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A note on moments of order statistics from the Pareto-Weibull distribution

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The order random variables have been widely used, because of their applications in different areas, such as sports, seismology, reliability, quality control, actuarial science, etc. In this paper, we consider the Pareto-Weibull distribution. We derive the exact expressions for moments and recurrence relations, which are useful for obtaining various statistical characteristics of the given distribution. Further, we calculate the moments at different values of the parameters.

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1. Introduction

Ordered Random variables are helpful in a variety of practical situations. Ordered random variables have been the focus of many scholars over the years owing to their wide range of applications; for instance, we may be interested in ordering the pricing of goods or the list of students according to their final test scores. Games also make use of ordered random variables when dealing with records and these variables are a logical option for dealing with severe events such as earthquakes, etc. The organization of data in ascending or descending order generates order statistics, which may be used to comprehend the features of the data, such as the maximum or lowest values of the data range, etc. The conventional aim of a statistician is a collection of n independent random variables $X_1, X_2, X_3, ..., X_n$, which, when ordered in ascending order of magnitude such that $X_{1:n} \leq X_{2:n} \leq ...... \leq X_{n:n}$, $X_{r:n}$ or $X_{(r)}$ is called the r^{th} order statistics in a sample of size n. In this case, it becomes clear that the well-developed theory of ordered random variables is an essential tool for dealing with observations $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ and distinct statistics based on $X_{r:n}$. Order statistics concern the characteristics and uses

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of these ordered random variables and their associated functions [1]. Several domains include statistical inference, actuarial science engineering, and quality control. The marginal and product moments derived from the order statistics are essential, David and Nagaraja (2004) [2], Joshi and Balakrishnan (1982) [3]. We refer to for a comprehensive examination of the moments of order statistics, Balakrishnan and Sultan [4]. Computing moments of order statistics for complex distributions is not a simple process. In this regard, recursive computational algorithms are thus simpler to implement. Identities and moment relations are essential for calculating higher-order moments since they minimize computation time and display a generic shape. Various recurrence relations of single and product moments and higher order moments of order statistics are obtained for example: Pareto distribution [5], Symmetric Generalized Log-logistic distribution [6], Truncated Logistic distribution [7], Symmetric Triangular distribution [8], Moments of order statistics recurrence relations and identities, specific continuous distributions [9], Exponentiated Burr XII distribution [10], Power Lindley distribution [11], Lindley distribution [12], Rayleigh and Weibull distributions [13], Length-biased Exponential distribution [14], Transmuted Power Function Distribution [15], Topp-Leone distribution [16], Half Logistic-Truncated Exponential distribution [17]. The pdf of r^{th} order statistics is given by

$$f_{r:n}(x) = \frac{n!}{(r-1!)(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), -\infty < x < \infty \tag{1}$$

The joint pdf of r^{th} and s^{th} order statistics $X_{(r)}$ and $X_{(s)}$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}f(x)[F(y)-F(x)]^{s-r-1} \times [1-F(y)]^{n-s}f(y), \quad x < y \tag{2}$$

The Pareto distribution is a well-known probability model for modeling and predicting various socioeconomic factors. Numerous applications of the Pareto distribution have been researched in actuarial sciences, economics, finance, life testing, climatology, biology, and physics. Although the distribution has various applications, income distribution research is one of the most significant and arguably essential. In his private economic works, Pareto (1897) introduced this term. Burroughs and Tebbens examine several uses of the distribution in modeling earthquakes, forest fire zones, and oil and gas field sizes (2001).

The Pareto Weibull Distribution was proposed by Saman Hanif et al. [18] and investigated a variety of statistical properties, including mean, variance, skewness, mgf, characteristic function, the quantile function, reliability analysis, and so on, and used various estimation methods, including the ML method, to estimate the unknown parameters of the distribution. Additional results and applications of the distribution are also provided.

The distribution is obtained by using the Pareto Weibull baseline distribution. The random variable X follows the Pareto Weibull distribution with pdf

$$f(x) = \frac{\alpha\theta}{\lambda^\theta \beta} x^{\theta-1} \left(1 + \frac{x^\theta}{\lambda^\theta \beta}\right)^{-\alpha-1}; \quad x > 0, \quad \alpha, \beta, \theta, \lambda > 0. \tag{3}$$

and the corresponding cdf is given by

$$F(x) = 1 - \left(1 + \frac{x^\theta}{\lambda^\theta \beta}\right)^{-\alpha}; \quad x > 0, \quad \alpha, \beta, \theta, \lambda > 0. \tag{4}$$

where β, λ are the scale parameters and α, θ are the shape parameters.

Using(3) and (4) we get the relation between cdf and pdf

$$\frac{\alpha\theta}{\lambda^\theta \beta} x^{\theta\,-1} \overline{F}(x) = \left(1 + \frac{x^\theta}{\lambda^\theta \beta}\right) f(x). \tag{5}$$

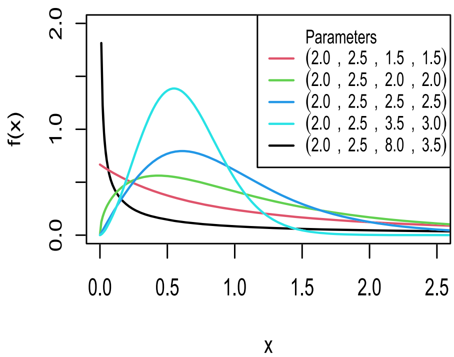
The reliability function for the Pareto-Weibull distribution is

$$R(t) = 1 - F(t) = \left(1 + \frac{t^\theta}{\lambda^\theta \beta}\right)^{-\alpha}.$$

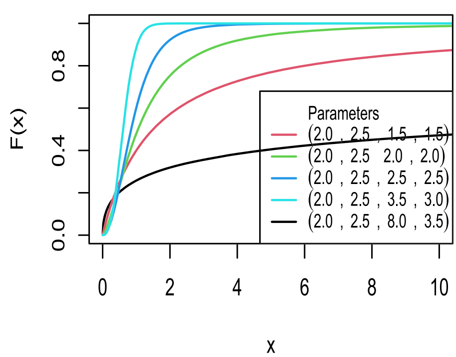
The Hazard function for the Pareto-Weibull distribution is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\alpha\theta}{\lambda^\theta \beta} t^{\theta-1} \left(1 + \frac{t^\theta}{\lambda^\theta \beta}\right)^{-1}.$$

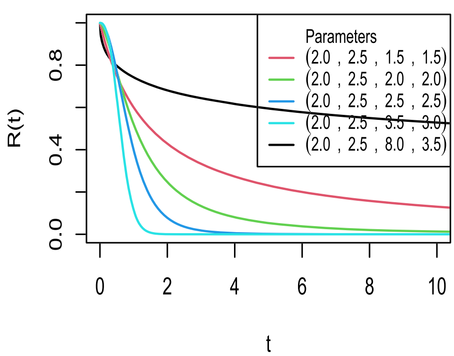
In this study, we have obtained the exact expressions for single, product, triple and quadrupole moments based on order statistics. Furthermore, we derived the recurrence relations for single and product moments, also obtained the moments and covariance at different values of parameters by using these relations. At last, we also reduce the results of order statistics for Pareto-exponential and Pareto-Rayleigh distribution.



