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Almost periodic vectors and representations in quasi-complete spaces

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Dedicated to professor A. T.-M. Lau with much appreciation for his lifetime contributions to mathematics

Let G be a topological group and E a quasi-complete space. We study approximation properties of almost periodic vectors of continuous, equicontinuous representations $\pi\colon G \longrightarrow \mathcal{B}(E)$. We extend an approximation theorem of Weyl and Maak from isometric Banach space representations to representations on quasi-complete spaces. We prove that if π is almost periodic, then E has a generalized direct sum decomposition $E = \oplus_{\theta\in\widehat{G}} E_\theta$, where each E_θ is linearly spanned by finite-dimensional, π-invariant subspaces. We show that on left translation invariant, quasi-complete subspaces of $L^p(G)$ (G locally compact, $1 \le p < \infty$), the left regular representation is almost periodic if and only if G is compact.

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1. Introduction

The study of almost periodic vectors of Banach space representations goes back to Weyl [22] and Maak [17]. These studies have been extended by several authors including Jacobs [13], DeLeeuw–Glicksberg [5, 6], Kaijser [14], Duncan–Ülger [8], Chou-Lau [4], Megrelishvili [18], Filali-Neufang-Monfared [9], Filali–Monfared [10].

In this paper we study approximation properties of almost periodic vectors and representations in quasi-complete spaces. A locally convex space E is *quasi-complete* if every bounded closed subset of E is complete. The importance of quasi-complete spaces is due, in part, to the fact that in such spaces, a set is relatively compact if and only if it is totally bounded (i.e., precompact). A great number of spaces of interest in analysis are quasi-complete; examples include Banach, Fréchet, and LF-spaces, together with their dual spaces under the w^*-topology. In addition, the space of continuous linear operators $\mathcal{B}(E)$ under the strong operator topology (SOT) is quasi-complete if E is quasi-complete and barreled. For more information on quasi-complete spaces we refer to [19, Section 34.3], [2, Section III.1.6].

Suppose G is a Hausdorff topological group and $\pi\colon G \longrightarrow \mathcal{B}(E)$ is a continuous, equicontinuous representation on a quasi-complete space E (for undefined terminology see Section 2). A vector $u \in E$ is called *almost periodic* if $\pi(G)u = \{\pi(x)u\colon x \in G\}$ is relatively compact in E. Since E is quasi-complete, it follows that u is almost periodic if and only if $\pi(G)u$ is totally bounded. The set of all such vectors, E_{ap}, is a π-invariant, closed linear subspace of E (in particular, E_{ap} is quasi-complete). The main result in Section 3 is Theorem 3.5 where we show that if M is any closed π-invariant subspace of E_{ap}, then finite-dimensional, π-invariant subspaces of M span a dense subspace of M. This result extends and unifies earlier approximation results for Banach space representations given in Weyl [22, p. 198-199], Maak [17, Haupsatz, p. 164]), and Filali-Monfared [10, Theorem 3.3].

In Theorem 4.1 we give a characterization of almost periodic representations in quasi-complete barreled spaces. In Theorem 4.3 we prove that if $\pi\colon G \longrightarrow \mathcal{B}(E)$ is an almost periodic representation, then E has a generalized direct sum decomposition (see Definition 4.2). This decomposition is an analogue of similar decomposition results for Hilbert and Banach space representations of compact groups ([12, Theorem 27.44], [10, Theorem 3.5]). As an application of our decomposition Theorem 4.3, we show in Theorem 4.4 that on a left translation invariant, quasi-complete subspace of $L^p(G)$ (G locally compact, $1 \leq p < \infty$), the left regular representation is almost periodic if and only if G compact. This theorem extends a result of Bochner and von Neumann for the (right) regular representation on $L^2(G)$, where G is a separable, metrizable, locally compact group [3, Theorem 40, p. 49].

2. Preliminaries and notation

Throughout this paper, we shall assume that all topological spaces are Hausdorff, and that G is a topological group, and E is a quasi-complete space. We shall denote by $\mathscr{U}$ a local base of $0 \in E$ consisting of absorbing, balanced, closed, convex neighborhoods of 0.

We shall assume that the space of continuous linear operators $\mathcal{B}(E)$ is equipped with the strong operator topology (SOT) (the topology of pointwise convergence). Let $\pi\colon G \longrightarrow \mathcal{B}(E)$ be a continuous representation (thus if $x_\alpha \to x \in G$, then $\pi(x_\alpha)u \to \pi(x)u$, for all $u \in E$). We call π *equicontinuous* if the family of operators $\{\pi(x)\colon x \in G\}$ is equicontinuous on E, that is, for every $V \in \mathscr{U}$, there is $U \in \mathscr{U}$, such that $\pi(x)U \subset V$, for all $x \in G$. Equicontinuous representations on quasi-complete spaces are the natural analogues of uniformly bounded representations on Banach spaces. A continuous, equicontinuous representation π is called *almost periodic* if every $u \in E$ is an almost periodic vector, i.e., if $E_{ap} = E$. Our definition of almost periodic representations in quasi-complete spaces extends naturally a similar definition for Banach space representations in [1, pp. 247-248]. We remark that if H is a Hilbert space and $\pi\colon G \longrightarrow \mathcal{B}(H)$ is an almost periodic representation, then all coefficient functions $G \longrightarrow \mathbb{C}$, $x \mapsto \langle \pi(x)u, v\rangle$ $(u, v \in H)$ are almost periodic (cf. Lemma 3.4 and Theorem 4.1).

We denote the set of all continuous bounded functions $f\colon G \longrightarrow E$, by $C^b(G,E)$. For each $U \in \mathscr{U}$, we define $U' \subset C^b(G,E)$ to be the set of all $f \in C^b(G,E)$ such that $f(G) \subset U$; the set of all such U' is denoted by $\mathscr{U}'$. With $\mathscr{U}'$ as a local base of $0 \in C^b(G,E)$, the space $C^b(G,E)$ is quasi-complete. The convergence in $C^b(G,E)$ is uniform in the sense that if $f_\alpha \to f$, then for every $U \in \mathscr{U}$, there is some α_0 such that if $\alpha \geq \alpha_0$, then $f_\alpha - f \in U'$, or equivalently, $f_\alpha(x) - f(x) \in U$, for all $x \in G$. For $U \in \mathscr{U}$, $\|\cdot\|_U$ is the corresponding seminorm on E defined by $\|u\|_U = \inf\{t > 0\colon u \in tU\}$ $(u \in E)$. Similarly, for $U' \in \mathscr{U}'$ we denote by $\|\cdot\|_{U'}$ the corresponding seminorm $C^b(G,E)$. We have $\|f\|_{U'} = \sup_{x\in G} \|f(x)\|_U$, for every $f \in C^b(G,E)$.

