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The Lie group of isometries of a pseudo-Riemannian manifold

Hung Pham

Dedicate to Professor Anthony To-Ming Lau with admiration on the occasion of his 80th birthday

We give an elementary proof of the Myers–Steenrod theorem, stating that the group of isometries of a connected Riemannian manifold M is a Lie group acting smoothly on M. Our proof follows the approach of Chu and Kobayashi, but replacing their use of a theorem of Palais with a topological condition detecting when a locally compact subspace of M is an embedded integral manifold of a given k-plane distribution.

Keywords and phrases: Lie group, locally compact group, isometries, pseudo-Riemannian manifold, integral manifold.

1. Introduction

Throughout this paper, let M be a (smooth) n-manifold (without boundary). Note that n-manifolds as well as Lie groups are *second countable* by definition; this is an important assumption (and requirement) that seems to be omitted in [2] and in [6].

Suppose that M is a connected Riemannian manifold. A well-known theorem of Myers and Steenrod [5] states that the group of isometries of a Riemannian manifold M, equipped with the compact-open topologies, admits a unique differential structure making it a Lie group acting smoothly on M. The original proof in [5] is complicated with difficult computation. Here, we will follow the approach of Chu and Kobayashi [2] (see also the textbook [4]). This approach uses the theory of principal bundles (see [3] or [7]) to reduce to the following theorem of Kobayashi; in fact, it even generalises the Myers–Steenrod theorem to pseudo-Riemann manifolds.

Theorem 1.1. *Let M be a* connected *smooth n-manifold that admits a smooth global frame $(E_1, \dots, E_n)$: that is*

> *each $E_j \in \mathfrak{X}(M)$ and, for every $p \in M$, $(E_{1p}, \ldots, E_{np})$ is a basis for T_pM.*

Denote by G the set of diffeomorphisms of M that leave each E_j invariant: that is

$$f_*(E_{jp}) = E_{jf(p)} \qquad (p \in M,\ f \in G,\ 1 \le j \le n)\,.$$

Then G is a group of diffeomorphisms of M that, equipped with the compact-open topology[1]*, is a topological group that acts continuously, freely, and properly on M with closed orbits, and there exists a unique smooth structure on G making it a Lie group of diffeomorphisms that acts smoothly on M.*[2]

The proof of this theorem in [4, Theorem I.3.2] uses a theorem of Palais (see [4, Theorem I.3.1]), and it goes by constructing a smooth structure on a certain normal subgroup H of G and then shows that it is possible to translate the topology and smooth structure on H to the other H-cosets to make G a Lie group. But it does not seem to address whether this new topology on G agrees with the compact-open one, and it does not even address whether there are countably or uncountably many H-cosets in G (for G to be a Lie group, there must be only countably many cosets of an open subgroup such as H). Thus without further argument, it is quite possible that H is just the trivial group, and then G becomes an uncountable discrete group.

To work around this issue, we shall adopt some element of the proof in [5], and introduce a topological condition guaranteeing that a locally compact subspace of a manifold is an integral submanifold for some given plane distribution. This is carried out in §2. This is then applied in §3 to prove Theorem 1.1; for the sake of the readers we include complete details, not just the part that requires §2.

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[1]Given two manifolds M and $\widetilde{M}$, recall that the **compact-open topology** on the set of continuous maps $M \to \widetilde{M}$ has a subbase consists of sets of the form

$$\Big\{ f : M \to \widetilde{M} \colon\ f \text{ is continuous, and } f(K) \subseteq V \Big\}$$

for compact sets K in M and open sets V in $\widetilde{M}$. Given any metric $\widetilde{d}$ that defines the topology of $\widetilde{M}$, the compact-open topology is the same as the topology of uniform convergence on compact subsets of M. See [1, Chapter VII] for more details.

[2]This implicitly means that the given topology on G already makes it a topological manifold.

2. Integral manifolds without Frobenius

Throughout this section, *let $S \subseteq TM$ be a k-plane distribution on M, and let N be a topological subspace of M*. This section is to give sufficient conditions for N to be an embedded integral manifold of S. Note that the k-plane distribution S is not assumed to be Frobenius, and so the Frobenius theorem is not applicable. See [3], especially §4.5 for k-plane distributions, integral manifolds, and the Frobenius theorem.

The following notions have their roots in the proof in [5], where M is a Riemannian manifolds and normal coordinate neighbourhoods (through geodesics) are used.

Definition 2.1.