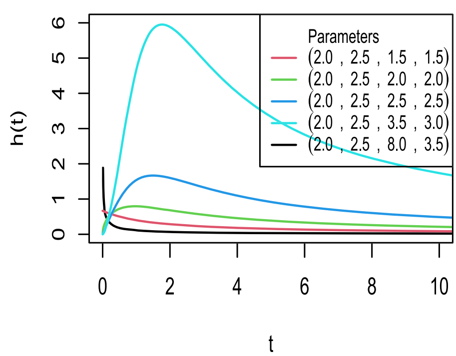
(a) The plot of pdf for P-W distribution



(b) The plot of cdf for P-W distribution



(c) The plot of reliability function for P-W distribution



(d) The plot of hazard function for P-W distribution

Figure (a), (b) displays various possible forms for the density and distribution functions of the Pareto-Weibull distribution. The suggested distribution can be shown in the Figure to be capable of capturing various dataset behaviours. Figures (c) and (d) above show the Pareto-Weibull distribution's reliability and hazard rate functions. The hazard rate function exhibits both increasing and decreasing behaviour, as can be shown in figure.

2. Single moments

Theorem 2.1 For the Pareto-Weibull distribution as given in (3) and $1 \leq r \leq n$, $\alpha\,,\beta\,,\lambda\,,\theta\, > 0$

$$E[X_{r:n}^j]=C_{r:n}\alpha\lambda^j\beta^{\frac{j}{\theta}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}B\left(\alpha(n-r+i+1)-\frac{j}{\theta},\frac{j}{\theta}+1\right). \quad (6)$$

Proof: Using binomial expansion in equation (1), then after that in view of equation (3) and (4), we get

$$E[X_{r:n}^j] = C_{r:n}\left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^\infty \frac{x^{j+\theta-1}}{\left(1+\frac{x^\theta}{\lambda^\theta\beta}\right)^{\alpha(n-r+i+1)+1}}dx.$$

Substituting $t = \frac{1}{1+\frac{x^\theta}{\lambda^\theta\beta}}$, we get

$$E[X_{r:n}^j] = C_{r:n}\alpha\lambda^j\beta^{\frac{j}{\theta}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^1 t^{\alpha(n-r+i+1)-\frac{j}{\theta}-1}(1-t)^{\frac{j}{\theta}}dt.$$

Now using the beta function, we get the expression as given in (6).

Special cases:
(1) For $r = 1$ in equation (6), we obtain an exact expression for single moments of the first order statistics, which is also denoted as sample minimum

$$E[X_{1:n}^j] = n\alpha\lambda^j\beta^{\frac{j}{\theta}}B\left(\alpha n-\frac{j}{\theta},\frac{j}{\theta}+1\right).$$

(2) In the other case, for sample maximum, putting $r = n$ in (6), we obtain the exact expression for single moments of the largest order statistics

$$E[X_{n:n}^j] = n\alpha\lambda^j\beta^{\frac{j}{\theta}}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}B\left(\alpha(i+1)-\frac{j}{\theta},\frac{j}{\theta}+1\right).$$

(3) If $r = n = j = 1$, we get

$$E(X) = \alpha\lambda\beta^{\frac{1}{\theta}}B\left(\alpha-\frac{1}{\theta},\frac{1}{\theta}+1\right).$$

Table 1: The first four moments and variances for the Pareto-Weibull Distribution

		$\alpha = 5$, $\beta = 0.5$, $\lambda = 0.5$, $\theta = 3.5$				
n	r	$E(X)$	$E\left(X^2\right)$	$E\left(X^3\right)$	$E\left(X^4\right)$	Variance
1	1	0.2422	0.0658	0.0196	0.0063	0.2379
2	1	0.1948	0.0421	0.0099	0.0025	0.0042
	2	0.2896	0.0895	0.0294	0.0102	0.0056
3	1	0.1724	0.0329	0.0068	0.0015	0.0032
	2	0.2396	0.0606	0.0161	0.0045	0.0032
	3	0.3147	0.1039	0.0360	0.0131	0.0049
4	1	0.1583	0.0277	0.0052	0.0011	0.0026
	2	0.2147	0.0485	0.0114	0.0028	0.0024
	3	0.2645	0.0727	0.0207	0.0061	0.0027
	4	0.3314	0.1144	0.0411	0.0154	0.0046
5	1	0.1482	0.0242	0.0043	8.0345e-04	0.0022
	2	0.1985	0.0414	0.0090	0.0020	0.0020
	3	0.2390	0.0591	0.0151	0.0040	0.0020
	4	0.2816	0.0817	0.0244	0.0075	0.0024
	5	0.3438	0.1225	0.0453	0.0174	0.0043
6	1	0.1405	0.0218	0.0036	6.4680e-04	0.0021
	2	0.1867	0.0366	0.0075	0.0016	0.0017
	3	0.2220	0.0510	0.0121	0.0029	0.0017
	4	0.2559	0.0673	0.0182	0.0050	0.0018
	5	0.2945	0.0890	0.0276	0.0088	0.0023
	6	0.3537	0.1292	0.0488	0.0191	0.0041
7	1	0.1343	0.0199	0.0032	5.3906e-04	0.0019
	2	0.1776	0.0331	0.0064	0.0013	0.0016
	3	0.2096	0.0454	0.0101	0.0023	0.0015
	4	0.2387	0.0584	0.0147	0.0038	0.0014
	5	0.2688	0.0739	0.0208	0.0060	0.0016
	6	0.3048	0.0950	0.0303	0.0099	0.0021
	7	0.3618	0.1349	0.0519	0.0206	0.0040
8	1	0.1292	0.0184	0.0028	4.6069e-04	0.0017
	2	0.1701	0.0303	0.0056	0.0011	0.0014
	3	0.1998	0.0412	0.0088	0.0019	0.0013
	4	0.2259	0.0523	0.0124	0.0030	0.0013
	5	0.2515	0.0646	0.0169	0.0045	0.0013
	6	0.2792	0.0795	0.0231	0.0068	0.0015
	7	0.3133	0.1002	0.0327	0.0109	0.0020
	8	0.3688	0.1399	0.0547	0.0220	0.0039
9	1	0.1249	0.0172	0.0025	4.0126e-04	0.0016
	2	0.1640	0.0282	0.0050	9.3608e-04	0.0013
	3	0.1918	0.0380	0.0077	0.0016	0.0012
	4	0.2158	0.0477	0.0108	0.0025	0.0011
	5	0.2385	0.0581	0.0144	0.0036	0.0012
	6	0.2618	0.0698	0.0189	0.0052	0.0013

Table 1: (Continued)

		$\alpha = 5,\ \beta = 0.5,\ \lambda = 0.5,\ \theta = 3.5$				
n	r	$E(X)$	$E\left(X^2\right)$	$E\left(X^3\right)$	$E\left(X^4\right)$	Variance
9	7	0.2878	0.0843	0.0251	0.0076	0.0015
	8	0.3206	0.1047	0.0348	0.0118	0.0019
	9	0.3748	0.1443	0.0571	0.0233	0.0038
10	1	0.1211	0.0162	0.0023	3.5475e-04	0.0015
	2	0.1587	0.0264	0.0046	8.1986e-04	0.0012
	3	0.1851	0.0353	0.0069	0.0014	0.0010
	4	0.2075	0.0441	0.0096	0.0021	0.0010
	5	0.2282	0.0531	0.0126	0.0030	0.0010
	6	0.2488	0.0630	0.0162	0.0042	0.0011
	7	0.2705	0.0744	0.0208	0.0059	0.0012
	8	0.2952	0.0886	0.0270	0.0084	0.0015
	9	0.3269	0.1087	0.0368	0.0127	0.0018
	10	0.3801	0.1482	0.0594	0.0245	0.0037