In a likewise manner, we can use each $U \in \mathscr{U}$ to define a neighborhood U' of $0 \in C^b(G \times G, E)$, turning $C^b(G \times G, E)$ into a quasi-complete space.

If $f \in C^b(G,E)$, and $a \in G$, the left and right translations of f are denoted by $L_a f$ and $R_a f$, respectively. The function $D_a f \in C^b(G \times G, E)$ is defined by $D_a f(x, y) = f(xay)$ $(x, y \in G)$. We put $\mathfrak{L}_f = \{L_a f\colon a \in G\}$, $\mathfrak{R}_f = \{R_a f\colon a \in G\}$, and $\mathfrak{D}_f = \{D_a f\colon a \in G\}$. For E quasi-complete, the relative compactness of $\mathfrak{L}_f$, $\mathfrak{R}_f$, and $\mathfrak{D}_f$ are equivalent, and characterize the almost periodicity of f ([3, 23]). The space of all almost periodic functions with values in E is denoted by $AP(G,E)$, forming a closed (and hence quasi-complete) subspace of $C^b(G,E)$. If $E = \mathbb{C}$, then $AP(G,E)$ coincides with space of scalar-valued almost periodic functions $AP(G)$.

We shall denote the two-sided invariant mean on $AP(G)$ by m ([11, Theorem 18.9, p. 252]). For quasi-complete spaces E, the existence and uniqueness of a vector-valued two-sided invariant mean M on $AP(G,E)$ is proved in [23, Theorems 4.6, 4.7]. Such a mean is a two-sided translation invariant map $M\colon AP(G,E) \longrightarrow E$, satisfying the following properties:

(i) If $f(x) = u$ $(u \in E)$ is a constant function, then $M(f) = u$.

(ii) If $\check{f}(x) = f(x^{-1})$, then $M(\check{f}) = M(f)$.
(iii) If $U \in \mathscr{U}$, $\|M(f)\|_U \leq m(\|f\|_U) \leq \|f\|_{U'}$.
(iv) If $U \in \mathscr{U}$, and $f - g \in U'$, then $M(f-g) \in 2U$.

A good reference for general properties of vector-valued almost periodic functions is Bochner–von Neumann [3]. We remark that this reference deals with groups G having no underlying topology, and with spaces E which are *topologically complete* (cf. [21]). However, it can be verified that with natural modifications of proofs, the results in [3] still hold for topological groups G and quasi-complete spaces E which are of interest in our paper. This allows us to use the results in [3] in the context of the present paper.

The collection (up to unitary equivalence) of all finite-dimensional, continuous, irreducible, unitary representations of G is denoted by $\widehat{G}$. Given $\sigma \in \widehat{G}$, the coefficients functions of σ in an orthonormal basis of H_σ are denoted by σ_{ij}, $1 \leq i, j \leq d_\sigma$, $d_\sigma = \dim H_\sigma$. The linear span of all such coefficient functions (as σ runs in $\widehat{G}$) form a dense linear subspace of $AP(G)$, which we denote by $T(G)$ (see [7, Théorème 16.2.1, p. 298]).

The main tool in our study is the convolution product

$$\phi * f(x) = M_y(\phi(y) f(y^{-1}x)) \qquad (x \in G), \tag{1}$$

where $\phi \in AP(G)$, $f \in AP(G, E)$, and M is the vector-valued invariant mean on $AP(G, E)$ ([3, Definition 6, p. 31]). For the convenience of our readers, we state some properties of E-valued almost periodic functions that will be used in our paper. These results can be proved without difficulty using results developed in [3].

Theorem 2.1. *Let $f \in AP(G, E)$, $\phi \in AP(G)$, and $U \in \mathscr{U}$. Then $\phi f \in AP(G, E)$, $\phi * f \in AP(G, E)$ and*

$$\|\phi f\|_{U'} \leq \|\phi\|_{\sup} \|f\|_{U'}, \quad \|\phi * f\|_{U'} \leq \|\phi\|_{\sup} \|f\|_{U'}. \tag{2}$$

Theorem 2.2. *Let $f \in AP(G, E)$ and $\mathcal{M}_f$ be the smallest closed left translation invariant subspace of $AP(G, E)$ containing f. For each $\sigma \in \widehat{G}$, let $d_\sigma = \dim H_\sigma$, and*

$$\mathcal{M}_{\sigma,f} = \operatorname{Span}\{\sigma_{ij} * f \colon 1 \leq i, j \leq d_\sigma\}. \tag{3}$$

(i) *Each $\mathcal{M}_{\sigma,f}$ is a finite-dimensional, left translation invariant subspace of $\mathcal{M}_f$.*

(ii) *The spaces* $\mathcal{M}_{\sigma,f}$ $(\sigma \in \widehat{G})$ *span a dense linear subspace of* $\mathcal{M}_f$*, i.e.,*

$$\overline{\operatorname{Span}} \bigcup_{\sigma \in \widehat{G}} \mathcal{M}_{\sigma,f} = \overline{AP(G) * f} = \mathcal{M}_f. \tag{4}$$

(Cf. [3, Theorem 27, p. 37].)

3. Almost periodic vectors in quasi-complete spaces

In this section E is a quasi-complete space, and G is a topological group.

Lemma 3.1. *Let* $\pi\colon G \longrightarrow \mathcal{B}(E)$ *be a continuous, equicontinuous representation. A vector* $u \in E$ *is almost periodic if and only if for every* $U \in \mathscr{U}$*, there is a finite number of subsets* $A_1, \ldots, A_n$ *of* G*, such that*

(i) $G = \bigcup_{i=1}^n A_i$,
(ii) *if* $x, y \in A_i$ *for some* $1 \le i \le n$*, then* $\pi(x)u - \pi(y)u \in U$.

Proof. Suppose $u \in E$ is almost periodic and $U \in \mathscr{U}$. Choose $V \in \mathscr{U}$ be such that $V + V \subset U$. By total boundedness of $\pi(G)u$, there are $x_1, \ldots, x_n \in G$ such that

$$\pi(G)u \subset \bigcup_{i=1}^{n} (\pi(x_i)u + V). \tag{5}$$

Define

$$A_i = \{x \in G\colon \pi(x)u \in \pi(x_i)u + V\}. \tag{6}$$

By (5), $G = \bigcup_{i=1}^n A_i$ and thus (i) holds. Furthermore, if $x, y \in A_i$, then

$$\pi(x)u - \pi(y)u = (\pi(x)u - \pi(x_i)u) + (\pi(x_i)u - \pi(y)u) \in V + V \subset U,$$

and therefore (ii) also holds.

