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Kohn–Rossi cohomology class, Sasakian space form and CR Frankel conjecture*

Der-Chen Chang[†], Shu-Cheng Chang[‡], Ting-Jung Kuo[‡], and Chien Lin[§,¶]

In this paper, we give a criterion of pseudo-Einstein contact forms and then affirm the CR analogue of Frankel conjecture in a closed, spherical, strictly pseudoconvex CR manifold of nonnegative pseudohermitian curvature on the space of smooth representatives of the first Kohn-Rossi cohomology group.

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1. Introduction

The well-known Riemann mapping theorem states that every simply connected domain Ω properly contained in $\mathbb{C}$ is biholomorphically equivalent to the open unit disc. In their paper of [6], Chern and Ji proved a generalization of the Riemann mapping theorem.

Proposition 1.1. *If Ω is a bounded, simply connected, strictly convex domain in $\mathbb{C}^{n+1}$ and its connected smooth boundary $\partial\Omega$ has a spherical CR structure, then it is biholomorphic to the unit ball and $M = \partial\Omega$ is the standard CR $(2n+1)$-sphere.*

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¶Corresponding author.

It is also known from Burns and Shnider ([1, Proposition 1.5.]) that if M is the compact **spherical** boundary of a Stein manifold, then either M is the standard CR sphere or $\pi_1(M)$ is infinite.

In Kaehler geometry, it was conjectured by Frankel ([8]) that a closed Kaehler manifold with positive bisectional curvature is biholomorphic to the complex projective space. The Frankel conjecture was proved in later 1970s independently by Mori ([22]) and Siu-Yau ([23]). Since Sasakian geometry (that is, its pseudohermitian torsion tensor vanishes) is an odd dimensional counterpart of Kaehler geometry, it is natural to ask for **CR analogue of Frankel conjecture** for Sasakian manifolds. In fact, this is proved by He and Sun ([17]):

Proposition 1.2. *The universal covering of any closed Sasakian* $(2n+1)$*-manifold of positive pseudohermitian bisectional curvature must be CR equivalent to the standard CR sphere* $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$.

Note that in view of Proposition 1.2, it involves the existence problem of transversely Kaehler-Einstein metrics (pseudo-Einstein contact structures) with positive pseudohermitian bisectional curvature and **Sasakian-Einstein** metrics in a closed Sasakian manifold.

From this inspiration, first by studying the existence theorem of pseudo-Einstein contact structures in a closed, strictly pseudoconvex CR $(2n+1)$-manifold of vanishing first Chern class for $n \geq 2$ as in Theorem 4.1 and Theorem 4.2, we are able to prove that such a manifold is Sasakian when it is spherical with nonnegative pseudohermitian curvature on the space of smooth representatives of the first Kohn-Rossi cohomology group. Then we affirm the CR Frankel conjecture as in Theorem 1.1 and Theorem 1.2.

More precisely, we first derive the key CR Bochner formulae as in Theorem 3.1 which are involved the CR Paneitz operator. This is one of main differences from Lee's key formula ([18]) as in (29). By using these formulae, we are able to obtain a pseudo-Eisntein contact form. Finally, we prove that any closed, spherical, strictly pseudoconvex CR $(2n+1)$-manifold (M, J, θ) of pseudo-Eisntein contact form θ with the positive constant Tanaka-Webster scalar curvature R must be **Sasakian space form** and manifolds always admit Riemannian metrics with positive Ricci curvature ([2]), so they must have finite fundamental group and the manifolds is a finite quotient of a standard CR sphere ([24]). Therefore the universal covering of M is globally CR equivalent to a standard CR sphere.

A strictly pseudoconvex CR $(2n+1)$-manifold is called *pseudo-Einstein* if its pseudohermitian Ricci curvature tensor is function-proportional to its

Levi metric

$$R_{\alpha\overline{\beta}} = \frac{R}{n} h_{\alpha\overline{\beta}} \tag{1}$$

for $n \geq 2$. It is equivalent to saying the following $(1,0)$-tensor W is vanishing ([18], [15], [4])

$$W_{\alpha} \doteqdot \left(R_{,\alpha} - inA_{\alpha\beta,}{}^{\beta} \right) = 0 \tag{2}$$

In particular, if the constant scalar curvature R is constant

$$A_{\alpha\beta,}{}^{\beta} = 0.$$

Hence the pseudo-Einstein condition (1) can be replaced by (2) for any $n \geq 1$. This is the main different point of view from the previous work by J. Lee ([18]). Here we come out with several key Bochner-type formulae as in Theorem 3.1. From this, we define ([15], [9], [3]) the quantity Q as the real part of covariant derivative of the $(1,0)$-tensor W by

$$Q := -\operatorname{Re}[\left(R_{,\alpha} - inA_{\alpha\beta,\overline{\beta}} \right)_{\overline{\alpha}}] = -\frac{1}{2}(W_{\alpha\overline{\alpha}} + W_{\overline{\alpha}\alpha}). \tag{3}$$

In particular as in [15] and [9] for $n = 1$, Q is the so-called CR Q-curvature in a closed strictly pseudoconvex CR 3-manifold.

Lee ([18]) showed an obstruction to the existence of a pseudo-Einstein contact form θ which is the vanishing of first Chern class $c_1(T_{1,0}M)$ for a closed, strictly pseudoconvex $(2n+1)$-manifold (M, J, θ) with $n \geq 2$. Thereafter, Lee conjectured that

Conjecture 1. *Any closed, strictly pseudoconvex CR $(2n+1)$-manifold of the vanishing first Chern class $c_1(T_{1,0}M)$ admits a global pseudo-Einstein structure for $n \geq 2$.*

Note that his pseudo-Einstein condition is less rigid than the Einstein condition in Riemannian geometry. Indeed, the CR contracted Bianchi identity no longer implies the pseudohermitian scalar curvature R to be a constant due to the presence of pseudohermitian torsion for $n \geq 2$

$$R_{\alpha\overline{\beta},\beta} = R_{\alpha} - i(n-1)A_{\alpha\beta,\overline{\beta}}.$$

To set up the method, we recall J. J. Kohn's Hodge theory for the $\overline{\partial}_b$ complex ([20]). Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold and $\eta \in \Omega^{0,1}(M)$ a smooth $(0,1)$-form on M with

$$\overline{\partial}_b \eta = 0.$$

Then there exists a smooth complex-valued function $\varphi = u + iv \in C^{\infty}_{\mathbb{C}}(M)$ and a smooth $(0,1)$-form $\gamma \in \Omega^{0,1}(M)$ for $\gamma = \gamma_{\overline{\alpha}}\theta^{\overline{\alpha}}$ such that

$$\left(\eta - \overline{\partial}_b \varphi\right) = \gamma \in \ker\left(\Box_b\right), \tag{4}$$

where $\Box_b = 2\left(\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b\right)$ is the Kohn-Rossi Laplacian.

Let the first Chern class $c_1(T^{1,0}M)$ of $T^{1,0}M$ be represented by Θ with

$$c_1(T^{1,0}M) = \frac{i}{2\pi}[d\omega_\alpha{}^\alpha] = \frac{i}{2\pi}[\Theta]$$

and

$$\Theta = R_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} + A_{\mu\alpha,\overline{\alpha}}\theta^\mu \wedge \theta - A_{\overline{\mu\alpha},\alpha}\theta^{\overline{\mu}} \wedge \theta,$$

which is the purely imaginary two-form. In this paper, we assume $c_1(T_{1,0}M) = 0$. Then there is a pure imaginary 1-form

$$\sigma = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}} - \sigma_\alpha\theta^\alpha + i\sigma_0\theta$$

with

$$d\omega_\alpha^\alpha = d\sigma = \Theta \tag{5}$$

for the pure imaginary Webster connection form ω_α^α. As in Lemma 3.3, we choose the $(0,1)$-form $\eta \in \Omega^{0,1}(M)$

$$\eta = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}}.$$

Then $\sigma_{\overline{\alpha}}\theta^{\overline{\alpha}}$ is $\overline{\partial}_b$-closed and the Kohn-Rossi solution is

$$\varphi_{\overline{\alpha}} = \sigma_{\overline{\alpha}} - \gamma_{\overline{\alpha}}. \tag{6}$$

By combining the CR Bochner-type estimates as in Theorem 3.1, we are able to prove the existence theorem of pseudo-Einstein contact structures $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ in a closed, strictly pseudoconvex CR $(2n+1)$-manifold of vanishing first Chern class as in Theorem 4.1 and Theorem 4.2 for $n \geq 2$. However, it follows from (13), (22), (35), and (47) that θ is also a pseudo-Einstein contact structure only if Q is the CR-pluriharmonic function. Moreover, by the contracted Bianchi identity, (3), and (47), then the CR-pluriharmonic function Q is equivalent to

$$A_{\alpha\beta,\overline{\alpha}\overline{\beta}} = 0$$

as in Theorem 1.1.