		$\alpha = 10,\ \beta = 0.5,\ \lambda = 0.5,\ \theta = 6$				
n	r	$E(X)$	$E\left(X^2\right)$	$E\left(X^3\right)$	$E\left(X^4\right)$	Variance
1	1	0.2844	0.0841	0.0258	0.0081	0.0032
2	1	0.2521	0.2521	0.0179	0.0050	0.1885
	2	0.3167	0.1023	0.0336	0.0113	0.0020
3	1	0.2352	0.0575	0.0145	0.0038	0.0022
	2	0.2858	0.0831	0.0246	0.0074	0.0014
	3	0.3321	0.1118	0.0382	0.0132	0.0015
4	1	0.2240	0.0521	0.0125	0.0031	0.0019
	2	0.2688	0.0735	0.0204	0.0058	0.0012
	3	0.3028	0.0928	0.0288	0.0090	0.0011
	4	0.3419	0.1182	0.0413	0.0146	0.0013
5	1	0.2157	0.0483	0.0112	0.0026	0.0018
	2	0.2571	0.0673	0.0179	0.0048	0.0012
	3	0.2863	0.0829	0.0243	0.0072	9.3231e-04
	4	0.3138	0.0994	0.0318	0.0102	9.2956e-04
	5	0.3489	0.1229	0.0437	0.0157	0.0012
6	1	0.2092	0.0454	0.0102	0.0023	0.0016
	2	0.2484	0.0627	0.0161	0.0042	9.9744e-04
	3	0.2747	0.0763	0.0214	0.0061	8.3991e-04
	4	0.2978	0.0894	0.0271	0.0083	7.1516e-04
	5	0.3218	0.1044	0.0341	0.0112	8.4476e-04
	6	0.3543	0.1266	0.0456	0.0166	0.0011
7	1	0.2038	0.0431	0.0094	0.0021	0.0016
	2	0.2413	0.0592	0.0148	0.0037	9.7431e-04
	3	0.2659	0.0715	0.0194	0.0053	7.9719e-04
	4	0.2865	0.0828	0.0241	0.0071	7.1775e-04
	5	0.3062	0.0945	0.0293	0.0092	7.4156e-04
	6	0.3280	0.1083	0.0360	0.0120	7.1600e-04
	7	0.3587	0.1297	0.0472	0.0173	0.0010

Table 1: (Continued)

n	r	$\alpha = 10,\ \beta = 0.5,\ \lambda = 0.5,\ \theta = 6$ $E(X)$	$E\left(X^2\right)$	$E\left(X^3\right)$	$E\left(X^4\right)$	Variance
8	1	0.1993	0.0412	0.0088	0.0019	0.0015
	2	0.2355	0.0564	0.0137	0.0034	9.3975e-04
	3	0.2588	0.0677	0.0179	0.0048	7.2256e-04
	4	0.2778	0.0778	0.0220	0.0062	6.2716e-04
	5	0.2952	0.0877	0.0262	0.0079	5.5696e-04
	6	0.3129	0.0985	0.0312	0.0099	5.9359e-04
	7	0.3331	0.1116	0.0376	0.0127	6.4439e-04
	8	0.3623	0.1322	0.0486	0.0180	9.3871e-04
9	1	0.1954	0.0396	0.0083	0.0018	0.0014
	2	0.2305	0.0540	0.0129	0.0031	8.6975e-04
	3	0.2529	0.0646	0.0167	0.0044	6.4159e-04
	4	0.2707	0.0739	0.0203	0.0056	6.2151e-04
	5	0.2866	0.0827	0.0240	0.0070	5.6044e-04
	6	0.3021	0.0918	0.0280	0.0086	5.3559e-04
	7	0.3183	0.1018	0.0328	0.0106	4.8511e-04
	8	0.3373	0.1144	0.0390	0.0134	6.2871e-04
	9	0.3655	0.1345	0.0498	0.0186	9.0975e-04
10	1	0.1920	0.0383	0.0079	0.0017	0.0014
	2	0.2262	0.0520	0.0122	0.0029	8.3356e-04
	3	0.2478	0.0620	0.0157	0.0040	5.9516e-04
	4	0.2647	0.0706	0.0190	0.0051	5.3391e-04
	5	0.2796	0.0787	0.0223	0.0064	5.2384e-04
	6	0.2936	0.0867	0.0257	0.0077	4.9904e-04
	7	0.3077	0.0951	0.0296	0.0092	4.2071e-04
	8	0.3228	0.1047	0.0341	0.0112	5.0016e-04
	9	0.3409	0.1168	0.0402	0.0139	5.8719e-04
	10	0.3682	0.1364	0.0509	0.0191	8.2876e-04

It is possible to verify the accuracy of the single moments of order statistics in equation (6) by applying $E\left(\sum_{r=1}^{n} X_{r:n}^{j}\right) = nE(X)^j$, given by David and Nagaragja (2003).

2.1 Recurrence relations

Theorem 2.2 For the distribution as given (3), we have for $r, n, i \in N$ and $r, n > 2$,

$$E[X_{r:n}^{j}] = \left(\frac{\alpha\theta}{\lambda^{\theta}\beta}\right) \sum_{v=1}^{\infty} (-1)^{v-1} \left(\frac{1}{\lambda^{\theta}\beta}\right)^{v-1} \left(\frac{v-r+1}{\theta v + j}\right) \times \left(E[X_{r:n}^{\theta v+j}] - E[X_{r-1:n}^{\theta v+j}]\right). \tag{7}$$

Proof: From (1), (5) and using the result $(1+x)^{-1} = \sum_{v=1}^{\infty}(-1)^{v-1}x^{v-1}$, we have

$$E[X_{r:n}^{j}] = C_{r:n}\left(\frac{\alpha\theta}{\lambda^{\theta}\beta}\right)\sum_{v=1}^{\infty}(-1)^{v-1}\left(\frac{1}{\lambda^{\theta}\beta}\right)^{v-1} \times \int_{0}^{\infty} x^{\theta v+j-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx$$

(8) $$E[X_{r:n}^{j}] = C_{r:n}\left(\frac{\alpha\theta}{\lambda^{\theta}\beta}\right)\sum_{v=1}^{\infty}(-1)^{v-1}\left(\frac{1}{\lambda^{\theta}\beta}\right)^{v-1} I_1,$$

where,

$$I_1 = \int_{0}^{\infty} x^{\theta v+j-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

After simplifying I_1, we get

$$I_1 = \frac{n-r+1}{\theta v+j}\int_{0}^{\infty} x^{\theta v+j}[F(x)]^{r-1}f(x)[1-F(x)]^{n-r}dx - \frac{r-1}{\theta v+j}\int_{0}^{\infty} x^{\theta v+j}[F(x)]^{r-2}f(x)[1-F(x)]^{n-r+1}dx.$$

Substituting I_1 in equation (8) we get expression (7).

