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Numerical solution of boundary value problem for the Bagley–Torvik equation using Hermite collocation method

Ayse Gul Kaplan and Mulla Veli Ablay

In this paper, the boundary value problem of Bagley–Torvik equation, which has an important place in fractional differential equations, is solved using the Hermite collocation method. Various definitions of fractional derivatives have been given in the literature but for boundary value problem, three different types of the mentioned equation were presented to show the accuracy and efficiency of the method. Obtained results were compared with exact solutions and some earlier results. It was seen that the presented method gave very high accuracy numerical results.

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1. Introduction

In several scientific disciplines, including dynamic systems, control theory, viscoelasticity, electrode-electrolyte polarization, electromagnetic waves, signal processing, diffusion wave and heat conduction, fractional differential equations are employed for modelling [1]. Fractional differential calculus technique helped to interpret these events by giving a new dimension to the mathematical approaches used to explain these models. The character of the event is now better understood, and some of the shortcomings of integer order differential equations in these occurrences are eliminated, thanks in large part to fractional differential equations. Bagley–Torvik equation has an important place in fractional differential equations.

The Bagley–Torvik equation, derived from modeling the motion of solid plates immersed in Newtonian fluid, was first proposed by Bagley and Torvik [2] and studied by Podlubny [1] and Trinks [3]. The Bagley–Torvik Equation with Caputo derivative is given as the following form;

$$P_4(x)y''(x) + P_3(x){}^{C}D^{3/2}y(x) + P_0(x)y(x) = g(x), \quad 0 \le x \le b \tag{1}$$

subject to the boundary conditions

$$y(0) = \lambda_0, \quad y(b) = \lambda_1 \tag{2}$$

where $P_4(x), P_3(x), P_0(x)$ and $g(x)$ are functions defined on the interval $0 \le x \le b$, ${}^{C}D^{3/2}$ is the derivative of y of order $3/2$ in the sense of Caputo fractional differential operator, b, λ_0, λ_1 are real constants, $y(x)$ is an unknown function of the independent variable x. The Bagley–Torvik equation plays a particularly important function in simulating the motion of a rigid plate submerged in a Newtonian fluid [4] and this equation is frequently encountered in various branches of applied mathematics and mechanics.

Many numerical methods have been developed to solve the Bagley–Torvik equation which are usually difficult to solve analytically. In the literature, the Bagley–Torvik equation has been solved using a variety of numerical techniques. These are a few of the techniques that were provided: a higher order numerical method [5], Lucas wavelet scheme [6], Legendre artificial neural network method [7], quadratic finite element method [8], via generalized Bessel polynomial on large domains [9], the bessel collocation method [10], simplified reproducing kernel method [11], fractional Taylor method [12], sinc operational matrix method [13], the generalized Taylor collocation method [14], hybridizable discontinuous Galerkin Methods [15], fractional linear multistep methods and a predictor-corrector method of Adams type [16], hybrid functions approximation [17], operational matrix of Haar wavelet operational method [18]. The asymptotic stability conditions were presented for the exact and discretized Bagley–Torvik equation in terms of its coefficients [19].

The main objective of this study is to numerically solve the boundary value problem of Bagley–Torvik equation, which has an important place in fractional differential equations using the Hermite Collocation Method (HCM). The suggested approach converts the aforementioned equations into linear algebraic systems with Hermite coefficients as the unknowns. It is easy to answer this algebraic problem since it may be stated by matrices and solved by matrix algebra. Thus, the numerical solutions of the fractional order linear equations are found in terms of truncated Hermite series.

2. Fractional calculus

Fractional calculus is an expansion of the integer order derivative and integral concepts of classical calculus to the real, rational, or complex order. Some fundamental topics in fractional calculus that are used throughout this article are provided in this part.

2.1. Gamma function

The generalization of the factorial for all real numbers is the simplest interpretation of the Gamma function, which is directly related to the fractional differential equation [20]. The Gamma Function or the second type of Euler integral is the generalized integral that depends on the parameter p and as defined

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \qquad \Gamma : (0, \infty) \longrightarrow \mathbb{R} \tag{3}$$

For $p > 0$, the gamma function is converging; for $p \leq 0$, it is diverging.