(i) Let (p_i) be a sequence that converges to p in M, and let $v \in T_pM\setminus\{0\}$. We says that v is a **direction of approach of** (p_i) if $p_i \neq p$ eventually and there exists a coordinate neighbourhood (U, φ) about p such that (identifying $\mathbb{R}^n$ with its tangent plane at each point) the $\mathbb{R}^n$-vectors $\varphi(p_i) - \varphi(p)$ have directions approach that of $\mathrm{d}\varphi_p(v)$. That is

$$\lim_{i\to\infty} \frac{\varphi(p_i) - \varphi(p)}{\|\varphi(p_i) - \varphi(p)\|} = \frac{\mathrm{d}\varphi_p(v)}{\|\mathrm{d}\varphi_p(v)\|}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

(ii) Let (p_i) and (q_i) be sequences that converges to p in M, and let $v \in T_pM\setminus\{0\}$. We says that v is a **direction of approach of** (p_i) **relative to** (q_i) towards p if $p_i \neq q_i$ eventually and there exists a coordinate neighbourhood (U, φ) about p such that the $\mathbb{R}^n$-vectors $\varphi(p_i) - \varphi(q_i)$ have directions approach that of $\mathrm{d}\varphi_p(v)$.

So if v is a direction of approach of a sequence of points, then so too is any positive multiple of it. Similar remark holds for the relative version. The following shows that these notions are independent of coordinates used:

Lemma 2.2. *Let $f : U \to \mathbb{R}^m$ be smooth where U is open in $\mathbb{R}^n$.*

(i) *Let (p_i) be a sequence that converges to p in U such that $p_i \neq p$ eventually, and let $v \in \mathbb{R}^n$ be such that $Jf_p(v) \neq 0$.*

$$\textit{If} \quad \lim_{i\to\infty} \frac{p_i - p}{\|p_i - p\|} = v \quad \textit{then} \quad \lim_{i\to\infty} \frac{f(p_i) - f(p)}{\|f(p_i) - f(p)\|} = \frac{Jf_p(v)}{\|Jf_p(v)\|}$$

(ii) *Let (p_i) and (q_i) be sequences that converges to p in U such that $p_i \neq q_i$ eventually, and let $v \in \mathbb{R}^n$ be such that $Jf_p(v) \neq 0$.*

$$\text{If} \quad \lim_{i\to\infty} \frac{p_i - q_i}{\|p_i - q_i\|} = v \quad \text{then} \quad \lim_{i\to\infty} \frac{f(p_i) - f(q_i)}{\|f(p_i) - f(q_i)\|} = \frac{Jf_p(v)}{\|Jf_p(v)\|}$$

Proof. Part (i) is a special case of (ii) (with all $q_i = p$), and so let us prove (ii). Let $\varepsilon > 0$. Then there exists an Euclidean ball B_ε centred at p in U such that if $x \in B_\varepsilon$, then the operator norm

$$\|Jf_x - Jf_p\| < \varepsilon\,.$$

Let i_0 be an index such that if $i \geq i_0$. Then $p_i, q_i \in B_\varepsilon$. Write $w_i := p_i - q_i$. Then

$$\|f(p_i) - f(q_i) - Jf_p(w_i)\| = \left\| \int_0^1 Jf_{tp_i+(1-t)q_i}(w_i)\mathrm{d}t - Jf_p(w_i) \right\| \leq \varepsilon \|w_i\|\,.$$

Since $w_i / \|w_i\| \to v$ as $i \to \infty$, we see from the above that

$$\frac{f(p_i) - f(q_i)}{\|w_i\|} \to Jf_p(v)$$

which shows that the direction of $f(p_i) - f(q_i)$ approaches that of $Jf_p(v)$. □

Definition 2.3. Let N be a topological subspace of M, let $p \in N$, and let $v \in T_pM$.

(i) The vector v is called an **approaching direction for** N **at** p if v is a direction of approach of a sequence (p_i) in N that converges to p. The **approaching direction set for** N **at** p, denoted by $\mathtt{A}_p^N$, is the set of all the approaching directions for N at p.

(ii) The vector v is called a **relative approaching direction for** N **at** p if v is a direction of approach of a sequence (p_i) relative to a sequence (q_i), both converge to p in N. The **relative approaching direction set for** N **at** p, denoted by $\mathtt{relA}_p^N$, is the set of all the relative approaching directions for N at p.

The following properties are obvious.

Lemma 2.4. *Let N be a subspace of M and let $p \in N$. Then*

(i) $\mathtt{A}_p^N \subseteq \mathtt{relA}_p^N$.

(ii) *If $v \in \mathtt{relA}_p^N$, then $-v \in \mathtt{relA}_p^N$.*

The following theorem will be the base to prove other results.

Theorem 2.5. *Let M be a n-manifold, let $S \subseteq TM$ be a k-plane distribution on M, and let N be a topological subspace of M. Suppose that the following two conditions hold for each $p \in N$:*

- $\mathtt{A}_p^N \subseteq S_p$, *and*
- *there exists a coordinate neighbourhood $(U, \xi = (x^1, \ldots, x^n))$ of p in M such that $\xi(p) = 0$, S_p is the tangent space at $p = 0$ for the coordinate k-plane $\xi^{-1}(\mathbb{R}^k) = \{x \in U : x^{k+1} = \ldots = x^n = 0\}$, and the projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ restricted to a homeomorphism from $\xi(U \cap N)$ onto an open neighbourhood of 0 in $\mathbb{R}^k$.*

Then N is an embedded k-submanifold of M and an integral manifold of S.

