Invariant subspaces in the dual of $A_{cb}(G)$ and $A_M(G)$

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Let G be a locally compact group. In this paper, we study various
invariant subspaces of the duals of the algebras $A_M(G)$ and $A_{cb}(G)$
obtained by taking the closure of the Fourier algebra $A(G)$ in the
multiplier algebra $MA(G)$ and completely bounded multiplier al-
gebra $M_{cb}A(G)$ respectively. In particular, we will focus on various
functorial properties and containment relationships between these
various invariant subspaces including the space of uniformly con-
tinuous functionals and the almost periodic and weakly almost
periodic functionals.

Amongst other results, we show that if $\mathcal{A}(G)$ is either $A_M(G)$ or
$A_{cb}(G)$, then $UCB(\mathcal{A}(G)) \subseteq AP(\mathcal{A}(G))$ if and only if G is discrete.
We also show that if $UCB(\mathcal{A}(G)) = \mathcal{A}(G)^*$, then every amenable
closed subgroup of G is compact.

Let $i : A(G) \to \mathcal{A}(G)$ be the natural injection. We show that if
X is any closed topologically introverted subspace of $\mathcal{A}(G)^*$ that
contains $L^1(G)$, then $i^*(X)$ is closed in $A(G)$ if and only if G is
amenable.

AMS 2000 subject classifications: Primary 43A07, 43A22, 46J10;
secondary 47L25.
Keywords and phrases: Fourier algebra, multipliers, Arens regularity,
uniformly continuous functionals, topologically invariant means.

1. Introduction

Let G be a locally compact group. The Fourier algebra of G is an algebra
$A(G)$ of continuous functions on G under pointwise operations consisting
of the set of all coefficient functions $u(x) = \langle \lambda(x)f, g \rangle$ of the left regular
representation of G on the Hilbert space $L^2(G)$. $A(G)$ becomes a Banach
algebra via the norm it inherits as the natural pre-dual of the group von

Neumann algebra

$$VN(G) = \overline{\text{span}\{\lambda(x)|x \in G\}}^{-WOT} \subseteq B(L^2(G))$$

where $\lambda(x)f(y) = f(x^{-1}y)$ for each $x, y \in G$ and $f \in L^2(G)$. If G is abelian, then $A(G)$ is isometrically isomorphic as a Banach algebra with $L^1(\hat{G})$ via the classical Fourier transform and $VN(G)$ can be identified with $L^\infty(\hat{G})$, where $\hat{G}$ is the Pontryagin dual of G.

For an abelian group G there are several distinguished C^*-subalgebras of $L^\infty(\hat{G})$ that have been very well studied. These include the spaces of uniformly continuous functions $UCB(\hat{G})$, the almost periodic functions $AP(\hat{G})$, the weakly almost periodic functions $WAP(\hat{G})$ and $C_0(\hat{G})$, the space of continuous functions vanishing at infinity. All of these spaces induce corresponding subspaces of $VN(G)$ via the Fourier transform. In particular, we refer to the subspaces corresponding to $UCB(\hat{G})$, $AP(\hat{G})$ and $WAP(\hat{G})$ in $VN(G)$ as the uniformly continuous, almost periodic and weakly almost periodic functionals on $A(G)$ respectively. The image of $C_0(\hat{G})$ is the reduced group C^*-algebra $C^*_\lambda(G)$. One common property that all of these spaces have in common is that they are invariant under the natural module action of $A(G)$ on $VN(G)$.

When G is non-abelian there is still a dual relationship between $A(G)$ and $L^1(G)$ but it is much more complex given the lack of a Pontryagin dual group. It is, however, possible to define analogous versions of all of the above subspaces of $VN(G)$ and indeed this can be done in the abstract for any commutative Banach algebra. For the Fourier algebra in particular, there is a significant body of work investigating the properties of the subspaces highlighted above with a focus on what their structures tell us about the underlying group. See for example [7], [9], [10], [12] and [13].

In this paper, we will focus our attention primarily on non-amenable groups and on analogous invariant subspaces in the duals of two Banach algebras that arise from the Fourier algebra, $A_M(G)$ and $A_{cb}(G)$, which are the closures of $A(G)$ in its multiplier and completely bounded multiplier algebra respectively. In particular, we will consider various functorial properties and containment properties associated with these subspaces. Amongst other results, if $\mathcal{A}(G)$ is either $A_M(G)$ or $A_{cb}(G)$, we will characterize discrete groups as those locally compact groups for which $UCB(\mathcal{A}(G)) \subseteq AP(\mathcal{A}(G))$. We also show that if $UCB(\mathcal{A}(G)) = \mathcal{A}(G)^*$, then every amenable closed subgroup of G is compact. In particular, G has an open compact subgroup.

$A(G)$ injects contractively into both $A_M(G)$ and $A_{cb}(G)$ with the range of the injection being proper precisely when G is non-amenable. If we let

$i : A(G) \to \mathcal{A}(G)$ be the natural injection, we will show that if X is any of the closed subspaces of $\mathcal{A}(G)^*$ mentioned above that contains $L^1(G)$, then $i^*(X)$ is closed in $A(G)$ if and only if G is amenable.

2. Preliminaries and notation

Throughout this paper, $\mathcal{A}$ will denote a Banach algebra. In this case, the dual $\mathcal{A}^*$ becomes a Banach $\mathcal{A}$-bimodule with respect to the module actions

$$\langle u \cdot T, v \rangle = \langle T, vu \rangle \qquad \text{and} \qquad \langle T \square u, v \rangle = \langle T, uv \rangle$$

for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$. Note that when $\mathcal{A}$ is commutative $\langle u \cdot T, v \rangle = \langle T \square u, v \rangle$ which will generally be the case for most of this paper. In fact, from this point we will assume that every Banach algebra $\mathcal{A}$ we consider is commutative unless stated otherwise.

2.1.

Remark. It is well known that there are two natural products that can be used to extend the multiplication of $\mathcal{A}$ to its second dual $\mathcal{A}^{**}$. In this paper we will choose the following Arens product:

A1) $\langle u \cdot T, v \rangle = \langle T, vu \rangle$ for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.
A2) $\langle n \odot T, u \rangle = \langle n, u \cdot T \rangle$ for every $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$ and $n \in \mathcal{A}^{**}$.
A3) $\langle m \odot n, T \rangle = \langle m, n \odot T \rangle$ for every $T \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

From here we will proceed with the following definitions and notational conventions.

2.2.

Definition. *We call the space*

$$UCB(\mathcal{A}) = \overline{\mathrm{span}\{v \cdot T \mid v \in \mathcal{A}, T \in \mathcal{A}^*\}}^{-\|\cdot\|_{\mathcal{A}^*}}$$

the uniformly continuous functionals on $\mathcal{A}$.
    We call $T \in \mathcal{A}^*$ a (weakly) almost periodic functional on $\mathcal{A}$ if

$$\{u \cdot T \mid u \in \mathcal{A}, \|u\|_{\mathcal{A}} \leq 1\}$$

is relatively (weakly) compact in $\mathcal{A}^*$ and we denote the space of all (weakly) almost periodic functionals on $\mathcal{A}$ by $AP(\mathcal{A})$ $(WAP(\mathcal{A}))$.

2.3.

Definition. *We say that a closed subspace $X \subseteq \mathcal{A}^*$ is invariant if $u \cdot T \in X$ for every $u \in \mathcal{A}$ and $T \in X$.*

Given a closed invariant subspace X. We say that X is topologically introverted if $m \odot T \in X$ for every $m \in X^*$ and $T \in X$.

2.4.