For $n = 1$, we refer to the authors' previous work where we established the following CR analogue Frankel conjecture in a closed spherical strictly pseudoconvex CR 3-manifold. That is

Proposition 1.3 ([4]). *Let (M, J, θ) be a closed spherical strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$ with the pluriharmonic CR Q-curvature. Assume that the CR Paneitz operator P_0 is nonnegative with kernel consisting of the CR pluriharmonic functions and the pseudohermitian curvature is $\frac{1}{2}$-positive*

$$R(x) > |A_{11}|(x)$$

with

$$A_{11,\overline{1}}(x) = 0$$

for all $x \in M$. Then (M, J, θ) is the Sasakian space form and the universal covering of M is CR equivalent to the standard CR 3-sphere.

Note that the CR Paneitz P_0 is always nonnegative for a closed pseudohermitian $(2n + 1)$-manifold (M, ξ, θ) with $n \geq 2$. Now it follows from Theorem 5.1 that if (M, J, θ) is a closed, spherical, strictly pseudoconvex CR $(2n+1)$-manifold with pseudo-Einstein contact form θ of positive constant Tanaka-Webster scalar curvature, then the universal covering of M must be globally CR equivalent to a standard CR sphere. Therefore by inspirations from Lee Conjecture 1 and results in [6], [5], [1] and [17], we affirm the CR analogue of Frankel conjecture via the nonnegativity of pseudohermitian curvature as in (7) and smooth representative of the first Kohn-Rossi cohomology group. In fact, as a consequence of Theorem 4.1 and Theorem 5.1, by comparing the above Proposition we have the following main theorem:

Theorem 1.1. *Let (M, J, θ) be a closed, spherical, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T^{1,0}M) = 0$, $n \geq 2$. Suppose that θ has the positive constant Tanaka-Webster scalar curvature R with*

$$A_{\alpha\beta,\overline{\alpha}\overline{\beta}} = 0$$

and the nonnegative pseudohermitian curvature

$$(Ric - \frac{1}{2}Tor)(\eta, \eta) \geq 0 \tag{7}$$

on the space of smooth representatives $(0,1)$*-form* $\eta = \rho_{\overline{\alpha}}\theta^{\overline{\alpha}} \in \Omega^{0,1}(M)$ *of the first Kohn-Rossi cohomology group* $H^{0,1}_{\overline{\partial}_b}(M)$ *(i.e.* $\eta \in \ker(\Box_b)$*). Then the universal covering of* M *is CR equivalent to the standard CR sphere* $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$*.*

We observe that the pseudohermitian curvature quantity (7) appears in the CR Bochner formula (50) as in the paper [2].

Furthermore, as a consequence of Theorem 4.2 and Theorem 5.1, we have

Theorem 1.2. *Let* (M, J, θ) *be a closed, spherical, strictly pseudoconvex CR* $(2n+1)$*-manifold of* $c_1(T^{1,0}M) = 0$*,* $n \geq 2$ *with* $d\omega_{\alpha}^{\alpha} = d\sigma, \sigma = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}} - \sigma_{\alpha}\theta^{\alpha} + i\sigma_0\theta$*. Assume that* θ *has the positive constant Tanaka-Webster scalar curvature* R *and* $\eta = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}}$ *satisfies*

(i)

$$\eta \in \ker(\Box_b),$$

(ii)

$$Tor'(\eta, \eta) = 0.$$

Here $Tor'(\eta, \eta) := 2\,\mathrm{Re}(i(A_{\overline{\alpha}\overline{\beta},\beta}\sigma_{\alpha})$*. Then the universal covering of* M *is CR equivalent to the standard CR sphere* $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$*.*

We briefly describe the methods used in our proofs. In section 2, we introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. In section 3, we will derive some crucial results such as the CR Bochner-type formula. In section 4, we give the existence theorems of pseudo-Einstein contact structures. In the final section, by applying results as in the previous sections, we then affirm the CR Frankel conjecture in a closed, spherical, strictly pseudoconvex CR $(2n+1)$-manifold.

2. Preliminaries

In this section, we recall some ingredients needed to prove main results in this paper. We first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold (see [18]). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ. A CR structure compatible with

ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$, and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{8}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. We also call such M a strictly pseudoconvex CR $(2n+1)$-manifold. The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form naturally induces a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha{}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}{}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where $\omega_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha{}^\beta + \theta \wedge \tau^\beta, \\ 0 &= \tau_\alpha \wedge \theta^\alpha, \\ 0 &= \omega_\alpha{}^\beta + \omega_{\bar{\beta}}{}^{\bar{\alpha}}. \end{aligned}$$

We can write (by the Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$, is

$$\begin{aligned} \Pi_\beta{}^\alpha &= \overline{\Pi_{\bar{\beta}}{}^{\bar{\alpha}}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha, \\ \Pi_0{}^\alpha &= \Pi_\alpha{}^0 = \Pi_0{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^0 = \Pi_0{}^0 = 0. \end{aligned}$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + W_\beta{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

Here $R_\gamma{}^\delta{}_{\alpha\bar\beta}$ is the pseudohermitian curvature tensor, $R_{\alpha\bar\beta} = R_\gamma{}^\gamma{}_{\alpha\bar\beta}$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the pseudohermitian torsion tensor. Furthermore, we denote

$$Tor(X,Y) := h^{\alpha\bar\beta}T_{\alpha\bar\beta}(X,Y) = i(A_{\overline{\alpha}\overline{\rho}}X^{\overline{\rho}}Y^{\overline{\alpha}} - A_{\alpha\rho}X^\rho Y^\alpha)$$

for any $X = X^\alpha Z_\alpha,\ Y = Y^\alpha Z_\alpha$ in $T_{1,0}$. We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\alpha = Z_\alpha u,\ u_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha u - \omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma u$. For a smooth real-valued function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle_{L_\theta} = du(Z)$ for all vector fields Z tangent to the contact plane. Locally, we denote $\nabla_b u = \sum_\alpha u_{\bar\alpha}Z_\alpha + u_\alpha Z_{\bar\alpha}$. We also denote $u_0 = Tu$. We can use the connection to define the subhessian as the complex linear map $(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$ by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular,

$$|\nabla_b u|^2 = 2\textstyle\sum_\alpha u_\alpha u_{\overline{\alpha}}, \quad |\nabla_b^2 u|^2 = 2\textstyle\sum_{\alpha,\beta}(u_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}} + u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \textstyle\sum_\alpha (u_{\alpha\bar\alpha} + u_{\bar\alpha\alpha}).$$

Definition 2.1 ([18], [6]). Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold with $n \geq 2$.

(i) We define the first Chern class $c_1(T_{1,0}M) \in H^2(M, \mathbf{R})$ for the holomorphic tangent bundle $T^{1,0}M$ by

$$\begin{aligned} c_1(T^{1,0}M) &= \frac{i}{2\pi}[d\omega_\alpha{}^\alpha] \\ &= \frac{i}{2\pi}[R_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} + A_{\alpha\mu,\overline{\alpha}}\theta^\mu \wedge \theta - A_{\overline{\alpha}\overline{\mu},\alpha}\theta^{\overline{\mu}} \wedge \theta]. \end{aligned} \tag{9}$$

(ii) We call a CR structure J spherical if the Chern curvature tensor

$$(10)\qquad \begin{array}{rcl} C_{\beta\overline{\alpha}\lambda\overline{\sigma}} & = & R_{\beta\overline{\alpha}\lambda\overline{\sigma}} - \frac{1}{n+2}[R_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + R_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}} + \delta^{\alpha}_{\beta}R_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}R_{\beta\overline{\sigma}}] \\ & & + \frac{R}{(n+1)(n+2)}[\delta^{\alpha}_{\beta}h_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}h_{\beta\overline{\sigma}}] \end{array}$$

vanishes identically.

Remark 2.2. 1. Note that $C_{\alpha\overline{\alpha}\lambda\overline{\sigma}} = 0$. Hence, $C_{\beta\overline{\alpha}\lambda\overline{\sigma}}$ is always vanishing for $n = 1$.

2. We observe that the spherical structure is CR invariant and a closed spherical CR $(2n+1)$-manifold (M, J) is locally CR equivalent to $(\mathbf{S}^{2n+1}, \widehat{J})$.

3. ([21]) In general, a spherical CR structure on a $(2n+1)$-manifold is a system of coordinate charts into S^{2n+1} such that the overlap functions are restrictions of elements of $PU(n+1, 1)$. Here $PU(n+1, 1)$ is the group of complex projective automorphisms of the unit ball in $\mathbf{C}^{n+1}$ and the holomorphic isometry group of the complex hyperbolic space $\mathbf{CH}^n$.