Proof. Since this is a local problem, it is sufficient for us to consider the case that $M = U$ is an open neighbourhood of $p = 0$ in $\mathbb{R}^n$, and (U, ξ) is the standard coordinate on M. Since $\mathrm{d}\pi_p = \pi$ under the natural identification $\mathbb{R}^n \equiv T_p(\mathbb{R}^n)$ and $\mathbb{R}^k \equiv T_{\pi(p)}\mathbb{R}^k$, it restricts to the identity map $S_p \to \mathbb{R}^k$. So we may and shall suppose (by shrinking M further if necessary) that if $x \in M$, then $\mathrm{d}\pi_x$ is injective on S_x, and so a linear bijection from S_x onto $\mathbb{R}^k \equiv T_{\pi(x)}\mathbb{R}^k$. For each $x \in M$ and each $v \in \mathbb{R}^k$, set

$$\eta(x, v) := (\mathrm{d}\pi_x|_{S_x})^{-1}(v),$$

where v is considered as an element of $T_{\pi(x)}\mathbb{R}^k$. Then $\eta : M \times \mathbb{R}^k \to S$ is smooth.

The hypothesis states that the restriction $\pi|_N$ of π maps N homeomorphically onto an open k-ball B centred at 0. Denote by φ the inverse of $\pi|_N$, so that $\varphi : B \to M$ is a topological embedding with image N. It is sufficient to prove that φ is a smooth embedding.

Take $a \in B$, and a nonzero vector $v \in \mathbb{R}^k \equiv T_a\mathbb{R}^k$. Then, by the definition of $\mathtt{A}_{\varphi(a)}^N$, for any sequence $t_i \to 0^+$, if

$$\lim_{i\to\infty} \frac{\varphi(a + t_i v) - \varphi(a)}{\|\varphi(a + t_i v) - \varphi(a)\|} = w$$

exists, then it belongs to $\mathtt{A}_{\varphi(a)}^N \subseteq S_{\varphi(a)}$, and by Lemma 2.2 applied to the map π, we see that $\mathrm{d}\pi_{\varphi(a)}(w)$ has the same direction as v. Thus w is the normalisation of $\left[(\mathrm{d}\pi_{\varphi(a)}|_{S_{\varphi(a)}})^{-1}(v)\right]$, and so

$$w = \frac{\eta(\varphi(a), v)}{\|\eta(\varphi(a), v)\|}.$$

A standard compactness argument then shows that

$$\lim_{t\to 0^+} \frac{\varphi(a+tv)-\varphi(a)}{\|\varphi(a+tv)-\varphi(a)\|} = \frac{\eta(\varphi(a),v)}{\|\eta(\varphi(a),v)\|}$$

Since π is an orthogonal (linear) projection from $\mathbb{R}^n \to \mathbb{R}^k$, and $\pi\circ\varphi = \mathrm{id}_B$, we see that

$$\lim_{t\to 0^+} \frac{\|\varphi(a+tv)-\varphi(a)\|}{\|tv\|} = \lim_{t\to 0^+} \frac{\|\varphi(a+tv)-\varphi(a)\|}{\|\pi(\varphi(a+tv)-\varphi(a))\|}$$

is equal to the cosecant of the acute angle between $\eta(\varphi(a),v)$ and $\pi(\eta(\varphi(a),v)) = v$, which is $\|\eta(\varphi(a),v)\| \,/\, \|v\|$. Hence,

$$\lim_{t\to 0^+} \frac{\varphi(a+tv)-\varphi(a)}{t} = \frac{\eta(\varphi(a),v)}{\|\eta(\varphi(a),v)\|} \cdot \frac{\|\eta(\varphi(a),v)\|}{\|v\|} \cdot \|v\| = \eta(\varphi(a),v)\,.$$

A similar argument shows that this is true with $t \to 0^-$ as well. Hence

$$\left.\frac{\mathrm{d}\varphi(a+tv)}{\mathrm{d}t}\right|_{t=0} = \eta(\varphi(a),v)$$

The above argument shows in particular that φ has continuous partial derivatives of first order everywhere on B, and so it is $\mathcal{C}^1$ on B. Moreover, its Jacobian satisfies

$$J\varphi_a(v) = \eta(\varphi(a),v) \quad (a\in B,\ v\in\mathbb{R}^k)\,,$$

and so φ is smooth with constant rank k. □

Here is the main theorem of this section.

Theorem 2.6. *Let M be an n-manifold, let $S \subseteq TM$ be a k-plane distribution on M, and let N be a locally compact subspace of M. Suppose that the following two conditions hold for each $p \in N$:*

(i) $\mathtt{A}_p^N$ *spans S_p linearly, and*
(ii) $\mathtt{relA}_p^N$ *is included in the cone generated by* $\mathtt{A}_p^N$.

Then N is an embedded k-submanifold of M and an integral manifold of S.