Remark. If X is topologically introverted then X^* can be made into a Banach algebra by mimicking what we did for $\mathcal{A}^{**}$ by defining

$$\langle m \odot n, T \rangle = \langle m, n \odot T \rangle$$

for $m, n \in X^*$ and $T \in X$.

It is a well-known criterion of Grothendieck that $T \in \mathcal{A}^*$ is weakly almost periodic if and only if given two nets $\{u_\alpha\}_{\alpha \in \Omega_1}$ and $\{v_\beta\}_{\beta \in \Omega_2}$ in $\mathcal{A}$ we have that

$$\lim_\alpha \lim_\beta \langle T, u_\alpha v_\beta \rangle = \lim_\beta \lim_\alpha \langle T, u_\alpha v_\beta \rangle$$

whenever both limits exist. From this one can show that if $\mathcal{A}$ is a commutative Banach algebra and X is topologically introverted, then X^* is commutative if and only if $X \subseteq WAP(\mathcal{A})$. (See [3] and [16].)

The following three propositions will prove useful.

2.5.

Proposition. *Let $\mathcal{A}$ be a commutative Banach algebra. Then each of $UCB(\mathcal{A})$, $AP(\mathcal{A})$ and $WAP(\mathcal{A})$ are invariant subspaces of $\mathcal{A}^*$.*

Proof. That $UCB(\mathcal{A})$ is invariant is obvious since for each $T \in \mathcal{A}^*$, we have $u \cdot T \in UCB(\mathcal{A})$.

Let $T \in (W)AP(\mathcal{A})$. Let $v \in \mathcal{A}$. Without loss of generality, we can assume that $\|v\|_{\mathcal{A}} = 1$. Since

$$\{u \cdot (v \cdot T) \mid u \in \mathcal{A}, \|u\|_{\mathcal{A}} \leq 1\} \subset \{u \cdot T \mid u \in \mathcal{A}, \|u\|_{\mathcal{A}} \leq 1\}$$

we have that $\{u \cdot (v \cdot T) \mid u \in \mathcal{A}, \|u\|_{\mathcal{A}} \leq 1\}$ is relatively (weakly) compact and hence $v \cdot T \in (W)AP(\mathcal{A})$. $\qquad\square$

2.6.

Proposition. *Let $\mathcal{A}$ be a commutative Banach algebra. Then $UCB(\mathcal{A})$ is topologically introverted.*

Proof. Let $m \in UCB(\mathcal{A})^*$ and let $T = u \cdot T_1 \in UCB(\mathcal{A})$. Let $v \in \mathcal{A}$. Then

$$
\begin{aligned}
\langle m \odot T, v \rangle &= \langle m, v \cdot T \rangle \\
&= \langle m, v \cdot (u \cdot T_1) \rangle \\
&= \langle m, uv \cdot T_1 \rangle \\
&= \langle m \odot T_1, uv \rangle \\
&= \langle u \cdot (m \odot T_1), v \rangle.
\end{aligned}
$$

It follows that $m \odot T = u \cdot (m \odot T_1) \in UCB(\mathcal{A})$. Moreover, if $T \in \text{span}\{u \cdot T_1 | u \in \mathcal{A}, T_1 \in \mathcal{A}^*\}$, then $m \odot T \in UCB(\mathcal{A})$ as well.

Finally, lets assume that $\{T_n\} \subset UCB(\mathcal{A})$ and that $\lim_{n \to \infty} T_n = T$. Let $u \in \mathcal{A}$ with $\|u\|_{\mathcal{A}} \le 1$. Then for $m \in UCB(\mathcal{A})^*$,

$$
|\langle m \odot T, u \rangle - \langle m \odot T_n, u \rangle| = |\langle m, u \cdot (T - T_n) \rangle| \le \|m\|_{UCB(\mathcal{A})^*} \|T - T_n\|_{\mathcal{A}^*}.
$$

It follows that $\lim_{n \to \infty} m \odot T_n = m \odot T$. Since $\text{span}\{u \cdot T_1 | u \in \mathcal{A}, T_1 \in \mathcal{A}^*\}$ is dense in $UCB(\mathcal{A})$, it follows that $m \odot T \in UCB(\mathcal{A})$ for every $m \in UCB(\mathcal{A})^*$ and $T \in UCB(\mathcal{A})$. That is, $UCB(\mathcal{A})$ is topologically introverted. $\qquad \square$

2.7.

Proposition. *Let X be a closed invariant subspace of $WAP(\mathcal{A})$. Then X is topologically introverted.*

Proof. Let $T \in X$. Let

$$
O(T) = \{u \cdot T \mid u \in \mathcal{A}, \|u\|_{\mathcal{A}} \le 1\}.
$$

Since $T \in WAP(\mathcal{A})$, the convex set $O(T)$ is relatively $\sigma(\mathcal{A}^*, \mathcal{A}^{**})$ compact. Hence its $\sigma(\mathcal{A}^*, \mathcal{A}^{**})$ closure $\overline{O(T)}^{\sigma(\mathcal{A}^*, \mathcal{A}^{**})}$ is weakly compact and hence is also a norm closed set in X. Moreover, the $\sigma(\mathcal{A}^*, \mathcal{A}^{**})$ and $\sigma(\mathcal{A}^*, \mathcal{A})$ topologies agree on $\overline{O(T)}^{\sigma(\mathcal{A}^*, \mathcal{A}^{**})}$.

Next, we let $m \in X^*$. Without loss of generality, we may assume that $\|m\|_{X^*} = 1$. Let $M \in \mathcal{A}^{**}$ be an extension of m to $\mathcal{A}^*$ with $\|m\|_{X^*} = \|M\|_{\mathcal{A}^*} = 1$. Then by Goldstine's Theorem, there exists a net $\{u_\alpha\}_{\alpha \in \Omega}$ in $\mathcal{A}$

such that $\|u_\alpha\| \leq 1$ and $\{u_\alpha\}_{\alpha \in \Omega}$ converges to M in the weak* topology on $\mathcal{A}^{**}$. It follows that if $u \in \mathcal{A}$ and $T \in X$, then

$$\langle M \odot T, u \rangle = \langle M, u \cdot T \rangle = \lim_{\alpha \in \Omega} \langle u_\alpha, u \cdot T \rangle = \lim_{\alpha \in \Omega} \langle u_\alpha \cdot T, u \rangle.$$

But this shows that

$$M \odot T \in \overline{O(T)}^{\sigma(\mathcal{A}^*, \mathcal{A})} \subset X.$$

Moreover, if $u \in \mathcal{A}$ and $T \in X$, then

$$\langle M \odot T, u \rangle = \langle M, u \cdot T \rangle = \langle m, u \cdot T \rangle = \langle m \odot T, u \rangle.$$

Hence $m \odot T \in X$. It follows that X is topologically introverted. $\square$

2.8.

Definition. *Let I be a closed ideal in $\mathcal{A}$. We let*

$$Z(I) = \{x \in \Delta(\mathcal{A}) \mid u(x) = 0 \text{ for all } u \in I\}.$$

Given a closed set $E \subseteq \Delta(\mathcal{A})$, we let

$$I(E) = \{u \in \mathcal{A} \mid u(x) = 0 \text{ for all } x \in E\}.$$

We say that a closed set $E \subseteq \Delta(\mathcal{A})$ is a set of spectral synthesis for $\mathcal{A}$ if the only closed ideal I is $\mathcal{A}$ with $Z(I) = E$ is $I(E)$.