Conversely, suppose for a given $U \in \mathscr{U}$, there exist subsets $A_1, \ldots, A_n$ of G satisfying conditions (i), (ii). Choose arbitrary points $x_i \in A_i$, for $i = 1, \ldots, n$. If $x \in G$, then by (i), $x \in A_i$ for some i, and hence by (ii), $\pi(x)u - \pi(x_i)u \in U$, or $\pi(x)u \in \pi(x_i)u + U$. It follows that $\pi(G)u = \bigcup_{i=1}^n (\pi(x_i)u + U)$. Since $U \in \mathscr{U}$ is arbitrary, $\pi(G)u$ is totally bounded and hence u is almost periodic. □

Theorem 3.2. *Let* $\pi\colon G \longrightarrow \mathcal{B}(E)$ *be a continuous, equicontinuous representation. Let* $u \in E$ *and* $f_u\colon G \longrightarrow E$ *by defined* $f_u(x) = \pi(x)u$ $(x \in G)$*. Then* $u \in E_{ap}$ *if and only if* $f_u \in AP(G, E)$*.*

Proof. First, let us suppose that f_u is almost periodic. Let $U \in \mathscr{U}$ be given, and choose $V \in \mathscr{U}$ such that $V + V \subset U$. Since $\{D_a f_u : a \in G\}$ is totally bounded in $C^b(G \times G, E)$, there is a finite number of functions $D_{a_1} f_u, \ldots, D_{a_n} f_u$ $(a_1, \ldots, a_n \in G)$, such that

$$\{D_a f_u : a \in G\} \subset \bigcup_{i=1}^{n} (D_{a_i} f_u + V'), \tag{7}$$

where $V' = \{h \in C^b(G \times G, E) : h(G \times G) \subset V\}$. Thus, for every $a \in G$, there corresponds some $1 \leq i \leq n$, such that $D_a f_u \in D_{a_i} f_u + V'$. This is equivalent to $f_u(cad) - f_u(ca_i d) \in V$ for all $(c, d) \in G \times G$. Define for each $1 \leq i \leq n$,

$$A_i = \{x \in G : D_x f_u - D_{a_i} f_u \in V'\}.$$

It follows from (7) that $\bigcup_{i=1}^{n} A_i = G$. Furthermore, if $x, y \in A_i$, then

$$D_x f_u - D_{a_i} f_u \in V', \quad D_y f_u - D_{a_i} f_u \in V',$$

and hence

$$D_x f_u - D_y f_u \in V' + V' \subset U'.$$

We have shown that if x, y are in the same A_i, then for all $c, d \in G$,

$$D_x f_u(c, d) - D_y f_u(c, d) = f_u(cxd) - f_u(cyd) = \pi(cxd)u - \pi(cyd)u \in U.$$

If we choose $c = d = e$, then $\pi(x)u - \pi(y)u \in U$, whenever x, y are the same A_i. This proves that $u \in E_{ap}$ by Lemma 3.1.

Conversely, suppose that $u \in E_{ap}$. The continuity of the representation π implies that function $f_u(x) = \pi(x)u$ is continuous. Since u is almost periodic, $f_u(G) = \pi(G)u$ is totally bounded in E, and in particular, $f_u(G)$ is bounded, i.e., $f_u \in C^b(G, E)$. To prove that $f_u \in AP(G, E)$ it suffices to show that $\mathfrak{R}_{f_u} = \{R_a f_u : a \in G\}$ is totally bounded in $C^b(G, E)$. Let $U \in \mathscr{U}$. By equicontinuity of π, there is $W \in \mathscr{U}$ such that $\pi(G)W \subset U$. Since $u \in E_{ap}$, Lemma 3.1 is applicable, and therefore upon choosing $a_1 \in A_1, \ldots, a_n \in A_n$, it follows that for every $a \in G$ there corresponds some a_i $(1 \leq i \leq n)$ with

$$\pi(a)u - \pi(a_i)u \in W.$$

Thus for all $x \in G$:

$$\pi(xa)u - \pi(xa_i)u \in \pi(x)W \subset U.$$

It follows that

$$R_a f_u(x) - R_{a_i} f_u(x) = f_u(xa) - f_u(xa_i) = \pi(xa)u - \pi(xa_i)u \in U.$$

Since $x \in G$ is arbitrary, we have shown that $R_a f_u - R_{a_i} f_u \in U'$. Since $U \in \mathscr{U}$ is arbitrary, total boundedness of $\mathfrak{R}_{f_u}$ follows. □

Lemma 3.3. *Let $\pi\colon G \longrightarrow \mathcal{B}(E)$ be a continuous, equicontinuous representation. For $\phi \in AP(G)$, let $\pi(\phi) : E_{ap} \longrightarrow E_{ap}$, be defined by*

$$\pi(\phi)u = M_y(\phi(y)\pi(y)u) \qquad (u \in E_{ap}), \tag{8}$$

where M is the E-valued invariant mean on $AP(G,E)$. Then:

(i) *$\pi(\phi) \in \mathcal{B}(E_{ap})$.*
(ii) *$\pi(\phi * \psi) = \pi(\phi)\pi(\psi)$, for all $\psi \in AP(G)$.*

Proof. (i) Recall that E_{ap} is a closed π-invariant subspace of E, and in particular E_{ap} is quasi-complete. The vector $\pi(\phi)u$ in (8) is well-defined since the function $y \mapsto \phi(y)\pi(y)u$, $G \longrightarrow E_{ap}$, is the product of two almost periodic functions ϕ and f_u (Theorem 3.2), and therefore itself is almost periodic and its mean in (8) exists and belongs to E_{ap}. It is easy to check that $\pi(\phi)$ is a linear map on E_{ap}. It remains to show that $\pi(\phi)$ is continuous. We may assume that $\phi \neq 0$. Suppose $\{u_\alpha\}_{\alpha\in I}$ is a net in E_{ap} and $u_\alpha \to u \in E_{ap}$. Let $U \in \mathscr{U}$ be arbitrary, and $\|\cdot\|_U$ and $\|\cdot\|_{U'}$ be the corresponding seminorms on E and $AP(G,E)$, respectively. It suffices to show that

