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The Kohn–Laplacian and Cauchy–Szegö projection on model domains*

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We study the Kohn–Laplacian and its fundamental solution on some model domains in $\mathbb{C}^{n+1}$, and further discuss the explicit kernel of the Cauchy–Szegö projections on these model domains using the real analysis method. We further show that these Cauchy–Szegö kernels are Calderón–Zygmund kernels under the suitable quasi-metric.

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1. Background and main results

In complex analysis, fundamental objects such as fundamental solutions for the Kohn–Laplacian, Cauchy–Szegö kernel, heat kernel, etc. are explicitly known in very few cases. But explicit solutions are very important for related analysis, especially for unbounded domains. In this paper, we discuss such formulae for some higher step case in higher dimension using a different approach. We first review the geometry of a general real hypersurface $\mathcal{M}$ in $\mathbb{C}^{n+1}$, which is unbounded and of high step, and study geometrically

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invariant formulas for the fundamental solutions of the Kohn–Laplacian, Cauchy–Szegö kernel and heat kernels. We refer to Theorems 5.3 and 6.1. This model domain was studied intensively by many mathematicians, especially Beals, Gaveau and Grenier (see *e.g.,* Beals [2], Beals–Gaveau–Greiner [3, 4, 5, 6] and Calin–Chang–Greiner [8]).

The main result in the current paper is the estimates for those kernels by applying real method in harmonic analysis. The first step is to establish the L^2 estimates. For the Cauchy–Szegö projection $\mathbf{S}$, this is automatic since by definition $\mathbf{S} : L^2(\partial\Omega_k) \to H^2(\Omega_k)$ is bounded. Then the next natural operator is $P(X_1, X_2)K_\lambda$. K_λ is the fundamental solution for the Kohn Laplacian which we derived in our Sections 4 and 5, and $P(X_1, X_2)$ is a quadratic polynomial in the "horizontal" vector fields X_1 and X_2. Then we can use David–Journé theorem (The famous $T(1)$ Theorem) (see David–Journé [14] and Nagel–Rosay–Stein–Wainger [24]). Due to the limitation of pages, we will put detailed calculations in a forthcoming paper.

The natural next step is to investigate whether the Cauchy–Szegö kernels on the boundary of model domains are Calderón–Zygmund kernels. Note that Diaz [13] studied this property for the Cauchy–Szegö kernels of Greiner and Stein [20] in domains in $\mathbb{C}^2$. In this paper, by choosing a suitable control metric, we provide another proof to show that that for the model domains Ω_k in $\mathbb{C}^2$ with $k \geq 1$, the Cauchy–Szegö Kernels on the boundary are Calderón–Zygmund kernels. We refer to Theorem 7.1 for detail. Our approach can be applied to model domains Ω_k in $\mathbb{C}^{n+1}$ for general $n > 1$ and $k \geq 1$, with full detailed calculations in a forthcoming paper.

Due to the notational complexity, we will not state the full details of our main theorems here and refer the details to each section. This paper is organized as follows:

- In Sections 2 and 3, we review the Cauchy–Riemann geometry and subRiemannian geometry in $\mathbb{C}^n$, and the Kohn–Laplacian on CR-manifolds in $\mathbb{C}^{n+1}$, respectively;
- In Section 4, we study the fundamental solution for Kohn–Laplacian on Siegel upper half space in $\mathbb{C}^{n+1}$;
- In Section 5, we derive the fundamental solution for Kohn–Laplacian on model domains (with higher steps) in $\mathbb{C}^{n+1}$ and prove our first main result Theorem 5.3;
- In Section 6, we apply the result in Section 5, and obtain the explicit Cauchy–Szegö kernels on the boundary of model domains, which is the second main result Theorem 6.1;
- In the last section, we prove that the Cauchy–Szegö kernels on the boundary are Calderón–Zygmund kernels which is the third main result Theorem 7.1.

2. Cauchy–Riemann geometry and subRiemannian geometry

Consider

$$\Delta_{\mathbf{X}} = \frac{1}{2}\sum_{j=1}^{m} X_j^2 + \cdots ,$$

where $\mathbf{X} = \{X_1, \ldots, X_m\}$ are m linearly independent vector fields on $\mathcal{M}_n$, an n-dimensional real manifold with $m \le n$. The subspace T_X spanned by $X_1, \ldots, X_m$ is called the horizontal subspace, and its complement is referred to as the missing directions.

$T_{\mathbf{X}} = T\mathcal{M}$ if and only if $\Delta_{\mathbf{X}}$ is elliptic. The operator $\Delta_{\mathbf{X}}$ is the usual Laplace–Beltrami operator. The Newtonian potential is

$$N(\mathbf{x}, \mathbf{x}_0) = \frac{1}{(2-n)|\Sigma_n(\mathbf{x}_0)| d^{n-2}(\mathbf{x}, \mathbf{x}_0)}, \quad n > 2,$$

where $|\Sigma_n(\mathbf{x}_0)|$ is the surface area of the induced unit ball with center $\mathbf{x}_0$, and $d(\mathbf{x}, \mathbf{x}_0)$ is the Riemannian distance between $\mathbf{x}$ and $\mathbf{x}_0$. Then

$$\Delta_X N(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) + \mathcal{O}\big(d^{-n+1}(\mathbf{x}, \mathbf{x}_0)\big).$$

When $T_{\mathbf{X}} \ne T\mathcal{M}$, the operator is non-elliptic. Assume $\mathbf{X}$ satisfies *bracket generating condition*: "the horizontal vector fields $\mathbf{X}$ and their brackets span $T\mathcal{M}$", then

(1). We know that from *Chow's Theorem* [12]: Given any two points $A, B \in \mathcal{M}$, there is a piecewise C^1 *horizontal curve* $\gamma : [0,1] \to \mathcal{M}$:

$$\gamma(0) = A, \quad \gamma(1) = B,$$

and

$$\dot\gamma(s) = \sum_{k=1}^{m} a_k(s) X_k.$$

This yields a distance and therefore a geometry which we shall call *subRiemannian.*

(2). By results of Fefferman–Phong [15] and Fefferman–Sanchez [16], we know that $\Delta_{\mathbf{X}}$ is *subelliptic*:

$$\big\|\mathcal{P}(X_j, X_k)u\big\|_{L^2_k} \le C\|f\|_{L^2_k}, \quad k \in \mathbb{Z}_+$$

where $\mathcal{P}(X_j, X_k)$ is any quadratic polynomial in X_j, X_k, $1 \le j, k \le m$. Hence, it is hypoelliptic, *i.e.*,

$$\Delta_X u = f, \ \ f \in C^\infty(\mathcal{M}_n) \ \Rightarrow \ u \in C^\infty(\mathcal{M}_n).$$

This recovered a theorem of Hörmander [22].

Set

$$X_j = \sum_{k=1}^{n} a_{jk}(x) \frac{\partial}{\partial x_k}, \quad j = 1, \dots, m.$$

Then

$$H = \frac{1}{2} \sum_{j=1}^{m} \Big(\sum_{k=1}^{n} a_{jk}(x) \xi_k \Big)^2$$

is the Hamiltonian function on the cotangent bundle $T^*\mathcal{M}$.

A *bicharacteristic curve* $(\mathbf{x}(s), \xi(s)) \in T^*\mathcal{M}$ is a solution of the Hamilton's system:

$$\dot{x}_j(s) = H_{\xi_j}, \qquad \dot{\xi}_j(s) = -H_{x_j},$$

with boundary conditions,

$$x_j(0) = x_j^{(0)}, \quad x_j(\tau) = x_j, \quad j = 1, \dots, n,$$

for given points $\mathbf{x}^{(0)}$, $\mathbf{x} \in \mathcal{M}$.

The projection $\mathbf{x}(s)$ of the bicharacteristic curve on $\mathcal{M}_n$ is a *geodesic*.

Remarks 2.1. *This new geometry has essential differences with the Riemannian geometry.*

(1) *Every point O of a Riemannian manifold is connected to every other point in a sufficiently small neighborhood by a unique geodesic. On a sub-Riemannian manifold there will be points arbitrarily near O which are connected to O by an infinite number of geodesics. This strange phenomenon was first pointed out by Gaveau (1977) and Strichartz (1986), and it brings up the question of what "local" means in subRiemannian geometry. Control theorists studying subRiemannian examples noticed that the Riemannian concepts of cut locus and conjugate locus behave badly in a subRiemannian context.*

(2) *In Riemannian geometry the unit ball is smooth. In subRiemannian geometry, among many distances, there is a shortest one, often referred to as the Carnot–Carathéodory distance. In subRiemannian geometry the Carnot–Carathéodory unit ball is singular.*

(3) *The exponential map is smooth in Riemannian geometry, but often singular in subRiemannian geometry. The singularities occur at points connected to an "origin" by an infinite number of geodesics. These singular points constitute a submanifold whose tangents yield the "missing directions", that is the directions in $T\mathcal{M}_n$ not covered by the horizontal directions.*

Suppose that we are in 3 dimensions, $\mathbf{x} = (x_1, x_2, t) = (x', t)$, with 2 vector fields,

$$X_1 = \frac{\partial}{\partial x_1} + 2kx_2|x'|^{2k-2}\frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x_2} - 2kx_1|x'|^{2k-2}\frac{\partial}{\partial t},$$

with $|x'|^2 = x_1^2 + x_2^2$. The differential operator one wants to invert is

$$\Delta_\lambda = \frac{1}{2}\big(X_1^2 + X_2^2\big) - \frac{1}{2}i\lambda\big[X_1, X_2\big].$$

The number given by the minimum number of brackets necessary to generate $T\mathcal{M}$ plus 1 is referred to as the "*step*" of the operator $\Delta_{\mathbf{X}}$. In particular, elliptic operator is *step* 1, one bracket generators are *step* 2, and everything else is referred to as higher step. Since

$$[X_1, X_2] = -2k(k+1)|x'|^{2(k-1)}\frac{\partial}{\partial t},$$

Δ_λ is step 2 at points $|x'| \neq 0$, and step $2k$ otherwise.

We first try to find the fundamental solution $K_\lambda(\mathbf{x}, \mathbf{x}_0)$ of Δ_λ which is the distribution solution of

$$\Delta_{\lambda,\mathbf{x}} K_\lambda(\mathbf{x}, \mathbf{x}_0) \;=\; \delta\big(\mathbf{x} - \mathbf{x}_0\big).$$

Before we go further discussion of the operator Δ_λ, let us review the geometry of a general real hypersurface $\mathcal{M}$ in $\mathbb{C}^n$. The beautiful interplay between real and complex geometry dominates the discussion.

Let us first work with $\mathbb{C}^n$ itself. A vector field L on $\mathbb{C}^n$ can be expressed as a first order different operator

$$L = \sum_{j=1}^{2n} a_j(\mathbf{x})\frac{\partial}{\partial x_j},$$

where $a_j \in C^\infty$. In order to allow us to express the differentiations in complex notation, we start by considering the complexified tangent bundle $\mathbb{C}T(\mathbb{C}^n) = \mathbb{C} \otimes T(\mathbb{C}^n)$. A section of this bundle is a complex vector field:

$$L = \sum_{j=1}^{n} c_j(\mathbf{x}) \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} d_j(\mathbf{x}) \frac{\partial}{\partial \bar{z}_j}.$$

We obtain two naturally defined integrable subbundles of $\mathbb{C}T(\mathbb{C}^n)$:

$$T^{(1,0)}(\mathbb{C}^n) = \Big\{ Z = \sum_{j=1}^{n} c_j(\mathbf{x}) \frac{\partial}{\partial z_j} \Big\}$$

where $c_j \in C^\infty(\mathbb{C}^n)$. $T^{(1,0)}(\mathbb{C}^n)$ is integrable in the sense of Frobenius: if $Z, W \in T^{(1,0)}(\mathbb{C}^n) \Rightarrow [Z, W] \in T^{(1,0)}(\mathbb{C}^n)$. Denoted

$$T^{(0,1)}(\mathbb{C}^n) = \overline{T^{(1,0)}(\mathbb{C}^n)}.$$

Hence,

$$T^{(1,0)}(\mathbb{C}^n) \cap T^{(0,1)}(\mathbb{C}^n) = \{0\}.$$

This splitting of the tangent bundle plays a crucial rule in all aspects of complex geometry.

Let $\mathcal{M}$ be a smooth real hypersurface of $\mathbb{C}^n$, or more generally, of a complex manifold. Again we start by tensoring with $\mathbb{C}$, writing $\mathbb{C}T\mathcal{M}$ for $\mathbb{C} \otimes T(\mathcal{M})$. We define

$$T^{(1,0)}(\mathcal{M}) = T^{(1,0)}(\mathbb{C}^n) \cap \mathbb{C}T\mathcal{M}.$$

As before, $T^{(0,1)}(\mathcal{M}) = \overline{T^{(1,0)}(\mathcal{M})}$. Again we have

$$T^{(1,0)}(\mathcal{M}) \cap T^{(0,1)}(\mathcal{M}) = \{0\}.$$

For an abstract real manifold $\mathcal{M}$ we say that the subbundle $T^{(1,0)}(\mathcal{M})$ defines a *CR structure on* $\mathcal{M}$ if it is integrable, and its intersection with its conjugate bundle is trivial. We call such a manifold a CR manifold. Its *horizontal subbundle* is the union

$$\mathcal{H}(\mathcal{M}) = T^{(1,0)}(\mathcal{M}) \cup T^{(0,1)}(\mathcal{M}).$$

We say that $\mathcal{M}$ is of *hypersurface type* if the fibres of $\mathcal{H}(\mathcal{M})$ have codimension 1 in $\mathbb{C}T\mathcal{M}$.

Given a CR manifold of hypersurface type, there is a non-vanishing differential 1-form η such that

$$\ker(\eta) = \mathcal{H}(\mathcal{M}).$$

We may assume that η is purely imaginary.