3. Multipliers of the fourier algebra

Let G be a locally compact group. We let $A(G)$ and $B(G)$ denote the Fourier and Fourier-Stieltjes algebras of G, which are Banach algebras of continuous functions on G and were introduced in [5]. A *multiplier* of $A(G)$ is a (necessarily bounded and continuous) function $v: G \to \mathbb{C}$ such that $vA(G) \subseteq A(G)$. For each multiplier v of $A(G)$, the linear operator M_v on $A(G)$ defined by $M_v(u) = vu$ for each $u \in A(G)$ is bounded via the Closed Graph Theorem. The *multiplier algebra* of $A(G)$ is the closed subalgebra

$$MA(G) := \{M_v : v \text{ is a multiplier of } A(G)\}$$

of $B(A(G))$, where $B(A(G))$ denotes the algebra of all bounded linear operators from $A(G)$ to $A(G)$. Throughout this paper we will generally use v in place of the operator M_v and we will write $\| v \|_{MA(G)}$ to represent the norm of M_v in $B(A(G))$.

3.1.

Definition. *Let $\mathcal{Q}_M(G)$ be the completion of $L^1(G)$ with respect to the norm*

$$\|f\|_{\mathcal{Q}_M(G)} = \sup \left\{ \int_G f(x)v(x)dx \mid v \in MA(G), \|v\|_{MA(G)} \leq 1 \right\}.$$

Then $\mathcal{Q}_M(G)^* = MA(G)$.

Let G be a locally compact group and let $VN(G)$ denote its group von Neumann algebra. The duality

$$A(G) = VN(G)_*$$

equips $A(G)$ with a natural operator space structure. With this operator space structures we can define the *cb-multiplier algebra* of $A(G)$ to be

$$M_{cb}A(G) := CB(A(G)) \cap M(A(G)),$$

where $CB(A(G))$ denotes the algebra of all completely bounded linear maps from $A(G)$ into itself. We let $\| v \|_{cb}$ denote the *cb*-norm of the operator M_v. It is well known that $M_{cb}(A(G))$ is a closed subalgebra of $CB(A(G))$ and is thus a (completely contractive) Banach algebra with respect to the norm $\| \cdot \|_{cb}$.

3.2.

Definition. *Let $\mathcal{Q}_{cb}(G)$ be the completion of $L^1(G)$ with respect to the norm*

$$\|f\|_{\mathcal{Q}_{cb}(G)} = \sup \left\{ \int_G f(x)v(x)dx \mid v \in M_{cb}A(G), \|v\|_{cb} \leq 1 \right\}.$$

Then $\mathcal{Q}_{cb}(G)^* = M_{cb}A(G)$.

It is known that in general,

$$A(G) \subseteq B(G) \subseteq M_{cb}(A(G)) \subseteq M(A(G))$$

and that for $v \in A(G)$

$$\| v \|_{A(G)} = \| v \|_{B(G)} \geq \| v \|_{cb} \geq \| v \|_M .$$

In case G is an amenable group, we have

$$B(G) = M_{cb}(A(G)) = M(A(G))$$

and that

$$\| v \|_{B(G)} = \| v \|_{cb} = \| v \|_M$$

for any $v \in B(G)$.

3.3.

Definition. *Given a locally compact group G let*

$$A_M(G) \overset{def}{=} A(G)^{-\|\cdot\|_M} \subseteq M(A(G)).$$

and

$$A_{cb}(G) \overset{def}{=} A(G)^{-\|\cdot\|_{cb}} \subseteq M_{cb}(A(G)).$$

3.4.

Remark. The algebra $A_{cb}(G)$ was introduced by the first author in [6] where it was denoted by $A_0(G)$. In that paper we show that in the case of $\mathbb{F}_2$, the free group on two generators, $A_{cb}(G)$ shares many of the properties characteristic of the Fourier-algebra of an amenable group. In particular, the algebra $A_{cb}(\mathbb{F}_2)$ is known to have a bounded approximate identity. The locally compact groups G for which $A_{cb}(G)$ has a bounded approximate identity are called *weakly amenable groups*. All amenable groups are weakly amenable, but many classical non-amenable groups such as $\mathbb{F}_2$ and $SL(2, \mathbb{R})$ are weakly amenable. We say that the locally compact group G is M-weakly amenable if there is an approximate identity $\{u_\alpha\}_{\alpha \in I}$ in $A(G)$ that is bounded in the norm $\| \cdot \|_M$.

3.5.

Remark. Let $\mathcal{A}(G)$ denote either $A_M(G)$ or $A_{cb}(G)$. Consider the following map and its adjoints:

$i_{\mathcal{A}} : A(G) \to \mathcal{A}(G)$
$i_{\mathcal{A}}^* : \mathcal{A}(G)^* \to VN(G)$
$i_{\mathcal{A}}^{**} : VN(G)^* \to \mathcal{A}(G)^{**},$

where $i_{\mathcal{A}}$ denotes the inclusion map. Since $i_{\mathcal{A}}$ has dense range, $i_{\mathcal{A}}^*$ is injective and as such is invertible with inverse $i_{\mathcal{A}}^{*-1}$ on $Range(i_{\mathcal{A}}^*)$. It is easy to see that $i_{\mathcal{A}}^*$ is simply the restriction map. That is

$$i_{\mathcal{A}}^*(T) = T_{|_{A(G)}}.$$

It will also be useful to view all of the above maps as embeddings. That is, when G is nonamenable $\mathcal{A}(G)^*$ can be viewed as a proper subset of $VN(G)$ and $VN(G)^*$ as a proper subset of $\mathcal{A}(G)^{**}$, of course with different norms.

<div align="center">3.6.</div>

Definition. *We let*

$$X_M^\lambda(G) = \{T \in \mathcal{Q}_M(G) \mid \langle u, T \rangle = 0 \text{ for every } u \in A_M(G)\}$$

and let

$$B_M^\lambda(G) = \overline{A_M(G)}^{\text{weak}^*} \subseteq MA(G).$$

   Let

$$\mathcal{Q}_M^\lambda(G) = \mathcal{Q}_M(G)/X_M^\lambda.$$

   We let

$$X_{cb}^\lambda(G) = \{T \in \mathcal{Q}_{A_{cb}(G)}(G) \mid \langle u, T \rangle = 0 \text{ for every } u \in A_{cb}(G)\}$$

and let

$$B_{cb}^\lambda(G) = \overline{A_{cb}(G)}^{\text{weak}^*} \subseteq M_{cb}A(G).$$

   Let

$$\mathcal{Q}_{cb}^\lambda(G) = \mathcal{Q}_{cb}(G)/X_{cb}^\lambda.$$

<div align="center">3.7.</div>

Remark. $\mathcal{Q}_M(G)$ and $\mathcal{Q}_{cb}(G)$ are natural analogs of the group C^*-algebra $C^*(G)$. Similarly, $\mathcal{Q}_M^\lambda(G)$ and $\mathcal{Q}_{cb}^\lambda(G)$ are natural analogs of the reduced group C^*-algebra $C_\lambda^*(G)$. In particular, we have that

$$(\mathcal{Q}_M^\lambda(G))^* = B_M^\lambda(G)$$

and

$$(\mathcal{Q}_{cb}^\lambda(G))^* = B_{cb}^\lambda(G).$$

   Moreover, we can identify $\mathcal{Q}_M^\lambda(G)$ and $\mathcal{Q}_{cb}^\lambda(G)$ with the closed subspaces of $A_M(G)^*$ and $A_{cb}(G)^*$ respectively generated by

$$\{\phi_f \mid f \in L^1(G)\}$$

where

$$\langle \phi_f, u \rangle = \int_G f(x)u(x) \, dx.$$

As such going forward, we will view $\mathcal{Q}_M^\lambda(G)$ and $\mathcal{Q}_{cb}^\lambda(G)$ as norm closed subspaces of $A_M(G)^*$ and $A_{cb}(G)^*$ respectively. Moreover, we will generally identify the functional ϕ_f with the function $f \in L^1(G)$ for notational convenience.

