z_+^{(0)}(t),t) - F(z_-^{(0)}(t),t)\big|t\right\}\mathrm{d}t.
\end{aligned}
$$

For the term I_1, one has by (92)

$$
I_1 \le \int_0^{\tau_-^{(0)}}\left\{\frac{M}{2}t + 3M^2t^2 + Mt\right\}\mathrm{d}t = \frac{3}{4}M(\tau_-^{(0)})^2 + M^2(\tau_-^{(0)})^3.
\tag{95}
$$

For the term I_2, making use of (80) and (92) yields

$$(96)\qquad \begin{aligned} I_2 \le & \int_{\tau_-^{(0)}}^{\tau} \left\{ Mt + 6M^2t^2 + M\tau^3 \right\} \mathrm{d}t \\ \le & \frac{1}{2}M\tau^2 + 2M^2\tau^3 + M\tau^4 - \frac{1}{2}M(\tau_-^{(0)})^2 - 2M^2(\tau_-^{(0)})^3. \end{aligned}$$

We put (95)–(96) into (94) and apply (69) again to obtain

$$(97)\qquad \begin{aligned} & \left|W^{(1)}(\zeta,\tau) - V^{(1)}(\zeta,\tau)\right| \le \frac{1}{4}M\tau^2 + \frac{3}{4}M(\tau_-^{(0)})^2 + M^2(\tau_-^{(0)})^3 \\ & \quad + \frac{1}{2}M\tau^2 + 2M^2\tau^3 + M\tau^4 - \frac{1}{2}M(\tau_-^{(0)})^2 - 2M^2(\tau_-^{(0)})^3 \\ \le & M\tau^2(1+2M\delta) \le M\tau^2 \sum_{j=0}^{1} \left(\frac{2}{3}\right)^j. \end{aligned}$$

Hence the inequalities in (91) are true for $n = 1$.

Assume that (91) holds for $n = k$. Due to the choice of $\widetilde{M}$ in (89), we see that

$$\left|W^{(k)}(\zeta,\tau)\right|; \left|V^{(k)}(\zeta,\tau)\right| \le M\tau^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \le 3M\tau^2 \le \widetilde{M}\tau^2,$$

which means that $(W^{(k)}, V^{(k)})^T \in \mathcal{S}(\overline{D}_\delta)$. Thus it suggests by (88) that

$$(98)\qquad |A_i(V^{(k)}, z, t)|; |B_i(W^{(k)}, z, t)|; |F(z,t)|; |F_z(z,t)| \le M \ (i = 1, 2, 3),$$

for any $(z, t) \in \overline{D}_\delta$. Combining with (69), (87) and (98) arrives at

$$(99)\qquad \begin{aligned} & \left|V^{(k+1)}(\zeta,\tau)\right| \le \left|b(\tau_-^{(k)})\right| + \int_{\tau_-^{(k)}}^{\tau} \left\{ \frac{\left|V^{(k)} - W^{(k)}\right|}{2t} + \left|B_1(W^{(k)})W^{(k)}\right| \right. \\ & \qquad \left. + \left|B_2(W^{(k)})V^{(k)}\right| + \left|B_3(W^{(k)})\right|t^2 + |F|t \right\} \mathrm{d}t \\ \le & \frac{1}{4}M\tau^2 + \int_0^{\tau} \left\{ \frac{Mt}{2} \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + 3M^2t^2 \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j + Mt \right\} \mathrm{d}t \\ \le & M\tau^2 \left\{ \frac{3}{4} + \left(\frac{1}{4} + M\delta\right) \sum_{j=0}^{k} \left(\frac{2}{3}\right)^j \right\} \le M\tau^2 \sum_{j=0}^{k+1} \left(\frac{2}{3}\right)^j. \end{aligned}$$

The inequality in (99) is also true for $W^{(k+1)}$. To estimate the difference between $W^{(k+1)}(\zeta,\tau)$ and $V^{(k+1)}(\zeta,\tau)$, one achieves by employing (84) again

$$\left|W^{(k+1)}(\zeta,\tau)-V^{(k+1)}(\zeta,\tau)\right| \le \left|b(\tau_-^{(k)})\right| + I_3 + I_4, \tag{100}$$

where

$$\begin{aligned} I_3 = \int_0^{\tau_-^{(k)}} \Big\{ & \frac{\left|W^{(k)}-V^{(k)}\right|}{2t} + \left|A_1(V^{(k)})\right|\cdot\left|W^{(k)}\right| \\ & + \left|A_2(V^{(k)})\right|\cdot\left|V^{(k)}\right| + \left|A_3(V^{(k)})\right|t^2 + |F|t\Big\}(z_+^{(k)}(t),t)\ \mathrm{d}t, \end{aligned}$$

and

$$\begin{aligned} I_4 = \int_{\tau_-^{(k)}}^{\tau} \Big\{ & 2\frac{\left|W^{(k)}-V^{(k)}\right|}{2t} + \left|A_1(V^{(k)})W^{(k)}\right| + \left|B_1(W^{(k)})W^{(k)}\right| \\ & + \left|A_2(V^{(k)})V^{(k)}\right| + \left|B_2(W^{(k)})V^{(k)}\right| + \left|A_3(V^{(k)})\right|t^2 \\ & + \left|B_3(W^{(k)})\right|t^2 + \left|F(z_+^{(k)}(t),t)-F(z_-^{(k)}(t),t)\right|t\Big\}\ \mathrm{d}t. \end{aligned}$$

By similar processes as before, we use the induction assumptions to obtain

$$\begin{aligned} I_3 &\le \int_0^{\tau_-^{(k)}} \left\{\frac{Mt}{2}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + 3M^2t^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + Mt\right\}\mathrm{d}t \\ &\le \frac{1}{2}M(\tau_-^{(k)})^2 + \frac{1}{4}M(\tau_-^{(k)})^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + M^2(\tau_-^{(k)})^3\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j, \end{aligned} \tag{101}$$

and

$$\begin{aligned} I_4 &\le \int_{\tau_-^{(k)}}^{\tau} \left\{Mt\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + 6M^2t^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + M\tau^3\right\}\mathrm{d}t \\ &\le \frac{1}{2}M\left[(\tau)^2-(\tau_-^{(k)})^2\right]\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \\ &\quad + 2M^2\left[(\tau)^3-(\tau_-^{(k)})^3\right]\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j + M\tau^4. \end{aligned} \tag{102}$$

Inserting (101)–(102) into (100) and applying (84) again gives

$$
\begin{aligned}
(103)\qquad & \big|W^{(k+1)}(\zeta,\tau)-V^{(k+1)}(\zeta,\tau)\big| \\
\le & \frac{1}{4}M\tau^2+\frac{1}{2}M(\tau_-^{(k)})^2+\frac{1}{4}M(\tau_-^{(k)})^2\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \\
& + M^2(\tau_-^{(k)})^3\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j+\frac{1}{2}M\big[(\tau)^2-(\tau_-^{(k)})^2\big]\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \\
& + 2M^2\big[(\tau)^3-(\tau_-^{(k)})^3\big]\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j+M\tau^4 \\
\le & M\tau^2\bigg\{\left(\frac{3}{4}+\delta^2\right)+\left(\frac{1}{2}+2M\delta\right)\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\bigg\}\le M\tau^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j,
\end{aligned}
$$

which along with (99) ends the proof of the induction step. The proof of the lemma is complete. □

It is easy to see by Lemma 3.2 that $(W^{(k)},V^{(k)})^T\in\mathcal{S}(\overline{D}_\delta)$ for each $k\ge1$, which together with (88) lead to

(104)

$$
\begin{aligned}
|A_i(V^{(k)},z,t)|;|B_i(W^{(k)},z,t)|;|F(z,t)|;|F_z(z,t)|;|F_{zz}(z,t)|\le M,\\
|A_{iz}(V^{(k)},z,t)|;|A_{iV}(V^{(k)},z,t)|;|B_{iz}(W^{(k)},z,t)|;|B_{iW}(W^{(k)},z,t)|\le M
\end{aligned}
$$

for any $(z,t)\in\overline{D}_\delta$. We now differentiate system (87) with respect to ζ to deduce

(105)

$$
\left\{
\begin{aligned}
& W_\zeta^{(k+1)}(\zeta,\tau)=\int_0^\tau\bigg\{\frac{W_z^{(k)}-V_z^{(k)}}{2t}+A_{11}(V^{(k)})W_z^{(k)} \\
&\qquad +A_{12}(W^{(k)},V^{(k)})V_z^{(k)}+A_{13}(W^{(k)},V^{(k)})+F_zt\bigg\}\frac{\partial z_+^{(k)}}{\partial\zeta}(z_+^{(k)}(t),t)\ dt,\\
& V_\zeta^{(k+1)}(\zeta,\tau)=\chi(\tau_-^{(k)})+\int_{\tau_-^{(k)}}^\tau\bigg\{\frac{V_z^{(k)}-W_z^{(k)}}{2t}+B_{11}(W^{(k)})V_z^{(k)} \\
&\quad +B_{12}(W^{(k)},V^{(k)})W_z^{(k)}+B_{13}(W^{(k)},V^{(k)})+F_zt\bigg\}\frac{\partial z_-^{(k)}}{\partial\zeta}(z_-^{(k)}(t),t)\ dt.
\end{aligned}
\right.
$$

where

$$
\begin{aligned}
&A_{11}(V^{(k)}) = A_1(V^{(k)}),\\
&A_{12}(W^{(k)},V^{(k)}) = A_2(V^{(k)}) + A_{1V}(V^{(k)})W^{(k)}\\
&\qquad\qquad\qquad\qquad + A_{2V}(V^{(k)})V^{(k)} + A_{3V}(V^{(k)})t^2,\\
&A_{13}(W^{(k)},V^{(k)}) = A_{1z}(V^{(k)})W^{(k)} + A_{2z}(V^{(k)})V^{(k)} + A_{3z}(V^{(k)})t^2,\\
&B_{11}(W^{(k)}) = B_2(W^{(k)}),\\
&B_{12}(W^{(k)},V^{(k)}) = B_1(W^{(k)}) + B_{1W}(W^{(k)})W^{(k)}\\
&\qquad\qquad\qquad\qquad + B_{2W}(W^{(k)})V^{(k)} + B_{3W}(W^{(k)})t^2,\\
&B_{13}(W^{(k)},V^{(k)}) = B_{1z}(W^{(k)})W^{(k)} + B_{2z}(W^{(k)})V^{(k)} + B_{3z}(W^{(k)})t^2,
\end{aligned}
$$

and

$$
\frac{\partial z_\pm^{(k)}}{\partial \zeta}(t;\zeta,\tau) = \exp\left\{\int_\tau^t \frac{\partial \lambda_\pm^{(k)}}{\partial z}(z_\pm^{(k)}(t;\zeta,\tau),t)\ \mathrm{d}t\right\}. \tag{106}
$$