Definition 2.3 (i)**.** Let (M, ξ, θ) be a closed pseudohermitian $(2n+1)$-manifold. Define

$$P\varphi = \textstyle\sum_{\alpha=1}^{n}(\varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})\theta^{\beta} = (P_{\beta}\varphi)\theta^{\beta},\ \beta = 1, 2, \cdots, n$$

which is an operator that characterizes CR-pluriharmonic functions ([18] for $n = 1$ and [13] for $n \geq 2$). Here $P_{\beta}\varphi = \sum_{\alpha=1}^{n}(\varphi_{\overline{\alpha}}{}^{\overline{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})$ and $\overline{P}\varphi = (\overline{P}_{\beta}\varphi)\theta^{\overline{\beta}}$, the conjugate of P. Moreover, we define

$$(11)\qquad P_0\varphi = \delta_b(P\varphi) + \overline{\delta}_b(\overline{P}\varphi)$$

which is the so-called CR Paneitz operator P_0. Here δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_{\alpha}\theta^{\alpha}) = \sigma_{\alpha},{}^{\alpha}$. Hence, P_0 is a real and symmetric operator and

$$\textstyle\int_M \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L^*_{\theta}} d\mu = -\int_M (P_0\varphi)\,\varphi d\mu.$$

(ii) We call the Paneitz operator P_0 with respect to (J, θ) essentially positive if there exists a constant $\Lambda > 0$ such that

$$(12)\qquad \int_M P_0\varphi \cdot \varphi d\mu \geq \Lambda \int_M \varphi^2 d\mu$$

for all real smooth functions $\varphi \in (\ker P_0)^{\perp}$ (i.e. perpendicular to the kernel of P_0 in the L^2 norm with respect to the volume form $d\mu = \theta \wedge d\theta$). We say that P_0 is nonnegative if

$$\int_M P_0\varphi\cdot\varphi d\mu \geq 0$$

for all real smooth functions φ.

Remark 2.4. 1. The space of kernel of the CR Paneitz operator P_0 is infinite dimensional, containing all CR-pluriharmonic functions. However, for a closed pseudohermitian $(2n+1)$-manifold (M,ξ,θ) with $n\geq 2$, it was shown ([13]) that

$$\ker P_\beta = \ker P_0. \tag{13}$$

2. ([13], [2]) The CR Paneitz P_0 is always nonnegative for a closed pseudohermitian $(2n+1)$-manifold (M,ξ,θ) with $n\geq 2$.

3. ([18]) A real-valued smooth function u is said to be CR-pluriharmonic if, for any point $x\in M$, there is a real-valued smooth function v such that

$$\overline{\partial}_b(u+iv)=0. \tag{14}$$

3. The Bochner-type formulae

In this section, we first derive some essential lemmas. Recall that the transformation law of the connection under a change of pseudohermitian structure was computed in [19, Sec. 5]. Let $\hat{\theta}=e^{2f}\theta$ be another pseudohermitian structure. Then we can define an admissible coframe by $\hat{\theta}^\alpha = e^f(\theta^\alpha + 2if^\alpha\theta)$. With respect to this local coframe, the connection 1-form and the pseudohermitian torsion are given by

$$\begin{aligned}\widehat{\omega}_\beta{}^\alpha = \omega_\beta{}^\alpha &+ 2(f_\beta\theta^\alpha - f^\alpha\theta_\beta) + \delta^\alpha_\beta(f_\gamma\theta^\gamma - f^\gamma\theta_\gamma)\\ &+ i(f^\alpha{}_\beta + f_\beta{}^\alpha + 4\delta^\alpha_\beta f_\gamma f^\gamma)\theta,\end{aligned} \tag{15}$$

and

$$\widehat{A}_{\alpha\beta} = e^{-2f}(A_{\alpha\beta}+2if_{\alpha\beta}-4if_\alpha f_\beta), \tag{16}$$

respectively. Thus the Webster curvature transforms as

$$\widehat{R} = e^{-2f}(R-2(n+1)\Delta_b f - 4n(n+1)f_\gamma f^\gamma). \tag{17}$$

Here covariant derivatives on the right side are taken with respect to the pseudohermitian structure θ and an admissible coframe θ^α. Note also that the dual frame of $\{\hat\theta, \hat\theta^\alpha, \hat\theta^{\overline{\alpha}}\}$ is given by $\{\widehat{T}, \widehat{Z}_\alpha, \widehat{Z}_{\overline{\alpha}}\}$, where

$$\widehat{T} = e^{-2f}(T + 2if^{\overline{\gamma}}Z_{\overline{\gamma}} - 2if^\gamma Z_\gamma), \quad \widehat{Z}_\alpha = e^{-f}Z_\alpha.$$

Now we derive the following transformation property for the CR-pluriharmonic operator and CR Paneitz operator.

Lemma 3.1. *Let θ and $\widehat{\theta}$ be contact forms in a $(2n+1)$-dimensional pseudohermitian manifold (M,ξ). If $\widehat{\theta} = e^{2f}\theta$, then we have*

$$\begin{aligned}\widehat{R}_\alpha - in\widehat{A}_{\alpha\beta},^\beta = e^{-3f}[&R_\alpha - inA_{\alpha\beta},^\beta - 2(n+2)P_\alpha f] \\ &+2ne^{-2f}(\widehat{R}_{\alpha\overline{\beta}} - \tfrac{\widehat{R}}{n}\widehat{h}_{\alpha\overline{\beta}})f^{\overline{\beta}}.\end{aligned} \tag{18}$$

Proof. By the contracted Bianchi identity, we have

$$\tfrac{n-1}{n}(R_\alpha - inA_{\alpha\beta},^\beta) = (R_{\alpha\overline{\beta}} - \tfrac{R}{n}h_{\alpha\overline{\beta}}),^{\overline{\beta}}.$$

Also, by [19, P 172]

$$(R_{\alpha\overline{\beta}} - \tfrac{R}{n}h_{\alpha\overline{\beta}}) - 2(n+2)(f_{\alpha\overline{\beta}} - \tfrac{1}{n}f_\gamma{}^\gamma h_{\alpha\overline{\beta}}) = \widehat{R}_{\alpha\overline{\beta}} - \tfrac{\widehat{R}}{n}\widehat{h}_{\alpha\overline{\beta}}. \tag{19}$$

Following the same computation as the proof of Lemma 5.4 in [15], by using (15), (16), and (17), we compute

$$\begin{aligned}\widehat{R}_\alpha &= \widehat{Z}_\alpha\widehat{R} = e^{-f}Z_\alpha e^{-2f}(R - 2(n+1)\Delta_b f - 2n(n+1)|\nabla_b f|^2) \\ &= e^{-3f}[R_\alpha - 2Wf_\alpha + 4(n+1)(\Delta_b f + n|\nabla_b f|^2)f_\alpha \\ &\qquad - 2(n+1)(f_\gamma{}^\gamma{}_\alpha + f_{\overline{\gamma}}{}^{\overline{\gamma}}{}_\alpha) - 4n(n+1)(f_{\gamma\alpha}f^\gamma + f_\gamma f^\gamma{}_\alpha)], \\ i\widehat{A}_{\alpha\beta,\overline{\gamma}} &= i(\widehat{Z}_{\overline{\gamma}}\widehat{A}_{\alpha\beta} - \widehat{\omega}_\alpha{}^l(\widehat{Z}_{\overline{\gamma}})\widehat{A}_{\beta l} - \widehat{\omega}_\beta{}^l(\widehat{Z}_{\overline{\gamma}})\widehat{A}_{\alpha l}) \\ &= ie^{-f}[(Z_{\overline{\gamma}} + 2f_{\overline{\gamma}})\widehat{A}_{\alpha\beta} + 2(\delta_{\alpha\gamma}\widehat{A}_{\beta l} + \delta_{\beta\gamma}\widehat{A}_{\alpha l})f^l] \\ &= ie^{-f}(Z_{\overline{\gamma}} + 2f_{\overline{\gamma}})e^{-2f}(A_{\alpha\beta} + 2if_{\alpha\beta} - 4if_\alpha f_\beta) \\ &\quad + 2e^{-3f}[\delta_{\beta\gamma}(iA_{\alpha l} - 2f_{\alpha l} + 4f_\alpha f_l) + \delta_{\alpha\gamma}(iA_{\beta l} - 2f_{\beta l} + 4f_\beta f_l)]f^l \\ &= e^{-3f}[iA_{\alpha\beta,\overline{\gamma}} - 2f_{\alpha\beta\overline{\gamma}} + 4(f_{\alpha\overline{\gamma}}f_\beta + f_\alpha f_{\beta\overline{\gamma}})] \\ &\quad + 2e^{-3f}[\delta_{\beta\gamma}(iA_{\alpha l} - 2f_{\alpha l} + 4f_\alpha f_l) + \delta_{\alpha\gamma}(iA_{\beta l} - 2f_{\beta l} + 4f_\beta f_l)]f^l.\end{aligned}$$