$$\lim_\alpha \|\pi(\phi)u_\alpha - \pi(\phi)u\|_U = 0. \tag{9}$$

For $\alpha \in I$, let $g_\alpha(y) = \pi(y)(u_\alpha - u)$, then by (8),

$$\begin{aligned}\|\pi(\phi)u_\alpha - \pi(\phi)u\|_U &= \|M_y[\phi(y)\pi(y)(u_\alpha - u)]\|_U = \|M_y[\phi(y)g_\alpha(y)]\|_U \\ &\le \|\phi g_\alpha\|_{U'} \le \|\phi\|_{\sup}\|g_\alpha\|_{U'} = \|\phi\|_{\sup}\sup_{y\in G}\|\pi(y)(u_\alpha - u)\|_U,\end{aligned} \tag{10}$$

where the first inequality follows from the properties of the mean M ([23, Theorem 4.7(v)]), and the second inequality from (2). Now, let $n \in \mathbb{N}$ be arbitrary. Since π is equicontinuous, there exists $V \in \mathscr{U}$ such that

$$\pi(y)V \subset \frac{1}{n\|\phi\|_{\sup}}U \qquad \text{for all } y \in G.$$

Since $u_\alpha \to u$, there exists $\alpha_0 \in I$ such that $u_\alpha - u \in V$ for all $\alpha \geq \alpha_0$. Therefore, for all $\alpha \geq \alpha_0$:

$$\pi(y)(u_\alpha - u) \in \frac{1}{n\|\phi\|_{\sup}} U \qquad \text{for all } y \in G.$$

Thus for all $y \in G$:

$$\|\pi(y)(u_\alpha - u)\|_U = \inf\{t > 0 \colon \pi(y)(u_\alpha - u) \in tU\} \leq \frac{1}{n\|\phi\|_{\sup}}. \tag{11}$$

It follows from (10) and (11) that if $\alpha \geq \alpha_0$:

$$\|\pi(\phi)u_\alpha - \pi(\phi)u\|_U \leq \|\phi\|_{\sup} \frac{1}{n\|\phi\|_{\sup}} = 1/n.$$

This proves (9) and hence completes the proof of (i).

(ii) Let m the invariant mean on $AP(G)$. Then for every $u \in E$, we have

$$\begin{aligned} \pi(\phi * \psi)u = M_x((\phi * \psi)(x)\pi(x)u) &= M_x(m_y[\phi(y^{-1})\psi(yx)]\pi(x)u) \\ &= M_x(M_y[\phi(y^{-1})\psi(yx)\pi(x)u]), \end{aligned}$$

where the last identity is an easy consequence of the properties of the mean M (cf. [23, Theorem 4.6]). The order of the means can be changed using a Fubini-type theorem for vector-valued almost periodic means ([3, Theorem 18, p. 30]), thus

$$\begin{aligned} \pi(\phi * \psi)u &= M_y[\phi(y^{-1})M_x(\psi(yx)\pi(x)u)] \\ (x \to y^{-1}x) \qquad &= M_y[\phi(y^{-1})M_x(\psi(x)\pi(y^{-1})\pi(x)u)] \\ &= M_y[\phi(y^{-1})\pi(y^{-1})M_x(\psi(x)\pi(x)u)] \\ (y \to y^{-1}) \qquad &= M_y(\phi(y)\pi(y)\pi(\psi)u) \\ &= \pi(\phi)\pi(\psi)u, \end{aligned}$$

and since $u \in E$ is arbitrary (ii) follows. □

Lemma 3.4. *Let E and F be quasi-complete spaces and $T \in \mathcal{B}(E, F)$ (the space of continuous linear operators from E to F). Let G be a topological group and M and M' be the invariant means on $AP(G, E)$ and $AP(G, F)$, respectively. Let $f \in AP(G, E)$, and $T \circ f \colon G \longrightarrow F$, $y \mapsto Tf(y)$. Then $T \circ f \in AP(G, F)$ and $T(M(f)) = M'(Tf)$.*

Proof. Since T is a continuous linear operator, it follows that $T \circ f \in AP(G, F)$. Given $U \in \mathscr{U}_F$, we choose $V \in \mathscr{U}_E$ such that $TV \subset U$. Using the definition of the almost periodic mean ([23, Theorem 4.6]), there are elements $a_1, \dots, a_n \in G$ such that

$$M(f) - \frac{1}{n}\sum_{k=1}^{n} f(xa_k y) \in V \qquad \text{for all } x, y \in G.$$

By applying T we find that

$$T(M(f)) - \frac{1}{n}\sum_{k=1}^{n} Tf(xa_k y) \in T(V) \subset U \qquad \text{for all } x, y \in G.$$

Since $U \in \mathscr{U}_F$ is arbitrary, it follows that $T(M(f)) = M'(Tf)$. □

Next we come to the main result of this section, which extends and unifies earlier approximation theorems by Weyl [22, p. 198-199], Maak [17, Haupsatz, p. 164]), and Filali-Monfared [10, Theorem 3.3]. The theorems of Weyl and Maak deal with isometric representations on Banach spaces. In [10], representations need not be isometric, however the group G is required to be compact.

Recall from the introduction that if $\sigma \in \widehat{G}$, then its coefficients functions σ_{ij} belong to $AP(G)$ and the linear span of all such functions (as σ runs in $\widehat{G}$) form a dense linear subspace of $AP(G)$ ([7, Théorème 16.2.1, p. 298]).

Theorem 3.5. *Suppose that G is a topological group, E a quasi-complete space and $\pi\colon G \longrightarrow \mathcal{B}(E)$ is a continuous, equicontinuous representation. For $u \in E_{ap}$, let E_u be the smallest closed π-invariant linear subspace of E_{ap} containing u. For each $\sigma \in \widehat{G}$, let*

$$E_{\sigma,u} = \operatorname{Span}\{\pi(\sigma_{ij})u \colon 1 \le i, j \le d_\sigma\}, \tag{12}$$

where $\pi(\sigma_{ij})u$ is defined as in (8). Then the following hold:

(i) *Each $E_{\sigma,u}$ is a finite-dimensional, π-invariant subspace of E_u.*

(ii) *The spaces $E_{\sigma,u}$ ($\sigma \in \widehat{G}$) span a dense linear subspace of E_u, i.e.,*

$$\overline{\operatorname{Span}} \bigcup_{\sigma \in \widehat{G}} E_{\sigma,u} = E_u. \tag{13}$$

(iii) *If $M \subset E_{ap}$ is any π-invariant closed linear space, then finite-dimensional, π-invariant subspaces in M span a dense linear subspace of M.*