Recalling the expressions of $\lambda_\pm$ in (66) gets

$$
\frac{\partial \lambda_+^{(k)}}{\partial z}(z_+^{(k)}(t;\zeta,\tau),t) = \frac{\kappa t^2}{1-t^2}\Big(C_{11}(V^{(k)})V_z^{(k)} + C_{12}(V^{(k)})\Big), \tag{107}
$$

where

$$
\begin{aligned}
C_{11}(V^{(k)}) =& \frac{c}{T_5(V^{(k)})} - \frac{\kappa t(1-t^2)(V^{(k)}+f)}{T_5^2(V^{(k)})},\\
C_{12}(V^{(k)}) =& \frac{c_z(V^{(k)}+f)+cf_z}{T_5(V^{(k)})} - \frac{\kappa t(1-t^2)(V^{(k)}+f)f_z}{T_5^2(V^{(k)})}\\
&+ \frac{\kappa t(1-t^2)(V^{(k)}+f)^2 c_z}{cT_5^2(V^{(k)})},
\end{aligned}
$$

and

$$
\frac{\partial \lambda_-^{(k)}}{\partial z}(z_-^{(k)}(t;\zeta,\tau),t) = -\frac{\kappa t^2}{1-t^2}\Big(D_{11}(W^{(k)})W_z^{(k)} + D_{12}(W^{(k)})\Big), \tag{108}
$$

where

$$
D_{11}(W^{(k)}) = \frac{c}{T_6(W^{(k)})} + \frac{\kappa t(1-t^2)(W^{(k)}+g)}{T_6^2(W^{(k)})},
$$

$$D_{12}(W^{(k)}) = \frac{c_z(W^{(k)}+g)+cg_z}{T_6(W^{(k)})} + \frac{\kappa t(1-t^2)(W^{(k)}+g)g_z}{T_6^2(W^{(k)})} - \frac{\kappa t(1-t^2)(W^{(k)}+g)^2 c_z}{cT_6^2(W^{(k)})}.$$

Thanks to (104) and (75), we can acquire the following estimates

$$\begin{aligned} &|A_{11}|;|B_{11}| \le M, \quad |A_{12}|;|B_{12}| \le 2M, \quad |A_{13}|;|B_{13}| \le 8M^2t^2, \\ &\frac{\kappa}{1-t^2}|C_{11}|; \frac{\kappa}{1-t^2}|C_{12}|; \frac{\kappa}{1-t^2}|D_{11}|; \frac{\kappa}{1-t^2}|D_{12}| \le K_0 \le M. \end{aligned} \tag{109}$$

Combing with (106)–(109) leads to

$$\begin{aligned} \left|\frac{\partial z_\pm^{(k)}}{\partial \zeta}(t;\zeta,\tau)\right| &\le \exp\left\{\int_0^\tau \left|\frac{\partial \lambda_\pm^{(k)}}{\partial z}\right| \mathrm{d}t\right\} \\ &\le \exp\left\{\int_0^\tau Mt^2\big(|W_z^{(k)}|+|V_z^{(k)}|+1\big)\,\mathrm{d}t\right\}. \end{aligned} \tag{110}$$

For the sequences $(W_\zeta^{(k)}, V_\zeta^{(k)})$ $(k \ge 0)$, we have

Lemma 3.3. *For any $(\zeta,\tau) \in \overline{D}_\delta$ and for all $k \ge 1$, the following inequalities hold*

$$\begin{aligned} &|W_\zeta^{(k)}(\zeta,\tau)|;|V_\zeta^{(k)}(\zeta,\tau)| \le M\tau^2 \sum_{j=0}^{k}\left(\frac{2}{3}\right)^j, \\ &|W_\zeta^{(k)}(\zeta,\tau) - V_\zeta^{(k)}(\zeta,\tau)| \le M\tau^2 \sum_{j=0}^{k}\left(\frac{2}{3}\right)^j. \end{aligned} \tag{111}$$

Proof. We proceed by induction again. It follows by the facts $W_z^{(0)} = 0$, $V_z^{(0)} = 0$ and (110) that

$$\begin{aligned} \left|\frac{\partial z_\pm^{(0)}}{\partial \zeta}(t;\zeta,\tau)\right| &\le \exp\left\{\int_0^\tau Mt^2\big(|W_z^{(0)}|+|V_z^{(0)}|+1\big)\,\mathrm{d}t\right\} \\ &\le \exp(M\delta^3). \end{aligned} \tag{112}$$

We combine (64), (105), (109) and (112) to deduce

$$|V_\zeta^{(1)}(\zeta,\tau)| \le |\chi(\tau_-^{(0)})| + \int_0^\tau \left\{|B_{13}(W^{(0)},V^{(0)})| + |F_z|t\right\}\left|\frac{\partial z_-^{(0)}}{\partial \zeta}\right| \mathrm{d}t \tag{113}$$

$$
\begin{aligned}
&\leq \widehat{K}\tau^2 + \int_0^\tau \Big\{8M^2t^2 + Mt\Big\}e^{M\delta^3}\ \mathrm{d}t\\
&\leq M\tau^2\Big(\frac{1}{4} + \frac{1}{2}e^{M\delta^3} + 3M\delta e^{M\delta^3}\Big) \leq M\tau^2\sum_{j=0}^{1}\Big(\frac{2}{3}\Big)^j.
\end{aligned}
$$

For the term $|W_\zeta^{(1)}(\zeta,\tau) - V_\zeta^{(1)}(\zeta,\tau)|$, one also gets

$$
\big|W_\eta^{(1)}(\zeta,\tau) - V_\eta^{(1)}(\zeta,\tau)\big| \leq \big|\chi(\tau_-^{(0)})\big| + I_5 + I_6 + I_7, \tag{114}
$$

where

$$
\begin{aligned}
I_5 &= \int_0^{\tau_-^{(0)}} \Big(|A_{13}^{(0)}| + |F_z|t\Big)\cdot\left|\frac{\partial z_+^{(0)}}{\partial\zeta}\right|\ \mathrm{d}t,\\
I_6 &= \int_{\tau_-^{(0)}}^\tau \left\{|A_{13}^{(0)}|\cdot\left|\frac{\partial z_+^{(0)}}{\partial\zeta}\right| + |B_{13}^{(0)}|\cdot\left|\frac{\partial z_-^{(0)}}{\partial\zeta}\right|\right\}\ \mathrm{d}t,\\
I_7 &= \int_{\tau_-^{(0)}}^\tau \Big|F_z(z_+^{(0)}(t),t)\frac{\partial z_+^{(0)}}{\partial\zeta}(z_+^{(0)}(t),t)\\
&\qquad\qquad - F_z(z_-^{(0)}(t),t)\frac{\partial z_-^{(0)}}{\partial\zeta}(z_-^{(0)}(t),t)\Big|\cdot t\ \mathrm{d}t.
\end{aligned}
$$

It is easily obtained the estimates of I_5 and I_6 by (109) and (112)

(115)

$$
\begin{aligned}
I_5 &\leq \int_0^{\tau_-^{(0)}} \Big(8M^2\tau^2 + Mt\Big)e^{M\delta^3}\ \mathrm{d}t \leq \Big(3M^2(\tau_-^{(0)})^3 + \frac{1}{2}M(\tau_-^{(0)})^2\Big)e^{M\delta^3},\\
I_6 &\leq \int_{\tau_-^{(0)}}^\tau 16M^2t^2e^{M\delta^3}\ \mathrm{d}t \leq 6M^2\Big(\tau^3 - (\tau_-^{(0)})^3\Big)e^{M\delta^3}.
\end{aligned}
$$

For the term I_7, one finds by (79) and (110) that

$$
\begin{aligned}
I_7 \leq &\int_0^\tau \Big\{t\big|F_z(z_+^{(0)}(t),t) - F_z(z_-^{(0)}(t),t)\big|\cdot\left|\frac{\partial z_+^{(0)}}{\partial\zeta}\right|\\
&\qquad + t|F_z(z_-^{(0)}(t),t)|\cdot\left|\frac{\partial z_+^{(0)}}{\partial\zeta}(z_+^{(0)}(t),t) - \frac{\partial z_-^{(0)}}{\partial\zeta}(z_-^{(0)}(t),t)\right|\Big\}\ \mathrm{d}t\\
\leq &\int_0^\tau \Big\{tM\big|z_+^{(0)}(t) - z_-^{(0)}(t)\big|e^{M\delta^3}
\end{aligned}
\tag{116}
$$

$$
+ tM \cdot e^{M\delta^3} \int_0^\tau \left(\left| \frac{\partial \lambda_+^{(0)}}{\partial z} \right| + \left| \frac{\partial \lambda_-^{(0)}}{\partial z} \right| \right) \mathrm{d}s \Big\} \,\mathrm{d}t
$$

$$
\leq \int_0^\tau \Big\{ M \cdot M\tau^4 \cdot e^{M\delta^3} + M \cdot e^{M\delta^3} \cdot M\tau^4 \Big\} \,\mathrm{d}t = 2M^2 e^{M\delta^3} \tau^5.
$$

We combine (114)–(116) and apply (64) again to obtain

$$
\begin{aligned}
(117) \qquad & \big| W_\eta^{(1)}(\zeta, \tau) - V_\eta^{(1)}(\zeta, \tau) \big| \\
\leq & \frac{1}{4} M\tau^2 + \Big(3M^2 (\tau_-^{(0)})^3 + \frac{1}{2} M (\tau_-^{(0)})^2 \Big) e^{M\delta^3} \\
& + 6M^2 \Big(\tau^3 - (\tau_-^{(0)})^3 \Big) e^{M\delta^3} + 2M^2 e^{M\delta^3} \tau^5 \\
\leq & M\tau^2 \Big\{ \Big(\frac{1}{4} + \frac{1}{2} e^{M\delta^3} \Big) + 8M\delta e^{M\delta^3} \Big\} \leq M\tau^2 \sum_{j=0}^{1} \Big(\frac{2}{3} \Big)^j.
\end{aligned}
$$

Hence all inequalities in (111) are true for $n = 1$.