Contracting the second equation with respect to the Levi metric $\widehat{h}_{\gamma\overline{\beta}} = h_{\gamma\overline{\beta}}$

yields

$$\begin{aligned} i\widehat{A}_{\alpha\beta},{}^{\beta} &= e^{-3f}[iA_{\alpha\beta},{}^{\beta} - 2f_{\alpha\beta}{}^{\beta} + 4(f_{\alpha}{}^{\beta}f_{\beta} + f_{\alpha}f_{\beta}{}^{\beta}) \\ &\quad +2(n+1)(iA_{\alpha\beta}-2f_{\alpha\beta} + 4f_{\alpha}f_{\beta})f^{\beta}]. \end{aligned}$$

Thus

$$\begin{aligned} \widehat{R}_{\alpha} - in\widehat{A}_{\alpha\beta},{}^{\beta} &= e^{-3f}[R_{\alpha} - inA_{\alpha\beta},{}^{\beta} - 2(n+1)(f_{\beta}{}^{\beta}{}_{\alpha} + f_{\overline{\beta}}{}^{\overline{\beta}}{}_{\alpha}) + 2nf_{\alpha\beta}{}^{\beta} \\ &\quad -2Rf_{\alpha} - 2n(n+1)iA_{\alpha\beta}f^{\beta} + 4(n+1)(f_{\beta}{}^{\beta} + f_{\overline{\beta}}{}^{\overline{\beta}})f_{\alpha} \\ &\quad -4n(n+1)f^{\beta}{}_{\alpha}f_{\beta} - 4n(f_{\alpha}{}^{\beta}f_{\beta} + f_{\beta}{}^{\beta}f_{\alpha})]. \end{aligned}$$

By using the commutation relations ([19, Lemma 2.3])

$$-2(n+1)f_{\beta}{}^{\beta}{}_{\alpha} + 2nf_{\alpha\beta}{}^{\beta} = -2f_{\overline{\beta}}{}^{\overline{\beta}}{}_{\alpha} + 2nR_{\alpha\overline{\beta}}f^{\overline{\beta}} - 2inA_{\alpha\beta}f^{\beta},$$

and

$$f_{\alpha\overline{\beta}} - f_{\overline{\beta}\alpha} = ih_{\alpha\overline{\beta}}f_{0},$$

and by (19)

$$[(R_{\alpha\overline{\beta}} - \tfrac{R}{n}h_{\alpha\overline{\beta}}) - 2(n+2)(f_{\alpha\overline{\beta}} - \tfrac{1}{n}f_{\gamma}{}^{\gamma}h_{\alpha\overline{\beta}})]f^{\overline{\beta}} = e^{f}(\widehat{R}_{\alpha\overline{\beta}} - \tfrac{\widehat{R}}{n}\widehat{h}_{\alpha\overline{\beta}})f^{\overline{\beta}},$$

we obtain the following transformation law:

$$\begin{aligned} &\widehat{R}_{\alpha} - in\widehat{A}_{\alpha\beta},{}^{\beta} - 2ne^{-2f}(\widehat{R}_{\alpha\overline{\beta}} - \tfrac{\widehat{R}}{n}\widehat{h}_{\alpha\overline{\beta}})f^{\overline{\beta}} \\ &= e^{-3f}[R_{\alpha} - inA_{\alpha\beta},{}^{\beta} - 2(n+2)(f_{\overline{\beta}}{}^{\overline{\beta}}{}_{\alpha} + inA_{\alpha\beta}f^{\beta})] \\ &= e^{-3f}[R_{\alpha} - inA_{\alpha\beta},{}^{\beta} - 2(n+2)P_{\alpha}f]. \end{aligned}$$

Then (18) follows easily. □

Lemma 3.2 ([18]). *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Then there is a pure imaginary 1-form*

$$\sigma = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}} - \sigma_{\alpha}\theta^{\alpha} + i\sigma_0\theta$$

with $d\omega_{\alpha}^{\alpha} = d\sigma$ such that

$$\sigma_{\overline{\beta},\overline{\alpha}} = \sigma_{\overline{\alpha},\overline{\beta}} \tag{20}$$

and

$$\begin{cases} R_{\alpha\overline{\beta}} = \sigma_{\overline{\beta},\alpha} + \sigma_{\alpha,\overline{\beta}} - \sigma_0 h_{\alpha\overline{\beta}}, \\ A_{\alpha\beta},{}^{\beta} = \sigma_{\alpha,0} + i\sigma_{0,\alpha} - A_{\alpha\beta}\sigma^{\beta}. \end{cases} \tag{21}$$

Lemma 3.3. *If (M, J, θ) is a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Then there exist $u \in C_{\mathbb{R}}^{\infty}(M)$ and $\gamma = \gamma_{\overline{\alpha}}\theta^{\overline{\alpha}} \in \Omega^{0,1}(M)$ such that*

$$W_\alpha = 2P_\alpha u + in\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) \tag{22}$$

and

$$\gamma_{\overline{\alpha},\overline{\beta}} = \gamma_{\overline{\beta},\overline{\alpha}} \text{ and } \gamma_{\overline{\alpha},\alpha} = 0. \tag{23}$$

Proof. By choosing

$$\eta = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}},$$

as in (4), where σ is chosen from Lemma 3.2, then from (20)

$$\overline{\partial}_b\eta = 0$$

and there exists

$$\varphi = u + iv \in C_{\mathbb{C}}^{\infty}(M)$$

and

$$\gamma = \gamma_{\overline{\alpha}}\theta^{\overline{\alpha}} \in \Omega^{0,1}(M) \cap \ker(\Box_b)$$

such that

$$\sigma_{\overline{\alpha}} = \varphi_{\overline{\alpha}} + \gamma_{\overline{\alpha}}. \tag{24}$$

Note that

$$\Box_b\gamma = 0 \Longrightarrow \overline{\partial}_b\gamma = 0 = \overline{\partial}_b^*\gamma \Longrightarrow \gamma_{\overline{\alpha},\overline{\beta}} = \gamma_{\overline{\beta},\overline{\alpha}} \text{ and } \gamma_{\overline{\alpha},\alpha} = 0$$

and

$$\sigma_\alpha = \left(\overline{\varphi}\right)_\alpha + \gamma_\alpha. \tag{25}$$

Here $\gamma_\alpha = \overline{\gamma_{\overline{\alpha}}}$. From the first equality in (21),

$$R = \sigma_{\overline{\mu},\mu} + \sigma_{\mu,\overline{\mu}} - n\sigma_0. \tag{26}$$

Therefore

$$\begin{array}{rcl}
\sigma_{\mu,\overline{\mu}\alpha} & = & (\overline{\varphi})_{,\mu\overline{\mu}\alpha} + \gamma_{\mu,\overline{\mu}\alpha} \text{ by } (25) \\
& = & (\overline{\varphi})_{,\mu\overline{\mu}\alpha} \text{ by } (23) \\
& = & (\overline{\varphi})_{,\overline{\mu}\mu\alpha} + in(\overline{\varphi})_{,0\alpha} \\
& = & (\overline{\varphi})_{,\overline{\mu}\mu\alpha} + in\left[(\overline{\varphi})_{,\alpha 0} + A_{\alpha\beta}(\overline{\varphi})_{,\overline{\beta}}\right]
\end{array}$$

and

$$\sigma_{\overline{\mu},\mu\alpha} = \varphi_{,\overline{\mu}\mu\alpha} \text{ by (24)} \quad \text{and (23)}.$$

It follows that

$$\begin{aligned}
W_\alpha &= \left(R_{,\alpha} - inA_{\alpha\beta,\overline{\beta}}\right) \\
&= \sigma_{\overline{\mu},\mu\alpha} + \sigma_{\mu,\overline{\mu}\alpha} - in\sigma_{\alpha,0} + inA_{\alpha\beta}\sigma_{\overline{\beta}} \text{ by (21) and (26)} \\
&= \varphi_{,\overline{\mu}\mu\alpha} + (\overline{\varphi})_{,\overline{\mu}\mu\alpha} + inA_{\alpha\beta}(\overline{\varphi})_{,\overline{\beta}} - in\gamma_{\alpha,0} + inA_{\alpha\beta}\left(\varphi_{\overline{\beta}} + \gamma_{\overline{\beta}}\right) \\
&= 2\left(u_{,\overline{\mu}\mu\alpha} + inA_{\alpha\beta}u_{\overline{\beta}}\right) + in\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) \\
&= 2P_\alpha u + in\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right). \qquad \square
\end{aligned}$$

We also recall Lemma 6.2 in [18] that states

Lemma 3.4. *If (M, J, θ) is a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$, then $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form if and only if*

$$\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}} = 0 \tag{27}$$

for all $\alpha, \beta \in I_n$.

Remark 3.5. Note that the conformal factor $e^{\frac{2u}{n+2}}$ is different from Lee's paper by $\frac{1}{n+2}$ due to the different setting between (24) and [18, (6.4)].