3. Product moments

Theorem 3.1 For the Pareto-Weibull distribution as given in (3) with $n \in N$, $1 \le r < s \le n$,

(9)
$$E[X_{r:n}^{j}Y_{s:n}^{l}] = C_{r,s:n}\left(\frac{\alpha^2\theta\lambda^{l+j}\beta^{\frac{l+j}{\theta}}}{(j+\theta)}\right)\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i} \times \binom{s-r-1}{t}B\left(\frac{j+l}{\theta}+2, \alpha b - \frac{l}{\theta}\right) \times {}_3F_2\left[\frac{j}{\theta}+1, 1-a\alpha+\frac{j}{\theta}, \frac{j+l}{\theta}+2; \frac{j}{\theta}+2, \frac{j}{\theta}+\alpha b+2; 1\right].$$

Proof: Using equation (2), (3) and expanding binomially, we get

$$E[X_{r:n}^j Y_{s:n}^l] = C_{r,s:n}\left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)^2 \sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}$$
$$\int_0^\infty\int_0^y \frac{x^{j+\theta-1}}{\left(1+\frac{x^\theta}{\lambda^\theta\beta}\right)^{\alpha(s-r-t+i)+1}} \times \frac{y^{l+\theta-1}}{\left(1+\frac{y^\theta}{\lambda^\theta\beta}\right)^{\alpha(n-s+t+1)+1}}dxdy$$

(10)

$$E[X_{r:n}^j Y_{s:n}^l] = C_{r,s:n}\left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)^2 \sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\phi_{j,l}(a,b).$$

Substituting, (s-r-t+i)=a and (n-s+t+1)=b in $\phi_{j,l}(a,b)$, we have

$$\phi_{j,l}(a,b) = \int_0^\infty\int_0^y \frac{x^{j+\theta-1}}{\left(1+\frac{x^\theta}{\lambda^\theta\beta}\right)^{\alpha a+1}} \times \frac{y^{l+\theta-1}}{\left(1+\frac{y^\theta}{\lambda^\theta\beta}\right)^{\alpha b+1}}dxdy. \tag{11}$$

Consider

$$B(y) = \int_0^y \frac{x^{j+\theta-1}}{\left(1+\frac{x^\theta}{\lambda^\theta\beta}\right)^{\alpha a+1}}dx,$$

substituting $1-t = \frac{1}{1+\frac{x^\theta}{\lambda^\theta\beta}}$ and simplifying, we get

$$B(y) = \frac{\lambda^{j+\theta}\beta^{\frac{j}{\theta}+1}}{\theta} B_{\frac{\frac{y^\theta}{\lambda^\theta\beta}}{1+\frac{y^\theta}{\lambda^\theta\beta}}}\left(\frac{j}{\theta}+1, \alpha a - \frac{j}{\theta}\right). \tag{12}$$

From equation (11),

(13)

$$\phi_{j,l}(a,b) = \frac{\lambda^{j+\theta}\beta^{\frac{j}{\theta}+1}}{\theta}\int_0^\infty \frac{y^{l+\theta-1}}{\left(1+\frac{y^\theta}{\lambda^\theta\beta}\right)^{\alpha b+1}} B_{\frac{\frac{y^\theta}{\lambda^\theta\beta}}{1+\frac{y^\theta}{\lambda^\theta\beta}}}\left(\frac{j}{\theta}+1, \alpha a - \frac{j}{\theta}\right)dy.$$

Where $B_x(p,q) = \int_0^x u^{p-1}(1-u)^{q-1}du$, we know that

$$B_x(p,q) = p^{-1}x^p {}_2F_1(p, 1-q, p+1, x) \tag{14}$$

and

$$\int_0^1 u^{a-1}(1-u)_2^{b-1}F_1(c,d,e,u)du = B(a,b)_3F_2(c,d,a;e,a+b;1). \tag{15}$$

Substituting equation (14) and (15) in (13), we get

$$\begin{aligned}\phi_{j,l}(a,b) = {} & \frac{\lambda^{j+\theta}\beta^{\frac{j}{\theta}+1}}{\theta}\int_0^\infty \frac{y^{l+\theta-1}}{\left(1+\frac{y^\theta}{\lambda^\theta\beta}\right)^{\alpha b+1}}\left(\frac{\frac{y^\theta}{\lambda^\theta\beta}}{1+\frac{y^\theta}{\lambda^\theta\beta}}\right)^{\frac{j}{\theta}+1}\left(\frac{j}{\theta}+1\right)^{-1} \\ & \times {}_2F_1\left[\frac{j}{\theta}+1, 1-a\alpha+\frac{j}{\theta},; \frac{j}{\theta}+2; \left(\frac{\frac{y^\theta}{\lambda^\theta\beta}}{1+\frac{y^\theta}{\lambda^\theta\beta}}\right)\right]dy.\end{aligned} \tag{16}$$

Setting $t = \frac{\frac{y^\theta}{\lambda^\theta\beta}}{1+\frac{y^\theta}{\lambda^\theta\beta}}$ in (16), we get

$$\begin{aligned}\phi_{j,l}(a,b) = {} & \frac{\lambda^{2\theta+l+j}\beta^{\frac{l+j}{\theta}+2}}{\theta(j+\theta)}\int_0^1 t^{\frac{j+l}{\theta}+1}(1-t)^{\alpha b-\frac{l}{\theta}-1} \\ & \times {}_2F_1\left[\frac{j}{\theta}+1, 1-a\alpha+\frac{j}{\theta}; \frac{j}{\theta}+2; t\right]dt\end{aligned}$$

$$\begin{aligned}\phi_{j,l}(a,b) = {} & \frac{\lambda^{2\theta+l+j}\beta^{\frac{l+j}{\theta}+2}}{\theta(j+\theta)}B\left(\frac{j+l}{\theta}+2, \alpha b-\frac{l}{\theta}\right) \\ & \times {}_3F_2\left[\frac{j}{\theta}+1, 1-a\alpha+\frac{j}{\theta}, \frac{j+l}{\theta}+2; \frac{j}{\theta}+2, \frac{j}{\theta}+\alpha b+2; 1\right].\end{aligned} \tag{17}$$

Substituting (17) in (10), we get (9).

Using the following identity, the accuracy of the product moments of order statistics in equation (9) may be verified