We now suppose that the inequalities in (111) are valid for $n = k$. Thus one has

$$
\big| W_\zeta^{(k)}(\zeta, \tau) \big|; \big| V_\zeta^{(k)}(\zeta, \tau) \big| \leq 3M\tau^2,
$$

Putting the above into (110) gives

$$
(118) \qquad \left| \frac{\partial z_\pm^{(k)}}{\partial \zeta}(t; \zeta, \tau) \right| \leq \exp \Big\{ \int_0^\tau Mt^2 \big(6M\tau^2 + 1 \big) \,\mathrm{d}t \Big\} \leq \exp(2M\delta^3).
$$

We sum up (104), (105), (109), (118) and employ (64) and the induction assumptions to conclude for $n = k + 1$

$$
\begin{aligned}
(119) \quad \big| V_\zeta^{(k+1)}(\zeta, \tau) \big| \leq & \frac{1}{4} M\tau^2 + \int_{\tau_-^{(k)}}^\tau \Big\{ \frac{1}{2} Mt \sum_{j=0}^{k} \Big(\frac{2}{3} \Big)^j + M^2 t^2 \sum_{j=0}^{k} \Big(\frac{2}{3} \Big)^j \\
& + 2M^2 t^2 \sum_{j=0}^{k} \Big(\frac{2}{3} \Big)^j + 8M^2 t^2 + Mt \Big\} e^{2M\delta^3} \,\mathrm{d}t \\
\leq & M\tau^2 \Big\{ \Big(\frac{1}{4} + (\frac{1}{2} + 3M\delta) e^{2M\delta^3} \Big) + \Big(\frac{1}{4} + M\delta \Big) e^{2M\delta^3} \sum_{j=0}^{k} \Big(\frac{2}{3} \Big)^j \Big\} \\
\leq & M\tau^2 \Big\{ 1 + \frac{2}{3} \sum_{j=0}^{k} \Big(\frac{2}{3} \Big)^j \Big\} = M\tau^2 \sum_{j=0}^{k+1} \Big(\frac{2}{3} \Big)^j.
\end{aligned}
$$

The above estimate also holds for $W_\zeta^{(k+1)}(\zeta,\tau)$. We next handle the term $|W_\zeta^{(k+1)}(\zeta,\tau)-V_\zeta^{(k+1)}(\zeta,\tau)|$. From (105), we deduce

$$\left|W_\zeta^{(k+1)}(\zeta,\tau)-V_\zeta^{(k+1)}(\zeta,\tau)\right|\le\left|\chi(\tau_-^{(k)})\right|+I_8+I_9+I_{10}+I_{11}, \tag{120}$$

where

$$I_8=\int_0^\tau\left\{\frac{\left|W_z^{(k)}-V_z^{(k)}\right|}{2t}+\left|A_{11}W_z^{(k)}\right|+\left|A_{12}V_z^{(k)}\right|+\left|A_{13}\right|\right\}\left|\frac{\partial z_+^{(k)}}{\partial\zeta}\right|\,\mathrm{d}t,$$
$$I_9=\int_{\tau_-^{(k)}}^\tau\left\{\frac{\left|V_z^{(k)}-W_z^{(k)}\right|}{2t}+\left|B_{11}V_z^{(k)}\right|+\left|B_{12}W_z^{(k)}\right|+\left|B_{13}\right|\right\}\left|\frac{\partial z_-^{(k)}}{\partial\zeta}\right|\,\mathrm{d}t,$$

and

$$I_{10}=\int_{\tau_-^{(k)}}^\tau t\left|F_z(z_+^{(k)}(t),t)\frac{\partial z_+^{(k)}}{\partial\zeta}-F_z(z_-^{(k)}(t),t)\frac{\partial z_-^{(k)}}{\partial\zeta}\right|\,\mathrm{d}t,$$
$$I_{11}=\int_0^{\tau_-^{(k)}}|F_zt|\cdot\left|\frac{\partial z_+^{(k)}}{\partial\zeta}\right|\,\mathrm{d}t.$$

It proceeds by (104), (109), (118) and the induction assumptions that

$$\begin{aligned}I_8;I_9\le&\int_0^\tau\left\{\frac12Mt\sum_{j=0}^k\left(\frac23\right)^j+3M\cdot Mt^2\sum_{j=0}^k\left(\frac23\right)^j+8M^2t^2\right\}e^{2M\delta^3}\,\mathrm{d}t\\ \le&M\tau^2\left\{3M\delta e^{2M\delta^3}+\left(\frac14+M\delta\right)e^{2M\delta^3}\sum_{j=0}^k\left(\frac23\right)^j\right\}.\end{aligned} \tag{121}$$

Moreover, one obtains

$$\begin{aligned}I_{10}\le&\int_0^\tau\left\{t\left|F_z(z_+^{(k)}(t),t)-F_z(z_-^{(k)}(t),t)\right|\cdot\left|\frac{\partial z_+^{(k)}}{\partial\zeta}\right|\right.\\&\left.+t|F_z(z_-^{(k)}(t),t)|\cdot\left|\frac{\partial z_+^{(k)}}{\partial\zeta}(z_+^{(k)}(t),t)-\frac{\partial z_-^{(k)}}{\partial\zeta}(z_-^{(k)}(t),t)\right|\right\}\,\mathrm{d}t\\ \le&\int_0^\tau\left\{tM\left|z_+^{(k)}(t)-z_-^{(k)}(t)\right|e^{2M\delta^3}+tM\cdot e^{2M\delta^3}2M\tau^3\right\}\,\mathrm{d}t\\ \le&\int_0^\tau\left\{tM\cdot M\tau^3\cdot e^{2M\delta^3}+2tM^2\tau^3e^{2M\delta^3}\right\}\,\mathrm{d}t=2M^2e^{2M\delta^3}\tau^5,\end{aligned} \tag{122}$$

and

$$I_{11} \leq \int_0^\tau Mte^{2M\delta^3}\,\mathrm{d}t = \frac{1}{2}M\tau^2 e^{2M\delta^3}. \tag{123}$$

Inserting (121)–(123) into (120) and making use of (64) yields

$$\begin{aligned}
&\left|W_\zeta^{(k+1)}(\zeta,\tau) - V_\zeta^{(k+1)}(\zeta,\tau)\right| \\
\leq& \frac{1}{4}M\tau^2 + 2M\tau^2\left\{3M\delta e^{2M\delta^3} + \left(\frac{1}{4} + M\delta\right)e^{2M\delta^3}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\} \\
&+ 2M^2 e^{2M\delta^3}\tau^5 + \frac{1}{2}M\tau^2 e^{2M\delta^3} \\
\leq& M\tau^2\left\{\left(\frac{1}{4} + \frac{1}{2}e^{2M\delta^3} + 8M\delta e^{2M\delta^3}\right) + \left(\frac{1}{2} + 2M\delta\right)e^{2M\delta^3}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\} \\
\leq& M\tau^2\left\{1 + \frac{2}{3}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\right\} = M\tau^2\sum_{j=0}^{k+1}\left(\frac{2}{3}\right)^j.
\end{aligned} \tag{124}$$

With the aid of (119) and (124) inequalities (111) follow. □

Based on Lemmas 3.2 and 3.3, we have

Lemma 3.4. *For any* $(\zeta,\tau) \in \overline{D}_\delta$ *and for all* $k \geq 0$*, the following inequalities hold*

$$\left|W^{(k+1)}(\zeta,\tau) - W^{(k)}(\zeta,\tau)\right|;\left|V^{(k+1)}(\zeta,\tau) - V^{(k)}(\zeta,\tau)\right| \leq M\tau^2\left(\frac{2}{3}\right)^k. \tag{125}$$

Proof. We also use the the argument of induction to prove the lemma. For $k = 0$, one gets by (87), (69) and (73)

$$\begin{aligned}
\left|V^{(1)}(\zeta,\tau) - V^{(0)}(\zeta,\tau)\right| \leq& |b(\tau_-^{(k)})| + |b(\tau)| + \int_{\tau_-^{(k)}}^{\tau}\left\{\frac{\left|b(t) - d(t)\right|}{2t}\right. \\
&\left. + \left|B_1(d)d\right| + \left|B_2(d)b\right| + \left|B_3(d)\right|t^2 + \left|Ft\right|\right\}(z_-^{(0)}(t),t)\,\mathrm{d}t \\
\leq& \frac{1}{4}M(\tau_-^{(k)})^2 + \frac{1}{4}M\tau^2 + \int_{\tau_-^{(k)}}^{\tau}\left\{K_0 t + 3M^2t^2 + Mt\right\}\mathrm{d}t \\
\leq& \frac{1}{4}M(\tau_-^{(k)})^2 + \frac{1}{4}M\tau^2 + \frac{1}{2}K_0\tau^2 + M^2\tau^3 + \frac{1}{2}M\tau^2 - \frac{1}{2}M(\tau_-^{(k)})^2
\end{aligned} \tag{126}$$

$$\leq M\tau^2\Big(\frac{7}{8}+M\delta\Big)\leq M\tau^2.$$

The derivation in (126) is also valid for W.