Finally, we let

$$\mathcal{Q}_M^\delta(G) \overset{def}{=} \overline{\text{span}\{\phi_x \mid x \in G\}}^{-\|\cdot\|_{A_M(G)^*}} \subseteq A_M(G)^*$$

and

$$\mathcal{Q}_{cb}^\delta(G) \overset{def}{=} \overline{\text{span}\{\phi_x \mid x \in G\}}^{-\|\cdot\|_{A_{cb}(G)^*}} \subseteq A_{cb}(G)^*$$

where $\phi_x(v) = v(x)$ for each $x \in G$ and each $v \in A_M(G)$ and $A_{cb}(G)$ respectively. In particular, if G is discrete, then $\mathcal{Q}_M^\lambda(G) = \mathcal{Q}_M^\delta(G)$ and $\mathcal{Q}_{cb}^\lambda(G) = \mathcal{Q}_{cb}^\delta(G)$.

4. Invariant subspaces of $A_{cb}(G)$ or $A_M(G)$ and their functorial properties

In this section we will focus our attention on some important invariant subspaces of the algebras $A(G)$, $A_{cb}(G)$, $A_M(G)$ and their closed ideals.

Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ and $A_M(G)$. We note that while by definition

$$UCB(\mathcal{A}(G)) = \overline{\text{span}\{v \cdot T \mid v \in \mathcal{A}(G), T \in \mathcal{A}(G)^*\}}^{-\|\cdot\|_{\mathcal{A}^*}}$$

it turns out that we don't actually need the linear span.

4.1.

Proposition. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ and $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$ with $Z(I)$ a set of spectral synthesis. Then*

$$UCB(I) = \overline{\{v \cdot T \mid v \in \mathcal{A}(G), T \in I^*\}}^{-\|\cdot\|_{I^*}}$$

Proof. Let

$$j(Z(I)) = \{u \in \mathcal{A}(G) \mid supp(u) \text{ is compact and disjoint from } Z(I)\}.$$

The assumption that $Z(I)$ is a set of spectral synthesis tells us that $j(Z(I))$ is dense in I.

Let $u_1, u_2 \in j(Z(I))$ and $T_1, T_2 \in I^*$. Then if $K_1 = supp(u_1)$ and $K_2 = supp(u_2)$ we have that $K = K_1 \cup K_2$ is a compact set disjoint from $Z(I)$. By the regularity, we can also find a $u \in A(G)$ with compact support disjoint from Z(I) such that $u(x) = 1$ if $x \in K$. In particular, $u \in j(Z(I))$. Moreover

$$u \cdot (u_1 \cdot T_1 + u_2 \cdot T_2) = uu_1 \cdot T_1 + uu_2 \cdot T_2 = u_1 \cdot T_1 + u_2 \cdot T_2.$$

Hence $j(Z(I)) \cdot I^*$ is a subspace of $UCB(I)$.

Next, we assume that $u \in I$ and $T \in I^*$. Choose $\{u_n\} \subset j(Z(I))$ such that $\lim_{n \to \infty} \|u_n - u\|_{\mathcal{A}(G)} = 0$. Since

$$\|u_n \cdot T - u \cdot T\|_{I^*} \leq \|u_n - u\|_{\mathcal{A}(G)} \|T\|_{I^*}$$

we have that $j(Z(I)) \cdot I^*$ is also dense in $UCB(I)$. $\qquad\square$

4.2.

Remark. Given an ideal I in $\mathcal{A}(G)$ and a $T \in I^*$ we say that $x \in G \setminus Z(I)$ is a support point for T if for every neighbourhood V of x, there exists a $u \in I$ with $supp(u) \subseteq V$ and $\langle T, u \rangle \neq 0$. We let $supp(T)$ denote the collection of all support points of T. We note that Proposition 4.1 shows that $UCB(I)$ is the closure in I^* of those $T \in I^*$ with compact support.

4.3.

Definition. *Let $\mathcal{A}(G)$ denote any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Given a closed ideal I in $\mathcal{A}(G)$ we define*

$$UCB_c(I) = \{T \in I^* \mid T \text{ has compact support}\}$$

or equivalently

$$UCB_c(I) = \{T \in I^* \mid T = u \cdot T_1, u \in I \text{ has compact support and } T_1 \in I^*\}.$$

<div align="center">**4.4.**</div>

Definition. *Let X be a closed invariant subspace of $\mathcal{A}^*$ containing a character $\phi \in \Delta(\mathcal{A})$. We say that $m \in X^*$ is a topological invariant mean at ϕ on X if*

$$\|m\|_{X^*} = \langle m, \phi \rangle = 1$$

and

$$\langle m, u \cdot T \rangle = \langle \phi, u \rangle \langle m, T \rangle$$

for all $u \in \mathcal{A}$ and $T \in X$. We denote the space of all topological invariant means at ϕ on X by $TIM_{\mathcal{A}}(X, \phi)$.

<div align="center">**4.5.**</div>

Definition. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $x \in G$ define the isometry $L_x : \mathcal{A}(G) \to \mathcal{A}(G)$ by*

$$L_x(u)(y) = u(xy),$$

for each $y \in G$.

The following two propositions are respectively [8, Proposition 4.3] and [8, Proposition 4.5].

<div align="center">**4.6.**</div>

Proposition. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $X \subseteq \mathcal{A}(G)^*$ be a closed invariant subspace of $\mathcal{A}(G)^*$. Let $x \in G$.*

 i) *If $Y = L_x^*(X)$, then Y is a closed invariant subspace of $\mathcal{A}(G)^*$ and*

$$u \cdot L_x^*(T) = L_x^*(L_x(u) \cdot T)$$

 for every $u \in \mathcal{A}(G)$ and $T \in X$.
 ii) *Let $x \in G$. Then $L_x^*(\phi_e) = \phi_x$ where e denotes the identity of G.*
 iii) *Let $m \in TIM_{\mathcal{A}(G)}(X, \phi_e)$. If $x \in G$, Then $\phi_x \in L_x^*(X)$ and $L_{x^{-1}}^{**}(m) \in TIM_{\mathcal{A}(G)}(L_x^*(X), \phi_x)$.*

4.7.

Proposition. *Let $\mathcal{A}(G)$ be either $A_{cb}(G)$ or $A_M(G)$. Let $\mathcal{A}(G) \cdot VN(G) = \{u \cdot T : u \in \mathcal{A}(G), T \in VN(G)\}$. Then*

    i) $\mathcal{A}(G) \cdot VN(G) \subseteq UCB(A(G))$
    ii) $i^*(v \cdot T) = v \cdot i^*(T)$ for each $v \in \mathcal{A}(G), T \in \mathcal{A}(G)^*$.
   iii) $i^*(UCB(\mathcal{A}(G))) \subseteq UCB(A(G))$.
   iv) If $\mathcal{A}(G)$ has a bounded approximate identity, then $\mathcal{A}(G) \cdot VN(G) = UCB(A(G))$
    v) $u \cdot T \in i^*(\mathcal{A}(G)^*)$ for each $u \in A(G), T \in VN(G)$.

We can now extend the result in Proposition 4.7 iii) to other important invariant subspaces.

4.8.