$$\sum_{r=1}^{r-1}\sum_{s=r+1}^{s-r-1} E[X_{r,s:n}] = \binom{n}{2}[E(X)]^2.$$

Table 2: Variances and covariances of order statistics for $n = 2, 3, 4, ...10$

$\alpha = 5,\ \beta = 0.5,\ \lambda = 0.5,\ \theta = 3.5$					$\alpha = 10,\ \beta = 0.5,\ \lambda = 0.5,\ \theta = 6$				
s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$	s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
2	1	2	0.0587	0.0023	2	1	2	0.0809	0.0011
		3	0.0431	0.0018			3	0.0682	9.7984e-04
		4	0.0355	0.0015			4	0.0611	8.8880e-04
		5	0.0307	0.0013			5	0.0563	8.4353e-04
		6	0.0274	0.0012			6	0.0528	8.3472e-04
		7	0.0249	0.0010			7	0.0500	8.2306e-04
		8	0.0230	0.0010			8	0.0477	7.6485e-04
		9	0.0214	9.1640e-04			9	0.0458	7.6030e-04
		10	0.0201	8.8143e-04			10	0.0441	6.6960e-04
3	1	3	0.0554	0.0011	3	1	3	0.0786	4.9008e-04
		4	0.0429	0.0010			4	0.0684	5.7280e-04
		5	0.0363	8.8020e-04			5	0.0623	5.4509e-04
		6	0.0320	8.0900e-04			6	0.0580	5.3276e-04
		7	0.0289	7.5072e-04			7	0.0547	5.0958e-04
		8	0.0265	6.8584e-04			8	0.0521	5.2116e-04
		9	0.0246	6.4418e-04			9	0.0499	4.8334e-04
		10	0.0230	5.8439e-04			10	0.0480	4.2240e-04
	2	3	0.0775	0.0021		2	3	0.0957	7.8582e-04
		4	0.0585	0.0017			4	0.0822	8.0736e-04
		5	0.0488	0.0014			5	0.0743	6.9227e-04
		6	0.0427	0.0013			6	0.0689	6.6452e-04
		7	0.0383	0.0011			7	0.0648	6.3833e-04
		8	0.0350	0.0010			8	0.0616	6.5260e-04
		9	0.0324	9.4480e-04			9	0.0589	6.0655e-04
		10	0.0302	8.2463e-04			10	0.0566	5.4764e-04
4	1	4	0.0532	7.3938e-04	4	1	4	0.0769	3.1440e-04
		5	0.0424	6.6688e-04			5	0.0681	4.1334e-04
		6	0.0366	6.4605e-04			6	0.0627	4.0024e-04
		7	0.0327	6.4259e-04			7	0.0588	4.1130e-04
		8	0.0297	5.1372e-04			8	0.0557	3.3446e-04
		9	0.0275	5.4658e-04			9	0.0532	3.0522e-04
		10	0.0256	4.7175e-04			10	0.0512	3.7760e-04
	2	4	0.0723	0.0011		2	4	0.0924	4.9728e-04
		5	0.0570	0.0011			5	0.0812	5.2202e-04
		6	0.0487	9.2347e-04			6	0.0744	4.2648e-04
		7	0.0432	8.0688e-04			7	0.0696	4.6755e-04
		8	0.0392	7.7441e-04			8	0.0659	4.7810e-04
		9	0.0361	7.0880e-04			9	0.0628	4.0365e-04
		10	0.0336	6.6975e-04			10	0.0603	4.2486e-04
	3	4	0.0897	0.0020		3	4	0.1042	6.7268e-04
		5	0.0688	0.0015			5	0.0905	6.5906e-04
		6	0.0581	0.0013			6	0.0824	5.9434e-04

Table 2: (Continued)

$\alpha = 5$, $\beta = 0.5$, $\lambda = 0.5$, $\theta = 3.5$					$\alpha = 10$, $\beta = 0.5$, $\lambda = 0.5$, $\theta = 6$				
s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$	s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
4	3	7	0.0512	0.0012			7	0.0767	5.1965e-04
		8	0.0461	9.6518e-04			8	0.0724	5.0536e-04
		9	0.0423	9.0956e-04			9	0.0690	5.3997e-04
		10	0.0393	8.9175e-04			10	0.0661	5.0734e-04
5	1	5	0.0515	5.4884e-04	5	1	5	0.0755	2.4227e-04
		6	0.0419	5.2275e-04			6	0.0676	2.7944e-04
		7	0.0366	5.0016e-04			7	0.0627	2.9644e-04
		8	0.0329	4.0620e-04			8	0.0591	2.6664e-04
		9	0.0302	4.1135e-04			9	0.0563	2.9836e-04
		10	0.0280	3.6498e-04			10	0.0540	3.1680e-04
	2	5	0.0690	7.5570e-04		2	5	0.0900	2.9781e-04
		6	0.0557	7.1685e-04			6	0.0803	3.6488e-04
		7	0.0484	6.6112e-04			7	0.0743	4.1394e-04
		8	0.0434	6.1985e-04			8	0.0699	3.8040e-04
		9	0.0397	5.8600e-04			9	0.0664	3.3870e-04
		10	0.0368	5.8466e-04			10	0.0636	3.5448e-04
	3	5	0.0833	0.0011		3	5	0.1003	4.0993e-04
		6	0.0664	0.0010			6	0.0888	4.0154e-04
		7	0.0572	8.5952e-04			7	0.0819	4.8142e-04
		8	0.0511	8.5030e-04			8	0.0768	4.0224e-04
		9	0.0465	7.5570e-04			9	0.0729	4.1886e-04
		10	0.0430	7.6018e-04			10	0.0697	4.1512e-04
	4	5	0.0987	0.0019		4	5	0.1101	6.1518e-04
		6	0.0768	0.0014			6	0.0964	5.6796e-04
		7	0.0654	0.0012			7	0.0883	5.7370e-04
		8	0.0579	0.0011			8	0.0825	4.9344e-04
		9	0.0524	9.3170e-04			9	0.0780	4.1738e-04
		10	0.0482	8.4850e-04			10	0.0745	4.8988e-04
6	1	6	0.0501	4.0515e-04	6	1	6	0.0743	1.8044e-04
		7	0.0413	3.6536e-04			7	0.0671	2.5360e-04
		8	0.0364	3.2736e-04			8	0.0626	2.3903e-04
		9	0.0331	4.0118e-04			9	0.0592	1.6966e-04
		10	0.0305	3.7032e-04			10	0.0566	2.2880e-04
	2	6	0.0666	5.6421e-04		2	6	0.0882	1.9188e-04
		7	0.0547	5.6752e-04			7	0.0794	2.5360e-04
		8	0.0480	5.0808e-04			8	0.0739	2.1205e-04
		9	0.0434	4.6480e-04			9	0.0699	2.6595e-04
		10	0.0400	5.1544e-04			10	0.0667	2.8768e-04
	3	6	0.0793	7.7860e-04		3	6	0.0976	2.7379e-04
		7	0.0646	7.1392e-04			7	0.0875	2.8480e-04
		8	0.0565	7.1584e-04			8	0.0813	3.2148e-04
		9	0.0509	6.8676e-04			9	0.0767	2.9891e-04

Table 2: (Continued)