2.2. Beta function

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \qquad \beta : (0, \infty)x(0, \infty) \longrightarrow \mathbb{R} \tag{4}$$

the function defined by its integral is called the Beta Function or the Euler integral of the first kind. The beta function is convergent for $p > 0$ and $q > 0$, divergent for $p \leq 0$ and $q \leq 0$.

2.3. Caputo fractional derivative

The most important difference of fractional analysis from classical analysis is that there is no single derivative definition as in classical analysis. The existence of more than one derivative definition in fractional calculus gives the opportunity to use the most suitable one for the type of problem and thus to obtain the best solution of the problem. The main ones are Riemann-Liouville, Caputo, Grünwald-Letnikov, Weyl, Riesz and Marchaud fractional derivatives. Although there are transitions between each other, they differ in terms of their definitions and physical interpretations of their definitions. It is Caputo's definition of fractional derivative that gives the initial conditions most appropriate to the physical states [22]. The following form is given for the Caputo fractional derivative from the f's *alpha*th order;

$$^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_{a}^{x}(x-t)^{m-\alpha-1}f^{(m)}(t)dt \tag{5}$$

where f function can be continuously differentiable m times, α any positive integer and m is a positive integer such that $m \in \mathbb{N}$, $m-1 < \alpha < m$.

3. Hermite polynomials

The Hermite polynomials are a well-known series of orthogonal polynomials in mathematics. Hermite polynomials were defined in an elusive way by Pierre-Simon Laplace in 1810 [23] and studied in detail by Pafnuty Chebyshev in 1859 [24]. Chebyshev's work was ignored, and his name was later finalized after Charles Hermite [25], who wrote on polynomials in 1864 and described them as new.

One of the solutions of

$$y'' - 2xy' + 2ny = 0 \tag{6}$$

equation is $H_n(x)$ Hermite polynomials which are shown by

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}, \quad n = 0, 1, \ldots \tag{7}$$

where $-\infty < x < \infty$. Hermite polynomials are orthogonal in terms of $w(x) = e^{-x^2}$ weight function between interval of $(-\infty, \infty)$ and satisfy the relation of

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0 \tag{8}$$

for $m \neq n$ and $m, n \in \mathbb{N}_0$.

4. Implementation of the HCM

The approximate solutions to the truncated Hermite series form are given below [26]-[27]:

$$y(x) = \sum_{n=0}^{N} a_n H_n(x^{\alpha}) \tag{9}$$

of the Bagley–Torvik equation which is different form of equation (1)

$$\sum_{k=0}^{4} P_k(x)^C D^{k\alpha} y(x) = g(x), \quad 0 \le x \le b \tag{10}$$

with the boundary conditions (2)

$$y(0) = \lambda_0, \quad y(b) = \lambda_1 \tag{11}$$

Here, N can be any positive integer such that $N \ge 2$, $0 < \alpha \le 1$, and a_n are the unknown Hermite coefficients. The collocation points can be used as

$$x_i = a + \left(\frac{b-a}{N}\right) i, \quad i = 0, 1, \cdots, N, \quad a = 0. \tag{12}$$

to find a solution in form (9). The approximate solution $y(x)$ given by (9) can be expressed in matrix form as:

$$y(x) = H(x^\alpha) A \tag{13}$$

where $H(x^\alpha) = [H_0(x^\alpha) \ H_1(x^\alpha) \ \cdots \ H_N(x^\alpha)]$ and $A = [a_0 \ a_1 \ \cdots \ a_N]^T$. Hermite polynomials given in (7) according to the odd and even values of N, x^α instead of x can be written as follows matrix form.

If N is odd

$$\underbrace{\begin{bmatrix} H_0(x^\alpha) \\ H_1(x^\alpha) \\ \vdots \\ H_{N-1}(x^\alpha) \\ H_N(x^\alpha) \end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ o_1 & 0 & \cdots & 2^{N-1} & 0 \\ 0 & o_2 & \cdots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^\alpha \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^\alpha)}$$

where $o_1 = (-1)^{\left(\frac{N-5}{2}\right)} \dfrac{2^0}{0!} \dfrac{(N-1)!}{\left(\frac{N-1}{2}\right)!}$ and $o_2 = (-1)^{\left(\frac{N-1}{2}\right)} \dfrac{2^1}{1!} \dfrac{N!}{\left(\frac{N-1}{2}\right)!}$.