Definition 2.2. *Let $\mathcal{M}$ be a CR manifold of hypersurface type. The Levi form λ is the Hermitian form defined by*

$$\lambda(Z, \bar{W}) = \big\langle \eta, [Z, \bar{W}] \big\rangle, \quad Z, W \in T^{(1,0)}(\mathcal{M}).$$

Definition 2.3. *A CR manifold of hypersurface type is pseudoconvex if all nonzero eigenvalues of λ have the same sign. It is called strongly pseudoconvex if λ is definite, that is, all eigenvalues have the same non-zero sign.*

Now we want to express the Levi form on a hypersurface in terms of partial derivatives. In a neighborhood of a given point, we suppose that

$$\mathcal{M} = \big\{ z \in \mathbb{C}^n : \ \rho(z) = 0, \ d\rho \neq 0 \big\}.$$

A complex vector field Z is tangent to $\mathcal{M}$ then $Z(\rho) = 0$ on $\mathcal{M}$. We may use

$$\eta = \big(\partial - \bar{\partial}\big)(\rho).$$

Let $Z, W \in T^{(1,0)}(\mathcal{M})$. Then

$$\langle \partial\rho, Z \rangle = \langle \partial\rho, W \rangle = 0.$$

By the Cartan formula for exterior derivatives,

$$\begin{aligned} \lambda(Z, \bar{W}) =& \big\langle \eta, [Z, \bar{W}] \big\rangle \\ =& \big\langle -d\eta, Z \wedge \bar{W} \big\rangle = \big\langle \partial\bar{\partial}\rho, Z \wedge \bar{W} \big\rangle. \end{aligned}$$

Hence we have interpreted the Levi form as the restriction of the complex Hessian of ρ to sections of $T^{(1,0)}(\mathcal{M})$.

Here we give a few examples.

Example 1. The zero set of

$$\rho(z) = \sum_{j=1}^{n+1} |z_j|^2 - 1$$

is the sphere $\mathbb{S}^{2n+1}$. The horizontal subbundle $\mathcal{H}_z$ on $\mathbb{S}^{2n+1}$ decomposes into the holomorphic subspace

$$T_z^{(1,0)}(\mathbb{S}^{2n+1}) = \text{span}\{Z_1, \dots Z_n\}$$

and its conjugate $T_z^{(0,1)}(\mathbb{S}^{2n+1})$, where

$$Z_j = \bar{z}_j \frac{\partial}{\partial z_{n+1}} - \bar{z}_{n+1} \frac{\partial}{\partial z_j}, \quad j = 1, \dots, n.$$

The annihilating contact 1-form is

$$\eta = \frac{1}{2} \sum_{j=1}^{n+1} \big(\bar{z}_j dz_j - z_j d\bar{z}_j\big).$$

The Levi form, after dividing out a nonzero factor, satisfies $\lambda(Z_j, \bar{Z}_k) = \delta_{jk} + \bar{z}_j z_k$. Hence $\mathbb{S}^{2n+1}$ is strongly pseudoconvex.

Example 2. The zero set of

$$\rho(z) = \text{Im}(z_{n+1}) - \sum_{j=1}^{n} |z_j|^2$$

is the boundary of the Siegel upper half space, which we identify with the Heisenberg group $\mathbf{H}_n$. To simplify notations, we use $\mathbf{H}_n$ to represent $\partial\widetilde{\Omega}_n$ from now on. Since $\mathbf{B}_{n+1} \approx \widetilde{\Omega}_n$ biholomorphically equivalent, the CR structures on the boundary and associated Levi forms must be equivalent. The horizontal subbundle of $\mathbf{H}_n$ decomposes into holomorphic and conjugate holomorphic subspaces $T_z^{(1,0)}(\mathbf{H}_n) = \text{span}\{Z_1, \dots, Z_n\}$ and $T_z^{(1,0)}(\mathbf{H}_n)$ with

$$Z_j = \frac{\partial}{\partial z_j} - 2i\bar{z}_j \frac{\partial}{\partial z_{n+1}}, \quad j = 1, \dots, n$$

and later is spanned by the conjugate vector fields $\bar{Z}_1, \dots, \bar{Z}_n$. An annihilating contact 1-form is

$$\eta = \frac{i}{2}\big(dz_{n+1} + d\bar{z}_{n+1}\big) + \sum_{j=2}^{n+1} \big(\bar{z}_j dz_j - z_j d\bar{z}_j\big).$$

Finally, the Levi form is $\lambda(Z_j, \bar{Z}_k) = \delta_{jk}$.

Example 3. The zero set of

$$\rho(z) = \mathrm{Im}(z_{n+1})$$

is a halfspace Σ. It's horizontal space $\mathcal{H}_z(\Sigma)$ decomposes into holomorphic and conjugate holomorphic subspaces, spanned respectively by the vector fields

$$Z_j = \frac{\partial}{\partial z_j}, \quad j = 1, \dots, n$$

and their conjugates $\bar{Z}_1, \dots, \bar{Z}_n$. In this case, the complex Hessian of the defining function is identically equal to zero, and hence the Levi form is $\lambda(Z_j, \bar{Z}_k) \equiv 0$.

3. Kohn–Laplacian

Let $\mathcal{M}$ be a CR-manifold in $\mathbb{C}^{n+1}$. Assume that $\{Z_1, \dots, Z_n\}$ is an orthonormal basis of $T^{(0,1)}(\mathcal{M})$. Denote $\mathcal{B}^{(0,q)}(\mathcal{M})$ the set of all $(0,q)$-forms on $\mathcal{M}$. An element $\phi \in \mathcal{B}^{(0,q)}(\mathcal{M})$ can be written as

$$\phi = \sum_{J \in \vartheta_q} \phi_J \bar{\omega}^J = \sum_{J \in \vartheta_q} \phi_J \bar{\omega}_{j_1} \wedge \bar{\omega}_{j_2} \wedge \dots \wedge \bar{\omega}_{j_q}$$

where ϑ_q is the set of all increasing q-tuples $J = (j_1, \dots, j_q)$ with $j_1 < j_2 < \dots < j_q$ and $\phi_J \in \mathbb{C}^\infty(\mathcal{M})$. The tangential Cauchy–Riemann operator

$$\bar{\partial}_b : \mathcal{B}^{(0,q)}(\mathcal{M}) \to \mathcal{B}^{(0,q+1)}(\mathcal{M}), \ \ q = 0, \dots, n$$

can be written as

$$\bar{\partial}_b \phi = \sum_{k=1}^{n} \sum_{J \in \vartheta_q} \bar{Z}_k(\phi_J) \bar{\omega}_k \wedge \bar{\omega}^J \ \in \ \mathcal{B}^{(0,q+1)}(\mathcal{M}).$$

The adjoint operator

$$\bar{\partial}_b^* : \mathcal{B}^{(0,q)}(\mathcal{M}) \to \mathcal{B}^{(0,q-1)}(\mathcal{M}), \ \ q = 1, \dots, n-1$$

can be written as

$$\bar{\partial}_b^* \phi = -\sum_{k=1}^{n} \sum_{J \in \vartheta_q} Z_k(\phi_J) \bar{\omega}_k \lrcorner \bar{\omega}^J \ \in \ \mathcal{B}^{(0,q-1)}(\mathcal{M})$$

where

$$\bar{\omega}_k \lrcorner \bar{\omega}^J = 0, \qquad \text{if } k \notin \vartheta_q$$

and

$$\bar{\omega}_k \lrcorner \bar{\omega}^J = (-1)^m \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_{m-1}} \wedge \bar{\omega}_{j_{m+1}} \wedge \cdots \wedge \bar{\omega}_{j_q}$$

if $k = j_m$. The Kohn Laplacian on $(0, q)$-forms is

$$\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$$

which is a self-adjoint operator defined $\mathcal{B}^{(0,q)}(\mathcal{M})$. Straightforward computation shows that the action of $\Box_b$ on $\mathcal{B}^{(0,q)}(\mathcal{M})$ is given by

$$\Box_b \Big(\sum_{J \in \vartheta_q} \phi_J \bar{\omega}^J \Big) = - \sum_{J \in \vartheta_q} \big(\mathcal{L}_{n-1-2q} \phi_J \big) \bar{\omega}^J$$

where, for $\lambda \in \mathbb{C}$,

$$\mathcal{L}_\lambda = \frac{1}{2} \sum_{k=1}^{n} \big(Z_k \bar{Z}_k + \bar{Z}_k Z_k \big) - i\lambda [Z_k, \bar{Z}_k].$$

Now let $\Omega_k = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im}(z_2) > |z_1|^{2k}\}$ be a domain in $\mathbb{C}^2$. The boundary $\partial\Omega_k$ is a CR manifold of hypersurface type. As before, we may change the coordinates

$$z_1 = x_1 + ix_2 \quad \text{and} \quad t = \operatorname{Re}(z_2),$$

then the operator Δ_λ is exactly the Kohn Laplacian with $1 - 2q$, $q = 0, 1, 2$.

In fact, the Kohn Laplacian Δ_λ has some connection with the classical mechanics. Consider a unit mass particle under the influence of force $F(x) = x$. Newton's law $\ddot{x} = x$ gives us an equation which describes the dynamics of an inverse pendulum in an unstable equilibrium, for small angle x. The potential energy

$$U(x) = -\int_0^x F(u)\, du = -\frac{x^2}{2}.$$

The Lagrangian $L : T\mathbb{R} \to \mathbb{R}$ is the difference between the kinetic and the potential energy

$$L(x, \dot{x}) = K - U = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2.$$

The momentum $p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$ and the Hamiltonian associated with the above Lagrangian is obtained using the Legendre transform: $H : T^*\mathbb{R} \to \mathbb{R}$

$$H(x,p) = p\dot{x} - L(x,\dot{x}) = \frac{1}{2}p^2 - \frac{1}{2}x^2.$$

Consider the following complexification

$$x = x_1 + ip_2, \quad p = p_1 + ix_2.$$

Hence $H : T^*\mathbb{C} \to \mathbb{C}$ and

$$\begin{aligned} H(x,p) &= \frac{1}{2}p^2 - \frac{1}{2}x^2 \\ &= \frac{1}{2}(p_1 + ix_2)^2 - \frac{1}{2}(x_1 + ip_2)^2 \\ &= \frac{1}{2}(p_1 + ix_2)^2 + \frac{1}{2}(p_2 - ix_1)^2. \end{aligned}$$

Replacing $\theta = i$,

$$H(x,p;\theta) = \frac{1}{2}(p_1 + \theta x_2)^2 + \frac{1}{2}(p_2 - \theta x_1)^2.$$

Quantizing, $p_1 \to \partial_{x_1}$, $p_2 \to \partial_{x_2}$, $\theta \to \partial_t$ and hence $H \to \Delta_0$, the Kohn Laplacian in the case $k = 1$ and $\lambda = 0$:

$$\Delta_0 = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial t}\Big)^2 + \frac{1}{2}\Big(\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial_t}\Big)^2.$$

In general, we shall look for K_λ in the form

$$K_\lambda(\mathbf{x},\mathbf{x}_0) \;=\; \int_{\mathbb{R}} \frac{E(\mathbf{x},\mathbf{x}_0,\tau)V_\lambda(\mathbf{x},\mathbf{x}_0,\tau)}{g(\mathbf{x},\mathbf{x}_0,\tau)}\,d\tau,$$

where the function g is a solution of the Hamilton–Jacobi equation:

$$\frac{\partial g}{\partial \tau} + \frac{1}{2}\big(X_1 g\big)^2 + \frac{1}{2}\big(X_2 g\big)^2 \;=\; 0,$$

given by a modified action integral of a complex Hamiltonian problem. The associated energy

$$E \;=\; -\frac{\partial g}{\partial \tau}$$

is the first invariant of motion, and the volume element V_λ is the solution of a transport equation, which is order 1 in the step 2 case, $k=1$, and order $2k$ in the higher step case, $k \geq 2$.

4. The step 2 case; $k = 1$

Here

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial t},$$

which are left-invariant with respect to the following Heisenberg group translation:

$$\mathbf{x} \circ \mathbf{y} = \big(x' + y', t + s + 2[x_2y_1 - x_1y_2]\big).$$

where $\mathbf{x} = (x', t)$ and $\mathbf{y} = (y', s)$. Moreover, one has $[X_1, X_2] = -4\frac{\partial}{\partial t} = -4T$.

In high dimensional case, the Siegel upper half space $\widetilde{\Omega}_n$ is defined as:

$$\widetilde{\Omega}_n = \Big\{(z', z_{n+1}) \in \mathbb{C}^{n+1} : \ \mathrm{Im}(z_{n+1}) > \sum_{j=1}^{n} |z_j|^2\Big\} \tag{1}$$

where $z' = (z_1, \dots, z_n) \in \mathbb{C}^n$.

The following elementary but useful identity is the key to discovering the biholomorphic maps:

$$\mathrm{Im}(\zeta) = \Big|\frac{i+\zeta}{2}\Big|^2 - \Big|\frac{i-\zeta}{2}\Big|^2. \tag{2}$$

With $\zeta = z_{n+1}$ we plug (2) into (1) and rewrite to obtain

$$\Big|\frac{i+z_{n+1}}{2}\Big|^2 > \sum_{j=1}^{n}|z_j|^2 + \Big|\frac{i-z_{n+1}}{2}\Big|^2. \tag{3}$$

After dividing by $\Big|\frac{i+z_{n+1}}{2}\Big|^2$ and changing notation, inequality (3) becomes

$$1 > \sum_{j=1}^{n+1} |w_j|^2,$$

the defining property of the unit ball. The explicit mapping is given by

$$w_j = \frac{2z_j}{i + z_{n+1}}, \quad 1 \leq j \leq n$$

and

$$w_{n+1} = \frac{i - z_{n+1}}{i + z_{n+1}}.$$

It is easy to check that this transformation $z \mapsto w$ is biholomorphic from $\widetilde{\Omega}_n$ to $\mathbf{B}_{n+1}$.

We now describe the analogous situation on $\partial\widetilde{\Omega}_n$ of the Siegel upper half space:

$$\partial\widetilde{\Omega}_n = \Big\{(z', z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im}(z_{n+1}) = \sum_{j=1}^{n} |z_j|^2\Big\}.$$

The variable z_{n+1} plays a different role, and hence for $\mathbf{z} \in \mathbb{C}^{n+1}$ we write $\mathbf{z} = (z', z_{n+1})$. Two natural families of biholomorphic self-maps of $\widetilde{\Omega}_n$ are the *dilations* $\delta_\rho : \widetilde{\Omega}_n \to \widetilde{\Omega}_n$, for $\rho > 0$, given by

$$\delta_\rho(z', z_{n+1}) = \big(\rho z', \rho^2 z_{n+1}\big),$$

and the *rotations* $R_A : \widetilde{\Omega}_n \to \widetilde{\Omega}_n$, for $A \in U(n)$, given by

$$R_A(z', z_{n+1}) = \big(A(z'), z_{n+1}\big).$$

To introduce an analogue of translation we consider

$$\tau_{\mathbf{x}} : \widetilde{\Omega}_n \to \widetilde{\Omega}_n,$$

for $\mathbf{x} = (z', t) \in \mathbb{C}^n \times \mathbb{R}$, given by

$$\tau_{\mathbf{x}}(w', w_{n+1}) = \Big(w' + z', w_{n+1} + t + 2i\langle z', w'\rangle + i|z'|^2, \Big).$$

All of the preceding maps extend to self-maps of the boundary $\partial\widetilde{\Omega}_n$. The action of the family $\{\tau_{\mathbf{x}} : \mathbf{x} = (z, t) \in \mathbb{C}^n \times \mathbb{R}\}$ on $\widetilde{\Omega}_n \times \partial\widetilde{\Omega}_n$ is faithful and the action on $\partial\widetilde{\Omega}_n$ is simply transitive. We obtain a useful identification of $\partial\widetilde{\Omega}_n$ with $\mathbb{C}^n \times \mathbb{R}$. By this method we equip $\mathbb{C}^n \times \mathbb{R}$ with a group law:

$$(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \circ \mathbf{y},$$

characterized by the identity $\tau_{\mathbf{x}} \cdot \tau_{\mathbf{y}} = \tau_{\mathbf{x}\circ\mathbf{y}}$. Here

$$\mathbf{x} \circ \mathbf{y} = \Big(x' + y', t + s + 2\sum_{j=1}^{n} \big[x_{n+j}y_j - x_j y_{n+j}\big]\Big).$$

The resulting space is the *Heisenberg group* $\mathbf{H}_n$ as we mentioned before.