$\alpha = 5, \beta = 0.5, \lambda = 0.5, \theta = 3.5$					$\alpha = 10, \beta = 0.5, \lambda = 0.5, \theta = 6$				
s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$	s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
6	3	10	0.0467	6.4712e-04			10	0.0731	3.4592e-04
6	4	6	0.0917	0.0012		4	6	0.1059	3.8946e-04
		7	0.0737	9.4424e-04			7	0.0944	4.2800e-04
		8	0.0639	8.2872e-04			8	0.0873	3.7638e-04
		9	0.0573	8.0356e-04			9	0.0821	3.2153e-04
		10	0.0523	6.7400e-04			10	0.0781	3.8408e-04
	5	6	0.1060	0.0018		5	6	0.1146	5.8626e-04
		7	0.0833	0.0014			7	0.1010	5.6640e-04
		8	0.0713	0.0011	6	5	8	0.0928	4.3192e-04
		9	0.0634	9.6070e-04			9	0.0870	4.1814e-04
		10	0.0577	9.2384e-04			10	0.0825	4.0944e-04
7	1	7	0.0489	3.1026e-04	7	1	7	0.0733	1.9694e-04
		8	0.0408	3.2164e-04			8	0.0665	1.1317e-04
		9	0.0362	2.5378e-04			9	0.0624	2.0418e-04
		10	0.0331	3.4245e-04			10	0.0592	1.2160e-04
	2	7	0.0647	4.4432e-04		2	7	0.0867	1.4569e-04
		8	0.0537	4.0767e-04			8	0.0786	1.5495e-04
		9	0.0476	4.0080e-04			9	0.0736	2.3185e-04
		10	0.0433	3.7165e-04			10	0.0698	1.9826e-04
	3	7	0.0764	5.6672e-04		3	7	0.0956	2.2167e-04
		8	0.0632	6.0266e-04			8	0.0864	1.9372e-04
		9	0.0558	5.9996e-04			9	0.0807	2.0193e-04
		10	0.0506	5.3045e-04			10	0.0765	2.5194e-04
	4	7	0.0872	8.3834e-04		4	7	0.1030	2.3245e-04
		8	0.0715	7.2553e-04			8	0.0928	2.6482e-04
		9	0.0628	6.9276e-04			9	0.0865	3.3619e-04
		10	0.0568	6.7125e-04			10	0.0817	2.5181e-04
	5	7	0.0984	0.0011		5	7	0.1102	3.6606e-04
		8	0.0797	9.0505e-04			8	0.0987	3.6888e-04
		9	0.0695	8.5970e-04			9	0.0916	3.7522e-04
		10	0.0625	7.7190e-04			10	0.0864	3.6708e-04
	6	7	0.1120	0.0017		6	7	0.1182	5.4640e-04
		8	0.0888	0.0013			8	0.1047	4.7301e-04
		9	0.0764	0.0011			9	0.0966	4.4157e-04
		10	0.0682	8.9960e-04			10	0.0907	3.5928e-04
8	1	8	0.0479	2.5104e-04	8	1	8	0.0723	9.3610e-05
		9	0.0403	2.5706e-04			9	0.0660	9.1580e-05
		10	0.0360	2.5128e-04			10	0.0621	1.2240e-04
	2	8	0.0631	3.6712e-04		2	8	0.0855	1.7835e-04
		9	0.0529	3.2160e-04			9	0.0779	1.5235e-04
		10	0.0472	3.5176e-04			10	0.0732	1.8264e-04

Table 2: (Continued)

$\alpha = 5$, $\beta = 0.5$, $\lambda = 0.5$, $\theta = 3.5$					$\alpha = 10$, $\beta = 0.5$, $\lambda = 0.5$, $\theta = 6$				
s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$	s	r	n	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
8	3	8	0.0742	5.1376e-04		3	8	0.0940	2.3676e-04
		9	0.0620	5.0892e-04			9	0.0855	1.9683e-04
		10	0.0551	4.5848e-04			10	0.0802	2.1016e-04
	4	8	0.0839	5.8808e-04		4	8	0.1009	2.5306e-04
		9	0.0697	5.1452e-04			9	0.0915	1.9289e-04
		10	0.0618	5.4600e-04			10	0.0857	2.5484e-04
	5	8	0.0935	7.4680e-04		5	8	0.1072	2.4904e-04
		9	0.0772	7.3690e-04			9	0.0970	3.2982e-04
		10	0.0680	6.3536e-04			10	0.0905	2.4512e-04
	6	8	0.1040	0.0010		6	8	0.1137	3.3633e-04
		9	0.0849	9.6692e-04			9	0.1022	3.0167e-04
		10	0.0743	8.5424e-04			10	0.0951	3.2592e-04
	7	8	0.1172	0.0017	8	7	8	0.1212	5.1787e-04
		9	0.0935	0.0012			9	0.1078	4.3741e-04
		10	0.0809	0.0010			10	0.0997	3.7444e-04
9	1	9	0.0470	1.8748e-04	9	1	9	0.0715	8.1300e-05
		10	0.0398	2.1241e-04			10	0.0656	1.4720e-04
	2	9	0.0618	3.3280e-04		2	9	0.0844	1.5225e-04
		10	0.0522	3.2097e-04			10	0.0773	1.8842e-04
	3	9	0.0723	4.1336e-04		3	9	0.0926	1.6505e-04
		10	0.0609	3.9081e-04			10	0.0846	1.2498e-04
	4	9	0.0814	5.1816e-04		4	9	0.0991	1.5915e-04
		10	0.0683	4.6825e-04			10	0.0904	1.6377e-04
	5	9	0.0900	6.1020e-04		5	9	0.1050	2.4770e-04
		10	0.0752	6.0142e-04			10	0.0955	1.8436e-04
	6	9	0.0989	7.7736e-04		6	9	0.1106	1.8245e-04
		10	0.0820	6.6728e-04			10	0.1004	3.1176e-04
	7	9	0.1089	0.0010		7	9	0.1166	2.6135e-04
		10	0.0893	8.7355e-04			10	0.1052	3.0507e-04
	8	9	0.1218	0.0016		8	9	0.1237	4.1685e-04
		10	0.0977	0.0012			10	0.1105	4.5748e-04
10	1	10	0.0462	1.6989e-04	10	1	10	0.0708	1.0560e-04
	2	10	0.0606	2.7813e-04		2	10	0.0834	1.1316e-04
	3	10	0.0707	3.4349e-04		3	10	0.0914	1.6004e-04
	4	10	0.0793	4.2925e-04		4	10	0.0976	1.3746e-04
	5	10	0.0872	4.6118e-04		5	10	0.1031	1.5128e-04
	6	10	0.0952	6.3112e-04		6	10	0.1083	1.9648e-04
	7	10	0.1036	7.8295e-04		7	10	0.1135	2.0486e-04
	8	10	0.1133	0.0011		8	10	0.1192	3.4504e-04
	9	10	0.1259	0.0016		9	10	0.1260	4.8062e-04

3.1 Recurrence relations

Theorem 3.2 For the Pareto-Weibull distribution as given in (3) and $n \in N$, $1 \le r < s \le n$, we have

$$\begin{aligned} E[X^j_{r:n}Y^l_{s:n}] &= \left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)\sum_{v=1}^{\infty}(-1)^{v-1}\left(\frac{1}{\lambda^\theta\beta}\right)^{v-1}\left(\frac{1}{\theta v+j}\right) \\ &\times\left[n\left(E[X^{\theta v+j}_{r:n-1}Y^l_{s-1:n-1}] - E[X^{\theta v+j}_{r-1:n-1}Y^l_{s-1:n-1}]\right)\right. \\ &\left.+r\left(E[X^{\theta v+j}_{r:n}Y^l_{s:n}] - E[X^{\theta v+j}_{r:n}Y^l_{s:n}]\right)\right]. \end{aligned} \tag{18}$$