Proposition. *Let $\mathcal{A}(G)$ be either $A_{cb}(G)$ or $A_M(G)$. Then*

    i) $i^*(AP(\mathcal{A}(G)) \subseteq AP(A(G))$.
    ii) $i^*(WAP(\mathcal{A}(G)) \subseteq WAP(A(G))$.
   iii) $i^*(\mathcal{Q}_{cb}^\delta(G)) \subseteq C_\delta^*(G)$ and $i^*(\mathcal{Q}_M^\delta(G)) \subseteq C_\delta^*(G)$.
   iv) $i^*(\mathcal{Q}_{cb}^\lambda(G)) \subseteq C_\lambda^*(G)$ and $i^*(\mathcal{Q}_M^\lambda(G)) \subseteq C_\lambda^*(G)$.

Proof.

    i) Let $T \in AP(\mathcal{A}(G))$. Let $\{u_n\}$ be a sequence in $A(G)$ with $\|u_n\|_{A(G)} \leq 1$ for each $n \in \mathbb{N}$. Then $\|u_n\|_{\mathcal{A}(G)} \leq 1$ for each $n \in \mathbb{N}$. It follows that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k} \cdot T\}$ converges to some $T_1 \in AP(\mathcal{A}(G))$. Since

$$i^*(u_{n_k} \cdot T) = u_{n_k} \cdot i^*(T)$$

and since i^* is norm continuous we have that $\{u_{n_k} \cdot i^*(T)\}$ converges to $i^*(T_1)$.

   ii) Let $T \in WAP(\mathcal{A}(G))$. Let $\{u_\alpha\}_{\alpha \in \Omega}$ be a net in in $A(G)$ with $\|u_\alpha\|_{A(G)} \leq 1$ for each $\alpha \in \Omega$. Then as before $\|u_\alpha\|_{\mathcal{A}(G)} \leq 1$ for each $\alpha \in \Omega$. It follows that there exists a subnet $\{u_{\alpha_\beta}\}$ such that $\{u_{\alpha_\beta} \cdot T\}$ converges weakly to some $T_1 \in WAP(\mathcal{A}(G))$.
    Let $m \in A(G)^{**}$. Then $i^{**}(m) \in \mathcal{A}(G)^{**}$. It follows that

$$\lim_\beta \langle m, u_{\alpha_\beta} \cdot i^*(T)\rangle \quad = \quad \lim_\beta \langle m, i^*(u_{\alpha_\beta} \cdot T)\rangle$$

$$
\begin{aligned}
&= \lim_\beta \langle i^{**}(m), u_{\alpha_\beta} \cdot T \rangle \\
&= \langle i^{**}(m), T_1 \rangle \\
&= \langle m, i^*(T_1) \rangle.
\end{aligned}
$$

This shows that $\{u_{n_k} \cdot i^*(T)\}$ converges weakly to $i^*(T_1)$ and hence $i^*(T) \in WAP(A(G))$.

iii) This follows immediately since, abusing notation, $i^*(\phi_x) = \phi_x$ for every $x \in G$.

iv) Again, this follows immediately since, once more abusing notation, $i^*(f) = f$ for every $f \in L^1(G)$. □

It is worth asking whether the containments in the previous proposition can ever be equalities. This is clearly the case if G is amenable since $A(G) = A_M(G) = A_{cb}(G)$. In [7, Lemma 4.7] the first author together with T. Miao showed that if $i^*(UCB(\mathcal{A}(G))) = UCB(A(G))$, then G must be amenable. In fact, the following is true:

4.9.

Theorem. *Let $\mathcal{A}(G)$ be either $A_{cb}(G)$ or $A_M(G)$. Let X be a closed invariant subspace of $\mathcal{A}(G)^*$ which contains $L^1(G)$. If $i^*(X)$ is closed in $VN(G)$, then G is amenable.*

Proof. Assume that $i^*(X)$ is closed in $VN(G)$. Since $i^*(L^1(G)) = L^1(G)$, we have that $C_\lambda^*(G) \subseteq i^*(X)$. Moreover, since i^* is continuous and injective, if $i^*(X)$ is closed, the restriction of i^* to X is an isomorphism. In particular, if we let $u \in \mathcal{A}(G)$, then u defines a continuous linear functional on $C_\lambda^*(G)$. Hence $\mathcal{A}(G) \subseteq B_\lambda(G)$ and the norm on $\mathcal{A}(G)$ is equivalent to the norm from $B(G)$. As such G is amenable [15]. □

4.10.

Remark. Let $\mathcal{A}(G)$ be either $A_{cb}(G)$ or $A_M(G)$. Let H be a closed subgroup of G. Then the restriction map $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is a contraction homomorphism. In general, we don't know if it is surjective. In fact, if $G = SL(2, \mathbb{R})$ and H is a closed subgroup of G isomorphic to $\mathbb{F}_2$, then the restriction map $R : A_M(G) \to A_M(H)$ is not surjective [1, Theorem 4.3]. However, there are cases when the restriction map is known to be surjective. For example, we say that $G \in [SIN]_H$ if there is a neighbourhood base at the identity consisting of sets that are invariant under the inner automorphisms by elements in H. If $G \in [SIN]_H$, then $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is surjective [1, Corollary 3.4].

If H is amenable, then $\mathcal{A}(H) = A(H)$ and the surjectivity of R follows from Herz's Extension Theorem [11]. In this case $A(H)$ is isometrically isomorphic to $\mathcal{A}(G)/I_{\mathcal{A}(G)}(H)$. Moreover, in this case R^* is a $*$-homomorphism from $VN(H)$ onto

$$VN_H(G) = \overline{\text{span}\{\phi_x \mid x \in H\}}^{-w*}$$

which is a von Neumann algebra in $\mathcal{A}(G)^*$. (See [10, Lemma 3.1]). Moreover,

$$VN_H(G) = I_{\mathcal{A}(G)}(H)^\perp.$$

In general, we let

$$\mathcal{A}_H^*(G) = \overline{\text{span}\{\phi_x \mid x \in H\}}^{-w*} = I_{\mathcal{A}(G)}(H)^\perp \subseteq \mathcal{A}(G)^*.$$

4.11.

Definition. *Let G be a locally compact group, H a closed subgroup and $C \geq 1$ be a constant. We say that $\mathcal{A}(H)$ is C-extendable in $\mathcal{A}(G)$ if for every $v \in \mathcal{A}(H)$ and every $\epsilon > 0$, there exists a $w \in \mathcal{A}(G)$ such that $v = w_{|_H}$ and*

$$\|w\|_{\mathcal{A}(G)} \leq (C + \epsilon)\|v\|_{\mathcal{A}(H)}.$$

Moreover, we also assume that if $v \in A(G)$, then we can choose $w \in A(G)$.
   We let

$$Ext_{\mathcal{A}(G)} = \{H | \mathcal{A}(H) \text{ is } C\text{-extendable for some } C \geq 1\}.$$

4.12.

Remark. $\mathcal{A}(G)$ be either $A_{cb}(G)$ or $A_M(G)$. Assume that $H \in Ext_{\mathcal{A}(G)}$. Then $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is surjective with kernel $I_{\mathcal{A}(G)}(H)$ and hence $\mathcal{A}(H)$ is isomorphic with $\mathcal{A}(G)/I_{\mathcal{A}(G)}(H)$. In particular, if $\mathcal{A}(H)$ is C-extendable in $\mathcal{A}(G)$, then for each $T \in \mathcal{A}(H)^*$

$$1/C\|T\|_{\mathcal{A}(H)^*} \leq \|R^*(T)\|_{\mathcal{A}(G)^*} \leq \|T\|_{\mathcal{A}(H)^*}$$

and R^* is an isomorphism of $\mathcal{A}(H)^*$ onto $\mathcal{A}_H^*(G)$. In case, $C = 1$, the natural identification of $\mathcal{A}(H)$ with $\mathcal{A}(G)/I_{\mathcal{A}(G)}(H)$ is an isometric algebra isomorphism.