Finally, we observe that the group of biholomorphic automorphisms of $\widetilde{\Omega}_n$ which fix the point at ∞ is generated by dilations, rotations, and translations.

4.1. Lagrangian formalism

In order to simplify notations, let us return to the case $\mathbb{C} \times \mathbb{R} \approx \mathbb{R}^3$. We shall associate a Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ with the Hamiltonian:

$$H(\mathbf{x}, \xi) \;=\; \frac{1}{2}(\xi_1 + 2x_2\theta)^2 + \frac{1}{2}(\xi_2 - 2x_1\theta)^2.$$

This can be done by using the *Legendre transform* in $(\dot{x}_1, \dot{x}_2, \dot{t})$. It is known that

$$H(\mathbf{x}, \dot{\mathbf{x}}) \;=\; \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \theta(\dot{t} - 2x_2\dot{x}_1 + 2x_1\dot{x}_2).$$

Using polar coordinates, the Lagrangian:

$$L \;=\; \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \theta(\dot{t} + 2r^2\dot{\phi}).$$

A computation shows

$$\begin{aligned} \frac{d}{ds}\frac{\partial L}{\partial \dot{r}} &= \ddot{r}, & \frac{\partial L}{\partial r} &= r\dot{\phi}\big(\dot{\phi} + 4\theta\big), \\ \frac{\partial L}{\partial \dot{\phi}} &= r^2\dot{\phi} + 2\theta r^2, & \frac{\partial L}{\partial \phi} &= 0, \\ \frac{\partial L}{\partial \dot{t}} &= \theta, & \frac{\partial L}{\partial t} &= 0, \end{aligned}$$

and hence $r(s)$, $\phi(s)$ and θ satisfy the *Euler–Lagrange system*

$$\begin{cases} \ddot{r} = r\dot{\phi}\big(\dot{\phi} + 4\theta\big) \\ r^2\big(\dot{\phi} + 2\theta\big) = C(\text{constant}) \\ \theta = \theta_0 = \text{constant} \end{cases} \tag{4}$$

If the geodesic starts at the origin, $r(0) = 0 \Rightarrow C = 0$. Then 2nd equation of (4) yields $\dot{\phi} = -2\theta$. The Euler–Lagrange system becomes

$$\begin{cases} \ddot{r} = -4\theta^2 r \\ \dot{\phi} = -2\theta \\ \theta = \theta_0(\text{constant}). \end{cases} \tag{5}$$

When $\theta_0 = 0$, the system (5) becomes

$$\begin{cases} \ddot{r} = 0 \\ \dot{\phi} = 0 \\ \theta_0 = 0. \end{cases}$$

Proposition 4.1. *Given a point $P(0, x')$, there is a* **unique** *geodesic between the origin and P. It is a straight line in the plane $\{t = 0\}$ of length $|x'| = \sqrt{x_1^2 + x_2^2}$, and it is obtained for $\theta = 0$.*

Now let us move the end point P away from the x' with $|x'| \neq 0$. Consider the boundary conditions:

$$\begin{aligned} x'(0) = 0, \quad t(0) = 0, \quad \phi(0) = \phi_0 \\ |x'(\tau)| = R, \quad t(\tau) = t, \quad \phi(\tau) = \phi_1. \end{aligned}$$

We may choose $\phi_0 = 0$. One has

$$\begin{cases} \dot{t} = -2r^2\dot{\phi} \\ \dot{\phi} = -2\theta \end{cases} \qquad \Rightarrow \qquad \dot{t} = 4\theta r^2 > 0.$$

This implies that $t(s)$ is increasing and if $t(0) = 0$, then $t(\tau) > 0$.

Lemma 4.2. *The following relations take place among the boundary conditions:*

$$\begin{aligned} \phi_1 &= -2\theta\tau, \quad \sin^2\phi_1 = 4\theta^2 R^2, \\ t &= \frac{1}{4\theta^2}\frac{\sin(2\phi_1) - 2\phi_1}{2}, \\ \frac{|t|}{R^2} &= -\mu(\phi_1) = \mu(2\theta\tau), \end{aligned}$$

where

$$\mu(z) = \frac{z}{\sin^2 z} - \cot z.$$

Lemma 4.3. *μ is a monotone increasing diffeomorphism of the interval $(-\pi, \pi)$ onto $\mathbb{R}$. On each interval $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$, μ has a unique critical point x_m. On this interval μ decreases strictly from $+\infty$ to $\mu(x_m)$ and then increases strictly from $\mu(x_m)$ to $+\infty$. Moreover*

$$\mu(x_m) + \pi < \mu(x_{m+1}), \quad m \in \mathbb{N}.$$

Theorem 4.1. *On* $\mathbf{H}_1$*, there are a* **finite** *number of geodesics connecting* $\mathbf{x}$ *to* $\mathbf{0}$ $\Leftrightarrow$ $x' \neq 0$*. They are parametrized by solutions* τ_m*,* $m = 1, \dots, N$*, of the transcendental equation*

$$|t| \;=\; \mu(\tau_m)|x'|^2 = \mu(\tau_m)(x_1^2 + x_2^2), \tag{6}$$

where

$$\mu(z) = \frac{z}{\sin^2 z} - \cot z.$$

The length

$$d_m(\mathbf{x}) = \sqrt{\nu(\tau_m)\big(|x'|^2 + |t|\big)}$$

where

$$\nu(z) = \frac{z^2}{z + \sin^2 z - \sin\, z \cos\, z}.$$

Here $\tau_1, \dots, \tau_N$ *are the solutions of equation* (6)*. The shortest* $d_1(\mathbf{x})$ *is called the Carnot–Carathéodory distance.*

It is easy to see that the number of geodesics increasing without bound as $\frac{|t|}{|x'|^2} \to \infty$. We have the following theorem.

Theorem 4.2. *Every point of the line* $(0, t)$ *is connected to the origin by an* **infinite** *number of geodesics with lengths*

$$d_m^2 = m\pi|t|, \qquad m \in \mathbb{N}.$$

For each length d_m*, the equations of geodesics are*

$$r^{(m)}(\phi) = \sqrt{\frac{|t|}{m\pi}}\sin(\phi - \phi_0)$$
$$t^{(m)}(\phi) = \frac{|t|}{2m\pi}\Big(\sin(2(\phi - \phi_0)) - 2(\phi - \phi_0)\Big).$$

The geodesic of that length are parametrized by the circle $\mathbb{S}^1$*.*

The projection of the m-th geodesic on the (x_1, x_2)-plane is a circle with radius

$$R_m = \frac{1}{2}\sqrt{\frac{|t|}{m\pi}}$$

and area

$$\sigma_m = \frac{|t|}{4m}.$$

4.2. Fundamental solution for the operator Δ_λ for $k = 1$

Consider the complex action function

$$g(x', t, \tau) = -\, it + \int_0^\tau \Big\{ \sum_{j=1}^{2} \dot{x}_j \xi_j - H \Big\} ds$$
$$= -\, it + \coth(2\tau)|x'|^2 \;=\; -it + \coth(2\tau)\big(x_1^2 + x_2^2\big).$$

Like the classical action, the complex action g satisfies the Hamilton–Jacobi equation:

$$\frac{\partial g}{\partial \tau} + H\big(\mathbf{x}, \nabla_{\mathbf{x}} g\big) = 0.$$

Set

$$f(\mathbf{x}, \tau) = \tau g(\mathbf{x}, \tau) = \tau\big(-\, it + \coth(2\tau)|x'|^2\big).$$

This is so called a "*modified complex distance*".

Theorem 4.3. *Let $\tau_1(\mathbf{x})$, $\tau_2(\mathbf{x})$,... denote the critical points of $f(\mathbf{x}, \tau)$, i.e.,*

$$\frac{\partial f}{\partial \tau}(\mathbf{x}, \tau_j(\mathbf{x})) = 0.$$

Then

$$f(\mathbf{x}, \tau_j(\mathbf{x})) = \frac{1}{2} d_j^2(\mathbf{x}).$$

It is expected that the fundamental solution has the following form:

$$K_\lambda(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}} \frac{E(\mathbf{x}, \mathbf{y}, \tau) V_\lambda(\mathbf{x}, \mathbf{y}, \tau)}{g(\mathbf{x}, \mathbf{y}, \tau)} d\tau \tag{7}$$

Since Δ_λ is left-invariant, we may set $\mathbf{x}_0 = \mathbf{0}$ in $K_\lambda(\mathbf{x}, \mathbf{x}_0)$:

$$K_\lambda(\mathbf{x}, \mathbf{0}) = \int_{\mathbb{R}} \frac{E(\mathbf{x}, \tau) V_\lambda(\mathbf{x}, \tau)}{g(\mathbf{x}, \tau)} d\tau,$$

where E and V_λ can be calculated explicitly:

$$E(\mathbf{x}, \tau) = -\frac{\partial g}{\partial \tau} = \frac{2(x_1^2 + x_2^2)}{\sinh^2(2\tau)},$$

and

$$V_\lambda(\mathbf{x}, \tau) = -\frac{1}{4\pi^2} e^{-2\lambda\tau} \frac{\sinh(2\tau)}{x_1^2 + x_2^2}.$$

Using contour integration, one obtains

$$K_\lambda(\mathbf{x}) = -\frac{\Gamma\big(\frac{1+\lambda}{2}\big)\Gamma\big(\frac{1-\lambda}{2}\big)}{4\pi^2} \times (|x'|^2 - it)^{-\frac{1+\lambda}{2}}(|x'|^2 + it)^{-\frac{1-\lambda}{2}}. \tag{8}$$

This is the famous Folland–Stein formula [17]. Moreover, if $\lambda \neq \pm(n+2\ell)$, $\ell \in \mathbb{Z}_+$,

$$\big\|\mathcal{P}(X_j, X_K)K_\lambda(f)\big\|_{L^p_k} \leq C\|f\|_{L^p_k}$$

and hence $\|K_\lambda(f)\|_{L^p_{k+1}} \leq C\|f\|_{L^p_k}$ for $k \in \mathbb{Z}_+$ and $1 < p < \infty$. Here $\mathcal{P}(X_j, X_K)$ is any quadratic polynomial in X_j, X_k. For detailed discussion, we refer readers to the books by Calin–Chang–Greiner [8]; Calin–Chang–Funrutani–Iwasaki [9] and Chang–Tie [11].

5. The higher step case; $k \geq 2$

As we know, there is no group structure on the boundary $\partial\Omega_k$ in this case. Moreover, solutions for Hamilton's system can not be written into elementary functions. Let us look at the simplest higher step case in $\mathbb{C}^2$, *i.e.,* $k = 2$. In this case, one has

$$X_1 = \frac{\partial}{\partial x_1} + 4x_2|x'|^2\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_2} - 4x_1|x'|^2\frac{\partial}{\partial t},$$

with $|x'|^2 = x_1^2 + x_2^2$. Since

$$\begin{aligned} &[X_1, X_2] = -16|x'|^2\frac{\partial}{\partial t}, \quad [X_1,[X_1,X_2]] = -32x_1\frac{\partial}{\partial t}, \\ &[X_1,[X_1,[X_1,X_2]]] = -32\frac{\partial}{\partial t}, \end{aligned} \tag{9}$$

hence $\partial\Omega_2$ is step 4 away from the (x_1, x_2)-plane. Denote $x_0 = t$. After length calculations, we have the following result.

Theorem 5.1. *Let $P(x_1, x_2, t)$ be a point of $\mathbb{R}^3$. Then*
(i). If $t = 0$, then there is a unique geodesic between the origin and P. It is a straight line in the (x_1, x_2)-plane of length $\sqrt{x_1^2 + x_2^2}$.
(ii). If $|x'| = 0$ and $t \neq 0$ there are infinitely many geodesics between the origin and P.

The subRiemannian geodesics that join the origin to a point $(0, 0, t)$ have lengths d_1, d_2, $d_3, \ldots$, where

$$(d_m)^4 = \frac{m^3K^4}{4Q}|t|$$

with

$$K = \int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}}$$

being the complete Jacobi integral and

$$k = \frac{\sqrt{2}}{4}\big(\sqrt{3}-1\big), \quad Q = \frac{1}{4}\frac{\Gamma(1/6)}{\Gamma(2/3)}\sqrt{\pi}.$$

For each length d_m, the geodesics of that length are parameterized by the circle $\mathbb{S}^1$.
(iii). If P is away from the t-axis and x'-plane with $0 < \frac{|t|}{(x_1^2+x_2^2)^2} < \infty$, then there are not less than $2m-1$ and not more than $2m+1$ subRiemannian geodesics between the origin and the point P, where the integer m is defined by

$$(m-\frac{1}{2})Q \;<\; \frac{3}{4}\frac{|t|}{(x_1^2+x_2^2)^2} \;\le\; (m+\frac{1}{2})Q.$$

We can also compute the lengths of geodesics connecting the origin and the point (x_1, x_2, t). Here is just state the result.