Lemma 3.6. *Let (M, J, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. If*

$$\int_M Ric\,(\gamma, \gamma)\, d\mu \geq 0,$$

then $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form and

$$\int_M Tor\,(\gamma, \gamma)\, d\mu = 0, \tag{28}$$

*where the smooth function $u \in C^\infty_{\mathbb{R}}(M)$ and $\gamma = \gamma_{\overline{\alpha}}\theta^{\overline{\alpha}} \in \Omega^{0,1}(M)$ with $\gamma_{\overline{\alpha},\alpha} = 0$ and $\gamma_{\overline{\alpha},\overline{\beta}} = \gamma_{\overline{\beta},\overline{\alpha}}$ are chosen as in **Lemma 3.3**.*

Proof. It is proved as in [18]

$$\int_M Ric(\gamma,\gamma)\,d\mu + \frac{1}{(n-1)}\sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu + \sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\beta}\right|^2 d\mu = 0. \tag{29}$$

It follows that if the pseudohermitian Ricci curvature is nonnegative

$$\gamma_{\overline{\beta},\alpha} = 0 = \gamma_{\overline{\beta},\overline{\alpha}} \tag{30}$$

and by complex conjugate

$$\gamma_{\beta,\overline{\alpha}} = 0. \tag{31}$$

Hence, by Lemma 3.4 that $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form. That is

$$\widetilde{R}_{\alpha\overline{\beta}} = \frac{\widetilde{R}}{n}\widetilde{h}_{\alpha\overline{\beta}}. \tag{32}$$

On the other hand, it follows from (18) that

$$\begin{array}{rcl} \widetilde{W_\alpha} & = & e^{-\frac{3u}{n+2}}\left[W_\alpha - 2(n+2)P_\alpha\left(\frac{u}{n+2}\right)\right] + \\ & & 2ne^{-\frac{2u}{n+2}}\left(\widetilde{R}_{\alpha\overline{\beta}} - \frac{\widetilde{R}}{n}\widetilde{h}_{\alpha\overline{\beta}}\right)\left(\frac{u}{n+2}\right)_{,\widetilde{\beta}} \end{array}$$

and then

$$W_\alpha = 2(n+2)P_\alpha\left(\frac{u}{n+2}\right) = 2P_\alpha u. \tag{33}$$

Thus, by **Lemma 3.3**, we obtain

$$\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) = 0.$$

Moreover, from the equality of Lemma 6.3 in [18] i.e.

$$\gamma_{\alpha,\overline{\beta}\beta} = i(1-n)\gamma_{\alpha,0}$$

and by (31)

$$\gamma_{\alpha,0} = 0. \tag{34}$$

This implies

$$A_{\alpha\beta}\gamma_{\overline{\beta}} = 0.$$

In particular

$$\int_M Tor\,(\gamma,\gamma)\,d\mu = 0. \qquad \square$$

In this paper, we have another criterion for $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ to be a pseudo-Einstein contact form.

Lemma 3.7. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Then $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form if and only if*

$$\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) = 0. \tag{35}$$

Proof. If $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form, then as the proof of Lemma 3.6, we have

$$\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) = 0. \tag{36}$$

Conversely, assume that $\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) = 0$, then

$$\begin{array}{rcl} 0 & = & ni\int_M \left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right)\gamma_{\overline{\alpha}}d\mu \\ & = & ni\int_M A_{\alpha\beta}\gamma_{\overline{\beta}}\gamma_{\overline{\alpha}}d\mu - \int_M \left(\gamma_{\alpha,\beta\overline{\beta}} - \gamma_{\alpha,\overline{\beta}\beta} - R_{\alpha\overline{\beta}}\gamma_\beta\right)\gamma_{\overline{\alpha}}d\mu \\ & = & ni\int_M A_{\alpha\beta}\gamma_{\overline{\beta}}\gamma_{\overline{\alpha}}d\mu + \int_M Ric\,(\gamma,\gamma)\,d\mu - \\ & & \sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu + \sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\beta}\right|^2 d\mu. \end{array}$$

Hence

$$\begin{array}{rcl} 0 & = & \int_M Ric\,(\gamma,\gamma)\,d\mu - \frac{n}{2}\int_M Tor\,(\gamma,\gamma)\,d\mu - \\ & & \sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu + \sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\beta}\right|^2 d\mu. \end{array}$$

Again by (29), we have

$$(n-1)\int_M Tor\,(\gamma,\gamma)\,d\mu + 2\sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu = 0.$$

On the other hand, it follows from (45) that

$$\begin{array}{l}\sum\limits_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu \\ = 2\sum\limits_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu + (n-1)\int_M Tor(\gamma,\gamma)d\mu.\end{array}$$

Hence,

$$\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu = 0.$$

It follows from (27) that $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form. □

In particular, if the pseudohermitian is vanishing, it is straightforward to obtain

$$\gamma_{\alpha,0} = 0.$$

Therefore, we recapture that $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form as following:

Corollary 3.8. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ and vanishing torsion $A_{\alpha\beta} = 0$ for $n \geq 2$. Then $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form.*

Proof. Since $\gamma_{\overline{\alpha},\alpha} = 0$ and $A_{\alpha\beta} = 0$, by the commutation relations ([18]) and (22),

$$\begin{array}{rcl} 0 & \leq & n\int_M |\gamma_{\alpha,0}|^2 d\mu \\ & = & n\int_M \gamma_{\alpha,0}\gamma_{\overline{\alpha},0} d\mu \\ & = & i\int_M \gamma_{\overline{\alpha},0}\left(R_{,\alpha} - 2u_{\overline{\beta}\beta\alpha}\right)d\mu \\ & = & -i\int_M \gamma_{\overline{\alpha}}\left(R_{,\alpha} - 2u_{\overline{\beta}\beta\alpha}\right)_0 d\mu \\ & = & -i\int_M \gamma_{\overline{\alpha}}\left(R_{,0\alpha} - 2u_{\overline{\beta}\beta0\alpha}\right)d\mu \\ & = & i\int_M \gamma_{\overline{\alpha},\alpha}\left(R_{,0} - 2u_{\overline{\beta}\beta0}\right)d\mu \\ & = & 0. \end{array}$$

Then

$$\gamma_{\alpha,0} = 0$$

and since $A_{\alpha\beta} = 0$

$$\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) = 0.$$

It follows from (35) that $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form. □

Next we come out with the following key Bochner-type formulae for $\gamma = \gamma_{\overline{\alpha}}\theta^{\overline{\alpha}}$.

Theorem 3.1. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Then*

(i)

$$\begin{array}{rcl} 0 & = & \int_M (Ric - \frac{1}{2}Tor)(\gamma,\gamma)\, d\mu + \sum\limits_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2\, d\mu + \\ & & \frac{1}{2(n-1)} \sum\limits_{\alpha,\beta} \int_M |\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}}|^2 d\mu \end{array} \tag{37}$$

(ii)

$$\frac{n}{2}\int_M Tor'(\gamma,\gamma)\, d\mu - \int_M (Q + P_0 u) u\, d\mu + \frac{n}{2(n-1)} \sum_{\alpha,\beta} \int_M |\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}}|^2 d\mu = 0. \tag{38}$$

(iii)

$$\int_M (Ric - \frac{1}{2}Tor - \frac{1}{2}Tor')(\gamma,\gamma)\, d\mu + \frac{1}{n}\int_M (Q + P_0 u) u\, d\mu + \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2\, d = 0. \tag{39}$$

Here $Ric(\gamma,\gamma) = R_{\alpha\overline{\beta}}\gamma_{\overline{\alpha}}\gamma_\beta$, $Tor(\gamma,\gamma) := i(A_{\overline{\alpha}\overline{\beta}}\gamma_\alpha\gamma_\beta - A_{\alpha\beta}\gamma_{\overline{\alpha}}\gamma_{\overline{\beta}})$ and $Tor'(\gamma,\gamma) := i(A_{\overline{\alpha}\overline{\beta},\beta}\gamma_\alpha - A_{\alpha\beta,\overline{\beta}}\gamma_{\overline{\alpha}})$.