Lemma 2.7. *Suppose that N is locally compact. Let $p\in N$, and let $(U, \xi = (x^1,\dots,x^n))$ be any coordinate neighbourhood of p in M such that $\xi(p) = 0$*

and S_p is the tangent space at p for the coordinate k-plane

$$\xi^{-1}(\mathbb{R}^k) = \left\{x \in U : x^{k+1} = \ldots = x^n = 0\right\}.$$

If $\mathtt{relA}_p^N \subseteq S_p$, then there exists an open neighbourhood V of p in U such that the coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ restricted to a homeomorphism from $V \cap N$ onto its image.

Proof. It is sufficient to work inside U, thinking of it as an open subset of $\mathbb{R}^n$ and ξ as the standard coordinate, and show the existence of an open neighbourhood V of p in U such that the coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is injective on $V \cap N$: indeed, by shrinking V further if necessary, using the local compactness of of N, we may require further that $\overline{V} \cap N$ is compact, and an injective continuous map from a compact space onto a Hausdorff space is a homeomorphism.

Assume towards a contradiction that for every open neighbourhood V of p with compact closure in U there are two distinct points in $V \cap N$ with the same π-image. But this with a compactness argument shows that there exist two sequences (p_i) and (q_i) that converge to p with $p_i \neq q_i$ and a nonzero vector $v \perp S_p$ and hence $v \notin S_p$ such that v is the approaching direction of (p_i) relative to (q_i). This contradicts the assumption that $\mathtt{relA}_p^N \subseteq S_p$. $\square$

Lemma 2.8. *Theorem 2.6 holds in the case where $k = n$.*

Proof. Since N is locally compact, replacing M by an open subset if necessary, we may and shall suppose that N is closed in M. Also, restricting to a coordinate neighbourhood, we may suppose that $M = \mathbb{R}^n$. We need to show that N is open in $\mathbb{R}^n$ (and so, by connectedness, must be all of $\mathbb{R}^n$ – but this is not important).

Assume towards a contradiction that N is not open. Then there exist $p \in N$ and a sequence (a_i) in $\mathbb{R}^n \setminus N$ that converges to p. Let us temporarily fix an i, and write $a = a_i$. Let δ be the distance from a to N (with respect to the Euclidean distance). Since N is closed, we see that $\delta > 0$ and there exists $b \in N$ such that $\|a - b\| = \delta$. Consider the closed ball

$$B := \{x \in \mathbb{R}^n : \|x - a\| \leq \delta\}.$$

Then $B^\circ \cap N = \emptyset$ and $b \in N \cap \partial B$.

We *claim* that $\mathtt{A}_b^N \subseteq T_b C$, where $C := \partial B$. Indeed, assume the contrary that there exists a vector $v \in \mathtt{A}_b^N \setminus T_b C$. Denote by L the hyperplane tangent to C at b; L divides $\mathbb{R}^n$ into two halves. Then there are two cases. Case 1: if

v points to the half that includes B, then by the definition of $v \in \mathtt{A}_b^N$, some element of N must be in B°; a contradiction. Case 2: if v points to the half that does not include B, then since $v \in \mathtt{A}_b^N \subseteq \mathtt{relA}_b^N$, we have $-v \in \mathtt{relA}_b^N$, and so, by condition (ii) of the hypothesis of Theorem 2.6, $-v$ lies in the cone generated by $\mathtt{A}_b^N$. Thus, in Case 2, some $w \in \mathtt{A}_b^N \setminus T_bC$ will point to the half of $\mathbb{R}^n$ (as divided by L) that includes B, and we arrive at a contradiction as in Case 1.

But the *claim* contradicts condition (i) of the hypothesis of Theorem 2.6 in the case where $k = n$, so N must be open as asserted. □

Proof of Theorem 2.6. Let $p \in N$. By Lemma 2.7, there exists a coordinate neighbourhood $(V, \xi = (x^1, \ldots, x^n))$ of p in M such that $\xi(p) = 0$ and S_p is the tangent space at p for the coordinate k-plane $\xi^{-1}(\mathbb{R}^k) = \{x \in V : x^{k+1} = \ldots = x^n = 0\}$ such that the coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is a homeomorphism from $V \cap N$ onto $\pi(V \cap N)$. Shrinking V further if necessary, we suppose that $\mathrm{d}\pi$ is injective on S_x for every $x \in V$. Working in V, we shall suppose that V is an open subset of $\mathbb{R}^n$ and ξ is the standard coordinate map.