Proof: From equation (2) and (5), we have

$$\begin{aligned} E[X^j_{r:n}Y^l_{s:n}] = C_{r,s:n}&\left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)\sum_{v=1}^{\infty}(-1)^{v-1}\left(\frac{1}{\lambda^\theta\beta}\right)^{v-1} \\ &\times\left[\int_0^\infty\int_0^y x^{\theta v+j-1}y^l[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}\right. \\ &\times[1-F(y)]^{n-s}f(y)dxdy \\ &-\int_0^\infty\int_0^y x^{\theta v+j-1}y^l[F(x)]^{r}[F(y)-F(x)]^{s-r-1} \\ &\left.\times[1-F(y)]^{n-s}f(y)dxdy\right]. \end{aligned}$$

$$\begin{aligned} E[X^j_{r:n}Y^l_{s:n}] = C_{r,s:n}&\left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)\sum_{v=1}^{\infty}(-1)^{v-1}\left(\frac{1}{\lambda^\theta\beta}\right)^{v-1} \\ &\times\left[\int_0^\infty y^l[1-F(y)]^{n-s}f(y)I_1(y)dy\right. \\ &\left.-\int_0^\infty y^l[1-F(y)]^{n-s}f(y)I_2(y)dy\right] \end{aligned} \tag{19}$$

where

$$I_1(y) = \int_0^y x^{\theta v+j-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}dx$$

and

$$I_2(y) = \int_0^y x^{\theta v+j-1}[F(x)]^{r}[F(y)-F(x)]^{s-r-1}dx.$$

Integrating by parts, treating $x^{\theta u+j-1}$ for integration and the remaining integrand for differentiation, we get

$$I_1(y) = \frac{s-r-1}{\theta v+j}\int_0^y x^{\theta v+j}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}f(x)dx$$
$$-\frac{r-1}{\theta v+j}\int_0^y x^{\theta v+j}[F(x)]^{r-2}[F(y)-F(x)]^{s-r-1}f(x)dx.$$
$$I_2(y) = \frac{s-r-1}{\theta v+j}\int_0^y x^{\theta v+j}[F(x)]^{r}[F(y)-F(x)]^{s-r-2}f(x)dx$$
$$-\frac{r}{\theta v+j}\int_0^y x^{\theta v+j}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}f(x)dx.$$

On substituting $I_1(y)$ and $I_2(y)$ in (19), we get (18).

4. Triple moments

Theorem 4.1 For the distribution with pdf and cdf as given in (3) and (4), we have for $r, s, t = 1, 2, 3, 4, ...n$ and $r < s < t$

$$\mu^{j,l,m}_{r,s,t:n} = C_{r,s,t:n}\frac{\alpha^3\theta^2\lambda^{l+j+m}\beta^{\frac{l+j+m}{\theta}}}{(j+\theta)((j+l)+\theta(n_1+2))} \tag{20}$$
$$\times\sum_{i_1=0}^{r-1}\sum_{i_2=0}^{s-r-1}\sum_{i_3=0}^{t-s-1}\sum_{n_1=0}^{\infty}\frac{(\frac{j}{\theta}+1)_{n_1}(1-a\alpha+\frac{j}{\theta})_{n_1}}{(\frac{j}{\theta}+2)_{n_1}}\frac{1}{n_1!}$$
$$\times(-1)^{i_1+i_2+i_3}\binom{r-1}{i_1}\binom{s-r-1}{i_2}\binom{t-s-1}{i_3}$$
$$\times B\left(\frac{j+l+m}{\theta}+n_1+3, \alpha c-\frac{m}{\theta}\right)$$
$$\times{}_3F_2\left[\frac{j+l}{\theta}+n_1+2, 1-\alpha b+\frac{l}{\theta}, \frac{j+l+m}{\theta}+n_1+3;\right.$$
$$\left.\frac{j+l}{\theta}+n_1+3, \frac{j+l}{\theta}+\alpha c+n_1+3; 1\right].$$

Proof: The triple moments of order statistics $E[X^j_{r:n}Y^l_{s:n}Z^m_{t:n}]$ are given by

$$\mu^{j,l,m}_{r,s,t:n} = C_{r,s,t:n}\int_0^\infty\int_0^z\int_0^y x^j y^l z^m[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$
$$\times[F(z)-F(y)]^{t-s-1}[1-F(z)]^{n-t}f(x)f(y)f(z)dxdydz$$
$$r, s, t = 1, 2, 3...nr < s < t$$

where

$$C_{r,s,t:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!}.$$

From (3), (4) and using binomial expansion, we get

$$\begin{aligned}\mu_{r,s,t:n}^{j,l,m} &= C_{r,s,t:n}\left(\frac{\alpha\theta}{\lambda^\theta\beta}\right)^3 \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{s-r-1}\sum_{i_3=0}^{t-s-1}(-1)^{i_1+i_2+i_3}\binom{r-1}{i_1} \\ &\times \binom{s-r-1}{i_2}\binom{t-s-1}{i_3} I_2\end{aligned} \tag{21}$$

where

(22)

$$I_2 = \int_0^\infty \int_0^z \int_0^y \frac{x^{j+\theta-1}}{\left(1+\frac{x^\theta}{\lambda^\theta\beta}\right)^{\alpha a+1}} \times \frac{y^{l+\theta-1}}{\left(1+\frac{y^\theta}{\lambda^\theta\beta}\right)^{\alpha b+1}} \times \frac{z^{m+\theta-1}}{\left(1+\frac{z^\theta}{\lambda^\theta\beta}\right)^{\alpha c+1}} dxdydz$$

and $a = i_1 + i_2 + 1$, $b = s - r - i_2 + i_3$ and $c = n - s - i_3$.

Consider

$$B(z) = \int_0^z \int_0^y \frac{x^{j+\theta-1}}{\left(1+\frac{x^\theta}{\lambda^\theta\beta}\right)^{\alpha a+1}} \times \frac{y^{l+\theta-1}}{\left(1+\frac{y^\theta}{\lambda^\theta\beta}\right)^{\alpha b+1}} dxdy.$$

From (12), (14), (15) and setting $t = \frac{\frac{y^\theta}{\lambda^\theta\beta}}{1+\frac{y^\theta}{\lambda^\theta\beta}}$, we get

$$\begin{aligned}B(z) &= \frac{\lambda^{2\theta+l+j}\beta^{\frac{l+j}{\theta}+2}}{\theta(j+\theta)} \int_0^{\frac{\frac{z^\theta}{\lambda^\theta\beta}}{1+\frac{z^\theta}{\lambda^\theta\beta}}} t^{\frac{j+l}{\theta}+1}(1-t)^{\alpha b-\frac{l}{\theta}-1} \\ &\times {}_2F_1\left[\frac{j}{\theta}+1, 1-a\alpha+\frac{j}{\theta}; \frac{j}{\theta}+2; t\right] dt\end{aligned}$$

Using the result, ${}_2F_1(a,b;c;z) = \sum_{n_1=0}^\infty \frac{(a)_{n_1}(b)_{n_1}}{(c)_{n_1}} \frac{z^{n_1}}{n_1!}$ and simplifying above expression, we get

$$\begin{aligned}B(z) &= \frac{\lambda^{2\theta+l+j}\beta^{\frac{l+j}{\theta}+2}}{\theta(j+\theta)} \sum_{n_1=0}^\infty \frac{(\frac{j}{\theta}+1)_{n_1}(1-a\alpha+\frac{j}{\theta})_{n_1}}{(\frac{j}{\theta}+2)_{n_1}} \frac{1}{n_1!} \\ &\times B_{\frac{\frac{z^\theta}{\lambda^\theta\beta}}{1+\frac{z^\theta}{\lambda^\theta\beta}}}\left(\frac{j+l}{\theta}+n_1+2, \alpha b - \frac{l}{\theta}\right)\end{aligned}$$