Assume that the inequalities in (125) hold for $n=k-1$. To establish (125) for $n=k$, we need to estimate the difference $|\tau_-^{(k)}-\tau_-^{(k-1)}|$. We recall (67) to find that

$$\begin{aligned}&\int_0^{\tau_-^{(k)}}\lambda_+(\bar{z}(t),t)\ \mathrm{d}t+\int_{\tau_-^{(k)}}^{\tau}\lambda_-^{(k)}(z_-^{(k)}(t;\zeta,\tau),t)\ \mathrm{d}t\\ =\zeta=&\int_0^{\tau_-^{(k-1)}}\lambda_+(\bar{z}(t),t)\ \mathrm{d}t+\int_{\xi_-^{(k-1)}}^{\tau}\lambda_-^{(k-1)}(z_-^{(k-1)}(t;\zeta,\tau),t)\ \mathrm{d}t,\end{aligned}$$

which means that

$$\begin{aligned}&\int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}}\Big(\lambda_-^{(k-1)}(z_-^{(k-1)}(t;\zeta,\tau),t)-\lambda_+(\bar{z}(t),t)\Big)\ \mathrm{d}t\\ =&\int_{\tau_-^{(k)}}^{\tau}\Big(\lambda_-^{(k)}(z_-^{(k)}(t),t)-\lambda_-^{(k-1)}(z_-^{(k-1)}(t),t)\Big)\ \mathrm{d}t.\end{aligned}\tag{127}$$

Here and below we assume, without loss of generality, $\tau_-^{(k)}\geq\tau_-^{(k-1)}$. Hence we combine (75), (108), (109) and (127) to acquire

$$\begin{aligned}&\frac{2}{3}\underline{k}\big|(\tau_-^{(k)})^3-(\tau_-^{(k-1)})^3\big|\\ \leq&\int_{\tau_-^{(k)}}^{\tau}\Big\{\big|\lambda_{-W}\big|\cdot\big|W^{(k)}(z_-^{(k)}(t),t)-W^{(k-1)}(z_-^{(k-1)}(t),t)\big|\\ &\qquad+\big|\lambda_{-z}\big|\cdot\big|z_-^{(k)}(t)-z_-^{(k-1)}(t)\big|\Big\}\ \mathrm{d}t\\ \leq&\int_{\tau_-^{(k)}}^{\tau}Mt^2\Big\{\big|W^{(k)}(z_-^{(k)}(t),t)-W^{(k-1)}(z_-^{(k-1)}(t),t)\big|\\ &\qquad+\big|z_-^{(k)}(t)-z_-^{(k-1)}(t)\big|\Big\}\ \mathrm{d}t.\end{aligned}\tag{128}$$

According to (111) and the induction assumptions, one achieves

$$\begin{aligned}&\big|W^{(k)}(z_-^{(k)}(t),t)-W^{(k-1)}(z_-^{(k-1)}(t),t)\big|\\ \leq&\big|W^{(k)}(z_-^{(k)}(t),t)-W^{(k)}(z_-^{(k-1)}(t),t)\big|\end{aligned}\tag{129}$$

$$
\begin{aligned}
&+\big|W^{(k)}(z_-^{(k-1)}(t),t)-W^{(k-1)}(z_-^{(k-1)}(t),t)\big|\\
\le&\big|W_z^{(k)}\big|\cdot\big|z_-^{(k)}(t)-z_-^{(k-1)}(t)\big|+Mt^2\Big(\frac{2}{3}\Big)^{k-1}\\
\le&3Mt^2\big|z_-^{(k)}(t)-z_-^{(k-1)}(t)\big|+Mt^2\Big(\frac{2}{3}\Big)^{k-1}.
\end{aligned}
$$

Next we estimate the difference $|z_-^{(k)}(t)-z_-^{(k-1)}(t)|$. It follows that for any $t\in[\tau_-^{(k)},\tau]$

$$
\begin{aligned}
&z_-^{(k)}(t)+\int_t^\tau\lambda_-^{(k)}(z_-^{(k)}(s),s)\,\mathrm{d}s\\
=&\,\zeta=z_-^{(k-1)}(t)+\int_t^\tau\lambda_-^{(k-1)}(z_-^{(k-1)}(s),s)\,\mathrm{d}s,
\end{aligned}
$$

from which and (129) one has

$$
\begin{aligned}
(130)\quad \big|z_-^{(k)}(t)-z_-^{(k-1)}(t)\big|\le&\int_t^\tau\Big|\lambda_-^{(k-1)}(z_-^{(k-1)}(s),s)-\lambda_-^{(k)}(z_-^{(k)}(s),s)\Big|\,\mathrm{d}s\\
\le&\int_t^\tau\Big\{\big|\lambda_{-W}\big|\cdot\big|W^{(k)}(z_-^{(k)}(s),s)-W^{(k-1)}(z_-^{(k-1)}(s),s)\big|\\
&+\big|\lambda_{-z}\big|\cdot\big|z_-^{(k)}(s)-z_-^{(k-1)}(s)\big|\Big\}\,\mathrm{d}s\\
\le&\int_{\tau_-^{(k)}}^\tau\Big\{Ms^2\Big(3Ms^2\big|z_-^{(k)}(s)-z_-^{(k-1)}(s)\big|+Ms^2\Big(\frac{2}{3}\Big)^{k-1}\Big)\\
&+Ms^2\big|z_-^{(k)}(s)-z_-^{(k-1)}(s)\big|\Big\}\,\mathrm{d}s.
\end{aligned}
$$

Set

$$
Z_-^{(k)}=\max_{t\in[\tau_-^{(k)},\tau]}\big|z_-^{(k)}(t)-z_-^{(k-1)}(t)\big|.
$$

One finds by (130) that

$$
\begin{aligned}
(131)\qquad Z_-^{(k)}\le&\int_{\tau_-^{(k)}}^\tau\Big\{Ms^2(1+3MS^2)Z_-^{(k)}+M^2s^4\Big(\frac{2}{3}\Big)^{k-1}\Big\}\,\mathrm{d}s\\
\le&M\tau^3Z_-^{(k)}+M^2\tau^5\Big(\frac{2}{3}\Big)^{k-1},
\end{aligned}
$$

which implies by the fact $M\delta^3 \le 1/2$ that

$$Z_-^{(k)} \le 2M^2\tau^5 \left(\frac{2}{3}\right)^{k-1}. \tag{132}$$

Inserting (132) into (129) gives

$$\left|W^{(k)}(z_-^{(k)}(t),t) - W^{(k-1)}(z_-^{(k-1)}(t),t)\right| \le Mt^2(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1}. \tag{133}$$

In a similar way, one also has for the function V

$$\left|V^{(k)}(z_-^{(k)}(t),t) - V^{(k-1)}(z_-^{(k-1)}(t),t)\right| \le Mt^2(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1}. \tag{134}$$

We put (133) and (132) into (128) to obtain

$$\begin{aligned} &\left|(\tau_-^{(k)})^3 - (\tau_-^{(k-1)})^3\right| \\ \le&\frac{3}{2\underline{k}}\int_0^\tau Mt^2\left\{Mt^2(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1} + 2M^2\tau^5\left(\frac{2}{3}\right)^{k-1}\right\}\,\mathrm{d}t \\ \le&\frac{2}{\underline{k}}M^2\tau^5\left(\frac{2}{3}\right)^{k-1}. \end{aligned} \tag{135}$$

We now consider the term $|V^{(k+1)}(\zeta,\tau) - V^{(k)}(\zeta,\tau)|$. It follows by (87) that

$$\begin{aligned} &\left|V^{(k+1)}(\zeta,\tau) - V^{(k)}(\zeta,\tau)\right| \\ \le &\int_{\tau_-^{(k)}}^\tau \left\{I_{12} + I_{13} + I_{14} + I_{15} + I_{16}\right\}\,\mathrm{d}t + I_{17}, \end{aligned} \tag{136}$$

where

$$\begin{aligned} I_{12} =& \frac{\left|V^{(k)}(z_-^{(k)}(t),t) - V^{(k-1)}(z_-^{(k-1)}(t),t)\right|}{2t} \\ &+ \frac{\left|W^{(k)}(z_-^{(k)}(t),t) - W^{(k-1)}(z_-^{(k-1)}(t),t)\right|}{2t}, \\ I_{13} =& \left|B_1(W^{(k)})W^{(k)}(z_-^{(k)}(t),t) - B_1(W^{(k-1)})W^{(k-1)}(z_-^{(k-1)}(t),t)\right|, \\ I_{14} =& \left|B_2(W^{(k)})V^{(k)}(z_-^{(k)}(t),t) - B_2(W^{(k-1)})V^{(k-1)}(z_-^{(k-1)}(t),t)\right|, \end{aligned}$$

$$I_{15} = \left|B_3(W^{(k)})(z_-^{(k)}(t),t) - B_3(W^{(k-1)})(z_-^{(k-1)}(t),t)\right|t^2,$$
$$I_{16} = \left|F(z_-^{(k)}(t),t) - F(z_-^{(k-1)}(t),t)\right|t,$$
$$I_{17} = \left|b(\tau_-^{(k)}) - b(\tau_-^{(k-1)}) - \int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}} \Theta^{(k-1)}(z_-^{(k-1)}(t),t)\ \mathrm{d}t\right|,$$

and

$$\begin{aligned}\Theta^{(k-1)} =& \bigg\{\frac{V^{(k-1)} - W^{(k-1)}}{2t} + B_1(W^{(k-1)})W^{(k-1)} \\ &+ B_2(W^{(k-1)})V^{(k-1)} + B_3(W^{(k-1)})t^2 + Ft\bigg\}(z_-^{(k-1)}(t),t).\end{aligned}$$

Next we estimate $I_{12}\cdots I_{17}$ term by term. For the term I_{12}, we use (133)–(134) to see that

$$I_{12} \le Mt(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1}. \tag{137}$$