Set $P := \pi(V \cap N)$. We *claim* that $\mathtt{A}_a^P$ spans $T_a\mathbb{R}^k$ and $\mathtt{relA}_a^P$ is contained in the cone generated by $\mathtt{A}_a^P$ for every $a \in P$. Indeed, if $x \in V \cap N$, then $\mathrm{d}\pi_x(\mathtt{A}_x^N) = \mathtt{A}_{\pi(x)}^P$ and $\mathrm{d}\pi_x(\mathtt{relA}_x^N) = \mathtt{relA}_{\pi(x)}^P$: The inclusions $\mathrm{d}\pi_x(\mathtt{A}_x^N) \subseteq \mathtt{A}_{\pi(x)}^P$ and $\mathrm{d}\pi_x(\mathtt{relA}_x^N) \subseteq \mathtt{relA}_{\pi(x)}^P$ are immediate from Lemma 2.2. To prove the reverse inclusions, say $\mathtt{A}_{\pi(x)}^P \subseteq \mathrm{d}\pi_x(\mathtt{A}_x^N)$ for example, take $\{x_i\} \in V \cap N$ such that $\pi(x_i)$ converges to $\pi(x)$ with an approaching direction $w \in T\mathbb{R}_{\pi(x)}^k$. Then $x_i \to x$, and assume the contrary that this convergent sequence did not have $v := (\mathrm{d}\pi|_{S_x})^{-1}(w)$ as its approaching direction. Since $\pi(x_i) \neq \pi(x)$ eventually, we have $x_i \neq x$ eventually, and so passing to a subsequence if necessary, we shall suppose that $(x_i - x)/\|x_i - x\|$ is kept away from $v/\|v\|$. Then, by compactness, passing to a further subsequence if necessary, $(x_i - x)/\|x_i - x\|$ converges to some $v' \neq v/\|v\|$. But then, Lemma 2.2 implies that $\mathrm{d}\pi_x(v')$ has the same direction as w, and so v' must have the same direction as v – a contradiction.

The *claim* then follows from the fact that $\mathrm{d}\pi$ is a linear bijection from S_x onto $T_{\pi(x)}\mathbb{R}^k$ for every $x \in V$ and conditions (i) and (ii) in the hypothesis of the theorem.

The special case, proved in Lemma 2.8, applied to the locally compact subspace P of $\mathbb{R}^k$, then implies that P is open in $\mathbb{R}^k$. Thus the hypothesis of Theorem 2.5 holds, and the current theorem then follows. □

As a sidenote, when N is locally Euclidean, the assumptions concerning $\mathtt{A}^N$ and $\mathtt{relA}^N$ can be relaxed with the use of Brower's invariance of domain theorem.

Theorem 2.9. *Let M be an n-manifold, let $S \subseteq TM$ be a k-plane distribution on M, and let N be a locally Euclidean subspace of M of dimension k. Suppose that $\mathtt{relA}_p^N \subseteq S_p$ for each $p \in N$. Then N is an embedded k-submanifold of M and an integral manifold of S.*

Proof. Let $p \in N$. By Lemma 2.7, there exists a coordinate neighbourhood $(V, \xi = (x^1, \ldots, x^n))$ of p in M such that $\xi(p) = 0$ and S_p is the tangent space at p for the coordinate k-plane $\xi^{-1}(\mathbb{R}^k) = \left\{x \in V : x^{k+1} = \ldots = x^n = 0\right\}$ such that the coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ restricted to a homeomorphism from $V \cap N$ onto its image. Since N is locally Euclidean of dimension k, the invariance of dimension theorem implies that $\pi(V \cap N)$ is an open subset of $\mathbb{R}^k$. Theorem 2.5 then applies. □

3. Proof of Theorem 1.1

First, note that it follows from the equivariant rank theorem and the condition that G acts smoothly and freely on M with closed orbits that, given any $p \in M$, the map

$$f \mapsto f(p), \;\; G \mapsto M$$

is a smooth embedding. This proves the uniqueness of the smooth structure on G.

It remains to prove the existence. For this, we shall go through a succession of lemmas, where each notation will keep the same meaning once it is introduced. For each smooth vector field $X \in \mathfrak{X}(M)$, let us denote by $\Phi_t^X(p) = \Phi^X(t, p)$ the maximal flow with generator X (see [3] for more details).

Definition 3.1. Define $\mathfrak{E}$ to be the linear span of $E_1, \ldots, E_n$ in $\mathfrak{X}(M)$. Set

$$\Psi(E, p) := \Phi_1^E(p)$$

for each $E \in \mathfrak{E}$ and $p \in M$ such that $\Phi_t^E(p)$ is defined for $0 \leq t \leq 1$. Then, for each $E \in \mathfrak{E}$, set

$$M_E := \{p \in M \mid \;\Psi(E, p) \text{ is defined}\}\,.$$

Lemma 3.2.

(i) Ψ *is a smooth M-valued function defined on an open neighbourhood* $\mathfrak{U}$ *of* $\{0\} \times M$ *in* $\mathfrak{E} \times M$.
(ii) $\Psi(tE, p) = \Phi_t^E(p)$ *whenever either side is defined.*
(iii) $\mathrm{d}\Psi_{(0,p)} : (E, v) \mapsto E_p + v$ *for each* $p \in M$, *where we identify* $T_0\mathfrak{E}$ *with* $\mathfrak{E}$ *naturally as for any linear manifold, and identify* $T_{(0,p)}\mathfrak{U}$ *with* $T_0\mathfrak{E} \oplus T_pM$.
(iv) *For every* $E \in \mathfrak{E}$,

$$M_E = \{p \in M\colon\ (E, p) \in \mathfrak{U}\} \quad \textit{is open in } M$$

and $\Psi(E, \cdot)$ *is a diffeomorphism from* M_E *onto* M_{-E} *with inverse* $\Psi(-E, \cdot)$.