Substituting $B(z)$ in (22), we get

$$I_2 = \frac{\lambda^{2\theta+l+j}\beta^{\frac{l+j}{\theta}+2}}{\theta(j+\theta)} \sum_{n_1=0}^{\infty} \frac{(\frac{j}{\theta}+1)_{n_1}(1-a\alpha+\frac{j}{\theta})_{n_1}}{(\frac{j}{\theta}+2)_{n_1}} \frac{1}{n_1!}$$
$$\times \int_0^\infty B_{\frac{\frac{z^\theta}{\lambda^\theta \beta}}{1+\frac{z^\theta}{\lambda^\theta\beta}}} \left(\frac{j+l}{\theta}+n_1+2, \alpha b-\frac{l}{\theta}\right) \frac{z^{m+\theta-1}}{\left(1+\frac{z^\theta}{\lambda^\theta\beta}\right)^{\alpha c+1}} dz.$$

Using (14) and setting $t = \frac{\frac{z^\theta}{\lambda^\theta \beta}}{1+\frac{z^\theta}{\lambda^\theta\beta}}$, we get

$$I_2 = \frac{\lambda^{3\theta+l+j+m}\beta^{\frac{l+j+m}{\theta}+3}}{\theta(j+\theta)((j+l)+\theta(n_1+2))} \sum_{n_1=0}^{\infty} \frac{(\frac{j}{\theta}+1)_{n_1}(1-a\alpha+\frac{j}{\theta})_{n_1}}{(\frac{j}{\theta}+2)_{n_1}} \frac{1}{n_1!}$$
$$\times \int_0^1 t^{\frac{j+l+m}{\theta}+n_1+2}(1-t)^{\alpha c-\frac{m}{\theta}-1}$$
$$\times {}_2F_1\left[\frac{j+l}{\theta}+n_1+2, 1-\alpha b+\frac{l}{\theta}; \frac{j+l}{\theta}+n_1+3; t\right] dt.$$

From (15), we have

$$I_2 = \frac{\lambda^{3\theta+l+j+m}\beta^{\frac{l+j+m}{\theta}+3}}{\theta(j+\theta)((j+l)+\theta(n_1+2))} \sum_{n_1=0}^{\infty} \frac{(\frac{j}{\theta}+1)_{n_1}(1-a\alpha+\frac{j}{\theta})_{n_1}}{(\frac{j}{\theta}+2)_{n_1}} \frac{1}{n_1!}$$
$$\times B\left(\frac{j+l+m}{\theta}+n_1+3, \alpha c-\frac{m}{\theta}\right) {}_3F_2\left[\frac{j+l}{\theta}+n_1+2, 1-\alpha b+\frac{l}{\theta},\right.$$
$$\left.\frac{j+l+m}{\theta}+n_1+3; \frac{j+l}{\theta}+n_1+3, \frac{j+l}{\theta}+\alpha c+n_1+3; 1\right].$$

Substituting I_2 in (21), we get (20).

Similar to the double moments, the validity of the triple moments of order statistics in equation (20) may be examined using given identity

$$\sum_{r=1}^{n-2}\sum_{s=r+1}^{n-1}\sum_{t=s+1}^{n} \mu_{r,s,t:n}^{j,l,m} = \binom{n}{3}[E(X)]^3.$$

5. Quadruple moments

In conjunction with the single, double, and triple moments of order statistics, the quadruple moments of order statistics may be utilised to derive inference procedures for the underlying distribution. The quadruple moments of order statistics are derived from the Pareto-Weibull distribution in the next section. Let $X_{1:n} \leq X_{2:n} \leq ...... \leq X_{n:n}$ represent the order statistics from the Pareto Weibull distribution provided in (3) with its cdf in (4). Then, the quadruple moments $E[X^j_{r:n} Y^l_{s:n} Z^m_{t:n} W^h_{u:n}]$ of order data is provided by

(23)

$$\begin{aligned}\mu^{j,l,m,h}_{r,s,t,u:n} = C_{r,s,t,u:n} &\int_0^{\infty}\int_0^{w}\int_0^{z}\int_0^{y} x^j y^l z^m w^h [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} \\ &\times [F(z)-F(y)]^{t-s-1}[F(w)-F(z)]^{u-t-1}[1-F(w)]^{n-u} \\ &\times f(x)f(y)f(z)dxdydzdw \\ &r,s,t,u = 1,2,3,4,....n, \quad r<s<t<u\end{aligned}$$

where

$$C_{r,s,t,u:n} = \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(u-t-1)!(n-u)!}.$$

By using the same argument as in the single, double and triple moments case, we get

(24)

$$\begin{aligned}&\mu^{j,l,m,h}_{r,s,t,u:n} \\ &= C_{r,s,t,u:n}\left(\frac{\alpha^4\theta^3\lambda^{l+j+m+h}\beta^{\frac{l+j+m+h}{\theta}}}{(j+\theta)((j+l)+\theta(n_1+2))((j+l+m)+\theta(n_1+n_2+3))}\right) \\ &\times \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{s-r-1}\sum_{i_3=0}^{t-s-1}\sum_{i_4=0}^{u-t-1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(\frac{j}{\theta}+1)_{n_1}(1-a\alpha+\frac{j}{\theta})_{n_1}}{(\frac{j}{\theta}+2)_{n_1}} \\ &\times \frac{(\frac{j+l}{\theta}+n_1+2)_{n_2}(1-b\alpha+\frac{l}{\theta})_{n_2}}{(\frac{j+l}{\theta}+n_1+3)_{n_2}}\frac{1}{n_1!}\frac{1}{n_2!}(-1)^{i_1+i_2+i_3+i_4}\binom{r-1}{i_1} \\ &\times \binom{s-r-1}{i_2}\binom{t-s-1}{i_3}\binom{u-t-1}{i_4} \\ &\times B\left(\frac{j+l+m+h}{\theta}+n_1+n_2+4, \alpha d-\frac{h}{\theta}\right) \\ &\times {}_3F_2\left[\frac{j+l+m}{\theta}+n_1+n_2+3, 1-\alpha c+\frac{l}{\theta}, \frac{j+l+m+h}{\theta}+n_1\right. \\ &\quad \left. +n_2+4; \frac{j+l+m}{\theta}+n_1+n_2+4, \frac{j+l+m}{\theta}+\alpha d+n_1+n_2+4; 1\right]\end{aligned}$$

where $a = i_1+i_2+1$, $b = s-r-i_2+i_3$, $c = t-s-i_3+i_4$, $d = n-t-i_4$.

The validity of the quadruple moments of order statistics in equation (24) can be checked by using

$$\sum_{r=1}^{n-3}\sum_{s=r+1}^{n-2}\sum_{t=s+1}^{n-1}\sum_{u=t+1}^{n}\mu_{r,s,t,u:n}^{j,l,m,h}=\binom{n}{4}[E(X)]^4.$$

Remarks
(1) For $\theta = 1$, we get all the above results for Pareto-Exponential distribution.
(2) For $\beta = 1, \theta = 2$ and $\lambda = \sqrt{2}\sigma$, we get all the above results for Pareto-Rayleigh distribution.

Conclusion

In this paper, expressions for the single, double, triple, and quadruple moments are obtained for the order statistics from the Pareto-Weibull distribution. The mean, variances, covariances, coefficient of skewness, kurtosis, L-moments, and the BLUEs of the location and scale parameters can be easily calculated using the result obtained in this paper. Further recurrence relations obtained in this paper can be used to calculate higher-order moments of order statistics from Pareto-Weibull distribution easily.

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