Theorem 5.2. *Let τ_j be the critical points of the modified complex action $f(\tau) = \tau g(\tau)$. Setting $\zeta_j = \digamma(i\tau_j)$, the lengths of the geodesics between the origin and the point (x_1, x_2, t), $|x'| \neq 0$ are given by*

$$\ell_j^4 = \nu(\zeta_j)\big(|t| + |x'|^4\big).$$

Here

$$\begin{aligned}\digamma(z) = {} & \frac{1+\sqrt{3}}{4^{1/3}}z - 3^{1/2}\,2^{-2/3}z - \frac{1}{2}\tan^{-1}\left(\frac{sd(2^{4/3}\,3^{1/4}z)}{2\cdot 3^{1/4}}\right) \\ & + \frac{1}{2}\Big\{uE(am^{-1}\omega, k') + \frac{i}{2}\log\frac{\theta_4(x-iy)}{\theta_4(x+iy)} \\ & + u\Big[\big(\frac{2\pi}{\Gamma(1/6)\Gamma(1/3)}\big)^2 - \frac{3-\sqrt{3}}{6}\Big]am^{-1}\omega\Big\}\end{aligned}$$

with

$$u = 2^{4/3}\,3^{1/4}\,z, \quad \text{and} \quad am^{-1}\omega = sn^{-1}(\sqrt{3}-1, k').$$

Here

$$am(v) = \int_0^v dn(\gamma)\,d\gamma$$

and θ_4 stands for Jacobi's zeta function.

5.1. Fundamental solution for the sub-Laplacian Δ_λ with $\lambda = 0$

Since the defining function for $\partial\Omega_k$ is $\{\mathrm{Im}(z_2) = |\phi(z_1)|^2 = |z_1|^{2k}\} \subset \mathbb{C}^2$ with $k = 2, 3, \ldots$, thus it is convenient to use polar coordinates $(r(\varsigma), \omega(\varsigma))$, $\varsigma \in [0, \tau]$, for the variable $z = x_1 + ix_2 \in \mathbb{C}$. As usual, set

$$r^2 = x_1^2 + x_2^2, \qquad \omega = \frac{1}{2i} \log \frac{x_1 + ix_2}{x_1 - ix_2}.$$

It follows that $(r^2)^{\cdot} = 2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$, and $r^2\dot{\omega} = x_1\dot{x}_2 - x_2\dot{x}_1$. Thus, $(r^2\dot{\omega})^{\cdot} = x_1\ddot{x}_2 - x_2\ddot{x}_1$. Since $\dot{x}_1 = \frac{\partial H}{\partial \xi_1}$ and $\dot{x}_2 = \frac{\partial H}{\partial \xi_2}$, we know that

$$\ddot{x}_1 = -4i\big(r^2\rho''(r^2) + \rho'(r^2)\big)\dot{x}_2, \qquad \ddot{x}_2 = 4i\big(r^2\rho''(r^2) + \rho'(r^2)\big)\dot{x}_1,$$

where $\rho(v) = v^k$ and $\rho'(v) = kv^{k-1}$. This yields

$$\begin{aligned} \big(r^2\dot{\omega}\big)^{\cdot} &= 2i\big(r^2\rho''(r^2) + \rho'(r^2)\big) \times \big(2x_1\dot{x}_1 + 2x_2\dot{x}_2\big) \\ &= 2i\big(r^2\rho''(r^2) + \rho'(r^2)\big) \times t\big(r^2\big)^{\cdot} = 2i\big(r^2\rho'(r^2)\big)^{\cdot}. \end{aligned}$$

Hence, there is a function $\Omega = \Omega(x, x_0, \tau)$, the *angular momentum*, constant on the bicharacteristic, such that

$$r^2(\varsigma)\dot{\omega}(\varsigma) = i\Big(2r^2(\varsigma)\rho'\big(r^2(\varsigma)\big) - \Omega\Big) = iW\big(r^2(\varsigma)\big)\Big|_{\varsigma = r^2}.$$

It follows that

$$W(v) = W(v, \Omega) = 2v \cdot \rho'(v) - \Omega = 2kv^k - \Omega. \tag{10}$$

Thus

$$\begin{aligned} \big(r\dot{r}\big)^2 + \big(r^2\dot{\omega}\big)^2 &= \big(x_1\dot{x}_1 + x_2\dot{x}_2\big)\big(x_1\dot{x}_2 - x_2\dot{x}_1\big)^2 \\ &= \big(x_1^2 + x_2^2\big)\big(\dot{x}_1^2 + x_2^2\big) = 2r^2E. \end{aligned}$$

Hence

$$\Big(\frac{1}{2}\big(r^2\big)^{\cdot}\Big)^2 = 2r^2E - \big(r^2(\varsigma)\dot{\phi}\big)^2 = 2r^2(\varsigma)E + W^2(r^2(\varsigma)).$$

Letting $v = r^2(\varsigma)$, we obtain

$$\frac{dv}{d\varsigma} = 2\sqrt{2Ev + W^2(v)} \qquad \Rightarrow \qquad 2\,d\varsigma = \frac{dv}{\sqrt{2Ev + W^2(v)}}. \tag{11}$$

Denote $r = r(\tau)$, $r_0 = r(0)$, $\omega = \omega(\tau)$ and $\omega_0 = \omega(0)$. The function E and Ω which are constant on each bicharacteristic, and determined implicitly by integrating (11)

$$\varsigma = \int_0^{\varsigma} dv = \frac{1}{2}\int_{r_0^2}^{r^2(\varsigma)} \frac{dv}{\sqrt{2Ev + W^2(v)}},$$

and

$$\omega(\varsigma) - \omega_0 = \int_0^{\varsigma} \dot{\omega}(v)dv = \frac{1}{2}\int_{r_0^2}^{r^2(\varsigma)} \frac{W(v)}{\sqrt{2Ev + W^2(v)}}\frac{dv}{v}.$$

On the other hand, from previous calculations, we know that

$$\begin{aligned} \dot{x}_1\xi_1 + \dot{x}_2\xi_2 - H &= \zeta_1\xi_1 + \zeta_2\xi_2 - \frac{1}{2}\big(\zeta_1^2 + \zeta_2^2\big) \\ &= \frac{1}{2}\big(\zeta_1^2 + \zeta_2^2\big) + \zeta_1\big(\xi_1 - \zeta_1\big) + \zeta_2\big(\xi_2 - \zeta_2\big) \\ &= E + 2i\rho'(r^2)\cdot r^2\dot{\phi} = E + 2\rho'(r^2)\cdot W(r^2). \end{aligned}$$

If follows that

$$\begin{aligned} &\int_0^{\tau} \Big[\sum_{j=1}^{2} \dot{x}_j(\varsigma)\xi_j(\varsigma) - H\big(x(\varsigma), t(\varsigma); \xi(\varsigma), \theta(\varsigma)\big)\Big]\, d\varsigma \\ &= E\tau + 2\int_0^{\tau} \rho'(r^2(\varsigma))W(r^2(\varsigma))d\varsigma = E\tau + \int_{|w|^2}^{|z|^2} \frac{\rho'(v)W(v)}{\sqrt{2Ev + W^2(v)}}dv. \end{aligned} \tag{12}$$

It is impossible to calculate W and Ω explicitly, but we know their analytic properties, and g and V_λ may be found in terms of E and Ω. When $\phi(z) = z^k$, one has $\rho(v) = v^k$ and $\rho'(v) = kv^{k-1}$. Moreover, from (10), we know that $W(v) = 2kv^k - \Omega$. Hence, $\rho'(v) = \frac{1}{2k}W'(v)$. In this case,

$$\begin{aligned} \int_{|w|^2}^{|z|^2} \frac{\rho'(v)W(v)}{\sqrt{2Ev + W^2(v)}}\, dv &= \frac{1}{2k}\int_{|w|^2}^{|z|^2} \frac{W'(v)W(v)}{\sqrt{2Ev + W^2(v)}}\, dv \\ &= \frac{1}{2k}\cdot \mathrm{sgn}(\tau)\cdot \sqrt{2Ev + W^2(v)}\Big|_{v=|w|^2}^{v=|z|^2} - \frac{E\tau}{k}. \end{aligned}$$

Combining the above result with (12), we have

$$g = -i(t - s) + \Big(1 - \frac{1}{k}\Big)E\tau$$

$$+\frac{1}{2k}\mathrm{sgn}(\tau)\Big\{\big(2E|z|^2+W(|z|^2)^2\big)^{\frac{1}{2}}-\big(2E|w|^2+W(|w|^2)^2\big)^{\frac{1}{2}}\Big\},$$

where one uses the principal branch of the square roots. We define

$$\mathcal{P}\;=\;\frac{2^{\frac{1}{k}}z\bar{w}}{\big[|z|^{2k}+|w|^{2k}-i(t-t_0)\big]^{\frac{1}{k}}}. \tag{13}$$

This expression in square brackets has non-negative real part and we take the principal branch of the root. Thus $\mathcal{P}^k=u_+$ band $\bar{\mathcal{P}}^k=u_-$. Note that $|\mathcal{P}|\le 1$ with equality only when $|z|=|w|$, $t=t_0$. We also define functions

$$\begin{aligned}F^{\pm}_{\lambda,\ell}(\mathcal{P}_+,\mathcal{P}_-)=\int_0^1\Big\{&\Big(\varsigma^{\frac{1}{k}}\mathcal{P}_\pm\Big)^{\ell}\varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}\\&\times\big(1-(\mathcal{P}_+\mathcal{P}_-)^k\varsigma\big)^{-\frac{1+\lambda}{2}}\big(1-\mathcal{P}^k_\pm\varsigma\big)^{-1}\Big\}d\varsigma,\end{aligned} \tag{14}$$

for $\ell=0,1,2,\dots,k$. These functions are holomorphic functions of their arguments so long as $\mathcal{P}^k_\pm$ and $(\mathcal{P}_+\mathcal{P}_-)^k$ do not belong to the interval $[1,\infty)$.

Remarks 5.1. *The expression (14) The functions $F^{\pm}_{\lambda,\ell}(\mathcal{P}_+,\mathcal{P}_-)$ of (14) can be identified as generalized hypergeometric functions of Appell [1] which are real analytic functions of z, $\bar{z}$, w, $\bar{w}$, t and t_0 is the region $|\mathcal{P}|<1$, i.e., everywhere except $t=t_0$ and $|z|=|w|$. Furthermore, they do not extend smoothly to the boundary but in pairs they do. More precisely, the functions $F^{+}_{\lambda,\ell}+F^{-}_{-\lambda,k-\ell}$, $\ell=0,1,\dots,k$, extend to an analytic function of z, $\bar{z}$, w, $\bar{w}$, t and t_0 except at the points where $\mathcal{P}^k_\pm=1$. In other words, $z^k=w^k$ and $t=t_0$. This is because*

$$\begin{aligned}&\Gamma\Big(\frac{1-\lambda}{2}\Big)\Gamma\Big(\frac{1+\lambda}{2}\Big)\big(F^{+}_{\lambda,\ell}+F^{-}_{-\lambda,k-\ell}\big)\\&=\int_0^1\int_0^1\Big\{\frac{\varsigma^{-\frac{1-\lambda}{2}}(1-\varsigma+)^{-\frac{1-\lambda}{2}}\sigma^{-\frac{1+\lambda}{2}}(1-\sigma)^{-\frac{1+\lambda}{2}}}{(1-\mathcal{P}^k_+\varsigma)(1-\mathcal{P}^k_-\sigma)}\\&\quad\times\Big[\frac{(\varsigma^{1/k}\mathcal{P}_+)^{\ell}\big(1-(\mathcal{P}_+\mathcal{P}_-(\varsigma\sigma)^{1/k}\big)^{k-\ell}}{1-(\mathcal{P}_+\mathcal{P}_-)^k\varsigma\sigma}\\&\qquad+\frac{(\sigma^{1/k}\mathcal{P}_-)^{k-\ell}\big(1-(\mathcal{P}_+\mathcal{P}_-(\varsigma\sigma)^{1/k}\big)^{\ell}}{1-(\mathcal{P}_+\mathcal{P}_-)^k\varsigma\sigma}\Big]\Big\}\frac{d\varsigma}{\varsigma}\frac{d\sigma}{\sigma}.\end{aligned} \tag{15}$$

The expression in braces is holomorphic in $\mathcal{P}+$ and $\mathcal{P}_-$ for $\mathcal{P}_+\mathcal{P}_-$ close to 1, i.e., $|\mathcal{P}|$ near 1, and for ς, $\sigma\in[0,1]$ because the possible zero of the denominator is cancelled by a zero in the numerator. This proves the argument of the Remark.

Using (15), we have

$$
\begin{aligned}
&K_+(z,w,t-t_0,\lambda)+K_+(\bar z,\bar w,t_0-t,-\lambda)\\
&=\frac{1}{4k\pi^2}(A)^{-\frac{1-\lambda}{2}}(\bar A)^{-\frac{1+\lambda}{2}}\Big\{\int_0^1\frac{\big(1-|\mathcal{P}|^{2k}\varsigma\big)^{-\frac{1+\lambda}{2}}}{\varsigma^{\frac{1+\lambda}{2}}(1-\varsigma)^{\frac{1-\lambda}{2}}}\frac{d\varsigma}{\big(1-\mathcal{P}\varsigma^{\frac{1}{k}}\big)}\\
&\qquad+\int_0^1\frac{\big(1-|\mathcal{P}|^{2k}\varsigma\big)^{-\frac{1-\lambda}{2}}}{\varsigma^{\frac{1-\lambda}{2}}(1-\varsigma)^{\frac{1+\lambda}{2}}}\frac{d\varsigma}{\big(1-\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}\big)}\Big\}.
\end{aligned}\tag{16}
$$

Summarizing what we discussed above, we have the following formula for K_λ:

$$
\begin{aligned}
&K_\lambda(z,w,t-t_0)\\
&=\frac{1}{8k\pi^2}(A)^{-\frac{1-\lambda}{2}}(\bar A)^{-\frac{1+\lambda}{2}}\Big\{\int_0^1\frac{\big(1-|\mathcal{P}|^{2k}\varsigma\big)^{-\frac{1+\lambda}{2}}}{\varsigma^{\frac{1+\lambda}{2}}(1-\varsigma)^{\frac{1-\lambda}{2}}}\frac{1+\mathcal{P}\varsigma^{\frac{1}{k}}}{1-\mathcal{P}\varsigma^{\frac{1}{k}}}d\varsigma\\
&\qquad+\int_0^1\frac{\big(1-|\mathcal{P}|^{2k}\varsigma\big)^{-\frac{1+\lambda}{2}}}{\varsigma^{\frac{1-\lambda}{2}}(1-\varsigma)^{\frac{1+\lambda}{2}}}\frac{1+\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}}{1-\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}}d\varsigma\Big\},
\end{aligned}\tag{17}
$$

for $-1<\mathrm{Re}(\lambda)<1$. Here

$$
A=\frac{1}{2}\Big(|z|^{2k}+|w|^{2k}+i(t-t_0)\Big),\tag{18}
$$

and

$$
\mathcal{P}=\frac{\bar z\cdot w}{A^{\frac{1}{k}}},\quad\text{if}\quad w\neq 0\qquad\text{and}\qquad \mathcal{P}=0,\quad\text{if}\quad w=0.\qquad\blacksquare
$$