Proof. These follow from the basic facts of the theory of (maximal) flows [3]. □

Definition 3.3. For each $p, q \in M$, define $L_{p,q} : T_pM \to T_qM$ by setting

$$L_{p,q}(E_p) = E_q \quad \text{for every} \quad E \in \mathfrak{E}.$$

Lemma 3.4. *Let* $f : M \to M$. *Then the following are equivalent:*

(i) f *is a local diffeomorphism and and* $\mathrm{d}f_p = L_{p,f(p)}$ *for every* $p \in M$.
(ii) f *is* $\mathcal{C}^1$ *and leaves each* $E \in \mathfrak{E}$ *invariant.*
(iii) $f(\Phi_t^E(p)) = \Phi_t^E(f(p))$ *whenever* $\Phi_t^E(p)$ *is defined.*
(iv) *If* $(E, p) \in \mathfrak{U}$, *then* $(E, f(p)) \in \mathfrak{U}$ *and* $f(\Psi(E, p)) = \Psi(E, f(p))$.
(v) *For each* $p \in M$, *there is a neighbourhood* $\mathfrak{V}$ *of* 0 *in* $\mathfrak{E}$ *such that if* $E \in \mathfrak{V}$ *then* $f(\Psi(E, p)) = \Psi(E, f(p))$.

In particular,

$$G = \{\textit{bijective } f : M \to M \textit{ such that } f(\Psi(E,p)) = \Psi(E, f(p)) \textit{ for } (E,p) \in \mathfrak{U}\}.$$

Proof. It is obvious that (i)⇒(ii) and that (iii)⇔(iv) and (iv)⇒(v). To see that (ii)⇒(iii), takes $p \in M$. Then $\gamma(t) := f(\Phi_t^E(p))$ is a $\mathcal{C}^1$ curve defined on an open interval with $\gamma(0) = f(p)$ and

$$\gamma'(t) = \mathrm{d}f(E_{\Phi_t^E(p)}) = E_{f(\Phi_t^E(p))} = E_{\gamma(t)}$$

where the second equality is due to f leaving E invariant. Thus γ is a flow line of E starting at $f(p)$, and so $\gamma(t) = \Phi_t^E(f(p))$.

To see that (v)⇒(i), it is sufficient to show that f is a local diffeomorphism; the rest then follows from the properties of flows. For each $p \in M$, part (iii) of Lemma 3.2 and the inverse function theorem imply that $\Psi(\cdot, p)$ is a diffeomorphism from an open neighbourhood of 0 in $\mathfrak{E}$ onto an open neighbourhood of p in M. Thus the local diffeomorphism property of f in the said neighbourhood of p follows from that of Ψ. $\square$

The following connectedness argument will be used several times.

Lemma 3.5. *Let $P \subseteq M$. Suppose that, for every $p \in P$ and $E \in \mathfrak{E}$, if $\Psi(E, p)$ is defined then it belongs to P. Then either $P = \emptyset$ or $P = M$.*

Proof. Since $\Psi(\cdot, p)$ is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{E}$ onto a neighbourhood of p, the hypothesis implies that P is open. This also shows that if $q \in \overline{P}$, then there exists $E \in \mathfrak{E}$ such that $p := \Psi(E, q) \in P$. Then $q = \Psi(-E, p)$, and the hypothesis again implies that $q \in P$. Thus, by the connectedness of M, if P is not empty, then $P = M$. $\square$

Below we shall write the identity map on M as either id_M or e_G, the identity element of G, interchangeably.

Lemma 3.6. *Let $p \in M$ and $f \in G$. If $f(p) = p$, then $f = \mathrm{id}_M$.*

Proof. Consider $P := \{q \in M \colon f(q) = q\}$. Then by Lemma 3.4(iv), we see that P satisfies the hypothesis of Lemma 3.5. This lemma then follows. $\square$

In the next two lemmas, let us fix an inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{E}$. This inner product induces a Riemannian metric on M, also denoted by $\langle\cdot|\cdot\rangle$ on each fibre of TM, and the associated distance function d on M. It then follows that each $L_{p,q}$ is a linear isometry from T_pM onto T_qM, and, from Lemma 3.4, that every $f \in G$ is a Riemannian isometry on M.

Lemma 3.7. *Let $p \in M$, and let (f_i) be a net in G. Suppose that $(f_i(p))$ converges to a point q in M. Then there exists an element $f \in G$ such that $f(p) = q$, (f_i) converges to f in G, and (f_i^{-1}) converges to f^{-1} in G.*

Proof. Denote by P the set of $a \in M$ such that $(f_i(a))$ is convergent in M. Then $p \in P$. For each $a \in P$, set $f(a) := \lim_i f_i(a)$.

Let $a \in P$ and let $E \in \mathfrak{V}$ such that $\Psi(E, a)$ is defined. Then by Lemma 3.4(iv)

$$f_i\left(\Psi(E, a)\right) = \Psi(E, f_i(a)) \to \Psi(E, f(a))\,, \tag{1}$$

and so $\Psi(E, a) \in P$. Thus $P = M$ by Lemma 3.5, and so f is globally defined on M with

$$f\left(\Psi(E, a)\right) = \Psi(E, f(a)) \tag{2}$$

whenever $\Psi(E, a)$ is defined.