If N is even

$$\underbrace{\begin{bmatrix} H_0(x^\alpha) \\ H_1(x^\alpha) \\ \vdots \\ H_{N-1}(x^\alpha) \\ H_N(x^\alpha) \end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & e_1 & \cdots & 2^{N-1} & 0 \\ e_2 & 0 & \cdots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^\alpha \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^\alpha)}.$$

where $e_1 = (-1)^{\left(\frac{N-2}{2}\right)} \dfrac{2^1}{1!} \dfrac{(N-1)!}{\left(\frac{N-2}{2}\right)!}$ and $e_2 = (-1)^{\left(\frac{N-4}{2}\right)} \dfrac{2^0}{0!} \dfrac{N!}{\left(\frac{N}{2}\right)!}$. The given matrix form is briefly expressed as

$$H^T(x^\alpha) = FX^T(x^\alpha) \tag{14}$$

or

$$H(x^\alpha) = X(x^\alpha)F^T. \tag{15}$$

Substitution of equation (15) into (13) yields

$$y(x) = X(x^\alpha)F^T A. \tag{16}$$

Now, the $k\alpha$-th order Caputo fractional derivative of equation (16) is written as

$${}^C D^{k\alpha} y(x) = {}^C D^{k\alpha} X(x^\alpha) F^T A \tag{17}$$

or equivalently:

$${}^C D^{k\alpha} X(x^\alpha) = X(x^\alpha)(B^T)^k \tag{18}$$

where the definition of matrix B is as follows:

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Gamma(\alpha+1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & \dfrac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} & 0 \end{bmatrix}$$

If, we substitute equation (18) into equation (17) we have:

$$ {}^{C}D^{k\alpha}y(x) = X(x^{\alpha})(B^{T})^{k}F^{T}A. \tag{19} $$

For the collocation points $x = x_i, \quad i = 0, 1, 2, \cdots, N$, equation (10) is rewritten as follows

$$ \sum_{k=0}^{m} P_k(x_i)^{C}D^{k\alpha}y(x_i) = g(x_i). \tag{20} $$

Equation (20)'s matrix form is provided as follows

$$ \underbrace{\begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}}_{P_k} \underbrace{\begin{bmatrix} {}^{C}D^{k\alpha}y(x_0) \\ {}^{C}D^{k\alpha}y(x_1) \\ \vdots \\ {}^{C}D^{k\alpha}y(x_N) \end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}}_{G}. $$

Therefore, the matrix form is equivalent to

$$ \sum_{k=0}^{m} P_k Y^{k\alpha} = G. \tag{21} $$

In equation (19), $x = x_i$ is written for obtaining of the matrix $Y^{k\alpha}$, equation (22) is found by

$$ {}^{C}D^{k\alpha}y(x_i) = X(x_i^{\alpha})(B^{T})^{k}F^{T}A. \tag{22} $$

This equation's matrix form can be expressed as follows

$$ \underbrace{\begin{bmatrix} {}^{C}D^{k\alpha}y(x_0) \\ {}^{C}D^{k\alpha}y(x_1) \\ \vdots \\ {}^{C}D^{k\alpha}y(x_N) \end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} X(x_0^{\alpha}) \\ X(x_2^{\alpha}) \\ \vdots \\ X(x_N^{\alpha}) \end{bmatrix}}_{X^{\alpha}} \left[(B^{T})^{k}F^{T}A\right]. \tag{23} $$

Equation (23) is rewritten as

$$ Y^{k\alpha} = X^{\alpha}(B^{T})^{k}F^{T}A. \tag{24} $$

Then, by writing equation (24) in equation (21), the following equation is obtained

$$\sum_{k=0}^{m} P_k X^{\alpha} (B^T)^k F^T A = G. \tag{25}$$

In addition, denoting

$$W = [w_{pq}] = \sum_{k=0}^{m} P_k X^{\alpha} (B^T)^k F^T, \quad p, q = 0, 1, 2, \cdots, N \tag{26}$$

equation (25) is briefly written by

$$WA = G \tag{27}$$

The augmented matrix of equation (27) is of type $(N+1) \times (N+1)$ and is represented as follows:

$$[W; G] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & g(x_{N-1}) \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(x_N) \end{bmatrix} \tag{28}$$

Now, the new form of equation (27) must be established under the specified boundary conditions (11).

$$y(0) = X^{\alpha}(0) B^T F^T A \;\text{ and }\; y(b) = X^{\alpha}(b) B^T F^T A. \tag{29}$$