For the term I_{13}, we recall (91), (104) and use (132)–(134) again to get

$$\begin{aligned} I_{13} \le& \left|B_1(W^{(k)})(z_-^{(k)}(t),t) - B_1(W^{(k-1)})(z_-^{(k-1)}(t),t)\right| \cdot \left|W^{(k)}\right| \\ &+ \left|B_1(W^{(k-1)})\right| \cdot \left|W^{(k)}(z_-^{(k)}(t),t) - W^{(k-1)}(z_-^{(k-1)}(t),t)\right| \\ \le& M\left|W^{(k)}(z_-^{(k)}(t),t) - W^{(k-1)}(z_-^{(k-1)}(t),t)\right| \cdot 3Mt^2 \\ &+ M\left|z_-^{(k)}(t) - z_-^{(k-1)}(t)\right| \cdot 3Mt^2 \\ &+ M\left|W^{(k)}(z_-^{(k)}(t),t) - W^{(k-1)}(z_-^{(k-1)}(t),t)\right| \\ \le& \left\{M\cdot Mt^2(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1} + M\cdot 2M^2\tau^5\left(\frac{2}{3}\right)^{k-1}\right\}\cdot 3Mt^2 \\ &+ M\cdot Mt^2(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1} \\ \le& M^2t^2\left(1+6M\tau^2\right)\left(\frac{2}{3}\right)^{k-1}. \end{aligned} \tag{138}$$

The above estimate is also valid for I_{14}. For the term I_{15}, one acquires

$$\begin{aligned} I_{15} \le& Mt^2\left|W^{(k)}(z_-^{(k)}(t),t) - W^{(k-1)}(z_-^{(k-1)}(t),t)\right| \\ &+ Mt^2\left|z_-^{(k)}(t) - z_-^{(k-1)}(t)\right| \end{aligned} \tag{139}$$

$$
\begin{aligned}
&\le Mt^2\left\{Mt^2(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1}+2M^2\tau^5\left(\frac{2}{3}\right)^{k-1}\right\}\\
&\le 2M^2t^2\tau^2\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
$$

For the term I_{16}, it suggests that

$$
\begin{aligned}
I_{16}\le & t|F_z|\cdot\left|z_-^{(k)}(t)-z_-^{(k-1)}(t)\right|\le tM\cdot 2M^2\tau^5\left(\frac{2}{3}\right)^{k-1}\\
\le & 2M^3\tau^6\left(\frac{2}{3}\right)^{k-1}.
\end{aligned} \tag{140}
$$

For the last term I_{17}, we have

$$
I_{17}\le\int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}}\left|b'(t)-\Theta^{(k-1)}(z_-^{(k-1)}(t),t)\right|\,\mathrm{d}t\le I_{18}+I_{19}. \tag{141}
$$

where

$$
\begin{aligned}
I_{18}&=\int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}}\left|b'(t)-\Theta(\bar z(t),t)\right|\,\mathrm{d}t,\\
I_{19}&=\int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}}\left|\Theta(\bar z(t),t)-\Theta^{(k-1)}(z_-^{(k-1)}(t),t)\right|\,\mathrm{d}t.
\end{aligned}
$$

Recalling (55) and (59) arrives at

$$
\Theta(\bar z(t),t)=a_1-\lambda_-(\bar z(t),t)(a_0'-ta_1')+\tilde\chi_0(t),
$$

which combined with (61), (56) and (57) yields

$$
\begin{aligned}
b'(t)-\Theta(\bar z(t),t)=&\Big(\tilde b_2'(t)-\hat a_0'\lambda_+(\bar z(t),t)+\hat a_1+t\hat a_1'\lambda_+(\bar z(t),t)\Big)\\
&-\Big(\hat a_1-\lambda_-(\bar z(t),t)(\hat a_0'-t\hat a_1')+\tilde\chi_0(t)\Big)\\
=&\big(\tilde b_2'(t)-\tilde\chi_0(t)\big)+(\hat a_0'-t\hat a_1')\big(\lambda_-(\bar z(t),t)-\lambda_+(\bar z(t),t)\big)\\
=&\big(\lambda_+(\bar z(t),t)-\lambda_-(\bar z(t),t)\big)\big(\tilde\chi_1(t)-\hat a_0'+t\hat a_1'\big),
\end{aligned} \tag{142}
$$

which indicates by the compatibility condition $(\mathcal{C})$ in (58) that

$$
\left|b'(t)-\Theta(\bar z(t),t)\right|\le 2\overline{K}K_0t^3. \tag{143}
$$

Hence we find by (135) and (143) that

$$
(144)\qquad I_{18} \leq \int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}} 2\overline{K}K_0 t^2\tau \ \mathrm{d}t \leq \overline{K}K_0\tau\big[(\tau_-^{(k)})^3 - (\tau_-^{(k-1)})^3\big] \\
\leq \frac{M^4\tau^6}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}.
$$

Furthermore, one applies the expression of $\Theta^{(k-1)}$ to conclude

$$
\begin{aligned}
&\Big|\Theta(\bar z(t),t) - \Theta^{(k-1)}(z_-^{(k-1)}(t),t)\Big| \\
=&\Big|\Theta^{(k-1)}(\bar z(t),t) - \Theta^{(k-1)}(z_-^{(k-1)}(t),t)\Big| \\
\leq&\bigg\{\frac{\big|V_z^{(k-1)} - W_z^{(k-1)}\big|}{2t} + \big|B_{11}V_z^{(k-1)}\big| + \big|B_{12}W_z^{(k-1)}\big| + \big|B_{13}\big| + |F_z|t\bigg\} \\
&\times\big(|\bar z(t)| + |z_-^{(k-1)}(t)|\big),
\end{aligned}
$$

from which and (109), (111) we obtain

$$
\begin{aligned}
(145)\qquad &\Big|\Theta(\bar z(t),t) - \Theta^{(k-1)}(z_-^{(k-1)}(t),t)\Big| \\
\leq&\bigg\{\frac{3}{2}Mt + M\cdot 3Mt^2 + 2M\cdot 3Mt^2 + 8M^2t^2 + Mt\bigg\}\cdot 2K_0t^3 \\
\leq&2M^2t^4(1+5M\delta) \leq 2M^2\tau^2t^2(1+5M\delta).
\end{aligned}
$$

Inserting (145) into the term I_{19} and employing (135) again gives

$$
\begin{aligned}
(146)\qquad I_{19} &\leq \int_{\tau_-^{(k-1)}}^{\tau_-^{(k)}} 2M^2t^4(1+5M\delta) \ \mathrm{d}t \\
&\leq M^2(1+5M\delta)\tau^2\big[(\tau_-^{(k)})^3 - (\tau_-^{(k-1)})^3\big] \\
&\leq \frac{2M^4(1+5M\delta)\tau^7}{\underline{k}}\left(\frac{2}{3}\right)^{k-1} \\
&\leq \frac{12M^4\tau^7}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
$$

one puts (144) and (146) into (141) to achieve

$$I_{17} \leq \frac{M^4\tau^6}{\underline{k}}\left(\frac{2}{3}\right)^{k-1} + \frac{12M^4\tau^7}{\underline{k}}\left(\frac{2}{3}\right)^{k-1} \leq \frac{2M^4\tau^6}{\underline{k}}\left(\frac{2}{3}\right)^{k-1}. \tag{147}$$

Combining with (136)–(140) and (147), we have

$$\begin{aligned}
&\left|V^{(k+1)}(\zeta,\tau) - V^{(k)}(\zeta,\tau)\right| \\
\leq &\int_0^\tau \left\{ Mt(1+6M^3\tau^5)\left(\frac{2}{3}\right)^{k-1} + 2M^2t^2\left(1+6M\tau^2\right)\left(\frac{2}{3}\right)^{k-1} \right. \\
&\left. + 2M^2t^2\tau^2\left(\frac{2}{3}\right)^{k-1} + 2M^3\tau^6\left(\frac{2}{3}\right)^{k-1}\right\} \mathrm{d}t + \frac{2M^4\tau^6}{\underline{k}}\left(\frac{2}{3}\right)^{k-1} \\
\leq &M\tau^2\left(\frac{1}{2} + 2M\delta + \frac{2\delta(M\delta)^3}{\underline{k}}\right)\left(\frac{2}{3}\right)^{k-1} \leq M\tau^2\left(\frac{2}{3}\right)^k,
\end{aligned} \tag{148}$$

by the choice of δ in (90). One can derive the same estimate for W and the proof of the lemma is finished. □

3.2.4. The existence and uniqueness of solutions. We now establish the existence and uniqueness of classical solutions to the problem (65), (60). According to Lemma 3.4, we see that the sequences $(W^{(k)}, V^{(k)})(\zeta,\tau)$ are uniformly convergent. Denote the limit functions by $(W,V)(\zeta,\tau)$ which are continuous on the region $\overline{D}_\delta$. Due to Lemma 3.2, we know that the limit functions $(W,V)(\zeta,\tau)$ satisfy

$$|W(\zeta,\tau)|;|V(\zeta,\tau)| \leq 3M\tau^2, \quad |W(\zeta,\tau) - V(\zeta,\tau)| \leq 3M\tau^2, \tag{149}$$

for any $(\zeta,\tau) \in \overline{D}_\delta$. Obviously, the functions (W,V) also satisfy the integral system (81) and the homogeneous initial conditions $W(\zeta,0) = V(\zeta,0) = 0$. Recalling the relation between τ_- and τ,ζ

$$\zeta - \int_{\tau_-}^{\tau} \lambda_-(z_i(t;\zeta,\tau),t)\,\mathrm{d}t = \bar{z}(\tau_-) = \int_0^{\tau_-} \lambda_+(\bar{z}(t),t)\,\mathrm{d}t, \tag{150}$$

one finds that $\zeta = \bar{z}(\tau_-)$ iff $\tau = \tau_-$, which along with (81) and (149) deduces $V(\bar{z}(\tau_-),\tau_-) = b(\tau_-)$. Thus the function V satisfies the boundary condition in (60). The boundary condition $W(\bar{z}(t),t) = d(t)$ also holds by (68). Moreover, it follows directly by (149) and the equation for W in (81) that

$W_\tau(\zeta,0)=0$. To derive the initial condition of V_τ, we differentiate (150) with respect to τ to obtain

$$\tau_{-\tau}=\frac{\lambda_-(\zeta,\tau)+\displaystyle\int_{\tau_-}^{\tau}\frac{\partial\lambda_-}{\partial z}\cdot\frac{\partial z_-}{\partial\tau}\,\mathrm{d}t}{\lambda_-(\bar z(\tau_-),\tau_-)-\lambda_+(\bar z(\tau_-),\tau_-)},\quad \frac{\partial z_-}{\partial\tau}(t;\zeta,\tau)=-\lambda_-\frac{\partial z_-}{\partial\zeta}(t;\zeta,\tau),$$

from which and (111), (118), (149) one gets

$$\tau_{-\tau}\le\frac{K_0}{\tau_-^2},$$

which together with (143) and (149) arrives at $V_\tau(\zeta,0)=0$. Therefore, the functions (W,V) satisfy all the conditions in (60).