Proof. (i) The fact that $E_{\sigma,u}$ is finite-dimensional follows from (12). Next, for all $x \in G$ we have

$$
\begin{aligned}
\pi(x)\pi(\sigma_{ij})u &= \pi(x)M_y(\sigma_{ij}(y)\pi(y)u) \\
\text{(Lemma 3.4)} \qquad &= M_y(\sigma_{ij}(y)\pi(xy)u) \\
&= M_y(\sigma_{ij}(x^{-1}y)\pi(y)u) \\
&= \sum_{k=1}^{d_\sigma} \sigma_{ik}(x^{-1})M_y(\sigma_{kj}(y)\pi(y)u) \\
&= \sum_{k=1}^{d_\sigma} \sigma_{ik}(x^{-1})\pi(\sigma_{kj})u \in E_{\sigma,u}. \qquad (14)
\end{aligned}
$$

Since $E_{\sigma,u}$ is generated by the vectors $\pi(\sigma_{ij})u$ $(1 \le i,j \le d_\sigma)$, the above calculation shows that $E_{\sigma,u}$ is π-invariant. It remains to show $E_{\sigma,u} \subset E_u$. Let $g\colon G \longrightarrow E$ be defined by $g(y) = \sigma_{ij}(y)\pi(y)u$. Since g is the product of two almost periodic functions σ_{ij} and f_u (Theorem 3.2), it follows that $g \in AP(G,E)$. Since E_u is a closed π-invariant linear subspace of E, we have $g(y) = \sigma_{ij}(y)\pi(y)u \in E_u$ for all $y \in G$, and therefore,

$$\pi(\sigma_{ij})u = M_y(\sigma_{ij}(y)\pi(y)u) = M(g) \in \overline{ch}(g(G)) \subset E_u,$$

where $\overline{ch}(g(G))$ is the closed convex hull of $g(G)$ in E, and the penultimate belonging follows from properties of the vector-valued mean (see [23, Theorem 4.6]). It follows that $E_{\sigma,u} \subset E_u$.

(ii) It follows from (i) that $\overline{\text{Span}} \bigcup_{\sigma\in\widehat{G}} E_{\sigma,u} \subset E_u$. To prove the reverse inclusion, it suffices to show that $u \in \overline{\text{Span}} \bigcup_{\sigma\in\widehat{G}} E_{\sigma,u}$ (since the latter space is π-invariant and obviously closed).

Consider the function $f_u \in AP(G,E)$ (Theorem 3.2). By Theorem 2.2(ii) there is a net $h_\alpha \in \text{Span} \bigcup_{\sigma\in\widehat{G}} \mathcal{M}_{\sigma,f_u}$ such that $h_\alpha \to f_u$ uniformly on G. Thus by evaluating at $x = e$, $h_\alpha(e) \to f_u(e) = u$ in E. So it remains to show that $h_\alpha(e) \in \text{Span} \bigcup_{\sigma\in\widehat{G}} E_{\sigma,u}$. By (3), each h_α is a linear combination of functions of the form $\sigma_{ij} * f_u$, where σ runs in $\widehat{G}$ and $1 \le i,j \le d_\sigma$. However,

$$
\begin{aligned}
(\sigma_{ij} * f_u)(x) = M_y(\sigma_{ij}(xy)f_u(y^{-1})) &= \sum_{k=1}^{d_\sigma} M_y(\sigma_{kj}(y)f_u(y^{-1}))\sigma_{ik}(x) \\
(y \to y^{-1}) \qquad &= \sum_{k=1}^{d_\sigma} M_y(\overline{\sigma}_{jk}(y)\pi(y)u)\sigma_{ik}(x),
\end{aligned}
$$

where $\overline{\sigma} \in \widehat{G}$ is the conjugate representation of σ, and $\overline{\sigma}_{jk}(y) = \overline{\sigma_{jk}(y)}$. Thus $h_\alpha(e)$ is a linear combination of vectors of the form

$$(\sigma_{ij} * f_u)(e) = \sum_{k=1}^{d_\sigma} M_y(\overline{\sigma}_{jk}(y)\pi(y)u)\sigma_{ik}(e) = M_y(\overline{\sigma}_{ji}(y)\pi(y)u) = \pi(\overline{\sigma}_{ji})u \in E_{\overline{\sigma},u}.$$

It follows that $h_\alpha(e) \in \operatorname{Span} \bigcup_{\sigma \in \widehat{G}} E_{\sigma,u}$, completing the proof of (ii).

(iii) For every $u \in M$, we know by (i) and (ii) that finite-dimensional, π-invariant linear subspaces of E_u span a dense subspace of E_u. Since $E_u \subset M$, and u is arbitrary, the result in (iii) follows. □

4. Almost periodic representations

In this section we study almost periodic representations, and prove the existence of generalized direct sum decompositions of their representation spaces. We apply our results to characterize almost periodicity of the left regular representations on $L^p(G)$ $(1 \leq p < \infty)$, when G is a locally compact group.

But let first G be still a topological group, and E a quasi-complete space. We recall from Section 2 that a continuous, equicontinuous representation $\pi\colon G \longrightarrow \mathcal{B}(E)$ is almost periodic if every $u \in E$ is an almost periodic vector; namely, if $E_{ap} = E$. A natural question is whether this definition is equivalent to almost periodicity of π as an operator-valued function. It follows from the following theorem that this is indeed the case if E is barreled. A locally convex space E is *barreled* if every closed, convex, balanced, absorbing subset of E is a neighborhood of $0 \in E$. Examples of barreled spaces include Fréchet spaces and LF-spaces. When E is quasi-complete and barreled, then $\mathcal{B}(E)$ under the strong operator topology is quasi-complete (Trèves [19, §34.3, Corollary 2, p. 356]). In the special case that E is a Hilbert space and G is a (discrete) group, the following result is due to Bochner–von Neumann [3, Theorem 36, p. 44].