5.2. Fundamental solution K_λ for the Kohn Laplacian Δ_λ when $|\mathrm{Re}(\lambda)|<1$

Let us start with a well-known identity:

$$
\frac{1}{(1-w)^\beta}=F(1,\beta,1,w)=\frac{1}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^1\frac{\varsigma^{\beta-1}}{(1-\varsigma)^\beta(1-w\varsigma)}d\varsigma\tag{19}
$$

for $\beta\notin\mathbf{Z}$. Hence we have

$$
\begin{aligned}
&\Gamma\big(\frac{1+\lambda}{2}\big)\Gamma\big(\frac{1-\lambda}{2}\big)\big(1-|\mathcal{P}|^{2k}\varsigma\big)^{-\frac{1+\lambda}{2}}\\
&=\int_0^1\sigma^{-\frac{1-\lambda}{2}}(1-\sigma)^{-\frac{1+\lambda}{2}}\big(1-|\mathcal{P}|^{2k}\varsigma\sigma\big)^{-1}d\sigma.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^1 \frac{\varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}}{\left(1-|\mathcal{P}|^{2k}\varsigma\right)^{-\frac{1+\lambda}{2}}\left(1-|\mathcal{P}|^{2k}\varsigma\right)^{\frac{1+\lambda}{2}}}\frac{1+\mathcal{P}\varsigma^{\frac{1}{k}}}{1-\mathcal{P}\varsigma^{\frac{1}{k}}}d\varsigma \\
&= \frac{1}{\Gamma\left(\frac{1+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)}\int_0^1 \frac{d\sigma}{\sigma^{\frac{1-\lambda}{2}}(1-\sigma)^{\frac{1+\lambda}{2}}} \\
&\quad\times\int_0^1 \frac{\varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}}{1-|\mathcal{P}|^{2k}\varsigma\sigma}\frac{1+\mathcal{P}\varsigma^{\frac{1}{k}}}{1-\mathcal{P}\varsigma^{\frac{1}{k}}}d\varsigma.
\end{aligned}
\tag{20}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1 \frac{\varsigma^{-\frac{1-\lambda}{2}}(1-\varsigma)^{-\frac{1+\lambda}{2}}}{\left(1-|\mathcal{P}|^{2k}\varsigma\right)^{-\frac{1-\lambda}{2}}\left(1-|\mathcal{P}|^{2k}\varsigma\right)^{\frac{1+\lambda}{2}}}\cdot\frac{1+\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}}{1-\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}}d\varsigma \\
&= \frac{1}{\Gamma\left(\frac{1+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)}\int_0^1 \frac{d\sigma}{\sigma^{\frac{1-\lambda}{2}}(1-\sigma)^{\frac{1+\lambda}{2}}} \\
&\quad\times\int_0^1 \frac{\varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}}{1-|\mathcal{P}|^{2k}\varsigma\sigma}\frac{1+\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}}{1-\bar{\mathcal{P}}\varsigma^{\frac{1}{k}}}d\varsigma.
\end{aligned}
\tag{21}
$$

Thus, the sum of (20) and (21) gives us

$$
\begin{aligned}
&\frac{2}{\Gamma\left(\frac{1+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)}\int_0^1\int_0^1 \varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}\sigma^{-\frac{1-\lambda}{2}}(1-\sigma)^{-\frac{1+\lambda}{2}} \\
&\times\prod_{\ell=1}^{k-1}\left(1-e^{\frac{2\ell\pi}{k}i}|\mathcal{P}|^2(\varsigma\sigma)^{\frac{1}{k}}\right)^{-1}\frac{d\varsigma d\sigma}{(1-\mathcal{P}\varsigma^{\frac{1}{k}})(1-\bar{\mathcal{P}}\sigma^{\frac{1}{k}})}.
\end{aligned}
$$

From the above and formula (17), we obtain for $k>1$,

$$
\begin{aligned}
K_\lambda &= \frac{1}{4k\pi^2}\frac{(A)^{-\frac{1-\lambda}{2}}(\bar{A})^{-\frac{1+\lambda}{2}}}{\Gamma\left(\frac{1+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)}\int_0^1 \frac{\varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}}{1-\mathcal{P}\varsigma^{\frac{1}{k}}}d\varsigma \\
&\quad\times\int_0^1\prod_{\ell=1}^{k-1}\left(1-e^{\frac{2\ell\pi}{k}i}|\mathcal{P}|^2(\varsigma\sigma)^{\frac{1}{k}}\right)^{-1}\frac{\sigma^{-\frac{1-\lambda}{2}}(1-\sigma)^{-\frac{1+\lambda}{2}}}{1-\bar{\mathcal{P}}\sigma^{\frac{1}{k}}}d\sigma,
\end{aligned}
\tag{22}
$$

and for $k = 1$,

$$K_\lambda = \frac{1}{4\pi^2} \frac{(A)^{-\frac{1-\lambda}{2}} (\bar{A})^{-\frac{1+\lambda}{2}}}{\Gamma\big(\frac{1+\lambda}{2}\big)\Gamma\big(\frac{1-\lambda}{2}\big)} \int_0^1 \frac{\varsigma^{-\frac{1+\lambda}{2}}(1-\varsigma)^{-\frac{1-\lambda}{2}}}{1-\mathcal{P}\varsigma} d\varsigma \times \int_0^1 \frac{\sigma^{-\frac{1-\lambda}{2}}(1-\sigma)^{-\frac{1+\lambda}{2}}}{1-\bar{\mathcal{P}}\sigma} d\sigma \tag{23}$$

Now we may state our result as following theorem.

Theorem 5.3. *Assume $(z,t) \neq (w,t_0)$. Then the fundamental solution K_λ for the sub-Laplacian Δ_λ is given by (22). In particular, when $k = 1$, K_λ is given by (23).*

Remarks 5.2. (1). *In fact, from what we have discussed, we know that $|\mathcal{P}| \leq 1$ and $\mathcal{P} = 1$ if and only if $(z,t) = (w,t_0)$. Therefore, it is easy to see that K_λ has a unique singularity at (w,t_0). Moreover, it is not difficult to show that $K_\lambda \in L^1_{loc}(\mathbb{R}^3)$ and $\Delta_\lambda K_\lambda = \delta(z,w,t-t_0)$. We omit the detail here.*

(2). *The integrand in (22) is real analytic in $\mathcal{P}$ and $\bar{\mathcal{P}}$ in the region*

$$\mathcal{W} = \Big\{ |\mathcal{P}| \leq 1,\ \ \mathcal{P} \neq 1, \bar{\mathcal{P}} \neq 1, \Big\}$$

when ς, $\sigma \in [0,1]$. Moreover,

$$F_\lambda(0,0) = \frac{\mathcal{B}\big(\frac{1-\lambda}{2}, \frac{1+\lambda}{2}\big)\mathcal{B}\big(\frac{1+\lambda}{2}, \frac{1-\lambda}{2}\big)}{\Gamma\big(\frac{1-\lambda}{2}\big)\Gamma\big(\frac{1+\lambda}{2}\big)} = \Gamma\big(\frac{1-\lambda}{2}\big)\Gamma\big(\frac{1+\lambda}{2}\big),$$

where $\mathcal{B}(\cdot,\cdot)$ denotes the beta function.

From Theorem 5.3, we obtain the fundamental solution for the operator Δ_0 as a corollary.

Corollary 5.3. *Assume $(z,t) \neq (w,t_0)$. Then the fundamental solution K_0 for the sub-Laplacian $\Delta_{0,k}$ has the following closed form:*

$$K_0 = \frac{1}{4k\pi^3|A|} \int_0^1 \int_0^1 \varsigma^{-\frac{1}{2}}(1-\varsigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}(1-\sigma)^{-\frac{1}{2}} \times \prod_{\ell=1}^{k-1} \Big(1 - e^{\frac{2\ell\pi}{k}i}|\mathcal{P}|^2(\varsigma\sigma)^{\frac{1}{k}}\Big)^{-1} \frac{d\varsigma d\sigma}{(1-\mathcal{P}\varsigma^{\frac{1}{k}})(1-\bar{\mathcal{P}}\sigma^{\frac{1}{k}})}. \tag{24}$$

6. The Cauchy–Szegö kernel on Ω_k and $\partial\Omega_k$

Let $f(\mathbf{z}, z_{n+1}) \in H^2(\Omega_k)$. Here $H^2(\Omega_k)$ is the space of all square integrable holomorphic functions on Ω_k. Then one has

$$f(\mathbf{z}, z_{n+1}) = \int_0^\infty e^{2\pi i\lambda z_{n+1}} \widetilde{f}(\mathbf{z}, \lambda) d\lambda$$

where $\widetilde{f}(\mathbf{z}, \lambda)$ is the Fourier transform of f with respect to the z_{n+1} variable. The "height function" on Ω_k can be written as

$$\rho = \mathrm{Im}(z_{n+1}) - \left(\sum_{j=1}^n |z_j|^2\right)^k = \mathrm{Im}(z_{n+1}) - |\mathbf{z}|^{2k},$$

where $|\mathbf{z}|^2 = \sum_{j=1}^n |z_j|^2$. Then

$$f(\mathbf{z}, z_{n+1}) = \int_0^\infty e^{2\pi i\lambda(\mathrm{Re}(z_{n+1})) - 2\pi\lambda(\rho + |\mathbf{z}|^{2k})} \widetilde{f}(\mathbf{z}, \lambda) d\lambda. \tag{25}$$

According to Plancherel's formula, we know that

$$e^{-2\pi\lambda(\rho+|\mathbf{z}|^{2k})} \widetilde{f}(\mathbf{z}, \lambda) = \int_{-\infty}^{+\infty} e^{2\pi i\lambda(\mathrm{Re}(z_{n+1}))} f(\mathbf{z}, z_{n+1}) d(\mathrm{Re}(z_{n+1})).$$

This implies that

$$\widetilde{f}(\mathbf{z}, \lambda) = \int_{-\infty}^{+\infty} e^{-2\pi i\lambda\bar{z}_{n+1}} f(\mathbf{z}, z_{n+1}) d(\mathrm{Re}(z_{n+1})). \tag{26}$$

The definition of $H^2(\Omega_k)$ and (25) imply that

$$\int_{\mathbb{C}^n} \int_0^\infty e^{-4\pi\lambda\left(\sum_{j=1}^n |z_j|^2\right)^k} |\widetilde{f}(\mathbf{z}, \lambda)|^2 d\lambda dV(\mathbf{z}) < \infty,$$

where $dV(\mathbf{z}) = dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n$. Let A_λ be the Bergman space of holomorphic functions in $\mathbb{C}^n$ with the norm

$$\|g\|_{A_\lambda(\Omega_k)} = \left\{\int_{\mathbb{C}^n} e^{-4\pi\lambda|\mathbf{z}|^{2k}} |g(\mathbf{z})|^2 dV(\mathbf{z})\right\}^{\frac{1}{2}} < \infty.$$

Then A_λ has a reproducing kernel $S_\lambda(\mathbf{z},\mathbf{w})$, *i.e.*, for $g \in A_\lambda$, one has

$$g(\mathbf{z}) = \int_{\mathbb{C}^n} S_\lambda(\mathbf{z},\mathbf{w}) dV(\mathbf{w}).$$

In particular,

$$\widetilde{f}(\mathbf{z},\lambda) = \int_{\mathbb{C}^n} S_\lambda(\mathbf{z},\mathbf{w}) \widetilde{f}(\mathbf{w},\lambda) dV(\mathbf{w}),$$

whenever $f \in H^2(\Omega_k)$. Therefore, such an f has the following integral representation:

$$f(\mathbf{z}, z_{n+1}) = \int_0^\infty e^{2\pi i \lambda z_{n+1}} d\lambda \int_{\mathbb{C}^n} S_\lambda(\mathbf{z},\mathbf{w}) \widetilde{f}(\mathbf{w},\lambda) dV(\mathbf{w}).$$

Substituting (26) for $\widetilde{f}(\mathbf{w},\lambda)$, we obtain

$$f(\mathbf{z}, z_{n+1}) = \int_{\partial\Omega_k} S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) f(\mathbf{w}, w_{n+1}) dV(\mathbf{w}) d(\mathrm{Re}(w_{n+1})).$$

Denote

$$A = \frac{i}{2}(\bar{w}_{n+1} - z_{n+1}),$$

and

$$S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) = \int_0^\infty e^{-4\pi\lambda A} S_\lambda(\mathbf{z},\mathbf{w}) d\lambda.$$

Then we have the following lemma.