Next we discuss the regularity of $(W,V)(\zeta,\tau)$. Based on the above analysis, we know that $W(\zeta,\tau)$ and $V(\zeta,\tau)$ possess one continuous derivative with respect to τ. In order to establish the existence of (W_ζ,V_ζ) in $\overline{D}_\delta$, we differentiate the integral system (81) with respect to ζ and consider the following system of integral equations

$$(151)\quad\begin{cases}W_\zeta(\zeta,\tau)=\displaystyle\int_0^\tau\left\{\frac{W_z-V_z}{2t}+A_{11}(V)W_z+A_{12}(W,V)V_z\right.\\ \qquad\qquad\left.+A_{13}(W,V)+F_zt\right\}\frac{\partial z_+}{\partial\zeta}(z_+(t),t)\,\mathrm{d}t,\\ V_\zeta(\zeta,\tau)=\chi(\tau_-)+\displaystyle\int_{\tau_-}^\tau\left\{\frac{V_z-W_z}{2t}+B_{11}(W)V_z+B_{12}(W,V)W_z\right.\\ \qquad\qquad\left.+B_{13}(W,V)+F_zt\right\}\frac{\partial z_-}{\partial\zeta}(z_-(t),t)\,\mathrm{d}t.\end{cases}$$

The coefficient functions in (151) are given in (105) but with the limit functions $(W,V,z_\pm)$ replacing $(W^{(k)},V^{(k)},z_\pm^{(k)})$. Similar arguments to Lemma 3.1, we solve the following ODE problem

$$\begin{cases}\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}=\dfrac{\sigma(t)-\chi(t)}{2t}+A_{11}(b(t))\sigma(t)+A_{12}(d(t),b(t))\chi(t)\\ \qquad\qquad+A_{13}(d(t),b(t))+F_z(\bar z(t),t)t,\\ \sigma(0)=\sigma'(0)=0,\end{cases}$$

and the solution $\sigma(t)$ satisfies the same estimate as $\chi(t)$. Set

$$(\widetilde{W}_z^{(0)}, \widetilde{V}_z^{(0)})(z,t) = (\sigma(t), \chi(t)).$$

Let us construct the sequences $(\widetilde{W}_z^{(k)}, \widetilde{V}_z^{(k)})(k \geq 1)$ as follows

(152)

$$\begin{cases} \widetilde{W}_\zeta^{(k+1)}(\zeta,\tau) = \displaystyle\int_0^\tau \left\{ \frac{\widetilde{W}_z^{(k)} - \widetilde{V}_z^{(k)}}{2t} + A_{11}(V)\widetilde{W}_z^{(k)} + A_{12}(W,V)\widetilde{V}_z^{(k)} \right. \\ \qquad\qquad\qquad \left. + A_{13}(W,V) + F_z t \right\} \dfrac{\partial z_+^{(k)}}{\partial \zeta}(z_+(t),t)\ \mathrm{d}t, \\ \widetilde{V}_\zeta^{(k+1)}(\zeta,\tau) = \chi(\tau_-) + \displaystyle\int_{\tau_-}^\tau \left\{ \frac{\widetilde{V}_z^{(k)} - \widetilde{W}_z^{(k)}}{2t} + B_{11}(W)\widetilde{V}_z^{(k)} \right. \\ \qquad\qquad \left. + B_{12}(W,V)\widetilde{W}_z^{(k)} + B_{13}(W,V) + F_z t \right\} \dfrac{\partial z_-^{(k)}}{\partial \zeta}(z_-(t),t)\ \mathrm{d}t, \end{cases}$$

where

$$\frac{\partial z_+^{(k)}}{\partial \zeta}(t;\zeta,\tau) = \exp\left\{ \int_\tau^t \frac{\kappa t^2}{1-t^2} \Big(C_{11}(V)\widetilde{V}_z^{(k)} + C_{12}(V) \Big)\ \mathrm{d}t \right\},$$

$$\frac{\partial z_-^{(k)}}{\partial \zeta}(t;\zeta,\tau) = \exp\left\{ -\int_\tau^t \frac{\kappa t^2}{1-t^2} \Big(D_{11}(W)\widetilde{W}_z^{(k)} + D_{12}(W) \Big)\ \mathrm{d}t \right\},$$

For the sequences $(\widetilde{W}_\zeta^{(k)}, \widetilde{V}_\zeta^{(k)})(k \geq 0)$, one can use the completely similar proof process of Lemma 3.3 to show the following lemma.

Lemma 3.5. *For any $(\zeta,\tau) \in \overline{D}_\delta$ and for all $k \geq 0$, the following inequalities hold*

$$\begin{aligned} &\big|\widetilde{W}_\zeta^{(k)}(\zeta,\tau)\big|; \big|\widetilde{V}_\zeta^{(k)}(\zeta,\tau)\big| \leq M\tau^2 \sum_{j=0}^k \left(\frac{2}{3}\right)^j, \\ &\big|\widetilde{W}_\zeta^{(k)}(\zeta,\tau) - \widetilde{V}_\zeta^{(k)}(\zeta,\tau)\big| \leq M\tau^2 \sum_{j=0}^k \left(\frac{2}{3}\right)^j. \end{aligned} \tag{153}$$

Based on Lemma 3.5, we have

Lemma 3.6. *For any $(\zeta,\tau)\in\overline{D}_\delta$ and for all $k\geq 0$, the following inequalities hold*

(154)
$$\big|\widetilde{W}_\zeta^{(k+1)}(\zeta,\tau)-\widetilde{W}_\zeta^{(k)}(\zeta,\tau)\big|;\big|\widetilde{V}_\zeta^{(k+1)}(\zeta,\tau)-\widetilde{V}_\zeta^{(k)}(\zeta,\tau)\big|\leq M\tau^2\Big(\frac{2}{3}\Big)^k.$$

Proof. The proof is also based on the argument of induction. We deduce for $n=0$ by (152)

$$\begin{aligned}\big|\widetilde{V}_\zeta^{(1)}(\zeta,\tau)-\widetilde{V}_\zeta^{(0)}(\zeta,\tau)\big|\leq&\ |\chi(\tau_-)|+|\chi(\tau)|\\&+\int_0^\tau\bigg\{\frac{\big|\sigma(t)-\chi(t)\big|}{2t}+\big|B_{11}(d)\chi(t)\big|+\big|B_{12}(d,b)\sigma(t)\big|\\&\qquad+\big|B_{13}(d,b)\big|+\big|F_z(\bar z(t),t)\big|t\bigg\}\bigg|\frac{\partial z_-^{(0)}}{\partial\zeta}\bigg|\,\mathrm{d}t\\\leq&2\widehat{K}\tau^2+\int_0^\tau\Big\{2K_0t+12M^2t^2\Big\}e^{2M\delta^3}\,\mathrm{d}t\\\leq&M\tau^2\bigg\{\frac12+\Big(\frac14+4M\delta\Big)e^{2M\delta^3}\bigg\}\leq M\tau^2.\end{aligned}$$

Here we used (64) and (109).

Assume that the inequalities in (154) are valid for $n=k-1$. Then for $n=k$, we calculate

(155)
$$\big|\widetilde{V}_\zeta^{(k+1)}(\zeta,\tau)-\widetilde{V}_\zeta^{(k)}(\zeta,\tau)\big|\leq I_{20}+I_{21},$$

where

$$\begin{aligned}I_{20}=&\int_{\tau_-}^\tau\bigg\{\frac{\big|\widetilde{V}_z^{(k)}-\widetilde{V}_z^{(k-1)}\big|+\big|\widetilde{W}_z^{(k)}-\widetilde{W}_z^{(k-1)}\big|}{2t}\\&+\big|B_{11}(W)\big|\cdot\big|\widetilde{V}_z^{(k)}-\widetilde{V}_z^{(k-1)}\big|+\big|B_{12}\big|\cdot\big|\widetilde{W}_z^{(k)}-\widetilde{W}_z^{(k-1)}\big|\bigg\}\bigg|\frac{\partial z_-^{(k)}}{\partial\zeta}\bigg|\,\mathrm{d}t,\\I_{21}=&\int_{\tau_-}^\tau\bigg\{\frac{\big|\widetilde{V}_z^{(k-1)}-\widetilde{W}_z^{(k-1)}\big|}{2t}+\big|B_{11}\widetilde{V}_z^{(k-1)}\big|+\big|B_{12}\widetilde{W}_z^{(k-1)}\big|\\&+|B_{13}|+|F_z|t\bigg\}\times\bigg|\frac{\partial z_-^{(k)}}{\partial\zeta}-\frac{\partial z_-^{(k-1)}}{\partial\zeta}\bigg|\,\mathrm{d}t.\end{aligned}$$

In view of the induction assumptions, we find that

$$
\begin{aligned}
I_{20} \leq & \int_0^\tau \left\{ Mt\left(\frac{2}{3}\right)^{k-1} + 3M \cdot Mt^2\left(\frac{2}{3}\right)^{k-1} \right\} e^{2M\delta^3}\, \mathrm{d}t \\
\leq & M\tau^2\left(\frac{1}{2} + M\delta\right) e^{2M\delta^3} \left(\frac{2}{3}\right)^{k-1},
\end{aligned} \tag{156}
$$

and

$$
\begin{aligned}
I_{21} \leq & \int_0^\tau \left\{ \frac{3Mt}{2} + M \cdot 3Mt^2 + 2M \cdot 3Mt^2 + 8M^2t^2 + Mt \right\} \\
& \qquad \times e^{2M\delta^3} M\tau^2 \cdot M\tau^2 \left(\frac{2}{3}\right)^{k-1} \mathrm{d}t \\
\leq & \left(2M\tau^2 + 6M^2\tau^3\right) \cdot e^{2M\delta^3} M^2\tau^4 \left(\frac{2}{3}\right)^{k-1}.
\end{aligned} \tag{157}
$$