Theorem 4.1. *Let G be a topological group, E a quasi-complete barreled space, and let $\mathcal{B}(E)$ have the strong operator topology. Let $\kappa\colon G \longrightarrow \mathcal{B}(E)$ be a function. Then $\kappa \in AP(G, \mathcal{B}(E))$ if and only if for every $u \in E$, $f_u \in AP(G, E)$, where $f_u\colon G \longrightarrow E$, $f_u(x) = \kappa(x)u$.*

Proof. First, consider the 'if' part of the theorem. Suppose that $f_u \in AP(G, E)$, for every $u \in E$. Continuity of the functions f_u imply that whenever $x_\alpha \to x \in G$, then $\kappa(x_\alpha)u \to \kappa(x)u$, for all $u \in E$, and therefore $\kappa\colon G \longrightarrow$

$\mathcal{B}(E)$ is continuous. To prove that κ is bounded, let $U \in \mathscr{U}$ and $u_i \in E$ $(i = 1, \dots, n)$, and consider the neighborhood of $0 \in \mathcal{B}(E)$ defined by

$$\mathcal{U} = \{T \in \mathcal{B}(E)\colon Tu_i \in U,\ i = 1, \dots, n\}. \tag{15}$$

By the assumption that f_{u_i} is bounded $(1 \leq i \leq n)$, there is $R_i > 0$ such that if $\alpha \in \mathbb{C}$ and $|\alpha| \geq R_i$, then $f_{u_i}(G) = \{\kappa(x)u_i\colon x \in G\} \subset \alpha U$. Thus if $R = \max\{R_1, \dots, R_n\}$, and if $\alpha \in \mathbb{C}$, $|\alpha| \geq R$, then $\bigcup_{i=1}^{n}\{\kappa(x)u_i\colon x \in G\} \subset \alpha U$. It follows that if $|\alpha| \geq R$, then

$$\kappa(G) \subset \{T \in \mathcal{B}(E)\colon Tu_i \in \alpha U,\ \text{for } i = 1, \dots, n\} = \alpha\mathcal{U},$$

and therefore κ is bounded.

Since $\mathcal{B}(E)$ is quasi-complete, to complete the proof of almost periodicity of κ it remains to show that $\mathfrak{L}_\kappa$ is totally bounded in $C^b(G, \mathcal{B}(E))$. For $u_1, \dots, u_n \in E$, and $U \in \mathscr{U}$, let $\mathcal{U}$ be the neighborhood of $0 \in \mathcal{B}(E)$ defined in (15). Let

$$f = (f_{u_1}, \dots, f_{u_n})\colon G \longrightarrow E \times \cdots \times E \quad (n\text{-times}).$$

Since each f_{u_i} is almost periodic, $\mathfrak{L}_{f_{u_i}}$ is totally bounded in $C^b(G, E)$, and therefore $\mathfrak{L}_{f_{u_1}} \times \cdots \times \mathfrak{L}_{f_{u_n}}$ is totally bounded in the product topology of $C^b(G, E) \times \cdots \times C^b(G, E)$. Since $\mathfrak{L}_f \subset \mathfrak{L}_{f_{u_1}} \times \cdots \times \mathfrak{L}_{f_{u_n}}$, it follows that $\mathfrak{L}_f$ is also totally bounded in the same topology. Thus corresponding to the neighborhood $U' \times \cdots \times U'$ of $(0, \dots, 0) \in C^b(G, E) \times \cdots \times C^b(G, E)$, there must exist elements $a_1, \dots, a_N \in G$, such that for each $x \in G$, we can find some $1 \leq j \leq N$ such that $L_x f - L_{a_j} f \in U' \times \cdots \times U'$, or equivalently,

$$L_x f_{u_i} - L_{a_j} f_{u_i} \in U' \qquad (i = 1, \dots, n).$$

By computing the function on the left side at an arbitrary $y \in G$, the above inclusion is easily seen to imply

$$L_x\kappa(y) - L_{a_j}\kappa(y) \in \mathcal{U} \qquad (y \in G),$$

which proves the total boundedness of $\mathfrak{L}_\kappa$.

The routine proof of the 'only if' part of the theorem is left for our readers. □

In Filali–Monfared [10, Theorem 3.5], the authors proved a decomposition result for continuous Banach space representations of compact groups.

Such representations are automatically almost periodic and equicontinuous. In our next result we shall extend this theorem to the more general setting of almost periodic representations on quasi-complete spaces. Before stating our result we need some definitions.

If G is a topological group and $\phi, \psi \in AP(G)$, then the convolution product $\phi * \psi \in AP(G)$ is defined by $(\phi * \psi)(x) = m_y(\phi(y)\psi(y^{-1}x))$ $(x \in G)$, where m is the invariant mean on $AP(G)$. The trace of a representation $\theta \in \widehat{G}$ is defined by $\mathrm{tr}(\theta) = \sum_{i=1}^{d_\theta} \theta_{ii} \in AP(G)$, $(d_\theta = \dim H_\theta)$. For $\theta, \sigma \in \widehat{G}$, we have

$$\sigma_{ij} * \theta_{kl} = \delta_{\sigma\theta}\delta_{jk}\frac{1}{d_\sigma}\sigma_{il}, \tag{16}$$

where δ is the Kronecker delta ([20, Theorem 21, p. 467]). It follows from (16) that

$$\pi(d_\theta \mathrm{tr}(\theta) * d_\sigma \mathrm{tr}(\sigma)) = \delta_{\theta\sigma}\pi(d_\theta \mathrm{tr}(\theta)). \tag{17}$$

Definition 4.2. Let E be topological vector space and $\{E_\alpha\}_{\alpha \in I}$ be a family of closed subspaces of E. We say that E is a *generalized direct sum* of $\{E_\alpha\}_{\alpha \in I}$ and we write $E = \oplus_{\alpha \in I} E_\alpha$ if the following conditions hold:

(a) for every $\alpha \in I$: $E_\alpha \cap \overline{\mathrm{Span}} \bigcup_{\beta \in I, \beta \neq \alpha} E_\beta = \{0\}$,
(b) Span $\bigcup_{\alpha \in I} E_\alpha$ is dense in E.

By comparison with algebraic direct sums (Köthe [15, §2.6, p. 54]), condition (a) states a *stronger* requirement that for each α, the intersection of E_α with the *closure* of the linear span of the union of all other E_β $(\beta \neq \alpha)$ must be $\{0\}$; and condition (b) states a *weaker* requirement that the linear span of all E_α is only dense in E.

Theorem 4.3. *Let π be a continuous, equicontinuous, almost periodic representation of a topological group G on a quasi-complete space E. For each $\theta \in \widehat{G}$, let*

$$P_\theta = \pi(d_\theta \mathrm{tr}(\theta)) \in \mathcal{B}(E), \tag{18}$$

$$E_\theta = \mathrm{Span} \bigcup_{u \in E} E_{\theta,u} = \mathrm{Span}\,\{\pi(\theta_{ij})u \colon 1 \leq i, j \leq d_\theta, u \in E\} \subset E, \tag{19}$$

where $\pi(d_\theta \mathrm{tr}(\theta))$ is defined as in (8), and $E_{\theta,u}$ as in (12).