Lemma 6.1. *Let U be a unitary transform on $\mathbb{C}^n$. If $S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1})$ is the Cauchy–Szegö kernel on Ω_k, then*

$$S(U(\mathbf{z}), z_{n+1}; U(\mathbf{w}), w_{n+1}) = S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}).$$

Proof. Let $f \in H^2(\Omega_k)$. Then $f \circ U^{-1} \in H^2(\Omega_k)$ for every unitary transform U and

$$\begin{aligned}
&\int_{\partial\Omega_k} S(U(\mathbf{z}), z_{n+1}; U(\mathbf{w}), w_{n+1}) f(\mathbf{w}, w_{n+1}) dV(\mathbf{w}) d(\mathrm{Re}(w_{n+1})) \\
&= \int_{\partial\Omega_k} S(U(\mathbf{z}), z_{n+1}; \mathbf{w}, w_{n+1}) f(U^{-1}(\mathbf{w}), w_{n+1}) dV(\mathbf{w}) d(\mathrm{Re}(w_{n+1})) \\
&= f(U^{-1}(\cdot), z_{n+1})\big|_{U(\mathbf{z})} = f(\mathbf{z}, z_{n+1})
\end{aligned}$$

$$= \int_{\partial\Omega_k} S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) f(\mathbf{w}, w_{n+1}) dV(\mathbf{w}) d(\mathrm{Re}(w_{n+1})).$$

The lemma follows immediately. □

Theorem 6.1. *The explicit Cauchy–Szegö kernel on* Ω_k *is as follows.*

$$\begin{aligned} & S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) \\ &= \frac{n!}{4\pi^{n+1}} \frac{1}{\left(A^{\frac{1}{k}} - \sum_{j=1}^n z_j \bar{w}_j\right)^{n+1} A^{\frac{k-1}{k}}} \\ &= \frac{n!}{4\pi^{n+1}} \frac{\left[\frac{i}{2}(s-t) + \frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) + \frac{1}{2}(\rho+\mu)\right]^{\frac{1-k}{k}}}{\left\{\left[\frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) - \frac{i}{2}(t-s) + \frac{1}{2}(\rho+\mu)\right]^{\frac{1}{k}} - \sum_{j=1}^n z_j \bar{w}_j\right\}^{n+1}}. \end{aligned}$$

Let us first compute $S_\lambda(\mathbf{z}, \mathbf{w})$. Assume that $\mathbf{w} = \mathbf{1} = (1, 0, \ldots, 0)$ is the "north pole" of the unit sphere in $\mathbb{C}^n$, then we have

$$S_\lambda(\mathbf{z}, \mathbf{1}) = \sum_{j=0}^{\infty} \frac{\mathbf{z}^j \mathbf{1}^j}{\|z_1^j\|^2_{A_\lambda(\Omega_k)}} = \sum_{j=0}^{\infty} \frac{z_1^j}{\|z_1^j\|^2_{A_\lambda(\Omega_k)}}$$

where

$$\begin{aligned} \|z_1^j\|^2_{A_\lambda(\Omega_k)} &= \int_{\mathbb{C}^n} e^{-4\pi\lambda|\mathbf{z}|^{2k}} |z_1|^{2j} dv(\mathbf{z}) \\ &= \int_{\partial B_n} |z_1|^{2j} d\sigma(\mathbf{z}) \int_0^\infty e^{-4\pi\lambda r^{2k}} r^{2j} r^{2n-1} dr \end{aligned}$$

with $r = |\mathbf{z}| = \left(\sum_{\ell=1}^n |z_\ell|^2\right)^{\frac{1}{2}}$. Set $u = 4\pi\lambda r^{2k}$. It follows that

$$\frac{du}{u} = 2k\frac{dr}{r}.$$

Hence we have

$$\begin{aligned} \|z_1^j\|^2_{A_\lambda(\Omega_k)} &= \frac{\pi^n 2(j!)}{(j+n-1)!} \cdot \frac{1}{2k(4\pi\lambda)^{\frac{j+n}{k}}} \int_0^\infty e^{-u} u^{\frac{j+n}{k}} \frac{du}{u} \\ &= \frac{\pi^n 2(j!)}{(j+n-1)!} \cdot \frac{1}{2k(4\pi\lambda)^{\frac{j+n}{k}}} \Gamma\left(\frac{j+n}{k}\right). \end{aligned}$$

Therefore,

$$S_\lambda(\mathbf{z},\mathbf{1}) = \frac{k}{\pi^n}\sum_{j=0}^{\infty}\frac{(j+n-1)!}{j!}\frac{z_i^j}{\Gamma\left(\frac{j+n}{k}\right)}(4\pi\lambda)^{\frac{j+n}{k}}.$$

Hence

$$\begin{aligned}
&S(\mathbf{z},z_{n+1};\mathbf{1},w_{n+1})\\
&= \frac{k}{\pi^n}\sum_{j=0}^{\infty}\frac{(j+n-1)!}{j!}\left(\int_0^\infty e^{-4\pi\lambda A}(4\pi\lambda)^{\frac{j+n}{k}}\,d\lambda\right)\frac{z_1^j}{\Gamma\left(\frac{j+n}{k}\right)}\\
&= \frac{k}{4\pi^{n+1}}\sum_{j=0}^{\infty}\frac{(j+n-1)!}{j!}\frac{\Gamma\left(\frac{j+n}{k}+1\right)}{\Gamma\left(\frac{j+n}{k}\right)}A^{-\frac{j+n}{k}-1}z_1^j\\
&= \frac{1}{4\pi^{n+1}}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\left(\frac{z_1}{A^{\frac{1}{k}}}\right)^j A^{-\frac{n}{k}-1}\\
&= \frac{n!}{4\pi^{n+1}}\sum_{j=0}^{\infty}\frac{(j+n)!}{j!n!}\left(\frac{z_1}{A^{\frac{1}{k}}}\right)^j A^{-\frac{n}{k}-1}\\
&= \frac{n!}{4\pi^{n+1}}\left(1-\frac{z_1}{A^{\frac{1}{k}}}\right)^{-n-1}A^{-\frac{n}{k}-1}.
\end{aligned}$$

Suppose $\mathbf{z} = r\cdot\mathbf{z}'$ with $\|\mathbf{z}'\| = 1$ and $\mathbf{z}'\in\partial B_n$. Then there exists a unitary transform U on ∂B_n such that $U(\mathbf{z}') = \mathbf{1}$. Hence, $\mathbf{z}' = U^{-1}(\mathbf{1})$. Therefore,

$$\begin{aligned}
&S(\mathbf{z},z_{n+1};\mathbf{w},w_{n+1})\\
&= S(r\cdot\mathbf{z}',z_{n+1};\mathbf{w},w_{n+1}) = S(r\cdot U^{-1}(\mathbf{1}),z_{n+1};\mathbf{w},w_{n+1})\\
&= S(r\cdot\mathbf{1},z_{n+1};U(\mathbf{w}),w_{n+1}) = \overline{S(U(\mathbf{w}),w_{n+1};r\cdot\mathbf{1},z_{n+1})}\\
&= \frac{n!}{4\pi^{n+1}}\frac{1}{\left(1-r\cdot\left(\frac{\overline{U(\mathbf{w})}}{A^{\frac{1}{k}}}\right)_1\right)^{n+1}}A^{-\frac{n}{k}-1}\\
&= \frac{n!}{4\pi^{n+1}}\frac{1}{\left(1-\left(r\mathbf{1}\cdot\left(\frac{\overline{U(\mathbf{w})}}{A^{\frac{1}{k}}}\right)\right)\right)^{n+1}A^{\frac{n}{k}+1}}\\
&= \frac{n!}{4\pi^{n+1}}\frac{1}{\left(1-\left(rU^{-1}(\mathbf{1})\cdot\left(\frac{\bar{\mathbf{w}}}{A^{\frac{1}{k}}}\right)\right)\right)^{n+1}A^{\frac{n}{k}+1}}
\end{aligned}$$

$$= \frac{n!}{4\pi^{n+1}} \frac{1}{\left(1 - \frac{\mathbf{z}\cdot\bar{\mathbf{w}}}{A^{\frac{1}{k}}}\right)^{n+1} A^{\frac{n}{k}+1}}$$
$$= \frac{n!}{4\pi^{n+1}} \frac{A^{\frac{1}{k}-1}}{\left(A^{\frac{1}{k}} - \sum_{j=1}^{n} z_j \bar{w}_j\right)^{n+1}}.$$

This tells us that

$$\begin{aligned} &S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) \\ &= \frac{n!}{4\pi^{n+1}} \frac{1}{\left(A^{\frac{1}{k}} - \sum_{j=1}^{n} z_j \bar{w}_j\right)^{n+1} A^{\frac{k-1}{k}}} \\ &= \frac{n!}{4\pi^{n+1}} \frac{\left[\frac{i}{2}(s-t) + \frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) + \frac{1}{2}(\rho+\mu)\right]^{\frac{1-k}{k}}}{\left\{\left[\frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) - \frac{i}{2}(t-s) + \frac{1}{2}(\rho+\mu)\right]^{\frac{1}{k}} - \sum_{j=1}^{n} z_j \bar{w}_j\right\}^{n+1}}. \end{aligned}$$

When both $(\mathbf{z}, z_{n+1})$ and $(\mathbf{w}, w_{n+1})$ in $\partial\Omega_k$, $\rho = \mu = 0$, then we have

$$S(\mathbf{z}, t; \mathbf{w}, s) = \frac{n!}{4\pi^{n+1}} \frac{\left[\frac{i}{2}(s-t) + \frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k})\right]^{\frac{1-k}{k}}}{\left\{\left[\frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) - \frac{i}{2}(t-s)\right]^{\frac{1}{k}} - \sum_{j=1}^{n} z_j \bar{w}_j\right\}^{n+1}}.$$

In particular, when $k = 1$, the Cauchy–Szegö for the Heisenberg group $\mathbf{H}_n$ is

$$S(\mathbf{z}, t; \mathbf{w}, s) = \frac{n!}{4\pi^{n+1}} \frac{1}{\left\{\left[\frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) - \frac{i}{2}(t-s)\right]^{\frac{1}{k}} - \sum_{j=1}^{n} z_j \bar{w}_j\right\}^{n+1}}.$$

We note that $|[\frac{1}{2}(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k}) - \frac{i}{2}(t-s)]^{\frac{1}{k}} - \sum_{j=1}^{n} z_j \bar{w}_j|^{\frac{1}{2}}$ can be considered as a generalization of the Korányi distance to the higher step operators (see Diaz [13]).

Remarks 6.2. (1). *The domain Ω_k is equivalent to the "ellipsoid"*

$$\mathbb{E}_k = \left\{(\mathbf{w}, w_{n+1}) \in \mathbb{C}^{n+1} : \Big(\sum_{j=1}^{n} |w_j|^2\Big)^k + |w_{n+1}|^2 < 1\right\}$$

via the "generalized Caley transform":

$$z_1 = \frac{w_1}{(1+w_{n+1})^{\frac{1}{k}}}, \ldots, z_n = \frac{w_n}{(1+w_{n+1})^{\frac{1}{k}}}, z_{n+1} = \frac{i(1-w_{n+1})}{1+w_{n+1}}.$$

Then the Cauchy–Szegö kernel for $\mathbb{E}_k$ *is*

$$S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) = \frac{n!}{4\pi^{n+1}} \frac{1}{\left[(1 - z_{n+1}\bar{w}_{n+1})^{\frac{1}{k}} - \sum_{j=1}^{n} z_j\bar{w}_j\right]^{n+1} (1 - z_{n+1}\bar{w}_{n+1})^{\frac{k-1}{k}}}. \tag{27}$$

In particular, when $k = 1$*, then the Cauchy–Szegö kernel for the unit ball* $\mathbb{B}_{n+1} \subset \mathbb{C}^{n+1}$ *is*

$$S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) = \frac{n!}{4\pi^{n+1}} \frac{1}{\left(1 - \sum_{j=1}^{n+1} z_j\bar{w}_j\right)^{n+1}}.$$

(2). *When* $k = 1$*, then the Cauchy–Szegö projection is closely related to the solvability of the Kohn Laplacian. Let us consider the Kohn Laplacian acting on functions, i.e.,* $q = 0$*. Then we know that*

$$\Box_b = -\frac{1}{2}\sum_{k=1}^{n}\left(Z_k\bar{Z}_k + \bar{Z}_kZ_k\right) - i[Z_k, \bar{Z}_k] = -\sum_{k=1}^{n} Z_k\bar{Z}_k.$$

In this case, the operator annihilates the boundary values of holomorphic functions on $\mathbf{H}_n$*. Now we need to deal with the Hans Lewy operator. In general, we can't expect this operator is hypoelliptic. Moreover, the equation*

$$\sum_{k=1}^{n} Z_k\bar{Z}_k(u) = f$$

is generally not even locally solvable. Following a method in Greiner–Kohn–Stein [19] and Greiner–Stein [20], we know that for any $f \in L^2(\mathbf{H}_n)$ *leads to the Cauchy–Szegö integral* $\mathcal{C}(f)$*, defined in* $C_0^\infty(\mathbf{H}_n)$ *by*

$$\mathcal{C}(f)(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) = \int_{\partial\widetilde{\Omega}_n} S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) f(\mathbf{w}, w_{n+1}) d\sigma,$$

with

$$S(\mathbf{z}, z_{n+1}; \mathbf{w}, w_{n+1}) = \frac{2^{n-1}n!}{\pi^{n+1}}\left\{i(\bar{w}_{n+1} - z_{n+1}) - 2\sum_{k=1}^{n} z_k\bar{w}_k\right\}^{-n-1}.$$

Here $d\sigma$ *is the Lebesgue measure defined on* $\partial\widetilde{\Omega}_n$ *which is identified as the*

Heisenberg group $\mathbf{H}_n$*. The restriction of* $\mathcal{C}(f)$ *to* $\partial\widetilde{\Omega}_n$ *is given by*

$$\mathcal{C}_b(f) \;=\; \lim_{\rho\to 0^+} f * S_\rho, \tag{28}$$

where

$$S_\rho(\mathbf{z},t) = \frac{2^{n-1}n!}{\pi^{n+1}}\Big(\rho^2+\sum_{k=1}^n |z_k|^2 - it\Big)^{-n-1}.$$

The convolution in (28) *is with respect to the Heisenberg group. Since* $f \in L^2$*, the limit in* (28) *exists in* L^2*-norm (see Korányi and Vági [23]).*

Let us consider the following equation:

$$\Box_b^{(0)} \;=\; \mathcal{L}_\lambda - i\big(\lambda-n\big)\frac{\partial}{\partial t}.$$

Thus

$$\Box_b^{(0)}\big(K_\lambda\big) \;=\; \mathcal{L}_\lambda\big(K_\lambda\big) - i\big(\lambda-n\big)\frac{\partial}{\partial t}\big(K_\lambda\big), \tag{29}$$

where

$$K_\lambda(\mathbf{z},t) \;=\; \frac{2^{2-n}\pi^{n+1}}{\Gamma\big(\frac{n+\lambda}{2}\big)\Gamma\big(\frac{n+\lambda}{2}\big)}\Big(\sum_{k=1}^n|z_k|^2-it\Big)^{-\frac{n+\lambda}{2}}\Big(\sum_{k=1}^n|z_k|^2+it\Big)^{-\frac{n-\lambda}{2}}.$$

Formal differentiation of (29) *with respect to the variable* λ *yields the following result*

$$\Box_b^{(0)}\Big[\frac{2^{n-2}(n-1)!}{\pi^{n+1}}\log\Big(\frac{|\mathbf{z}|^2-it}{|\mathbf{z}|^2+it}\Big)\cdot\Big(\sum_{k=1}^n|z_k|^2-it\Big)^{-n}\Big]$$
$$=\;\delta-\frac{2^{n-1}n!}{\pi^{n+1}}\Big(\sum_{k=1}^n|z_k|^2-it\Big)^{-(n+1)}.$$

Denote

$$\Psi \;=\; \frac{2^{n-2}(n-1)!}{\pi^{n+1}}\log\Big(\frac{|\mathbf{z}|^2-it}{|\mathbf{z}|^2+it}\Big)\cdot\Big(\sum_{k=1}^n|z_k|^2-it\Big)^{-n}.$$

Then we have the following identity

$$\Box_b^{(0)}K \;=\; K\Box_b^{(0)} \;=\; \mathbf{I}-\mathcal{C}_b,$$

where $K(f) \;=\; f*\Psi$ *with* $f\in C_0^\infty(\mathbf{H}_n)$*.*

7. Sharp estimates of Cauchy–Szegö kernel

Consider the triple $(\partial\Omega_k, d, \mu)$, where μ is the Lebesgue measure on $\mathbb{C}\times\mathbb{R}$. We use new coordinates $(\mathbf{z}, t)$ on the boundary $\partial\Omega_k$ to identify it as $\mathbb{C}\times\mathbb{R}$. Here $\mathbf{z} = x + iy$ and $t = \mathrm{Re}(z_2)$.