The matrix form of equation (29) is written as

$$U_j A = \lambda_j \;\text{ or }\; [U_j; \lambda_j], \quad j = 0, 1 \tag{30}$$

where

$$U_0 = X^{\alpha}(0) B^T F^T \equiv [u_{00} \;\; u_{01} \;\; u_{02} \;\; \cdots \;\; u_{0N}] \tag{31}$$

and

$$U_1 = X^{\alpha}(b) B^T F^T \equiv [u_{10} \;\; u_{11} \;\; u_{12} \;\; \cdots \;\; u_{1N}]. \tag{32}$$

The augmented matrix obtained by deleting the last two rows of the matrix (28) and adding the condition matrix is of type $(N+1)\times(N+1)$ and is as follows:

$$
(33) \qquad [\widetilde{W};\widetilde{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-2)0} & w_{(N-2)1} & \cdots & w_{(N-2)N} & ; & g(x_{N-2}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \end{bmatrix}
$$

This augmented matrix is briefly written as

$$
(34) \qquad \widetilde{W}A = \widetilde{G}
$$

If $det(\widetilde{W}) \neq 0$, Hermite coefficients matrix A, which is the solution of equation (34) is determined as follows:

$$
(35) \qquad A = (\widetilde{W})^{-1}\widetilde{G}.
$$

Finally, the required solution of the form is obtained by substituting these coefficients into the truncated Hermite series:

$$
(36) \qquad y(x) = \sum_{n=0}^{N} a_n H_n(x^{\alpha}).
$$

5. Numerical examples and comparisons

The Bagley–Torvik Equation's boundary value problem was solved for three different situations in this section. The resulting solutions were compared with exact solutions and a few numerical solutions from the literature in order to demonstrate the efficacy and correctness of the proposed method. Utilizing MatlabR2009b, all numerical computations were carried out.

5.1. Example 1

As the first example, the following boundary value problem of the Bagley–Torvik equation was solved

$$
D^2y(x) +^{C} D^{\frac{3}{2}}y(x) + y(x) = x^3 + 5x + \frac{8x^{3/2}}{\sqrt{\pi}}, \quad x \in [0,1]
$$

Table 1: Comparison of HCM solution with exact solution for Example 1

x	HCM	Exact Solution	SRKM[11]	Absolute Error
0.1	-0.0990000000005	-0.0990000000000	-0.098986845	4.6704E-13
0.2	-0.1920000000004	-0.1920000000000	-0.191526228	3.8591E-13
0.3	-0.2730000000003	-0.2730000000000	-0.272291301	2.8599E-13
0.4	-0.3360000000002	-0.3360000000000	-0.335193435	1.9679E-13
0.5	-0.3750000000001	-0.3750000000000	-0.374195895	1.2257E-13
0.6	-0.3840000000001	-0.3840000000000	-0.383275259	6.6502E-14
0.7	-0.3570000000000	-0.3570000000000	-0.356413889	2.5480E-14
0.8	-0.2880000000000	-0.2880000000000	-0.287596280	5.6621E-15
0.9	-0.1710000000000	-0.1710000000000	-0.170805515	4.5519E-15

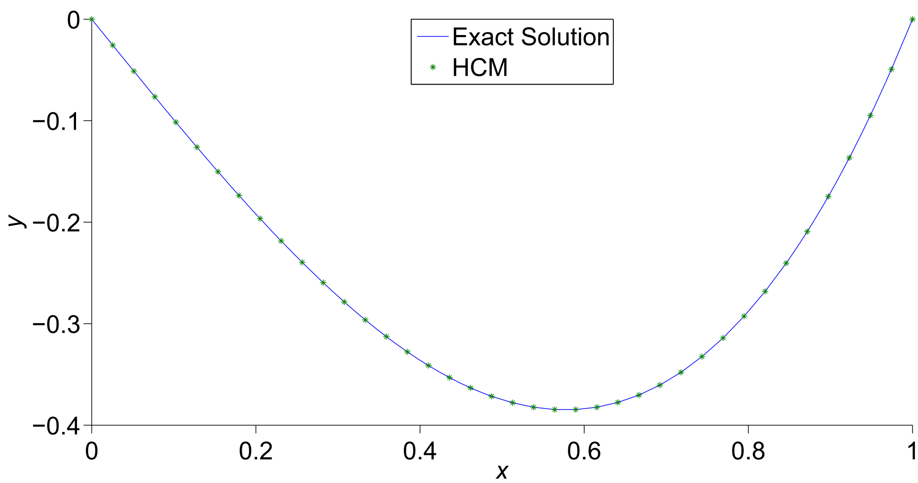

Figure 1: Graphs of exact and approximate solution graphs for Example 1.