The following proposition was established by Kaniuth and Lau [10, Lemma 3.2] for $A(G)$ for all closed subgroups H.

<center>**4.13.**</center>

Proposition. *Let $\mathcal{A}(G)$ either $A_{cb}(G)$ or $A_M(G)$. Let $H \in Ext_{\mathcal{A}(G)}$. Let $R : \mathcal{A}(G) \to \mathcal{A}(H)$ be the restriction map. Then*

    i) $R^*(UCB_c(\mathcal{A}(H)) = UCB_c(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$
   ii) $R^*(UCB(\mathcal{A}(H)) = UCB(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$
  iii) $R^*(AP(\mathcal{A}(H)) = AP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$
  iv) $R^*(WAP(\mathcal{A}(H)) = WAP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$

Proof.

  i) If $T \in UCB_c(\mathcal{A}(H))$, then there exists a $u \in \mathcal{A}(H) \cap C_{00}(H)$ and a $T_1 \in \mathcal{A}(H)^*$ such that $T = u \cdot T_1$. Since $H \in Ext_{\mathcal{A}(G)}$, there exists a $v \in \mathcal{A}(G) \cap C_{00}(G)$ such that $R(v) = u$. For $w \in \mathcal{A}(G)$,

$$\begin{aligned}
\langle R^*(T), w \rangle &= \langle R^*(u \cdot T_1), w \rangle \\
&= \langle u \cdot T_1, R(w) \rangle \\
&= \langle T_1, uR(w) \rangle \\
&= \langle T_1, R(vw) \rangle \\
&= \langle R^*(T_1), vw \rangle \\
&= \langle v \cdot R^*(T_1), w \rangle.
\end{aligned}$$

Hence $R^*(T) = v \cdot R^*(T_1) \in UCB_c(\mathcal{A}(G))$. But we also have $R^*(T) \in \mathcal{A}_H^*(G)$.
Next assume that $T \in UCB_c(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$. In particular, there exists $T_1 \in \mathcal{A}(H)^*$ such that $T = R^*(T_1)$.
Let $K = supp(T)$. Since $T \in UCB_c(\mathcal{A}(G))$, K is compact. As such there exists $u \in \mathcal{A}(G) \cap C_{00}(G)$ such that $u(x) = 1$ for each x in some open neighbourhood of K. It follows that $u \cdot T = T$. Note also that $v = R(u) \in \mathcal{A}(H) \cap C_{00}(H)$. In particular $v \cdot T_1 \in UCB_c(\mathcal{A}(H))$. We claim that $R^*(v \cdot T_1) = T$.
Let $w \in \mathcal{A}(G)$. Then

$$\begin{aligned}
\langle R^*(v \cdot T_1), w \rangle &= \langle v \cdot T_1, R(W) \rangle \\
&= \langle T_1, vR(w) \rangle \\
&= \langle T_1, R(uw) \rangle \\
&= \langle R^*(T_1), uw) \\
&= \langle u \cdot T, w \rangle \\
&= \langle T, w \rangle.
\end{aligned}$$

ii) Since R^* is continuous, and $UCB_c(\mathcal{A}(H))$ is dense in $UCB(\mathcal{A}(H))$, it follows from i) that $R^*(UCB(\mathcal{A}(H)) \subseteq UCB(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$.

Since $H \in Ext_{\mathcal{A}(G)}$, R^* is an isomorphism and hence $R^*(UCB(\mathcal{A}(H))$ has closed range. It now follows from i) that $R^*(UCB(\mathcal{A}(H)) = UCB(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$

iii) Let $T \in AP(\mathcal{A}(H))$. Let $\|u\|_{\mathcal{A}(G)} \leq 1$. Then $\|R(u)\|_{\mathcal{A}(H)} \leq 1$. Moreover, if $v = R(u)$, then for each $w \in \mathcal{A}(G)$, we have

$$
\begin{aligned}
\langle R^*(v \cdot T), w \rangle &= \langle v \cdot T, R(w) \rangle \\
&= \langle T, R(uw) \rangle \\
&= \langle R^*(T), uw \rangle \\
&= \langle u \cdot R^*(T), w \rangle
\end{aligned}
$$

It follows that

$$\{u \cdot R^*(T) \mid \|u\|_{\mathcal{A}(G)} \leq 1\} \subseteq R^*(\{v \cdot T \mid \|v\|_{\mathcal{A}(H)} \leq 1\}).$$

Since $\{v \cdot T \mid \|v\|_{\mathcal{A}(H)} \leq 1\}$ is relatively compact, so is $R^*(\{v \cdot T \mid \|v\|_{\mathcal{A}(H)} \leq 1\}$ and hence so is $\{u \cdot R^*(T) \mid \|u\|_{\mathcal{A}(G)} \leq 1\}$. That is

$$R^*(AP(\mathcal{A}(H)) \subseteq AP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G).$$

Next suppose that $T \in AP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$. It follows that there exists a $T_1 \in \mathcal{A}(H)^*$ such that $T = R^*(T_1)$. Let $\{u_n\} \subset \mathcal{A}(H)$ with $\|u_n\|_{\mathcal{A}(H)} \leq 1$. For each $n \in \mathbb{N}$, choose $v_n \in \mathcal{A}(G)$, such that $\|v_n\|_{\mathcal{A}(G)^*} \leq C + 1$ and $v_{n_{|H}} = u_n$. Then from our calculation above we see that

$$v_n \cdot T = R^*(u_n \cdot T_1).$$

Since $T \in AP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$, it follows that there exists a subsequence $\{v_{n_k} \cdot T\}$ of $\{v_n \cdot T\}$, that converges in norm to some $S \in \mathcal{A}_H^*(G)$. As R^* is an isomorphism onto $\mathcal{A}_H^*(G)$, there exists an $S_1 \in \mathcal{A}(H)^*$ so that $\{u_{n_k} \cdot T_1\}$ converges in norm to S_1. This shows that $T_1 \in AP(\mathcal{A}(H))$ and hence that $R^*(AP(\mathcal{A}(H)) = AP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$.

iv) That $R^*(WAP(\mathcal{A}(H)) \subseteq WAP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$ follows in a manner similar to the proof of iii) above.

Suppose that $T \in WAP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$. It follows as before that that there exists a $T_1 \in \mathcal{A}(H)^*$ such that $T = R^*(T_1)$. Let $\{u_\alpha\}_{\alpha \in \Omega} \subseteq \mathcal{A}(H)$ with $\|u_n\|_{\mathcal{A}(H)} \leq 1$. For each $\alpha \in \Omega$, choose $v_\alpha \in \mathcal{A}(G)$, such that $\|v_\alpha\|_{\mathcal{A}(G)} \leq C + 1$ and $v_{\alpha_{|H}} = u_\alpha$.

Since $T \in WAP(\mathcal{A}(G)) \cap \mathcal{A}_H^*(G)$, there exists a subnet $\{v_{\alpha_\beta} \cdot T\}$ of $\{v_\alpha \cdot T\}$ that converges weakly to some $S \in \mathcal{A}(G)^*$. However, since $\mathcal{A}_H^*(G)$ is weakly closed, we have that $S \in \mathcal{A}_H^*(G)$. In particular, $\{v_{\alpha_\beta} \cdot T\}$ converges to S in the $\sigma(\mathcal{A}_H^*(G), \mathcal{A}_H^*(G)^*)$ topology. Moreover $S = R^*(S_1)$ for some $S_1 \in \mathcal{A}(H)^*$. Since $R^* : \mathcal{A}(H)^* \to \mathcal{A}_H^*(G)^*$ is an isomorphism it follows that $\{u_{\alpha_\beta} \cdot T_1\}$ converges to S_1 weakly in $\mathcal{A}(H)^*$. Hence $T_1 \in WAP(\mathcal{A}(H))$. $\square$

Note: In the previous proposition, we do not know if, in general, without the assumption that $H \in Ext_{\mathcal{A}(G)}$ to obtain the equalities.