Proof. From the equality (22)

$$W_\alpha = 2P_\alpha u + in\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right),$$

we are able to get

$$\begin{array}{rcl} \left(R_{,\alpha} - inA_{\alpha\beta,\overline{\beta}}\right)\gamma_{\overline{\alpha}} & = & W_\alpha \gamma_{\overline{\alpha}} \\ & = & 2\left(u_{\overline{\beta}\beta\alpha} + inA_{\alpha\beta}u_{\overline{\beta}}\right)\gamma_{\overline{\alpha}} + in(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0})\gamma_{\overline{\alpha}} \\ & = & 2\left(u_{\overline{\beta}\beta\alpha} + inA_{\alpha\beta}u_{\overline{\beta}}\right)\gamma_{\overline{\alpha}} + inA_{\alpha\beta}\gamma_{\overline{\beta}}\gamma_{\overline{\alpha}} - \\ & & \left(\gamma_{\alpha,\beta\overline{\beta}} - \gamma_{\alpha,\overline{\beta}\beta} - R_{\alpha\overline{\beta}}\gamma_\beta\right)\gamma_{\overline{\alpha}}. \end{array}$$

Taking the integration over M of both sides and its conjugation, we have, by the fact that $\gamma_{\alpha,\overline{\alpha}} = 0$,

$$\begin{aligned}
&\textstyle\int_M \left(Ric - \frac{n}{2}Tor - \frac{n}{2}Tor'\right)(\gamma,\gamma)\,d\mu - \sum\limits_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu \\
&\textstyle+\sum\limits_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\,d\mu - n\int_M Tor\,(d_b u,\gamma)\,d\mu \\
&= 0.
\end{aligned} \tag{40}$$

Here $Tor\,(d_b u,\gamma) = i(A_{\overline{\alpha}\overline{\beta}}u_\beta\gamma_\alpha - A_{\alpha\beta}u_{\overline{\beta}}\gamma_{\overline{\alpha}})$.

On the other hand, it follows from equality (22) that

$$\left(R,_\alpha - inA_{\alpha\beta,\overline{\beta}}\right)u_{\overline{\alpha}} = W_\alpha u_{\overline{\alpha}} = \left[2P_\alpha u + in\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right)\right]u_{\overline{\alpha}}. \tag{41}$$

By the fact that $\gamma_{\alpha,\overline{\alpha}} = 0$ again, we see that

$$\begin{aligned}
\textstyle\int_M \gamma_{\alpha,0}u_{\overline{\alpha}}d\mu &= \textstyle-\int_M \gamma_\alpha u_{\overline{\alpha}0}d\mu \\
&= \textstyle-\int_M \gamma_\alpha\left(u_{0\overline{\alpha}} - A_{\overline{\alpha}\overline{\beta}}u_\beta\right)d\mu \\
&= \textstyle\int_M A_{\overline{\alpha}\overline{\beta}}u_\beta\gamma_\alpha d\mu.
\end{aligned} \tag{42}$$

It follows from (41) and (42) that

$$\begin{aligned}
&\textstyle 2\int_M Qud\mu + 2\int_M (P_0 u)\,ud\mu \\
&= \textstyle in\int_M \left[\left(A_{\alpha\beta}u_{\overline{\beta}}\gamma_{\overline{\alpha}} - A_{\overline{\alpha}\overline{\beta}}u_\beta\gamma_\alpha\right) - conj\right]d\mu \\
&= \textstyle -2n\int_M Tor\,(d_b u,\gamma)\,d\mu.
\end{aligned}$$

That is

$$\int_M Qud\mu + \int_M (P_0 u)\,ud\mu = -n\int_M Tor\,(d_b u,\gamma)\,d\mu. \tag{43}$$

Thus by (40),

$$\begin{aligned}
&\textstyle\int_M \left(Ric - \frac{n}{2}Tor - \frac{n}{2}Tor'\right)(\gamma,\gamma)\,d\mu - \sum\limits_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu \\
&\textstyle+\sum\limits_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\,d\mu + \int_M (Q + P_0 u)ud\mu \\
&= 0.
\end{aligned} \tag{44}$$

On the other hand, since

$$\begin{aligned}&\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu\\&=2\sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu+\left(\int_M \gamma_{\alpha,\overline{\beta}}\gamma_{\beta,\overline{\alpha}}d\mu+\mathrm{conj}\right)\end{aligned}$$

and by commutation relations,

$$\begin{aligned}&\int_M \gamma_{\alpha,\overline{\beta}}\gamma_{\beta,\overline{\alpha}}d\mu\\&=-\int_M \gamma_{\alpha}\gamma_{\beta,\overline{\alpha}\overline{\beta}}d\mu\\&=i(n-1)\int_M A_{\overline{\alpha}\overline{\rho}}\gamma_{\alpha}\gamma_{\rho}d\mu.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu\\&=2\sum_{\alpha,\beta}\int_M \left|\gamma_{\alpha,\overline{\beta}}\right|^2 d\mu+(n-1)\int_M Tor(\gamma,\gamma)d\mu.\end{aligned}\tag{45}$$

This and (44) implies

$$\begin{aligned}0 &= \int_M(Ric-\tfrac{1}{2}Tor)(\gamma,\gamma)\,d\mu-\tfrac{1}{2}\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu-\\&\tfrac{n}{2}\int_M Tor'(\gamma,\gamma)\,d\mu+\sum_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\,d\mu+\int_M(Q+P_0u)ud\mu.\end{aligned}\tag{46}$$

(29) and (45) implies

$$\begin{aligned}0 &= \int_M \left(Ric-\tfrac{1}{2}Tor\right)(\gamma,\gamma)\,d\mu+\sum_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\,d\mu+\\&\tfrac{1}{2(n-1)}\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu.\end{aligned}$$

By combining (46) and (37),

$$\begin{aligned}0 &= -\tfrac{n}{2}\int_M Tor'(\gamma,\gamma)\,d\mu+\int_M(Q+P_0u)ud\mu-\\&\tfrac{n}{2(n-1)}\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta}+\gamma_{\beta,\overline{\alpha}}|^2 d\mu.\end{aligned}$$

By combining (44) and (29),

$$\begin{aligned}0 &= \int_M(Ric-\tfrac{1}{2}Tor-\tfrac{1}{2}Tor')(\gamma,\gamma)\,d\mu+\\&\tfrac{1}{n}\int_M(Q+P_0u)ud\mu+\sum_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\,d\mu.\end{aligned}$$

□

4. Pseudo-Einstein contact structures

Now, with the help of the lemmas in the last section, we are able to give the existence theorems for pseudo-Einstein contact structures as in Theorem 4.1 and Theorem 4.2.

Lemma 4.1. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Then*

(i)

$$Q_{\mathrm{ker}} = 0. \tag{47}$$

(ii)

$$Q^{\perp} + P_0 u^{\perp} = 0, \tag{48}$$

if $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form. Here $Q = Q_{\mathrm{ker}} + Q^{\perp}$. $Q^{\perp}$ is in $(\ker P_0)^{\perp}$ which is perpendicular to the kernel of self-adjoint Paneitz operator P_0 in the L^2 norm with respect to the volume form $d\mu = \theta \wedge d\theta$.

Proof. (i) We observe that the equality (22) still holds if we replace u by $(u + CQ_{\mathrm{ker}})$. It follows from the Bochner-type formula (46) that

$$\begin{aligned}
&\int_M (Ric - \tfrac{1}{2}Tor)(\gamma,\gamma)\, d\mu - \tfrac{n}{2}\int_M Tor'(\gamma,\gamma)\, d\mu - \tfrac{1}{2}\sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}}|^2 d\mu \\
&+\sum_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\, d\mu + \int_M (P_0 u)\, u d\mu + \int_M Q u d\mu + C\int_M (Q_{\mathrm{ker}})^2 d\mu \\
&= 0.
\end{aligned}$$

However, if $\int_M (Q_{\mathrm{ker}})^2 d\mu$ is not zero, this will lead to a contradiction by choosing the constant $C << -1$ or $C >> 1$, and the proof is complete.

(ii) If $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form, it follows from Lemma 3.7 that

$$\left(A_{\alpha\beta}\gamma_{\overline{\beta}} - \gamma_{\alpha,0}\right) = 0.$$

Then from Lemma 3.3

$$W_\alpha = 2P_\alpha u.$$

Hence

$$(W_\alpha)_{\overline{\alpha}} = 2(P_\alpha u)_{\overline{\alpha}}.$$

Taking its conjugacy in both sides

$$-Q = P_0 u$$

and then from (47)

$$Q^{\perp} + P_0 u^{\perp} = 0. \qquad \square$$

We observe that Q is vanishing when it is pseudo-Einstein. Our first goal is to justify the case whether a contact form θ is pseudo-Einstein whenever Q is the CR-pluriharmonic function consisting of infinite dimensional kernel of the CR Paneitz operator P_0 in a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, J, θ) for $n \geq 2$. The following proposition is due to (29) and Lemma 3.6 that

Proposition 4.2 ([18])**.** *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$. Suppose that*

$$\int_M Ric\,(\gamma, \gamma)\, d\mu \geq 0. \tag{49}$$

Then

(i) $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form.

(ii) θ is also a pseudo-Einstein contact form if Q is the CR-pluriharmonic function (i.e. $Q^{\perp} = 0$).