$$y(0) = 0, \quad y(1) = 0.$$

The exact solution of this problem was given by $y(x) = x^3 - x$. The approximate solution of this problem was obtained by applying the HCM for $N = 6$ and $\alpha = 0.5$. In Table 1, the obtained results were compared with exact solutions and result of Simplified Reproducing Kernel Method (SRKM) [11], also the absolute error is calculated for Example 1. In [11], SRKM using the properties of the new kernel space created by the Reproducing Kernel Hilbert Space Method and the Simplified Kernel Multiplication Method compatible with Bagley–Torvik limit values. In Figure 1, graphs of exact and approximate solution were depicted for Example 1. Graph of absolute error was shown in Figure 2. It can be said that the presented method gives numerical results with very high accuracy.

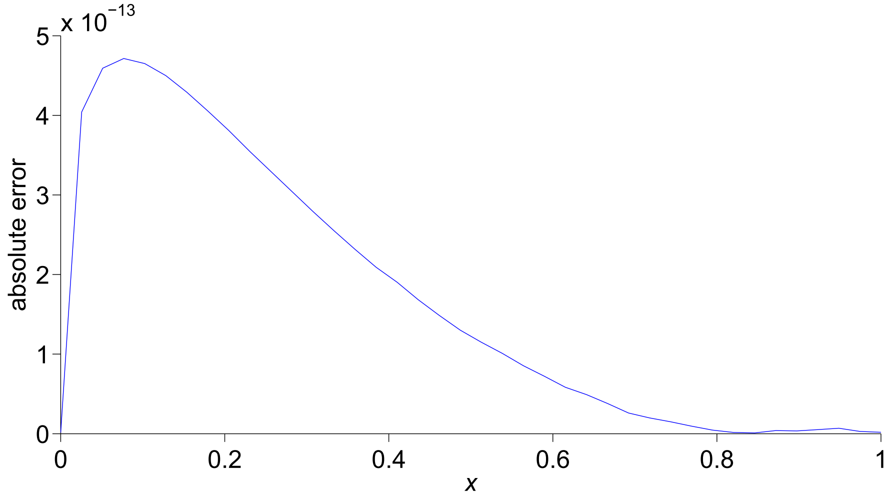

Figure 2: Graphs of absolute error for Example 1.

5.2. Example 2

As the second example, the following boundary value problem of the Bagley–Torvik equation was solved

$$^{C}D^{\frac{3}{2}}y(x) + y(x) = x^4 - 8x + \frac{64x^{5/2}}{5\sqrt{\pi}}, \quad x \in [0,2]$$

$$y(0) = 0, \quad y(2) = 0.$$

The exact solution of this problem was given by $y(x) = x^4 - 8x$. The approximate solution of this problem was obtained by applying the HCM for $N = 8$ and $\alpha = 0.5$. In Table 2, the obtained results were compared with exact solutions and results of simplified reproducing kernel method [11], also the absolute error was calculated. In Figure 3, graphs of exact and approximate solution was depicted for Example 2. Graph of absolute error was shown for Example 2 in Figure 4. It can be said that the presented method gives numerical results with very high accuracy.

Table 2: Comparison of HCM solution with exact solution

x	HCM	Exact Solution	SRKM[11]	Absolute Error
0.2	-1.5984000000288	-1.598400000000	-0.60136047	2.8754E-11
0.4	-3.1744000000344	-3.174400000000	-3.17998495	3.4392E-11
0.6	-4.6704000000350	-4.670400000000	-4.67817629	3.4971E-11
0.8	-5.9904000000327	-5.990400000000	-5.99986988	3.2657E-11
1	-7.0000000000285	-7.000000000000	-7.01062878	2.8505E-11
1.2	-7.5264000000232	-7.526400000000	-7.53763082	2.3225E-11
1.4	-7.3584000000174	-7.358400000000	-7.36970130	1.7351E-11
1.6	-6.2464000000114	-6.246400000000	-6.25699104	1.1394E-11
1.8	-3.9024000000055	-3.902400000000	-3.91298649	5.4818E-12