(i) *If $\sigma \in \widehat{G}$, then $P_\theta P_\sigma = \delta_{\theta\sigma} P_\theta$.*

(ii) *P_θ is a continuous projection and $E_\theta = P_\theta(E)$ is a closed π-invariant subspace of E.*

(iii) *$E = \oplus_{\theta\in\hat{G}} E_\theta$(generalized direct sum).*

(iv) *Every nonzero E_θ $(\theta \in \widehat{G})$, is linearly spanned by d_θ-dimensional, π-invariant subspaces of E, on each of which π is equivalent to $\overline{\theta}$.*

Proof. (i) follows from Lemma 3.3(ii) and (17):

$$P_\theta P_\sigma = \pi(d_\theta \mathrm{tr}(\theta))\pi(d_\sigma \mathrm{tr}(\sigma)) = \pi(d_\theta \mathrm{tr}(\theta) * d_\sigma \mathrm{tr}(\sigma)) = \delta_{\theta\sigma}\pi(d_\theta \mathrm{tr}(\theta)) = \delta_{\theta\sigma} P_\theta.$$

(ii) It follows from (i) that each P_θ is a continuous projection on E, and thus $P_\theta(E)$ is a closed subspace of E. Moreover, for each $u \in E$,

$$P_\theta u = \pi(d_\theta \mathrm{tr}(\theta))u = d_\theta \sum_{i=1}^{d_\theta} \pi(\theta_{ii})u \in E_\theta,$$

and therefore $P_\theta(E) \subset E_\theta$. To prove the reverse inclusion, let $1 \le i, j \le d_\theta$, and observe that by (16) we have

$$P_\theta \pi(\theta_{ij}) = \pi(d_\theta \mathrm{tr}(\theta))\pi(\theta_{ij}) = \pi(d_\theta \mathrm{tr}(\theta) * \theta_{ij}) = \pi(\theta_{ij}),$$

from which it follows that for each $u \in E$, $\pi(\theta_{ij})u = P_\theta\pi(\theta_{ij})u \in P_\theta(E)$. This shows that $E_\theta \subset P_\theta(E)$, and hence, $P_\theta(E) = E_\theta$. Since E_θ is spanned by $E_{\theta,u}$ as u varies in E, its π-invariance follows from Theorem 3.5(i).

(iii) It follows from (i) that $P_\theta(E) \cap P_\sigma(E) = \{0\}$ if θ is not equivalent to σ. Thus, using (ii), we have

$$E_\theta \cap \mathrm{Span} \bigcup_{\substack{\sigma\in\widehat{G},\\ \sigma\neq\theta}} E_\sigma = \{0\}.$$

Suppose that

$$x \in E_\theta \cap \overline{\mathrm{Span}} \bigcup_{\substack{\sigma\in\widehat{G},\\ \sigma\neq\theta}} E_\sigma.$$

Since P_θ is a projection on E_θ, we have $P_\theta(x) = x$. Now let

$$x_\alpha \in \mathrm{Span} \bigcup_{\substack{\sigma\in\widehat{G},\\ \sigma\neq\theta}} E_\sigma$$

be such that $x_\alpha \to x$ in the topology of E. Since each x_α can be written as a finite sum of elements from subspaces E_σ $(\sigma \neq \theta)$, and since $P_\theta P_\sigma = 0$, it follows that $P_\theta(x_\alpha) = 0$ for all α. Thus

$$x = P_\theta(x) = P_\theta(\lim_\alpha x_\alpha) = \lim_\alpha P_\theta(x_\alpha) = 0.$$

This proves that $E_\theta \cap \overline{\text{Span}} \bigcup_{\sigma\in\hat{G},\sigma\neq\theta} E_\sigma = \{0\}$, and hence condition (a) of a generalized direct sum is verified.

The fact that $\text{Span} \bigcup_{\theta\in\widehat{G}} E_\theta$ is dense in E follows from (13), since by assumption every $u \in E$ is almost periodic, and by definition, E_θ is spanned by the subspaces $E_{\theta,u}$ as u varies in E. Thus the condition (b) of a generalized direct sum is also verified. This complete the proof of (iii).

(iv) Let $\{\xi_1, \xi_2, \ldots, \xi_{d_\theta}\}$ be an orthonormal basis of H_θ (the representation space of θ), and θ_{ij} be the corresponding coefficient functions of θ. Thus the coefficient functions of $\overline{\theta}$ in the same basis are the complex conjugates $\overline{\theta}_{ij}$.

Let $u \in E_\theta$ be any nonzero vector. Since by (i) and (ii), $P_\theta = \pi(d_\theta \text{tr}(\theta))$ is a projection onto E_θ, it follows that

$$u = \pi(d_\theta \text{tr}(\theta))u = d_\theta \sum_{i=1}^{d_\theta} \pi(\theta_{ii})u, \tag{20}$$

and hence there must exist an index $1 \leq k \leq d_\theta$, such that $\pi(\theta_{kk})u \neq 0$. Let us fix this vector u and the index k for the rest of the proof of (iv). Let $u_i \in E_\theta$ be defined by $u_i = \pi(\theta_{ik})u$, $i = 1, 2, \ldots, d_\theta$. Let F be the subspace of E_θ spanned by $\{u_1, \ldots, u_{d_\theta}\}$. We claim that the linear map V defined by

$$V\colon H_\theta \longrightarrow F, \quad V(\xi_i) = u_i \qquad (i = 1, \ldots, d_\theta), \tag{21}$$

is a linear isomorphism under which the representation $\overline{\theta}$ is equivalent to (π, F), where (π, F) denotes π when its representation space is restricted to F.

It follows from (14) that for any $x \in G$ and $1 \leq j \leq d_\theta$,

$$\pi(x)\pi(\theta_{jk}) = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}\pi(\theta_{ik}). \tag{22}$$

The invariance of F under π follows from (22) since for each $1 \leq j \leq d_\theta$, and $x \in G$:

$$\pi(x)u_j = \pi(x)\pi(\theta_{jk})u = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}\pi(\theta_{ik})u = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}u_i \in F. \tag{23}$$

Next, we observe that for any $x \in G$ and $1 \le j \le d_\theta$, using the definitions of u_i and ξ_i, and using (22) and (21) we can write

$$\begin{aligned} V\overline{\theta}(x)\xi_j &= V(\sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}\xi_i) = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}V\xi_i = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}u_i \\ &= \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)}\pi(\theta_{ik})u = \pi(x)\pi(\theta_{jk})u = \pi(x)u_j = \pi(x)V\xi_j. \end{aligned}$$

Since ξ_j is arbitrary element of the basis of H_θ, it follows that for all $x \in G$:

$$V\overline{\theta}(x) = \pi(x)V. \tag{24}$$

Now we can show that V is injective, in fact (24) shows that $\ker V$ is invariant under $\overline{\theta}(x)$ for all $x \in G$, and since $\overline{\theta} \in \widehat{G}$ is irreducible, it follows that $\ker V = \{0\}$ (since $V \neq 0$). Moreover, the relation (24) also shows that $\overline{\theta}$ and (π, F) are equivalent representations.