In general, we hope to replace the nonnegative assumption (49) by more natural pseudohermitian curvatures (51) which is a combination of pseudohermitian Ricci curvature and torsion. In fact, the CR analogue of Bochner formula states that

$$\begin{array}{rcl} \frac{1}{2}\Delta_b|\nabla_b u|^2 & = & |(\nabla^H)^2 u|^2 + (1 + \frac{2}{n}) < \nabla_b u, \nabla_b \Delta_b u >_{L_\theta} \\ & & +[2Ric - (n+2)Tor]((\nabla_b u)_{\mathbf{C}}, (\nabla_b u)_{\mathbf{C}}) \\ & & -\frac{4}{n} < Pu + \overline{P}u, d_b u >_{L_\theta^*} . \end{array} \tag{50}$$

Here $(\nabla_b u)_{\mathbf{C}} = u_{\bar{\alpha}} Z_\alpha$ is the corresponding complex $(1,0)$-vector field of $\nabla_b u$ and $d_b u = u_\alpha \theta^\alpha + u_{\overline{\alpha}} \theta^{\overline{\alpha}}$. We refer this pseudohermitian curvature quantity to our previous results as in [2].

More precisely, it follows from Lemma 4.1 and the CR Bochner-type formulae (46), (37), one can derive the following:

Theorem 4.1. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Assume that*

$$\int_M (Ric - \frac{1}{2}Tor)\,(\gamma, \gamma)\, d\mu \geq 0. \tag{51}$$

Then

(i) $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ *is a pseudo-Einstein contact form.*

(ii) θ *is also a pseudo-Einstein contact form if* Q *is the CR-pluriharmonic function (i.e.* $Q^{\perp} = 0$*).*

Proof. It follows from (37) that

$$\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}} = 0.$$

Hence, by Lemma 3.4, $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form. On the other hand, if Q is the CR-pluriharmonic function, then by (48) and (47),

$$u^{\perp} = 0$$

for $u = u_{\mathrm{ker}} + u^{\perp}$. Thus by (33),

$$W_{\alpha} = 0.$$

Then θ is also a pseudo-Einstein contact form. $\square$

Corollary 4.3. *Let* (M, J, θ) *be a closed, strictly pseudoconvex CR* $(2n+1)$*-manifold of* $c_1(T_{1,0}M) = 0$ *for* $n \geq 2$*. Assume that*

$$\int_M (Ric - \frac{1}{2}Tor)\,(\gamma,\gamma)\,d\mu \geq 0.$$

Then

$$\int_M Tor'\,(\gamma,\gamma)\,d\mu = 0. \tag{52}$$

Here $Tor'\,(\gamma,\gamma) := i(A_{\overline{\alpha}\overline{\beta},\beta}\gamma_{\alpha} - A_{\alpha\beta,\overline{\beta}}\gamma_{\overline{\alpha}}) = 2\,\mathrm{Re}(i(A_{\overline{\alpha}\overline{\beta},\beta}\gamma_{\alpha})$*.*

Proof. It follows from (37) and the assumption that

$$\int_M (Ric - \frac{1}{2}Tor)\,(\gamma,\gamma)\,d\mu = 0$$

and

$$0 = \sum_{\alpha,\beta}\int_M |\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}}|^2 d\mu = \sum_{\alpha,\beta}\int_M |\gamma_{\alpha,\beta}|^2\,d\mu.$$

Hence by (46), we have

$$\int_M Qud\mu + \int_M (P_0u)\,ud\mu - \frac{n}{2}\int_M Tor'\,(\gamma,\gamma)\,d\mu = 0.$$

Finally, it follows from (48) that

$$\int_M Qud\mu + \int_M (P_0 u)\, ud\mu = 0$$

and then

$$\int_M Tor'(\gamma, \gamma)\, d\mu = 0. \qquad \square$$

Now we want to relate the existence of pseudo-Einstein contact forms with the first Kohn-Rossi cohomology group $H^{0,1}_{\overline{\partial}_b}(M)$. By combining the Bochner formulae (38), we have

Theorem 4.2. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$ with*

$$d\omega_\alpha^\alpha = d\sigma$$

for $\sigma = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}} - \sigma_\alpha\theta^\alpha + i\sigma_0\theta$. Assume that

$$\eta = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}} \in \ker(\Box_b)\,. \tag{53}$$

Then $\widetilde{\theta}$ is pseudo-Einstein if and only if

$$\int_M Tor'(\eta, \eta)\, d\mu = 0. \tag{54}$$

In fact, θ is also pseudo-Einstein.

Remark 4.4. We observe that $\eta = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}}$ is a smooth representative of the first Kohn-Rossi cohomology group $H^{0,1}_{\overline{\partial}_b}(M)$ if and only if

$$\sigma_{\alpha,\overline{\alpha}} = 0 \quad \text{and} \quad \sigma_{\alpha,\beta} = \sigma_{\beta,\alpha}.$$

However, $\sigma_{\alpha,\beta} = \sigma_{\beta,\alpha}$ holds if $d\omega_\alpha^\alpha = d\sigma$. If $\sigma_{\alpha,\beta} = 0$, then

$$\int_M Tor'(\eta, \eta)\, d\mu = i\int_M (A_{\alpha\beta}\sigma_{\overline{\alpha},\overline{\beta}} - A_{\overline{\alpha}\overline{\beta}}\sigma_{\alpha,\beta})d\mu = 0.$$

Proof. It follows from (14), (6) and (53) that

$$\sigma_{\overline{\alpha}} = \gamma_{\overline{\alpha}}$$

and

$$u^{\perp} = 0.$$

Here we use the fact that the Kohn-Rossi cohomology group $H^{0,1}_{\overline{\partial}_b}(M)$ has a unique smooth representative $\gamma \in \ker(\Box_b)$. This implies

$$\int_M (Q + P_0 u) u d\mu = \int_M (Q^{\perp} + P_0 u^{\perp}) u^{\perp} d\mu = 0.$$

It follows from Bochner formula (38) that

$$\frac{n}{2} \int_M Tor'(\gamma, \gamma)\, d\mu + \frac{n}{2(n-1)} \sum_{\alpha,\beta} \int_M |\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}}|^2 d\mu = 0. \tag{55}$$

Then

$$\int_M Tor'(\eta, \eta)\, d\mu = 0$$

if and only if

$$\int_M \sum_{\alpha,\beta} |\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}}|^2 d\mu = 0.$$

That is

$$\gamma_{\overline{\alpha},\beta} + \gamma_{\beta,\overline{\alpha}} = 0.$$

All these imply that $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form as well as θ due to $u^{\perp} = 0$. $\square$

We observe that if the first Kohn-Rossi cohomology group $H^{0,1}_{\overline{\partial}_b}(M)$ is vanishing, it follows from Lemma 3.4 that $\widetilde{\theta} = e^{\frac{2u}{n+2}}\theta$ is a pseudo-Einstein contact form. As a consequence of Theorem 4.2, we have

Corollary 4.5. *Let (M, J, θ) be a closed, strictly pseudoconvex CR $(2n+1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$ with $d\omega_{\alpha}^{\alpha} = d\sigma$ for some $\sigma = \sigma_{\overline{\alpha}}\theta^{\overline{\alpha}} - \sigma_{\alpha}\theta^{\alpha} + i\sigma_0\theta$. Assume that either*

(i) the first Kohn-Rossi cohomology group $H^{0,1}_{\overline{\partial}_b}(M)$ is vanishing or

(ii)

$$d\sigma = \Theta = id(f\theta) \tag{56}$$

for some smooth, real-valued function f. Then $\widetilde{\theta}$ is the pseudo-Einstein contact form.

5. The CR analogue of Frankel conjecture

We affirm the CR analogue of Frankel conjecture in a closed, spherical, strictly pseudoconvex CR $(2n+1)$-manifold.

Lemma 5.1. *Let (M, J, θ) be a closed, spherical, strictly pseudoconvex CR $(2n+1)$-manifold with the pseudo-Einstein contact form θ for $n \geq 2$. Then*

$$\begin{array}{rcl} 0 & = & \frac{n+2}{n+1}\int_M k\sum_{\alpha,\gamma}|A_{\alpha\gamma}|^2 d\mu + \int_M \sum_{\alpha,\gamma,\sigma}|A_{\alpha\gamma,\sigma}|^2 d\mu \\ & & +\frac{1}{n-1}[\int_M \sum_{\alpha,\gamma,\beta}|A_{\alpha\gamma,\overline{\beta}}|^2 d\mu - n\int_M \sum_{\alpha} A_{\overline{\alpha}\overline{\beta},\beta}A_{\alpha\gamma,\overline{\gamma}}d\mu]. \end{array}$$

Here $k := \frac{R}{n}$.