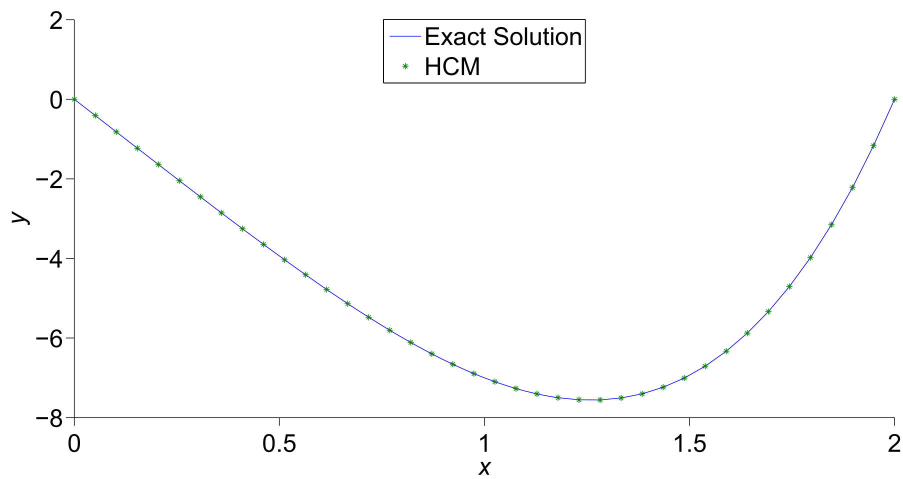

Figure 3: Graphs of exact and approximate solution graphs for Example 2.

5.3. Example 3

As the third example, the following boundary value problem of the Bagley–Torvik equation was solved

$$^{C}D^{\frac{3}{2}}y(x) + y(x) = \frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x, \quad x \in [0,1]$$

$$y(0) = 0, \quad y(1) = 0.$$

The exact solution of this problem was given by $y(x) = x^2 - x$. The approximate solution of this problem was obtained by applying the HCM for $N = 4$ and $\alpha = 0.5$. In Table 3, the obtained results were compared with

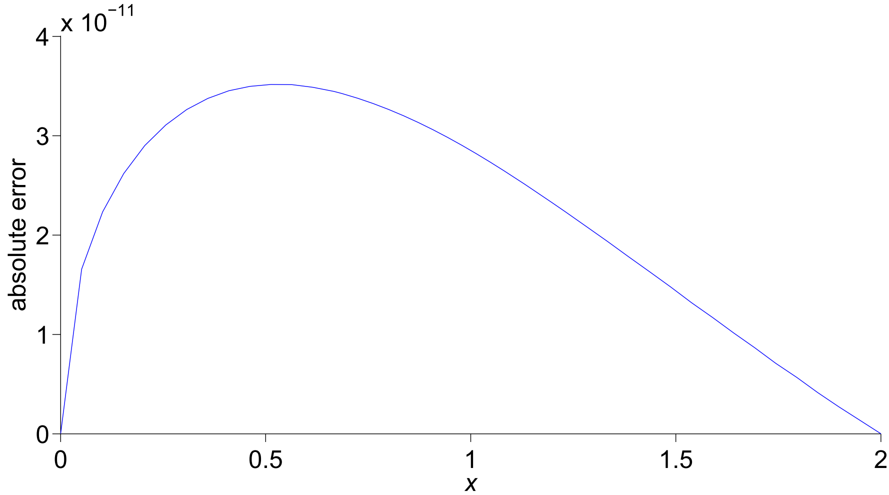

Figure 4: Graphs of absolute error for Example 2.

Table 3: Comparison of HCM solutions with exact solutions for Example 2

x	HCM	Exact Solution	Absolute Error	BCM[10]
0.2	-0.1600000000000	-0.1600000000000	4.9960E-16	2.2037E-14
0.4	-0.2400000000000	-0.2400000000000	4.9960E-16	2.0808E-14
0.6	-0.2400000000000	-0.2400000000000	4.4408E-16	1.5371E-14
0.8	-0.1600000000000	-0.1600000000000	4.4409E-16	8.0494E-15

exact solutions, also the calculated absolute errors were compared with results of Bessel Collocation Method (BCM) [10]. In [10], BCM transforms Bessel functions of the first kind into systems of linear equations that can be solved by generalizing to obtain numerical solutions of Bagley–Torvik equations. On the other hand, HCM transformed hermite polynomials into systems of linear equations. In Figure 5, graphs of exact and approximate solution was depicted for Example 3. Graph of absolute error was shown for Example 3 in Figure 6. It can be said that the presented method gives numerical results with very high accuracy.