4.14.

Remark. Let $\mathcal{M}(G)$ denote either $MA(G)$ or $M_{cb}A(G)$, and let $\mathcal{A}(G)$ denote respectively $A_M(G)$ or $A_{cb}(G)$. Let H be an open subgroup of G. For each $u \in \mathcal{M}(H)$ we define u° by

$$u^\circ = \left\{ \begin{array}{ll} u(x) & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{array} \right.$$

Then $u^\circ \in \mathcal{M}(G)$ with $\|u^\circ\|_{\mathcal{M}(G)} = \|u\|_{\mathcal{M}(H)}$.

Define $\Gamma : \mathcal{M}(H) \to \mathcal{M}(G)$ by $\Gamma(u) = u^\circ$. Then Γ is an isometric algebra isomorphism from $\mathcal{M}(H)$ onto the ideal $1_H \mathcal{M}(G)$ of $\mathcal{M}(G)$. Moreover, if $u \in \mathcal{A}(H)$, then $\Gamma(u) = u^\circ \in \mathcal{A}(G)$

Note that if $T \in \mathcal{A}(G)^*$ and $w \in \mathcal{A}(G)$, then

$$\begin{aligned} \langle R^* \circ \Gamma^*(T), w \rangle &= \langle \Gamma^*(T), R(w) \rangle \\ &= \langle T, \Gamma(R(w)) \rangle \\ &= \langle T, 1_H w \rangle \\ &= \langle 1_H \cdot T, w \rangle. \end{aligned}$$

That is $R^* \circ \Gamma^*(T) = 1_H \cdot T$.

If $T \in \mathcal{A}(G)^*$, $v \in \mathcal{A}(G)$ and $u = R(v)$, then for $w \in \mathcal{A}(H)$, we have

$$\begin{aligned} \langle \Gamma^*(v \cdot T), w \rangle &= \langle v \cdot T, w^\circ \rangle \\ &= \langle T, vw^\circ \rangle \\ &= \langle T, (uw)^\circ \rangle \\ &= \langle \Gamma^*(T), uw \rangle \\ &= \langle u \cdot \Gamma^*(T), w \rangle \end{aligned}$$

Hence $\Gamma^*(v \cdot T) = R(v) \cdot \Gamma^*(T)$.

Finally, with H open, we can define a projection P_H from $\mathcal{A}(G)$ onto the ideal $I_{\mathcal{A}(G)}(G \setminus H)$ by $P_H(u) = 1_H u$. The kernel of P_H is the ideal $I_{\mathcal{A}(G)}(H)$. Moreover, $P_H = \Gamma \circ R$.

4.15.

Proposition. *Let $\mathcal{A}(G)$ denote either $A_M(G)$ or $A_{cb}(G)$. Let H be an open subgroup of G. Then*

i) $\Gamma^*(\mathcal{A}(G)^*) = \mathcal{A}(H)^*$ *and* $\Gamma^*_{|_{1_H \cdot \mathcal{A}(G)^*}}$ *is an isometric isomorphism from* $1_H \cdot \mathcal{A}(G)^*$ *onto* $\mathcal{A}(H)^*$

ii) $\Gamma^*(UCB(\mathcal{A}(G))) = UCB(\mathcal{A}(H))$ *and* $\Gamma^*_{|_{1_H \cdot UCB(\mathcal{A}(G))}}$ *is an isometric isomorphism from* $1_H \cdot UCB(\mathcal{A}(G))$ *onto* $UCB(\mathcal{A}(H))$.

iii) $\Gamma^*(AP(\mathcal{A}(G))) = AP(\mathcal{A}(H))$ *and* $\Gamma^*_{|_{1_H \cdot AP(\mathcal{A}(G))}}$ *is an isometric isomorphism from* $1_H \cdot AP(\mathcal{A}(G))$ *onto* $AP(\mathcal{A}(H))$.

iv) $\Gamma^*(WAP(\mathcal{A}(G))) = WAP(\mathcal{A}(H))$ *and* $\Gamma^*_{|_{1_H \cdot WAP(\mathcal{A}(G))}}$ *is an isometric isomorphism from* $1_H \cdot WAP(\mathcal{A}(G))$ *onto* $WAP(\mathcal{A}(H))$.

v) $\Gamma^*(Q_M^\delta(G)) = Q_M^\delta(H)$ *and* $\Gamma^*_M|_{1_H \cdot Q_M^\delta(G)}$ *is an isometric isomorphism from* $1_H \cdot Q_M^\delta(G)$ *onto* $Q_M^\delta(H)$.

vi) $\Gamma^*(Q_{cb}^\delta(G)) = Q_{cb}^\delta(H)$ *and* $\Gamma^*_{|_{1_H \cdot Q_{cb}^\delta(G)}}$ *is an isometric isomorphism from* $1_H \cdot Q_{cb}^\delta(G)$ *onto* $Q_{cb}^\delta(H)$.

vii) $\Gamma^*(Q_M^\lambda(G)) = Q_M^\lambda(H)$ *and* $\Gamma^*_{|_{1_H \cdot Q_M^\lambda(G)}}$ *is an isometric isomorphism from* $1_H \cdot Q_M^\lambda(G)$ *onto* $Q_M^\lambda(H)$.

viii) $\Gamma^*(Q_{cb}^\lambda(G)) = Q_{cb}^\lambda(H)$ *and* $\Gamma^*_{|_{1_H \cdot Q_{cb}^\lambda(G)}}$ *is an isometric isomorphism from* $1_H \cdot Q_{cb}^\lambda(G)$ *onto* $Q_{cb}^\lambda(H)$.

Proof.

i) We first show that Γ^* is surjective. Assume that $T \in \mathcal{A}(H)^*$. Let $S = R^*(T)$. We claim that $T = \Gamma^*(S)$. Let $u \in \mathcal{A}(G)$. Then

$$
\begin{aligned}
\langle \Gamma^*(S), u \rangle &= \langle S, \Gamma(u) \rangle \\
&= \langle R^*(T), u^\circ \rangle \\
&= \langle T, R(u^\circ) \rangle \\
&= \langle T, u \rangle.
\end{aligned}
$$

Next we show that $ker(\Gamma^*) = 1_{G\backslash H} \cdot \mathcal{A}(G)^*$. Let $1_{G\backslash H} \cdot T \in 1_{G\backslash H} \cdot \mathcal{A}(G)^*$ and let $u \in \mathcal{A}(H)$. Then

$$
\begin{aligned}
\langle \Gamma^*(1_{G\backslash H} \cdot T), u \rangle &= \langle 1_{G\backslash H} \cdot T, \Gamma(u) \rangle \\
&= \langle 1_{G\backslash H} \cdot T, u^\circ \rangle \\
&= \langle T, 1_{G\backslash H} u^\circ \rangle \\
&= 0.
\end{aligned}
$$

Assume that $T \in \mathcal{A}(G)^*$, but $T \notin 1_{G\backslash H} \cdot \mathcal{A}(G)^*$. Then there exists $x \in H \cap supp(T)$. In particular, since H is an open neighbourhood of x, there exists $v \in \mathcal{A}(G)$ such that $v(x) \neq 0$ with $supp(v) \subseteq H$ and $\langle T, v \rangle \neq 0$. Let $u = R(v)$. Then $u^\circ = v$ so

$$
\langle \Gamma^*(T), u \rangle = \langle T, \Gamma(u) \rangle = \langle T, u^\circ \rangle = \langle T, v \rangle \neq 0.
$$