We now introduce the quasi-distance on $\partial\Omega_k$ as follows: for any $(\mathbf{z}, t)$, $(\mathbf{w}, s) \in \partial\Omega_k$,

$$d((\mathbf{z}, t), (\mathbf{w}, s)) := h^2((\mathbf{z}, t), (\mathbf{w}, s))\rho^{2-2k}((\mathbf{z}, t), (\mathbf{w}, s)), \tag{30}$$

where

$$\rho((\mathbf{z}, t), (\mathbf{w}, s)) := |\mathbf{z}| + |\mathbf{w}| + |\sigma|^{\frac{1}{2k}} \approx |\mathbf{z}| + |\mathbf{w}| + |t-s|^{\frac{1}{2k}},$$

and

$$h((\mathbf{z}, t), (\mathbf{w}, s)) = |\mathbf{z}-\mathbf{w}|^2\rho^{2k-2}((\mathbf{z}, t), (\mathbf{w}, s)) + |\sigma((\mathbf{z}, t), (\mathbf{w}, s))|$$

with

$$\sigma((\mathbf{z}, t), (\mathbf{w}, s)) = t - s + 2\mathrm{Im}(\mathbf{z}^k\overline{\mathbf{w}}^k).$$

Based on Proposition 9.6 in [5], we see that this quasi-metric d satisfies the quasi-triangle inequality:

$$d((\mathbf{z}, t), (\mathbf{u}, r)) \lesssim d((\mathbf{z}, t), (\mathbf{w}, s)) + d((\mathbf{w}, s), (\mathbf{u}, r)).$$

Recall the functions in (18) and (13)

$$A(\mathbf{z}, t; \mathbf{w}, s) = \frac{1}{2}\big(|\mathbf{z}|^{2k} + |\mathbf{w}|^{2k} - i(t-s)\big);$$

$$\mathcal{P}(\mathbf{z}, t; \mathbf{w}, s) = \frac{\mathbf{z}\overline{\mathbf{w}}}{A(\mathbf{z}, t; \mathbf{w}, s)^{\frac{1}{k}}}.$$

By Lemma 9.3 in [5] and the estimate on Page 242 in [6], we have the following properties.

Lemma 7.1. *The functions h, ρ, A and $\mathcal{P}$ satisfy*

$$\begin{aligned} &|\mathbf{z}^k - \mathbf{w}^k|^2 \lesssim h((\mathbf{z}, t), (\mathbf{w}, s)) \lesssim \rho^{2k}((\mathbf{z}, t), (\mathbf{w}, s)) \approx |A(\mathbf{z}, t; \mathbf{w}, s)|;\\ &|1 - P(\mathbf{z}, t; \mathbf{w}, s)| \approx \frac{h((\mathbf{z}, t), (\mathbf{w}, s))}{|A(\mathbf{z}, t; \mathbf{w}, s)|}, \quad (\mathbf{z}, t), (\mathbf{w}, s) \in \partial\Omega_k. \end{aligned} \tag{31}$$

Based on these notation, we see that the Cauchy–Szegö kernel $S(\mathbf{z}, t; \mathbf{w}, s)$ on $\partial\Omega_k$ can be expressed as

$$S(\mathbf{z}, t; \mathbf{w}, s) = \frac{1}{4\pi^2} A^{-\frac{k+1}{k}}(\mathbf{z}, t; \mathbf{w}, s)\big(1 - \mathcal{P}(\mathbf{z}, t; \mathbf{w}, s)\big)^{-2}. \tag{32}$$

Moreover, based on (31), we see that

$$d((\mathbf{z}, t), (\mathbf{w}, s)) \lesssim \rho^{2k+2}((\mathbf{z}, t), (\mathbf{w}, s)). \tag{33}$$

Theorem 7.1. *For both $(\mathbf{z}, t)$ and $(\mathbf{w}, s)$ in $\partial\Omega_k$ with $(\mathbf{z}, t) \neq (\mathbf{w}, s)$, the Cauchy–Szegö projection associated with the kernel $S(\mathbf{z}, t; \mathbf{w}, s)$ is a Calderón–Zygmund operator on $(\partial\Omega_k, d, \mu)$.*

Proof. We first consider the size estimate of $S(\mathbf{z}, t; \mathbf{w}, s)$. Note that for $(\mathbf{z}, t) \neq (\mathbf{w}, s)$, by Lemma 7.1, we have

$$\begin{aligned} |S(\mathbf{z}, t; \mathbf{w}, s)| &= \frac{1}{4\pi^2} \left| A^{-\frac{k+1}{k}}(\mathbf{z}, t; \mathbf{w}, s)\big(1 - \mathcal{P}(\mathbf{z}, t; \mathbf{w}, s)\big)^{-2} \right| \\ &\approx |A(\mathbf{z}, t; \mathbf{w}, s)|^{-\frac{k+1}{k}} |A(\mathbf{z}, t; \mathbf{w}, s)|^2 h^{-2}((\mathbf{z}, t), (\mathbf{w}, s)) \\ &= |A(\mathbf{z}, t; \mathbf{w}, s)|^{-\frac{1}{k}+1} h^{-2}((\mathbf{z}, t), (\mathbf{w}, s)) \\ &\approx \rho^{-(2-2k)}((\mathbf{z}, t), (\mathbf{w}, s)) h^{-2}((\mathbf{z}, t), (\mathbf{w}, s)) \\ &= \frac{1}{d((\mathbf{z}, t), (\mathbf{w}, s))}. \end{aligned} \tag{34}$$

Next, we verify the regularity conditions. Consider the difference

$$|S(\mathbf{z}, t; \mathbf{w}_1, s_1) - S(\mathbf{z}, t; \mathbf{w}_0, s_0)|,$$

where

$$d((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0)) \leq c d((\mathbf{z}, t), (\mathbf{w}_0, s_0)), \tag{35}$$

for some small c.

To continue, we consider the following substitution: fix $(\mathbf{z}, t)$, by the change of variables

$$\begin{cases} \mathbf{w}' = \mathbf{w}, \\ s' = s - 2\mathrm{Im}(\mathbf{z}^k \overline{\mathbf{w}}^k), \end{cases} \tag{36}$$

we have

$$S(\mathbf{z},t;\mathbf{w},s) = S(\mathbf{z},t;\mathbf{w}',s' + 2\mathrm{Im}(z^k\overline{\mathbf{w}'}^k)) =: \widetilde{S}(\mathbf{z},t;\mathbf{w}',s'),$$

and thus,

$$\begin{cases} \dfrac{\partial \widetilde{S}}{\partial \mathbf{w}'} = \dfrac{\partial S}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k \dfrac{\partial S}{\partial s}, \\ \dfrac{\partial \widetilde{S}}{\partial s'} = \dfrac{\partial S}{\partial s}. \end{cases} \tag{37}$$

Let

$$\begin{aligned} s'_\alpha &= s'(\mathbf{w}_\alpha, s_\alpha), & \alpha &= 0,1, \\ (\mathbf{w}'_\nu, s'_\nu) &= (1-\nu)(\mathbf{w}_0, s'_0) + \nu(\mathbf{w}_1, s'_1), & 0 &\le \nu \le 1. \end{aligned}$$

Let $(\mathbf{w}_\nu, s_\nu)$ denote the point whose $(\mathbf{w}', s')$ coordinates are $(\mathbf{w}'_\nu, s'_\nu)$. Therefore,

$$\begin{aligned} \mathbf{w}_\nu &= (1-\nu)\mathbf{w}_0 + \nu \mathbf{w}_1, \\ s_\nu &= s'_\nu + 2\mathrm{Im}(\mathbf{z}^k \overline{\mathbf{w}}_\nu^k) \\ &= (1-\nu)\big(s_0 - 2\mathrm{Im}(\mathbf{z}^k\overline{\mathbf{w}}_0^k)\big) + \nu\big(s_1 - 2\mathrm{Im}(\mathbf{z}^k\overline{\mathbf{w}}_1^k)\big) \\ &\quad + 2\mathrm{Im}\left[\mathbf{z}^k\left((1-\nu)\overline{\mathbf{w}}_0 + \nu\overline{\mathbf{w}}_k\right)^k\right]. \end{aligned}$$

By [5, (9.34), (9.35), (9.37)], we have

$$\begin{aligned} &\rho((\mathbf{z},t),(\mathbf{w}_\nu,s_\nu)) \approx \rho((\mathbf{z},t),(\mathbf{w}_0,s_0)), \\ &h((\mathbf{z},t),(\mathbf{w}_\nu,s_\nu)) \approx h((\mathbf{z},t),(\mathbf{w}_0,s_0)), \quad 0 \le \nu \le 1, \end{aligned} \tag{38}$$

and

$$\begin{aligned} &|\mathbf{w}_1 - \mathbf{w}_0| \lesssim d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))\rho^{\frac{1-k}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0)), \\ &|s'_1 - s'_0| \le h((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0)) \\ &\qquad\qquad + h^{\frac12}((\mathbf{z},t),(\mathbf{w}_0,s_0))h^{\frac12}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0)), \\ &h((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0)) \lesssim h((\mathbf{z},t),(\mathbf{w}_0,s_0)). \end{aligned} \tag{39}$$

Since

$$\begin{cases} \dfrac{\partial A}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\dfrac{\partial A}{\partial s} = k\mathbf{w}^{k-1}\left(\overline{\mathbf{w}}^k - \overline{\mathbf{z}}^k\right) = \mathcal{O}(\rho^{k-1}h^{\frac12}), \\ \dfrac{\partial \mathcal{P}}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\dfrac{\partial \mathcal{P}}{\partial s} = -\dfrac{1}{k}\dfrac{\mathcal{P}}{A}\left(\dfrac{\partial A}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\dfrac{\partial A}{\partial s}\right), \end{cases} \tag{40}$$

for $0 \le \nu \le 1$, we have

$$\begin{aligned}
&\frac{\partial \widetilde{S}}{\partial \mathbf{w}'}\Big|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)} \\
&= \left(\frac{\partial S}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\frac{\partial S}{\partial s}\right)\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} \\
&= \frac{1}{4\pi^2}\Big\{-\frac{k+1}{k}A^{-\frac{k+1}{k}-1}(1-\mathcal{P})^{-2}\left(\frac{\partial A}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\frac{\partial A}{\partial s}\right)\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} \\
&\quad + 2A^{-\frac{k+1}{k}}(1-\mathcal{P})^{-3}\left(\frac{\partial \mathcal{P}}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\frac{\partial \mathcal{P}}{\partial s}\right)\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}\Big\} \\
&= \frac{1}{4\pi^2}A^{-\frac{k+1}{k}}(1-\mathcal{P})^{-2}\left[C_1\frac{1}{A} + C_2\frac{\mathcal{P}}{A(1-\mathcal{P})}\right] \\
&\quad \times \left(\frac{\partial A}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\frac{\partial A}{\partial s}\right)\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} \\
&= S\left[C_1\frac{1}{A} + C_2\frac{\mathcal{P}}{A(1-\mathcal{P})}\right]\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} k{\mathbf{w}_\nu}^{k-1}\left(\overline{\mathbf{w}_\nu}^k - \overline{\mathbf{z}}^k\right).
\end{aligned}$$

By using (36), (34), (31) and (38), we obtain that

$$\begin{aligned}
&\left|\frac{\partial \widetilde{S}}{\partial \mathbf{w}'}\Big|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)}(\mathbf{w}'_1 - \mathbf{w}'_0)\right| \\
&= \left|\left(\frac{\partial S}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\frac{\partial S}{\partial s}\right)\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}(\mathbf{w}'_1 - \mathbf{w}'_0)\right| \\
&= \left|\left(\frac{\partial S}{\partial \mathbf{w}} + ik\mathbf{w}^{k-1}\overline{\mathbf{z}}^k\frac{\partial S}{\partial s}\right)\Big|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}(\mathbf{w}_1 - \mathbf{w}_0)\right| \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{1}{|A|} + \frac{|\mathcal{P}|}{h}\right)\Big|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)}\left|k{\mathbf{w}_\nu}^{k-1}\left(\overline{\mathbf{w}_\nu}^k - \overline{\mathbf{z}}^k\right)\right| \\
&\quad \times \frac{d^{\frac{1}{4}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))} \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{1}{|A|} + \frac{1}{h}\right)\Big|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)}\left|k{\mathbf{w}_\nu}^{k-1}\left(\overline{\mathbf{w}_\nu}^k - \overline{\mathbf{z}}^k\right)\right| \\
&\quad \times \frac{d^{\frac{1}{4}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}
\end{aligned}$$

$$\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\,\frac{1}{h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left|k{\mathbf{w}_\nu}^{k-1}\left(\overline{\mathbf{w}_\nu}^k-\overline{\mathbf{z}}^k\right)\right|$$
$$\times\,\frac{d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}$$

$$\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{\rho^{k-1}((\mathbf{z},t),(\mathbf{w}_0,s_0))h^{\frac12}((\mathbf{z},t),(\mathbf{w}_0,s_0))}{h((\mathbf{z},t),(\mathbf{w}_0,s_0))}$$
$$\times\,\frac{d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}$$

$$= \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{\rho^{\frac{k-1}{2}}((\mathbf{z},t),(\mathbf{w}_0,s_0))}{d^{\frac14}((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}$$

$$= \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac14}.$$

Moreover, we have

$$\frac{\partial\widetilde{S}}{\partial\overline{\mathbf{w}'}}\bigg|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)}$$
$$=\left(\frac{\partial S}{\partial\overline{\mathbf{w}}}-ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial S}{\partial s}\right)\bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}$$
$$=\frac{1}{4\pi^2}\bigg\{-\frac{k+1}{k}A^{-\frac{k+1}{k}-1}(1-\mathcal{P})^{-2}\left(\frac{\partial A}{\partial\overline{\mathbf{w}}}-ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial A}{\partial s}\right)\bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}$$
$$+2A^{-\frac{k+1}{k}}(1-\mathcal{P})^{-3}\left(\frac{\partial\mathcal{P}}{\partial\overline{\mathbf{w}}}-ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial\mathcal{P}}{\partial s}\right)\bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}\bigg\}$$
$$=S\bigg\{\frac{C_1}{A}\left(\frac{\partial A}{\partial\overline{\mathbf{w}}}-ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial A}{\partial s}\right)$$
$$+\frac{C_2}{1-\mathcal{P}}\left(\frac{\partial\mathcal{P}}{\partial\overline{\mathbf{w}}}-ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial\mathcal{P}}{\partial s}\right)\bigg\}\bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}.$$