Finally the claim that E_θ is spanned by such finite-dimensional subspaces as F is immediate from the above arguments. In fact, in the identity (20), each nonzero vector $\pi(\theta_{ii})u$ belongs to a π-invariant subspace F of E_θ constructed as above, and thus u belongs to the linear span of such subspaces. Since $u \in E_\theta$ is arbitrary, the claim follows. □

As an application of the decomposition Theorem 4.3 we can prove the following theorem which extends a result by Bochner–von Neumann [3, Theorem 40, p. 49] from $L^2(G)$ to quasi-complete subspaces of $L^p(G)$ $(1 \le p < \infty)$, on which the left regular representation is continuous and equicontinuous. Our proof is inspired by the argument of Bochner and von Neumann.

Theorem 4.4. *Let G be a locally compact group equipped with a left Haar measure μ. Let $1 \le p < \infty$, and $E \subset L^p(G)$ be a quasi-complete space (under a suitable locally convex topology) such that E is left translation invariant, and the left regular representation*

$$\lambda\colon G \longrightarrow \mathcal{B}(E), \quad (\lambda(x)u)(y) = u(x^{-1}y) \qquad (u \in E,\ x, y \in G),$$

is continuous and equicontinuous. Then λ is an almost periodic representation if and only if G is compact.

Remark. Examples of spaces E satisfying the conditions of the theorem include the Banach spaces $L^p(G)$ and $L^p(G) \cap L^r(G)$ $(1 \leq p < r \leq \infty)$, where the latter space is equipped with the norm $\|f\| = \|f\|_p + \|f\|_r$.

Proof. If G is compact, then for every $u \in E$, the vector-valued continuous function $G \longrightarrow E$, $x \mapsto \lambda(x)u$, is almost periodic ([23, Corollary 3.7]) and therefore λ is almost periodic by Theorem 3.2.

To prove the converse, let λ be almost periodic. By Theorem 4.3, we know that E is a generalized direct sum of closed subspaces E_θ $(\theta \in \widehat{G})$, and each nonzero E_θ is spanned by its finite-dimensional, λ-invariant subspaces. Let $\theta \in \widehat{G}$ be chosen such that $E_\theta \neq \{0\}$, and let $u \in E_\theta$ be a nonzero vector. By our discussion following the identity (20) (with π replaced with λ), there exists some $1 \leq k \leq d_\theta$, such that the vectors $u_j = \lambda(\theta_{jk})u$, $j = 1, \ldots, d_\theta$, form a basis for a λ-invariant subspace F of E_θ. With π replaced with λ, it follows from (23) that for every $x \in G$, $1 \leq j \leq d_\theta$,

$$\lambda(x)u_j = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)} u_i,$$

or

$$u_j(x^{-1}y) = \sum_{i=1}^{d_\theta} \overline{\theta_{ij}(x)} u_i(y) \qquad y\text{-}a.e.$$

Replacing x with x^{-1}, we find that for all $x \in G$ and all $1 \leq j \leq d_\theta$,

$$u_j(xy) = \sum_{i=1}^{d_\theta} \theta_{ji}(x) u_i(y) \qquad y\text{-}a.e. \tag{25}$$

We define a measurable function $f(x, y) \geq 0$ on $G \times G$ by

$$f(x, y) = \sum_{j=1}^{d_\theta} |u_j(xy) - \sum_{i=1}^{d_\theta} \theta_{ji}(x) u_i(y)|.$$

By (25), for every $x \in G$, $f(x, y) = 0$ (y-a.e.), and hence

$$\int_G \int_G f(x, y)\, d\mu(y)\, d\mu(x) = 0.$$

By Fubini's theorem we find that the positive measurable function $y \mapsto \int_G f(x,y)\, d\mu(x)$, has zero integral, and therefore this function must be 0, for almost all y. In particular, there is some $y_0 \in G$ such that $\int_G f(x,y_0)\, d\mu(x) = 0$. Since $f \geq 0$, we find that $f(x,y_0) = 0$, for almost all $x \in G$. Thus we have shown that there is some $y_0 \in G$ such that for every $1 \leq j \leq d_\theta$:

$$u_j(xy_0) = \sum_{i=1}^{d_\theta} \theta_{ji}(x) u_i(y_0) \qquad x\text{-}a.e. \tag{26}$$

Let $\Phi \colon G \longrightarrow \mathbb{C}^{d_\theta}$ be defined by $\Phi(x) = (u_1(x), \ldots, u_{d_\theta}(x))$. The identities in (26) for $1 \leq j \leq d_\theta$, are equivalent to

$$\Phi(xy_0) = \theta(x)\Phi(y_0) \qquad x\text{-}a.e.,$$

or since $\theta(x)$ is unitary,

$$\Phi(y_0) = \overline{\theta(x)}^{\,T} \Phi(xy_0) \qquad x\text{-}a.e. \tag{27}$$

It follows from (26) that for some $1 \leq i_0 \leq d_\theta$, $u_{i_0}(y_0) \neq 0$ since otherwise $u_j = 0$ contradicting that u_j is an element of a basis for F. For this u_{i_0}, it follows from (27) that

$$u_{i_0}(y_0) = \sum_{j=1}^{d_\theta} \overline{\theta_{ji_0}(x)} u_j(xy_0) \qquad x\text{-}a.e.,$$

and hence using Hölder's inequality and the fact that $|\theta_{ji_0}(x)| \leq 1$, we get

$$|u_{i_0}(y_0)|^p \leq d_\theta^{p/p'} \sum_{j=1}^{d_\theta} |u_j(xy_0)|^p \qquad x\text{-}a.e.,$$

where $1/p + 1/p' = 1$. Integrating with respect to x, and recalling $u_j \in L^p(G)$, gives

$$|u_{i_0}(y_0)|^p \mu(G) \leq d_\theta^{p/p'} \Delta(y_0)^{-1} \sum_{j=1}^{d_\theta} \int_G |u_j(x)|^p \, d\mu(x) < \infty.$$

Thus $\mu(G) < \infty$, and G is compact ([11, Theorem 15.9, p. 195]). □

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