Proof. Since θ is pseudo-Einstein, it follows that

$$R_{\alpha\overline{\beta}} = \frac{R}{n}h_{\alpha\overline{\beta}} := kh_{\alpha\overline{\beta}}. \tag{57}$$

Here $k := \frac{R}{n}$. Since J is spherical, it follows from (10) and (57) that

$$\begin{array}{rcl} R_{\beta\overline{\alpha}\lambda\overline{\sigma}} & = & \frac{k}{n+2}[h_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + h_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}} + \delta^{\alpha}_{\beta}h_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}h_{\beta\overline{\sigma}}] \\ & & -\frac{nk}{(n+1)(n+2)}[\delta^{\alpha}_{\beta}h_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}h_{\beta\overline{\sigma}}] \\ & = & \frac{k}{n+2}[h_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + h_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}}] \\ & & +\frac{k}{(n+1)(n+2)}[\delta^{\alpha}_{\beta}h_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}h_{\beta\overline{\sigma}}]. \end{array} \tag{58}$$

Again by [18, (2.15)],

$$A_{\alpha\rho,\beta\overline{\gamma}} = ih_{\beta\overline{\gamma}}A_{\alpha\rho,0} + R_{\alpha}{}^{\kappa}{}_{\beta\overline{\gamma}}A_{\kappa\rho} + R_{\rho}{}^{\kappa}{}_{\beta\overline{\gamma}}A_{\alpha\kappa} + A_{\alpha\rho,\overline{\gamma}\beta}.$$

Contracting both sides by $h^{\beta\overline{\gamma}}$

$$\begin{array}{rcl} A_{\alpha\rho,\gamma}{}^{\gamma} & = & inA_{\alpha\rho,0} + R_{\alpha}{}^{\kappa}{}_{\gamma}{}^{\gamma}A_{\kappa\rho} + R_{\rho}{}^{\kappa}{}_{\gamma}{}^{\gamma}A_{\alpha\kappa} + A_{\alpha\rho,\overline{\gamma}}{}^{\overline{\gamma}} \\ & = & inA_{\alpha\rho,0} + R_{\alpha\overline{\kappa}}A^{\overline{\kappa}}{}_{\rho} + R_{\rho\overline{\kappa}}A^{\overline{\kappa}}{}_{\alpha} + A_{\alpha\rho,\overline{\gamma}}{}^{\overline{\gamma}} \\ & = & inA_{\alpha\rho,0} + kh_{\alpha\overline{\kappa}}A^{\overline{\kappa}}{}_{\rho} + kh_{\rho\overline{\kappa}}A^{\overline{\kappa}}{}_{\alpha} + A_{\alpha\rho,\overline{\gamma}}{}^{\overline{\gamma}} \\ & = & inA_{\alpha\rho,0} + 2kA_{\alpha\rho} + A_{\alpha\rho,\overline{\gamma}}{}^{\overline{\gamma}}. \end{array}$$

That is

$$A_{\alpha\gamma,\sigma}{}^{\sigma} = inA_{\alpha\gamma,0} + 2kA_{\alpha\gamma} + A_{\alpha\gamma,\overline{\sigma}}{}^{\overline{\sigma}} \tag{59}$$

for all α, γ. Next we claim that

$$inA_{\alpha\gamma,0} = -\frac{nk}{n+1}A_{\alpha\gamma} + \frac{n}{n-1}(A_{\alpha\beta,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\beta}). \tag{60}$$

Again from [18, (2.9)],

$$A_{\alpha\rho,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\rho} = ih_{\rho\overline{\beta}}A_{\alpha\gamma,0} - ih_{\gamma\overline{\beta}}A_{\alpha\rho,0} + R_{\alpha\overline{\beta}\rho\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma} - R_{\alpha\overline{\beta}\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho}$$

Contracting both sides by $h^{\rho\overline{\beta}}$,

$$inA_{\alpha\gamma,0} - i\delta^{\rho}_{\gamma}A_{\alpha\rho,0} + R_{\alpha}{}^{\rho}{}_{\rho\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma} - R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho} = A_{\alpha\beta,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\beta}.$$

Hence

$$i(n-1)A_{\alpha\gamma,0} + R_{\alpha\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma} - R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho} = A_{\alpha\beta,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\beta}$$

and thus

$$i(n-1)A_{\alpha\gamma,0} + kA_{\alpha\gamma} - R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho} = A_{\alpha\beta,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\beta}.$$

On the other hand,

$$\begin{array}{rcl} R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho} & = & \frac{k}{n+2}[h_{\alpha\overline{\rho}}h_{\gamma\overline{\sigma}} + h_{\gamma\overline{\rho}}h_{\alpha\overline{\sigma}}]A^{\overline{\sigma}}{}_{\rho} \\ & & +\frac{k}{(n+1)(n+2)}[\delta^{\rho}_{\alpha}h_{\gamma\overline{\sigma}} + \delta^{\rho}_{\gamma}h_{\alpha\overline{\sigma}}]A^{\overline{\sigma}}{}_{\rho} \\ & = & \frac{2k}{n+1}A_{\alpha\gamma}. \end{array}$$

All these imply

$$i(n-1)A_{\alpha\gamma,0} + \tfrac{n-1}{n+1}kA_{\alpha\gamma} = A_{\alpha\beta,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\beta}$$

for $n \geq 2$. Thus (60) follows. Next, from (59) and (60), we obtain

$$\begin{array}{rcl} A_{\alpha\gamma,\sigma}{}^{\sigma} & = & inA_{\alpha\gamma,0} + 2kA_{\alpha\gamma} + A_{\alpha\gamma,\overline{\sigma}}{}^{\overline{\sigma}} \\ & = & \frac{n+2}{n+1}kA_{\alpha\gamma} + \frac{n}{n-1}(A_{\alpha\beta,\overline{\beta}\gamma} - A_{\alpha\gamma,\overline{\beta}\beta}) + A_{\alpha\gamma,\overline{\sigma}}{}^{\overline{\sigma}}. \end{array}$$

We integrate both sides with $A^{\alpha\gamma}$ to get

$$\begin{array}{rcl} 0 & = & \frac{n+2}{n+1}\int_M k\sum_{\alpha,\gamma}|A_{\alpha\gamma}|^2 d\mu + \int_M \sum_{\alpha,\gamma,\sigma}|A_{\alpha\gamma,\sigma}|^2 d\mu \\ & & +\frac{1}{n-1}[\int_M \sum_{\alpha,\gamma,\beta}|A_{\alpha\gamma,\overline{\beta}}|^2 d\mu - n\int_M \sum_{\alpha} A_{\overline{\alpha}\overline{\beta},\beta}A_{\alpha\gamma,\overline{\gamma}}d\mu], \end{array}$$

and we finish the proof of the lemma. □

Theorem 5.1. *Let (M, J, θ) be a closed, spherical, strictly pseudoconvex CR $(2n+1)$-manifold with pseudo-Einstein contact form θ of positive constant Tanaka-Webster scalar curvature. Then the universal covering of M must be globally CR equivalent to a standard CR sphere.*

Proof. Since

$$R_{\alpha\overline{\beta},\beta} = R_\alpha - i(n-1)A_{\alpha\beta,\overline{\beta}},$$

if $R_{\alpha\overline{\beta}} = \frac{R}{n}h_{\alpha\overline{\beta}}$ and R is constant, then

$$A_{\alpha\gamma,\overline{\gamma}} = 0.$$

It follows from Lemma 5.1 that if $k > 0$, one has

$$\begin{array}{l} \frac{n+2}{n+1}k\int_M \sum_{\alpha,\gamma}|A_{\alpha\gamma}|^2 d\mu + \int_M \sum_{\alpha,\gamma,\sigma}|A_{\alpha\gamma,\sigma}|^2 d\mu \\ +\frac{1}{n-1}\int_M \sum_{\alpha,\gamma,\beta}|A_{\alpha\gamma,\overline{\beta}}|^2 d\mu \ = \ 0 \end{array}$$

and

$$A_{\alpha\gamma} = 0.$$

Moreover, it follows from (58) that

$$R_{\beta\overline{\alpha}\lambda\overline{\sigma}} = \frac{R}{n(n+1)}[h_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + h_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}}].$$

Hence (M,θ) is a closed, Sasakian CR $(2n+1)$-manifold of positive constant pseudohermitian bisectional curvature. Hence manifolds always admit Riemannian metrics with positive Ricci curvature ([2]), so they must have finite fundamental group. It follows from ([24]) that the universal covering of M is CR equivalent to a CR standard Sphere $\mathbf{S}^{2n+1}$ in $\mathbb{C}^{n+1}$. □

Then the proofs of Theorem 1.1 and Theorem 1.2 are therefore completed.

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Der-Chen Chang
Department of Mathematics and Statistics
Georgetown University
Washington D. C. 20057
USA
Graduate Institute of Business Administration
College of Management
Fu Jen Catholic University
Taipei 242
Taiwan
E-mail address: chang@georgetown.edu

Shu-Cheng Chang
Department of Mathematics
National Taiwan University
Taipei 10617, Taiwan
Mathematical Science Research Center
Chongqing University of Technology
Chongqing 400054
P.R. China
E-mail address: scchang@math.ntu.edu.tw

Ting-Jung Kuo
Department of Mathematics
National Taiwan Normal University
Taipei 11677
Taiwan
E-mail address: tjkuo1215@ntnu.edu.tw

Chien Lin
Mathematical Science Research Center
Chongqing University of Technology
Chongqing 400054
P.R. China
E-mail address: chienlin@cqut.edu.cn

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