Hence,

$$\left|\frac{\partial\widetilde{S}}{\partial\overline{\mathbf{w}'}}\bigg|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)}(\overline{\mathbf{w}'_1}-\overline{\mathbf{w}'_0})\right|$$
$$=\left|\left(\frac{\partial S}{\partial\overline{\mathbf{w}}}-ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial S}{\partial s}\right)\bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}(\overline{\mathbf{w}_1}-\overline{\mathbf{w}_0})\right|$$

$$
\begin{aligned}
&= |S|\Bigg|\frac{C_1}{A}\left(\frac{\partial A}{\partial \overline{\mathbf{w}}} - ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial A}{\partial s}\right) \\
&\quad + \frac{C_2}{1-\mathcal{P}}\left(\frac{\partial \mathcal{P}}{\partial \overline{\mathbf{w}}} - ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial \mathcal{P}}{\partial s}\right)\Bigg|\Bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} |\overline{\mathbf{w}_1} - \overline{\mathbf{w}_0}| \\
&\lesssim \left\{\frac{1}{d}\left|\frac{1}{A}\left(\frac{\partial A}{\partial \overline{\mathbf{w}}} - ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial A}{\partial s}\right)\right|\right\}\Bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} |\overline{\mathbf{w}_1} - \overline{\mathbf{w}_0}| \\
&\quad + \left\{\frac{1}{d}\left|\frac{1}{1-\mathcal{P}}\left(\frac{\partial \mathcal{P}}{\partial \overline{\mathbf{w}}} - ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial \mathcal{P}}{\partial s}\right)\right|\right\}\Bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} |\overline{\mathbf{w}_1} - \overline{\mathbf{w}_0}| \\
&=: I_1 + I_2.
\end{aligned}
$$

For I_1, by (39), (38), (31), (33), we obtain that

$$
\begin{aligned}
I_1 &\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left|\frac{k}{A}\overline{\mathbf{w}_\nu}^{k-1}(\mathbf{z}^k + \mathbf{w}_\nu^k)\right| |\overline{\mathbf{w}_1} - \overline{\mathbf{w}_0}| \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\frac{1}{|A|}|\mathbf{w}_\nu^{k-1}|(|\mathbf{z}|^k + |\mathbf{w}_\nu|^k)\ \frac{d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))} \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{\rho^{2k-1}((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho^{2k}((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))} \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{\rho^{k-1}((\mathbf{z},t),(\mathbf{w}_0,s_0))}{h^{\frac12}((\mathbf{z},t),(\mathbf{w}_0,s_0))}\cdot\frac{d^{\frac14}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))} \\
&\approx \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac14}.
\end{aligned}
$$

Also, note that

$$
\begin{aligned}
&\left(\frac{\partial \mathcal{P}}{\partial \overline{\mathbf{w}}} - ik\overline{\mathbf{w}}^{k-1}\mathbf{z}^k\frac{\partial \mathcal{P}}{\partial s}\right)\Bigg|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)} \\
&= \frac{\mathbf{z}}{A^{\frac1k}} - \left(\frac{\mathbf{z}\cdot\overline{\mathbf{w}}}{kA^{\frac1k+1}}\right)\left(k\overline{\mathbf{w}_\nu}^{k-1}\mathbf{w}_\nu^k - ik\overline{\mathbf{w}_\nu}^{k-1}\mathbf{z}_\nu^k\cdot i\right) \\
&= \frac{\mathbf{z}}{A^{\frac1k+1}}\left\{|\mathbf{z}|^{2k} + |\mathbf{w}_\nu|^{2k} - i(t-s_\nu) - |\mathbf{w}_\nu|^{2k} - \overline{\mathbf{w}_\nu}^k\mathbf{z}^k\right\} \\
&= \frac{\mathbf{z}}{A^{\frac1k+1}}\left\{\mathbf{z}^k(\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k) - i(t-s_\nu)\right\}
\end{aligned}
$$

$$= \frac{\mathbf{z}}{A^{\frac{1}{k}+1}} \Big\{ \mathrm{Re}\Big(\mathbf{z}^k (\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k) \Big) - i(t - s_\nu + 2\mathrm{Im}\ (\mathbf{z}^k \overline{\mathbf{w}_\nu}^k)) - i\mathrm{Im}\Big(\mathbf{z}^k (\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k) \Big) \Big\}.$$

By (31) and (38), we have

$$\begin{aligned} |\mathbf{z}^k (\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k)| &= |\mathbf{z}^k| |\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k| \\ &\lesssim |\mathbf{z}^k| h^{\frac{1}{2}}((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \\ &\lesssim \rho^k((\mathbf{z}, t), (\mathbf{w}_0, s_0)) h^{\frac{1}{2}}((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \end{aligned}$$

and that

$$|(t - s_\nu + 2\mathrm{Im}\mathbf{z}^k \overline{\mathbf{w}_\nu}^k)| = \sigma((\mathbf{z}, t), (\mathbf{w}_\nu, s_\nu)) \lesssim h((\mathbf{z}, t), (\mathbf{w}_0, s_0)).$$

Therefore, we can obtain that

$$\begin{aligned} I_2 &= \frac{1}{d((\mathbf{z}, t), (\mathbf{w}_\nu, s_\nu))} \bigg| \frac{1}{(1 - \mathcal{P})A} \frac{\mathbf{z}}{A^{\frac{1}{k}}} \Big\{ \mathrm{Re}\Big(\mathbf{z}^k (\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k) \Big) \\ &\qquad - i(t - s_\nu + 2\mathrm{Im}\mathbf{z}^k \overline{\mathbf{w}_\nu}^k) - i\mathrm{Im}\Big(\mathbf{z}^k (\overline{\mathbf{z}}^k - \overline{\mathbf{w}_\nu}^k) \Big) \Big\} \bigg| |\overline{\mathbf{w}_1} - \overline{\mathbf{w}_0}| \\ &\lesssim \frac{1}{d((\mathbf{z}, t), (\mathbf{w}_0, s_0))} \cdot \frac{d^{\frac{1}{4}}((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0))} \\ &\quad \times \frac{\rho((\mathbf{z}, t), (\mathbf{w}_0, s_0))}{h((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \rho^2((\mathbf{z}, t), (\mathbf{w}_0, s_0))} \\ &\quad \times \Big(h^{\frac{1}{2}}((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \rho^k((\mathbf{z}, t), (\mathbf{w}_0, s_0)) + h((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \Big) \\ &\lesssim \frac{1}{d((\mathbf{z}, t), (\mathbf{w}_0, s_0))} \cdot \frac{d^{\frac{1}{4}}((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0))} \\ &\quad \times \frac{\rho((\mathbf{z}, t), (\mathbf{w}_0, s_0))}{h((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \rho^2((\mathbf{z}, t), (\mathbf{w}_0, s_0))} \\ &\quad \times h^{\frac{1}{2}}((\mathbf{z}, t), (\mathbf{w}_0, s_0))\, \rho^k((\mathbf{z}, t), (\mathbf{w}_0, s_0)) \\ &= \frac{1}{d((\mathbf{z}, t), (\mathbf{w}_0, s_0))} \cdot \frac{d^{\frac{1}{4}}((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0))}{\rho^{\frac{k-1}{2}}((\mathbf{w}_1, s_1), (\mathbf{w}_0, s_0))} \cdot \frac{\rho^{k-1}((\mathbf{z}, t), (\mathbf{w}_0, s_0))}{h^{\frac{1}{2}}((\mathbf{z}, t), (\mathbf{w}_0, s_0))} \end{aligned}$$

$$\approx \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{4}}.$$

Moreover, note that

$$\begin{aligned}
&\left.\frac{\partial \widetilde{S}}{\partial s'}\right|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)} = \left.\frac{\partial S}{\partial s}\right|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}\\
&= \left.\left\{-\frac{k+1}{k}A^{-\frac{k+1}{k}-1}\frac{\partial A}{\partial s}(1-\mathcal{P})^{-2} + 2A^{-\frac{k+1}{k}}(1-\mathcal{P})^{-3}\frac{\partial \mathcal{P}}{\partial s}\right\}\right|_{(\mathbf{z},t;\mathbf{w}_\nu,s_\nu)}\\
&= -\frac{k+1}{k}\frac{i}{A^{\frac{1}{k}+2}(1-\mathcal{P})^2} - \frac{2}{k}\, i\, \frac{\mathbf{z}\overline{\mathbf{w}_\nu}}{A^{\frac{2}{k}+2}(1-\mathcal{P})^3}.
\end{aligned}$$

As a consequence, by (38), (39), (30), we further obtain that we obtain that

$$\begin{aligned}
&\left|\left.\frac{\partial \widetilde{S}}{\partial s'}\right|_{(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu)}(s'_1-s'_0)\right|\\
&= \frac{1}{4\pi^2}\left| -\frac{k+1}{k}\frac{i}{A^{\frac{1}{k}+2}(1-\mathcal{P})^2} - \frac{2}{k}\, i\, \frac{\mathbf{z}\overline{\mathbf{w}_\nu}}{A^{\frac{2}{k}+2}(1-\mathcal{P})^3}\right| |s'_1-s'_0|\\
&= \frac{1}{4\pi^2 A^{\frac{1}{k}+1}(1-\mathcal{P})^2}\left|\frac{k+1}{k}\frac{1}{A} + \frac{2}{k}\frac{\mathbf{z}\overline{\mathbf{w}_\nu}}{A^{\frac{1}{k}+1}(1-\mathcal{P})}\right| |s'_1-s'_0|\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{1}{|A|} + \frac{|\mathbf{z}\overline{\mathbf{w}_\nu}|}{|A|^{\frac{1}{k}}h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right) |s'_1-s'_0|\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{1}{h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right.\\
&\qquad \left. + \frac{\rho^2((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho^2((\mathbf{z},t),(\mathbf{w}_0,s_0))h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right) |s'_1-s'_0|\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\ \frac{1}{h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\ |s'_1-s'_0|\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\ \frac{1}{h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\big(h((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))\\
&\qquad + h^{\frac{1}{2}}((\mathbf{z},t),(\mathbf{w}_0,s_0))h^{\frac{1}{2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))\big)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{h((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{h((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{2}}\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{4}}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}.
\end{aligned}
$$

To sum up, we have

$$
\begin{aligned}
&\left|\left\langle \nabla_{\mathbf{w}',\overline{\mathbf{w}'},s'}\widetilde{S}(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu),\ (\mathbf{w}'_1-\mathbf{w}'_0,\overline{\mathbf{w}'_1}-\overline{\mathbf{w}'_0},s'_1-s'_0)\right\rangle\right|\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{4}}.
\end{aligned}
$$

From (30) we can see that $d((\mathbf{z},t),(\mathbf{w}_0,s_0))\lesssim \rho^{2k+2}((\mathbf{z},t),(\mathbf{w}_0,s_0))$. On the one hand, if

$$
d((\mathbf{z},t),(\mathbf{w}_0,s_0))\ll \rho^{2k+2}((\mathbf{z},t),(\mathbf{w}_0,s_0)),
$$

then

$$
h((\mathbf{z},t),(\mathbf{w}_0,s_0))\ll \rho^{2k}((\mathbf{z},t),(\mathbf{w}_0,s_0)).
$$

Therefore,

$$
|z-w_0|\ll \rho((\mathbf{z},t),(\mathbf{w}_0,s_0)),\quad |\sigma((\mathbf{z},t),(\mathbf{w}_0,s_0))|\ll \rho^{2k}((\mathbf{z},t),(\mathbf{w}_0,s_0)),
$$

which imply that

$$
\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))\approx |w_0|\le \rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0)).
$$

Combining with (35), in this case, we have

$$
\begin{aligned}
&\left|\left\langle \nabla_{\mathbf{w}',\overline{\mathbf{w}'},s'}\widetilde{S}(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu),\ (\mathbf{w}'_1-\mathbf{w}'_0,\overline{\mathbf{w}'_1}-\overline{\mathbf{w}'_0},s'_1-s'_0)\right\rangle\right|\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{4}}\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{4}}\\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{2k+2}}.
\end{aligned}
$$

On the other hand, if $d((\mathbf{z},t),(\mathbf{w}_0,s_0)) \approx \rho^{2k+2}((\mathbf{z},t),(\mathbf{w}_0,s_0))$, then by (33), we have

$$\begin{aligned}
&\left|\left\langle \nabla_{\mathbf{w}',\overline{\mathbf{w}'},s'}\widetilde{S}(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu),\ (\mathbf{w}'_1-\mathbf{w}'_0,\overline{\mathbf{w}'_1}-\overline{\mathbf{w}'_0},s'_1-s'_0)\right\rangle\right| \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{4}} \\
&= \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d^{\frac{1}{2k+2}}((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{\rho((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}} \\
&\quad\times\left(\frac{\rho((\mathbf{z},t),(\mathbf{w}_0,s_0))}{d^{\frac{1}{2k+2}}((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{k-1}{2}}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{2k+2}} \\
&\le \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{2k+2}}.
\end{aligned}$$

Consequently, for

$$d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0)) \le c d((\mathbf{z},t),(\mathbf{w}_0,s_0)),$$

with some small c, we have

$$\begin{aligned}
&|S(\mathbf{z},t;\mathbf{w}_1,s_1)-S(\mathbf{z},t;\mathbf{w}_0,s_0)| \\
&= \left|\widetilde{S}(\mathbf{z},t;\mathbf{w}'_1,s'_1)-\widetilde{S}(\mathbf{z},t;\mathbf{w}'_0,s'_0)\right| \\
&= \left|\int_0^1 \left\langle \nabla_{\mathbf{w}',\overline{\mathbf{w}'},s'}\widetilde{S}(\mathbf{z},t;\mathbf{w}'_\nu,s'_\nu),\ (\mathbf{w}'_1-\mathbf{w}'_0,\overline{\mathbf{w}'_1}-\overline{\mathbf{w}'_0},s'_1-s'_0)\right\rangle d\nu\right| \\
&\lesssim \int_0^1 \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{2k+2}} d\nu \\
&\lesssim \frac{1}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\left(\frac{d((\mathbf{w}_1,s_1),(\mathbf{w}_0,s_0))}{d((\mathbf{z},t),(\mathbf{w}_0,s_0))}\right)^{\frac{1}{2k+2}}.
\end{aligned}$$

This finishes the proof of Theorem 7.1. □

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