Here we used the following result by the expression $\frac{\partial z_-^{(k)}}{\partial \zeta}$ in (50) and the estimates (109), (118)

$$
\begin{aligned}
\left| \frac{\partial z_-^{(k)}}{\partial \zeta} - \frac{\partial z_-^{(k-1)}}{\partial \zeta} \right| \leq & e^{2M\delta^3} \int_0^\tau \frac{\kappa t^2}{1-t^2} \left| D_{11}(W) \right| \cdot \left| \widetilde{W}_z^{(k)} - \widetilde{W}_z^{(k-1)} \right| \mathrm{d}t \\
\leq & e^{2M\delta^3} M\tau^2 \cdot M\tau^2 \left(\frac{2}{3}\right)^{k-1}.
\end{aligned}
$$

Now inserting (156) and (157) into (155) yields

$$
\begin{aligned}
& \left| \widetilde{V}_\zeta^{(k+1)}(\zeta,\tau) - \widetilde{V}_\zeta^{(k)}(\zeta,\tau) \right| \\
\leq & M\tau^2\left(\frac{1}{2} + M\delta\right) e^{2M\delta^3} \left(\frac{2}{3}\right)^{k-1} \\
& + \left(2M\tau^2 + 6M^2\tau^3\right) \cdot e^{2M\delta^3} M^2\tau^4 \left(\frac{2}{3}\right)^{k-1} \\
\leq & M\tau^2\left(\frac{1}{2} + 2M\delta\right) e^{2M\delta^3} \left(\frac{2}{3}\right)^{k-1} \leq M\tau^2 \left(\frac{2}{3}\right)^{k},
\end{aligned} \tag{158}
$$

by the choice of δ in (90). The inequality for W in (154) can be derived in a similar way. This completes the proof of the lemma. □

According to Lemmas 3.5 and 3.6, it is known that $(\widetilde{W}_\zeta^{(k)}, \widetilde{V}_\zeta^{(k)})$ are uniformly convergent, which imply that the functions $(W_\zeta, V_\zeta)(\zeta,\tau)$ are continuous and satisfy

$$
\begin{aligned}
&\big|W_\zeta(\zeta,\tau)\big|; \big|V_\zeta(\zeta,\tau)\big| \le 3M\tau^2,\\
&\big|W_\zeta(\zeta,\tau) - V_\zeta(\zeta,\tau)\big| \le 3M\tau^2.
\end{aligned} \tag{159}
$$

Thus the functions $(W,V)(z,t)$ are C^1-continuous. Since $(W,V)(z,t)$ satisfy the integral system (81) and have the required differentiability properties, it is a smooth solution of the mixed-type boundary problem (65), (60).

We assert that the solution $(W,V)(z,t)$ is unique. To show this assertion, we consider the difference of solutions $\widehat{W} = W_2 - W_1$ and $\widehat{V} = V_2 - V_1$, where (W_1, V_1) and (W_2, V_2) are two smooth solutions of problem (65), (60). It is not difficult to check by (81) and (109) that the functions $(\widehat{W}, \widehat{V})(z,t)$ satisfy the following homogeneous integral inequality system

$$
\begin{aligned}
&\big|\widehat{W}(\zeta,\tau)\big|; \big|\widehat{V}(\zeta,\tau)\big| \le \int_0^\tau \left\{ \frac{|\widehat{W} - \widehat{V}|}{2t} + \widehat{M}\big(|\widehat{W}| + |\widehat{V}|\big) \right\} \mathrm{d}t,\\
&\big|\widehat{W}(\zeta,\tau) - \widehat{V}(\zeta,\tau)\big| \le \int_0^\tau \left\{ \frac{|\widehat{W} - \widehat{V}|}{t} + \widehat{M}\big(|\widehat{W}| + |\widehat{V}|\big) \right\} \mathrm{d}t
\end{aligned} \tag{160}
$$

for some positive constant $\widehat{M}$. Clearly, the functions $(\widehat{W}, \widehat{V})$ also satisfy the inequalities as in (91). We repeat the insertion of these in the right side of (160) to see that there exists a positive constant M^* such that for arbitrary $k \ge 0$

$$
\big|\widehat{W}\big|; \big|\widehat{V}\big| \le M^* \left(\frac{2}{3}\right)^k.
$$

which means that there holds $\widehat{W} = \widehat{V} \equiv 0$. Hence we obtain the uniqueness of classical solutions of the mixed-type boundary problem (65), (60).

Thanks to the two problems (45), (53) and (63), (60) are equivalent by (59), the proof of Theorem 3.1 is complete.

4. Solutions in terms of self-similar variables

In the previous Section 3, we have established a classical solution $(\tilde{R}, \tilde{S})(z,t)$ in the region $\overline{D} := \{(z,t)|\ t \in [0,\delta], \tilde{z}(\delta) - \overline{K}\delta^3 + Kt^3 \le z \le \tilde{z}(t)\}$ for the mixed-type boundary problem (45), (53). Based on this result, in this section we construct a classical solution to system (13) in the self-similar (ξ,η) plane.

Due to (44) and (41), we first obtain the functions $(\bar{R},\bar{S})(z',t')$ in the region $\overline{D}' := \{(z',t')|\ t'\in[0,\delta^2], \tilde{z}(\delta)-\overline{K}\delta^3+K(\sqrt{t'})^3\le z\le \tilde{z}(\sqrt{t'})\}$. In order to obtain a solution in (ξ,η) plane, it is necessary to construct the coordinate functions $\xi=\xi(z',t')$ and $\eta=\eta(z',t')$. Recalling the coordinate transformation (37) and using (4), (10), (12) and (17), one has

$$\begin{aligned}\frac{\partial\xi}{\partial t'} &= -\frac{c\sin\theta(z',t')}{J\sqrt{1-t'}}, \quad \frac{\partial\eta}{\partial t'} = \frac{c\cos\theta(z',t')}{J\sqrt{1-t'}},\\ \frac{\partial\xi}{\partial z'} &= -\frac{2\sqrt{1-t'}\varpi_\eta}{J}, \quad \frac{\partial\eta}{\partial z'} = \frac{2\sqrt{1-t'}\varpi_\xi}{J},\end{aligned}\tag{161}$$

where $J(z',t')$ is defined by

$$J=-\frac{c(z',t')\sqrt{\kappa+1-t'}[\bar{R}(z',t')+\bar{S}(z',t')]}{\kappa\sqrt{1-t'}},$$

and

$$\begin{aligned}\varpi_\xi(z',t') =& \frac{\cos\theta(z',t')\sqrt{\kappa+1-t'}\bar{R}\bar{S}}{\kappa}\left(\frac{V-W}{2t}(z',t')-a_1\right)\\ &-\frac{\cos\theta(z',t')(1-t')}{c(z',t')}-\frac{\sin\theta(z',t')\sqrt{\kappa+1-t'}(\bar{R}+\bar{S})}{2\kappa\sqrt{1-t'}},\\ \varpi_\eta(z',t') =& \frac{\sin\theta(z',t')\sqrt{\kappa+1-t'}\bar{R}\bar{S}}{\kappa}\left(\frac{V-W}{2t}(z',t')-a_1\right)\\ &-\frac{\sin\theta(z',t')(1-t')}{c(z',t')}+\frac{\cos\theta(z',t')\sqrt{\kappa+1-t'}(\bar{R}+\bar{S})}{2\kappa\sqrt{1-t'}},\end{aligned}$$

which are well-defined by (149). The function $\theta(z',t')$ in (161) is defined by

$$\theta(z',t')=\begin{cases}\hat{\theta}(\hat{\xi}(z'))\\ \quad+\displaystyle\int_0^{t'}\frac{c(z',s)\sqrt{s}(\bar{R}-\bar{S})-2\kappa(1-s)\sqrt{\kappa+1-s}}{2c(z',s)\sqrt{1-s}(\kappa+1-s)(\bar{R}+\bar{S})}\,\mathrm{d}s,\\ \text{if } z'\in[\hat{z}_1,z_2],\\ \tilde{\theta}(\tilde{\xi}(z'))\\ \quad+\displaystyle\int_{\tilde{t}'}^{t'}\frac{c(z',s)\sqrt{s}(\bar{R}-\bar{S})-2\kappa(1-s)\sqrt{\kappa+1-s}}{2c(z',s)\sqrt{1-s}(\kappa+1-s)(\bar{R}+\bar{S})}\,\mathrm{d}s,\\ \text{if } z'\in[z_2,\tilde{z}_3],\end{cases}\tag{162}$$

where $\hat{z}_1 = \tilde{z}(\delta) - \overline{K}\delta^3$, $\tilde{z}_3 = \tilde{z}(\delta)$, $\tilde{t}' = (\tilde{z}^{-1}(z'))^2$ and $\tilde{z}^{-1}$ represents the inverse of $\tilde{z}(\cdot)$. The integrand function in (162) follows from the following relations

$$\begin{aligned}
\theta_{t'}(z',t') &= \frac{c(z',t')\sqrt{t'}(\bar{R}-\bar{S}) - 2\kappa(1-t')\sqrt{\kappa+1-t'}}{2c(z',t')\sqrt{1-t'}(\kappa+1-t')(\bar{R}+\bar{S})}, \\
\theta_{z'}(z',t') &= -\frac{\sqrt{1-t'}c(z',t')(T_2{}'\bar{S}+T_1{}'\bar{R})}{\kappa c^2(z',t')(\kappa+1-t')(\bar{R}+\bar{S})} \\
&\quad -\frac{\sqrt{1-t'}\kappa(1-t')\sqrt{\kappa+1-t'}\frac{T_2-T_1}{t}(z',t')}{\kappa c^2(z',t')(\kappa+1-t')(\bar{R}+\bar{S})},
\end{aligned} \tag{163}$$

which are derived by (13), (15) and (39). By using (161), we define the functions $\xi = \xi(z',t')$ and $\eta = \eta(z',t')$ for any $(z',t') \in \overline{D}'$

$$\xi(z',t') = \begin{cases} \hat{\xi}(z') + \displaystyle\int_0^{t'} \frac{\kappa\sin\theta(z',s)}{\sqrt{\kappa+1-s}(\bar{R}+\bar{S})(z',s)}\,\mathrm{d}s, & z' \in [\hat{z}_1, z_2], \\ \tilde{\xi}(z') + \displaystyle\int_{\tilde{t}'}^{t'} \frac{\kappa\sin\theta(z',s)}{\sqrt{\kappa+1-s}(\bar{R}+\bar{S})(z',s)}\,\mathrm{d}s, & z' \in [z_2, \tilde{z}_3], \end{cases} \tag{164}$$

and

$$\eta(z',t') = \begin{cases} \varphi(\hat{\xi}(z')) - \displaystyle\int_0^{t'} \frac{\kappa\cos\theta(z',s)}{\sqrt{\kappa+1-s}(\bar{R}+\bar{S})(z',s)}\,\mathrm{d}s, & z' \in [\hat{z}_1, z_2], \\ \psi(\tilde{\xi}(z')) - \displaystyle\int_{\tilde{t}'}^{t'} \frac{\kappa\cos\theta(z',s)}{\sqrt{\kappa+1-s}(\bar{R}+\bar{S})(z',s)}\,\mathrm{d}s, & z' \in [z_2, \tilde{z}_3]. \end{cases} \tag{165}$$