6. Conclusion

Boundary value problem of the Bagley–Torvik equation was solved numerically using Hermite collocation method. Three different types of the mentioned equation were presented to illustrate the accuracy of the method. The obtained numerical results showed that the HCM solutions demonstrate

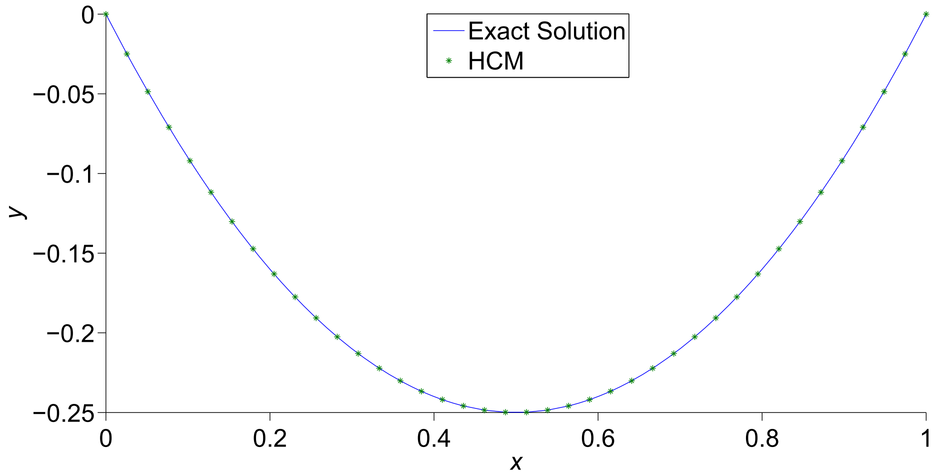

Figure 5: Graphs of exact and approximate solution graphs for Example 3.

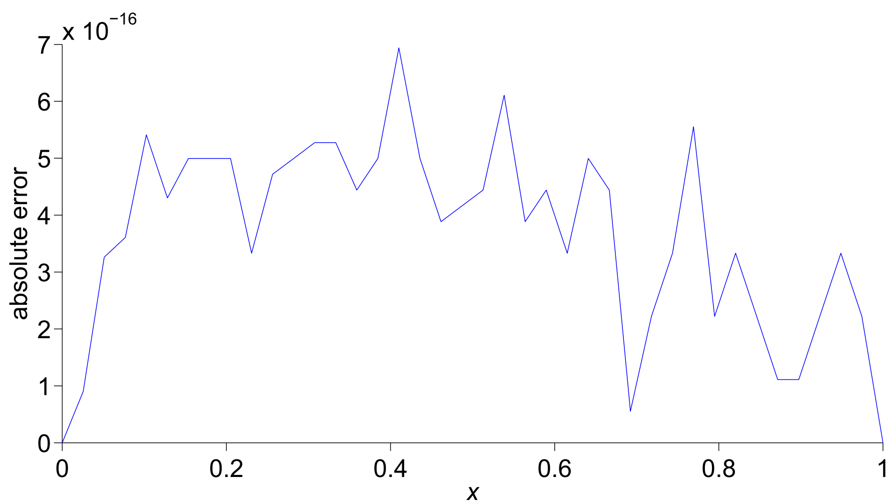

Figure 6: Graphs of absolute error for Example 3.

excellent approximations in comparison with the exact solutions. Consequently, this presented method has the simplicity of implementation and very high accuracy. Therefore, it can be said that the presented method is very effective numerical technique for the fractional order differential equations.

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Ayse Gul Kaplan
Osmaniye Korkut Ata University
Mathematics Department
80000 Osmaniye
Turkey
E-mail address: aysegulkaplan@osmaniye.edu.tr

Mulla Veli Ablay
Osmaniye Korkut Ata University
Graduate School of Sciences
80000 Osmaniye
Turkey
E-mail address: veliablay@gmail.com

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