Moreover, by the uniform continuity of Ψ on compact subsets of $\mathcal{U}$, we see from (1) and (2) that, for each $a \in M$,

$$f_i(\Psi(E, a)) \to f(\Psi(E, a))$$

uniformly for E in a compact neighbourhood of 0 in $\mathfrak{U}$. Thus $(f_i(x))$ converges to $f(x)$ uniformly for x in a neighbourhood of a. Since $a \in M$ is arbitrary, (f_i) converges to f uniformly on each compact subset of M.

The same argument applies to (f_i^{-1}) with $\lim_i f_i^{-1}(q) = p$ (using $d(f_i(p), q) = d(p, f_i^{-1}(q)))$ shows that there exists a map $g : M \to M$ such that (f_i^{-1}) converges to g uniformly on each compact subset of M. For each $a \in M$, one then sees that $(f_i(a))$ converges to $f(a)$, and so $(f_i^{-1}(f_i(a)))$ converges to $g(f(a))$, using local compactness of M and uniform convergence of (f_i^{-1}) to g on compact subsets of M. Similarly, $g(f(a)) = a$. Thus f is bijective and $g = f^{-1}$.

Finally, by (2), f satisfies condition (v) of Lemma 3.4, and so $f \in G$. □

Lemma 3.8. *Let (p_i) be a net that converges to p in M. Let (f_i) be a net in G, and $f \in G$. Then the following are equivalent:*

(i) *(f_i) converges to f in G.*
(ii) *$(f_i(p_i))$ converges to $f(p)$ in M.*
(iii) *$(f_i(p))$ converges to $f(p)$ in M.*

Note that Lemma 3.6 is a special case of this lemma (relative to the simple fact that G with the compact-open topology is Hausdorff).

Proof. Condition (i) implies condition (ii) since (f_i) converges to f uniformly on compact subsets of M.

To see that (ii) implies (iii), we note that $d(f_i(p_i), f_i(p)) = d(p_i, p) \to 0$ as $i \to \infty$ since each f_i is a Riemannian isometry.

To see that (iii) implies (i), we use Lemma 3.7 to obtain a function $h \in G$ such that (f_i) converges to h in G. Since $h(p) = f(p)$, Lemma 3.6 shows that $h = f$. □

Let us summarise what we have obtained about the given group G and its action on M so far in the following, which is immediate from the previous lemmas:

Corollary 3.9. *G equipped with the compact-open topology is a locally compact metrisable group that acts continuously, freely, and properly on M, and the G-orbits are closed.* □

Let us now work towards putting a smooth structure on G.

Lemma 3.10. *Let $p_0 \in M$. If G admits a smooth structure making the map*

$$f \mapsto f(p_0), \ G \to M,$$

a (smooth) embedding, then G is a Lie group and the action of G on M is smooth.

Proof. Denote by P the set of $p \in M$ such that $f \mapsto f(p)$ is an embedding. Then $p_0 \in P$. Let $p \in P$ and $E \in \mathfrak{V}$ such that $\Psi(E, p)$ is defined. Then, by Lemma 3.4(iv), we see that

$$\{f(p) \colon f \in G\} \subseteq M_E$$

and, by Lemma 3.2, $\Psi(E, \cdot)$ is a diffeomorphism from the open set M_E onto the open set M_{-E}. Thus

$$f \mapsto \Psi(E, f(p)) = f(\Psi(E, p))$$

is an embedding, and so $\Psi(E, p) \in P$. By Lemma 3.5, $P = M$.

Moreover, for each $p \in M = P$, let $\mathfrak{V}$ be an open neighbourhood of 0 in $\mathfrak{E}$ that is mapped diffeomorphically by $\Psi(\cdot, p)$ onto a neighbourhood D of p. Then since

$$(f, E) \mapsto \Psi(E, f(p))$$

is smooth on $G \times \mathfrak{V}$, we see that $(f, x) \mapsto f(x)$ is smooth on $G \times D$. Thus the map $(f, x) \mapsto f(x)$, $G \times M \to M$, is smooth. Since $f \mapsto f(p_0)$ is assumed to be an embedding of G into M, we see then that

$$(f, g) \mapsto fg = f \circ g$$

being the composition of $(f, g) \mapsto (f, g(p_0))$, $(f, x) \mapsto f(x)$, and then $h(p_0) \mapsto h$, is smooth.

Finally, the inverse of G is smooth is due to the implicit function theorem applied to the map $(f, g) \mapsto fg$. $\square$

Definition 3.11. Let $p \in M$. Define A_p to be the set of those vectors v in T_pM for which there exists a sequence (f_i) in G such that $(f_i(p))$ converges to p from the direction v. Let S_p be the linear span of A_p in T_pM.