By the arbitrariness of (z',t'), the above process determines a region $\overline{\Omega}$ in the self-similar (ξ,η) plane

$$\overline{\Omega} = \{(\xi,\eta) |\ \xi = \xi(z',t'), \eta = \eta(z',t'), (z',t') \in \overline{D}'\},$$

which is corresponded to the region $\overline{D}'$ in the (z',t') plane. Moreover, we obtain by (161) that the Jacobian of the map $(z',t') \mapsto (\xi,\eta)$ is

$$j := \frac{\partial(\xi,\eta)}{\partial(z',t')} = -\frac{\kappa\sqrt{1-t'}}{c(z',t')\sqrt{\kappa+1-t'}(\bar{R}(z',t')+\bar{S}(z',t'))},$$

which along with the facts $\bar{R} > 0, \bar{S} > 0$ lead to $j < 0$ in $\overline{D}'$. This means that the map $(z', t') \mapsto (\xi, \eta)$ is an one-to-one mapping for $t' \in [0, \delta^2]$. Therefore, we have the functions $z' = z'(\xi, \eta)$, $t' = t'(\xi, \eta)$ defined on $\overline{\Omega}$. Furthermore, it follows by (164) and (165) that the images of $t' = 0, z' \in [\hat{z}_1, z_2]$ and $z' = \tilde{z}(\sqrt{t'}), z' \in [z_2, \tilde{z}_3]$ are, respectively, $\eta = \varphi(\xi)$ $(\xi \in [\xi_P, \hat{\xi}_C])$ and $\eta = \psi(\xi)$ $(\xi \in [\xi_P, \tilde{\xi}_E])$, where $\hat{\xi}_C = \hat{\xi}(\hat{z}_1)$ and $\tilde{\xi}_E = \hat{\xi}(\tilde{z}_3)$.

We now define the functions $(c, \theta, \varpi)(\xi, \eta)$ for any $(\xi, \eta) \in \overline{\Omega}$

(166)
$$c = \sqrt{\frac{2\kappa(1 - t'(\xi,\eta))z'(\xi,\eta)}{\kappa + 1 - t'(\xi,\eta)}}, \quad \varpi = \sqrt{1 - t'(\xi,\eta)}, \quad \theta = \theta(z'(\xi,\eta), t'(\xi,\eta)),$$

and denote

$$\alpha = \theta(\xi,\eta) + \arcsin \varpi(\xi,\eta), \qquad \beta = \theta(\xi,\eta) - \arcsin \varpi(\xi,\eta).$$

It is observed by the construction process that the functions $(\theta, \varpi)(\xi, \eta)$ defined in (166) satisfy the boundary conditions on $\widehat{PC}$ and $\widehat{PE}$. Next we check that they satisfy system (13). We only consider the first equation, the second equation of (13) can be checked analogously. Making use of (161) gives

$$\begin{aligned}(167)\quad \bar{\partial}^+\varpi &= \cos\alpha\varpi_\xi + \sin\alpha\varpi_\eta \\ &= \cos\alpha\left\{\frac{\cos\theta\sqrt{\kappa+1-t'}}{\kappa}\cdot\frac{\bar{R}-\bar{S}}{2\sqrt{t'}} - \frac{\cos\theta(1-t')}{c}\right. \\ &\quad \left. - \frac{\sin\theta\sqrt{\kappa+1-t'}(\bar{R}+\bar{S})}{2\kappa\sqrt{1-t'}}\right\} + \sin\alpha\left\{\frac{\sin\theta\sqrt{\kappa+1-t'}}{\kappa}\cdot\frac{\bar{R}-\bar{S}}{2\sqrt{t'}}\right. \\ &\quad \left. - \frac{\sin\theta(1-t')}{c} + \frac{\cos\theta\sqrt{\kappa+1-t'}(\bar{R}+\bar{S})}{2\kappa\sqrt{1-t'}}\right\} \\ &= \frac{\sqrt{\kappa+1-t'}\bar{R}}{\kappa} - \frac{(1-t')\sqrt{t'}}{c}.\end{aligned}$$

We combine (161) and (167) to calculate

$$\begin{aligned}\bar{\partial}^+\theta &= \cos\alpha\theta_\xi + \sin\alpha\theta_\eta \\ &= (\cos\alpha {z'}_\xi + \sin\alpha {z'}_\eta)\theta_{z'} + (\cos\alpha {t'}_\xi + \sin\alpha {t'}_\eta)\theta_{t'} \\ &= \left(\cos\alpha\frac{\eta_{t'}}{j} + \sin\alpha\frac{-\xi_{t'}}{j}\right)\theta_{z'} + \left(\cos\alpha\frac{-\eta_{z'}}{j} + \sin\alpha\frac{\xi_{z'}}{j}\right)\theta_{t'}\end{aligned}$$

$$=\frac{c(\cos\alpha\cos\theta+\sin\alpha\sin\theta)}{jJ\sqrt{1-t'}}\theta_{z'}+\frac{-2\sqrt{1-t'}(\cos\alpha\varpi_\xi+\sin\alpha\varpi_\eta)}{jJ}\theta_{t'}$$
$$=\frac{c\sqrt{t'}}{\sqrt{1-t'}}\theta_{z'}-\sqrt{1-t'}\left(\frac{\sqrt{\kappa+1-t'}\bar{R}}{\kappa}-\frac{(1-t')\sqrt{t'}}{c}\right)(2\theta_{t'}).$$

Putting (163) into the above and applying the relations by the expression of T_1', T_2' in (40)

$$T_2'\bar{S}+T_1'\bar{R}=2\sqrt{\kappa+1-t'}\bar{R}\bar{S}+\frac{\kappa\sqrt{t'}(1-t')}{c}(\bar{R}-\bar{S}),$$
$$T_2'-T_1'=\sqrt{\kappa+1-t'}(\bar{R}-\bar{S})-\frac{2\kappa\sqrt{t'}(1-t')}{c},$$

one has

$$\begin{aligned}\bar{\partial}^+\theta=&\frac{-1}{c(\kappa+1-t^2)(\bar{R}+\bar{S})}\Bigg\{\\&\frac{c\sqrt{t'}}{\kappa}\left(2\sqrt{\kappa+1-t'}\bar{R}\bar{S}+\frac{\kappa\sqrt{t'}(1-t')}{c}(\bar{R}-\bar{S})\right)\\&+(1-t')\sqrt{\kappa+1-t'}\left(\sqrt{\kappa+1-t'}(\bar{R}-\bar{S})-\frac{2\kappa\sqrt{t'}(1-t')}{c}\right)\\&+c\sqrt{t'}(\bar{R}-\bar{S})\left(\frac{\sqrt{\kappa+1-t'}}{\kappa}\bar{R}-\frac{\sqrt{t'}(1-t')}{c}\right)\\&-2\kappa(1-t')\sqrt{\kappa+1-t'}\left(\frac{\sqrt{\kappa+1-t'}}{\kappa}\bar{R}-\frac{\sqrt{t'}(1-t')}{c}\right)\Bigg\}\\=&-\frac{\sqrt{t'}\bar{R}}{\kappa\sqrt{\kappa+1-t'}}+\frac{1-t'}{c}.\end{aligned}$$

which together with (167) arrives at

$$\begin{aligned}&\bar{\partial}^+\theta+\frac{\cos\omega}{\kappa+\varpi^2}\bar{\partial}^+\varpi\\=&-\frac{\sqrt{t'}\bar{R}}{\kappa\sqrt{\kappa+1-t'}}+\frac{1-t'}{c}+\frac{\sqrt{t'}}{\kappa+1-t'}\left(\frac{\sqrt{\kappa+1-t'}\bar{R}}{\kappa}-\frac{(1-t')\sqrt{t'}}{c}\right)\\=&\frac{1-t'}{c}-\frac{(1-t')t'}{c(\kappa+1-t')}=\frac{(1-t')(\kappa+1-2t')}{c(\kappa+1-t')}\\=&\frac{\varpi^2(\kappa-1+2\varpi^2)}{c(\kappa+\varpi^2)},\end{aligned}$$

which is the desired result.

Finally, we define the functions $(\rho, u, v)(\xi, \eta)$ by (10) and (166)

$$\rho = \left(\frac{c^2(\xi,\eta)}{A\gamma}\right)^{\frac{1}{\gamma-1}}, u = \xi - c(\xi,\eta)\frac{\cos\theta(\xi,\eta)}{\varpi(\xi,\eta)}, v = \eta - c(\xi,\eta)\frac{\sin\theta(\xi,\eta)}{\varpi(\xi,\eta)}.$$

It is not difficult to check that the functions $(\rho, u, v)(\xi, \eta)$ defined as above satisfy the 2-D isentropic self-similar Euler equations (2).

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (12071106, 12171130), and Natural Science Foundation of Zhejiang province of China (LY21A010017).

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Received November 26, 2022