In other words, if N is an G-orbit in M and if $p \in N$, then A_p is nothing but $\mathtt{A}_p^N$ as defined in Definition 2.3.

Lemma 3.12.

(i) *Let $p \in M$, and let (f_i) be a sequence in G that converges to $e_G = \mathrm{id}_M$. Suppose that $q = \Psi(E, p)$ for some $E \in \mathfrak{E}$. Then if $(f_i(p))$ converges to p from the direction $v \in T_pM$, then $(f_i(q))$ converges to q from the direction $\mathrm{d}\Psi(E, \cdot)_p(v)$.*

(ii) *The spaces S_p $(p \in M)$ together form a vector subbundle S of TM.*

Proof. To prove (i), note that if $\Psi(E, p)$ is defined, then $p \in M_E$, and $\Psi(E, \cdot)$ is a diffeomorphism from M_E onto M_{-E} with inverse $\Psi(-E, \cdot)$. Also,

$$f_i(q) = f_i(\Psi(E, p)) = \Psi(E, f_i(p))$$

and so the assertion follows from Lemma 2.2(i).

To prove (ii), we see from (i) that if $\Psi(E, p)$ is defined, then the linear bijection $\mathrm{d}\Psi(E, \cdot)_p$ is a bijection from A_p onto $A_{\Psi(E,p)}$, and so it restrict to a linear bijection from S_p onto $S_{\Phi(E,p)}$. Thus, for each $v \in A_p$, if we set

$$Z^v_{\Psi(E,p)} := \mathrm{d}\Psi(E, \cdot)_p(v)\,,$$

then as E varies, we obtain a smooth vector field Z^v on a neighbourhood V of p, whose value at each x in that neighbourhood belongs to A_x, and if we let v varies in a maximal linear independent subset of A_p, then the corresponding Z^v form a collection of vector fields on V whose values at each $x \in V$ form a linear basis for S_x. This shows that $S = \bigcup_{p \in M} S_p$ is a vector subbundle of TM. □

A final piece needed to apply Theorem 2.6 is the following connection to Definition 2.1.

Lemma 3.13. *Let $p \in M$ and let $v \in T_pM$. Then $v \in A_p$ if there exist sequences (f_i) and (g_i) in G such that v is the direction of approach of $(f_i(p))$ relative to $(g_i(p))$.*

Proof. Set $h_i := g_i^{-1} f_i$ and choose a closed Euclidean coordinate ball U centred at p with positive radius. Part of the assumption is that $f_i(p) \to p$ and $g_i(p) \to p$ and that $f_i \neq g_i$ eventually, so that both $f_i \to e_G$ and $g_i \to e_G$ by Lemma 3.8. Thus $(h_i(p))$ converges to p, as well as $h_i(p) \neq p$ eventually.

Take $\varepsilon > 0$. Then since $\mathrm{d}(g_i)_x = L_{x, g_i(x)}$ by Lemma 3.4, and since $g_i(x) \to x$ uniformly for x on compact subsets of U, we see that, shrinking U if necessary, there exists i_0 such that for all $i \geq i_0$ and all $x \in U$, we have the operator norm

$$\|\mathrm{id} - J(g_i)_x\| < \varepsilon\,.$$

By increasing i_0 if necessary, let us suppose also that $f_i(p), g_i(p), h_i(p) \in U$ for all $i \geq i_0$. Then, using the fundamental theorem of calculus on the restriction of $\mathrm{id} - g_i$ on the coordinate line segment joining p and $h_i(p)$, we obtain

$$|h_i(p) - p - (f_i(p) - g_i(p))| = |(\mathrm{id} - g_i)(h_i(p)) - (\mathrm{id} - g_i)(p)| \leq \varepsilon\, |h_i(p) - p|$$

here we identify points of U with points of $\mathbb{R}^n$ using the given coordinate. From this and the fact that $(f_i(p) - g_i(p))/\, \|f_i(p) - g_i(p)\| \to v/\, \|v\|$, it follows that $(h_i(p) - p)/\, \|h_i(p) - p\| \to v/\, \|v\|$, that is $v \in A_p$. $\square$

We can now complete the proof of Theorem 1.1: Fix an $p_0 \in M$, and let N be the G-orbit of p_0 in M. Then, by Corollary 3.9, N is closed and the map

$$f \mapsto f(p_0),\ G \to N, \tag{3}$$

is a homeomorphism.

For each $p \in N$, $\mathtt{A}_p^N$ as defined in Definition 2.3 is nothing but A_p. Thus each $\mathtt{A}_p^N$ spans the fibre S_p of the vector bundle S. Lemma 3.13 shows that $\mathtt{relA}_p^N = \mathtt{A}_p^N$ for every $p \in N$. Thus, by Theorem 2.6, N is an embedded submanifold of M. Therefore, one can put on G a smooth structure making the homeomorphism in (3) a diffeomorphism. This then allows us to apply Lemma 3.10 to complete the proof.

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Hung Pham
School of Mathematics and Statistics
Victoria University of Wellington
Wellington 6140
New Zealand
E-mail address: hung.pham@vuw.ac.nz

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