Finally, we show that $\Gamma^*_{|_{1_H \cdot \mathcal{A}(G)^*}}$ is an isometry. Clearly, $\|\Gamma^*\|_{B(\mathcal{A}(G)^*, \mathcal{A}(H)^*)} \leq 1$.
Let $T \in 1_H \cdot \mathcal{A}(G)^*$ such that $\|T\|_{\mathcal{A}(G)^*} = 1$. Let $\epsilon > 0$. Then there exists $v \in \mathcal{A}(G)$ with $\|v\|_{\mathcal{A}(G)} = 1$ such that

$$
|\langle T, v \rangle| > 1 - \epsilon.
$$

Note that since $T \in 1_H \cdot \mathcal{A}(G)^*$, we have that $T = 1_H \cdot T$ so that if $w = 1_H v$, then $\|w\|_{\mathcal{A}(G)} \leq 1$ and

$$
|\langle T, v \rangle| = |\langle T, w \rangle| > 1 - \epsilon.
$$

Let $u = R(v)$. Then $\|u\|_{\mathcal{A}(H)} \leq 1$ and

$$
|\langle \Gamma^*(T), u \rangle| = |\langle T, \Gamma(u) \rangle| = |\langle T, w \rangle| > 1 - \epsilon.
$$

ii) We need only show that $\Gamma^*(UCB(\mathcal{A}(G))) = UCB(\mathcal{A}(H))$.
The first observation we will make is that for any $T \in \mathcal{A}(G)^*$, we can write

$$
T = 1_H \cdot T + 1_{G\backslash H} \cdot T
$$

and that $\Gamma^*(T) = \Gamma^*(1_H \cdot T)$. Moreover

$$
UCB(\mathcal{A}(G)) = 1_H \cdot UCB(\mathcal{A}(G)) \oplus 1_{G\backslash H} \cdot UCB(\mathcal{A}(G))
$$

Assume that $T = v \cdot T_1 \in UCB(\mathcal{A}(G))$ with $v \in \mathcal{A}(G)$. Let $u = R(v)$. Then $\Gamma^*(T) = u \cdot \Gamma^*(T_1) \in UCB(\mathcal{A}(H))$. Since Γ^* is continuous and

$\{u \cdot T \mid T \in \mathcal{A}(G)^*, u \in \mathcal{A}(G)\}$ is dense in $UCB(\mathcal{A}(G))$, we get that $\Gamma^*(UCB(\mathcal{A}(G))) \subseteq UCB(\mathcal{A}(H))$.

Next assume that $T = u \cdot T_1$ where $T_1 \in \mathcal{A}(H)^*, u \in \mathcal{A}(H)$. Then there exists an $S \in 1_H \cdot \mathcal{A}(G)^*$ such that $T_1 = \Gamma^*(S)$. If $v = u^\circ$, then

$$\Gamma^*(v \cdot S) = R(v) \cdot \Gamma^*(S) = u \cdot T_1.$$

Hence, $\Gamma^* : 1_H \cdot UCB(\mathcal{A}(G)) \to UCB(\mathcal{A}(H)$ has dense range. But as Γ^* is isometric on $1_H \cdot UCB(\mathcal{A}(G))$, we have that $\Gamma^*(UCB(\mathcal{A}(G))) = UCB(\mathcal{A}(H))$.

iii) It is easy to see that $T \in AP(\mathcal{A}(G))$ if and only if both $1_H \cdot T$ and $1_{G \setminus H} \cdot T$ are in $AP(\mathcal{A}(G))$. Let $T \in 1_H \cdot AP(\mathcal{A}(G))$. Let $S = \Gamma^*(T)$. Now

$$\{u \cdot S \| u\|_{\mathcal{A}(H)} \le 1\} = \Gamma^*(\{u^\circ \cdot T \| u\|_{\mathcal{A}(H)} \le 1\}).$$

But $\{u^\circ \cdot T \| u\|_{\mathcal{A}(H)} \le 1\})$ is relatively compact. Hence, $\{u \cdot S \| u\|_{\mathcal{A}(H)} \le 1\}$ is also relatively compact and $S \in AP(\mathcal{A}(H))$.

For the converse, suppose that $S \in AP(\mathcal{A}(H))$ and choose $T \in 1_H \cdot \mathcal{A}(G)^*$, so that $S = \Gamma^*(T)$. Since $1_H \cdot \mathcal{A}(G)^*$ is invariant, Γ^* maps $\{v \cdot T \| v\|_{\mathcal{A}(G)} \le 1\}$ isometrically onto $\{R(v) \cdot S \| v\|_{\mathcal{A}(G)} \le 1\}$. Since $\{R(v) \cdot S \| v\|_{\mathcal{A}(G)} \le 1\}$ is relatively compact, so is $\{v \cdot T \| v\|_{\mathcal{A}(G)} \le 1\}$.

iv) This follows in a similar manner to iii) above.

v) This follows from i) since, abusing notation,

$$\Gamma^*(\phi_x) = \begin{cases} \phi_x & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

for every $x \in G$.

vi) See v).

vii) This follows from i) since, abusing notation, $\Gamma^*(\phi_f) = \phi_{f_{|H}}$ and that $L^1(H)$ is isometrically isomorphic with $1_H L^1(G)$.

viii) See vii).   □

5. Containment results

In this section, we will establish various containment results for invariant subspaces of $A(G)$, $A_{cb}(G)$, and $A_M(G)$ and their ideals.

5.1.

Lemma. *Let $\mathcal{A}(G)$ denote any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a non-zero closed ideal in $\mathcal{A}(G)$ with $E = Z(I)$. Then*

$$\overline{\operatorname{span}\{\phi_x \mid x \in G \setminus E\}}^{-\|\cdot\|_{I^*}} \subseteq UCB(I).$$

If G is discrete and $Z(I)$ is a set of spectral synthesis, then

$$\overline{\operatorname{span}\{\phi_x \mid x \in G \setminus E\}}^{-\|\cdot\|_{I^*}} = UCB(I).$$

Proof. Given any $x \in G \setminus E$, we can find an open neighbourhood U of x for which $U \cap E = \emptyset$. Then we can choose $u \in A(G) \cap C_{00}(G) \subseteq \mathcal{A}(G)$ so that $u(x) = 1$ and $u(y) = 0$ for every $y \notin U$ and hence $u \in I$. If $v \in I$, then

$$\langle u \cdot \phi_x, v \rangle = \langle \phi_x, vu \rangle = v(x)u(x) = v(x) = \langle \phi_x, v \rangle.$$

Hence $\phi_x = u \cdot \phi_x \in UCB(I)$. It follows that $\overline{\operatorname{span}\{\phi_x \mid x \in G \setminus E\}}^{-\|\cdot\|_{I^*}} \subseteq UCB(I)$.

Next assume that G is discrete and that $E = Z(I)$ is a set of spectral synthesis. Let $T = u \cdot T_1 \in I^*$ where $u \in I$ and $T_1 \in I^*$.

Let $x \in G \setminus E$. If

$$J = \{v \in I \mid v(x) = 0\},$$

then J is a closed ideal in I of co-dimension 1. Hence $J^\perp$ is one dimensional. Clearly $\phi_x \in J^\perp$. It follows that

$$J^\perp = \{\alpha \phi_x \mid \alpha \in \mathbb{C}\}.$$

Next we note that if $w = 1_{\{x\}}$, then $w \cdot T_1 \in J^\perp$. It follows that $w \cdot T_1 = \alpha \phi_x$ for some $\alpha \in \mathbb{C}$.

Let $v \in C_{00}(G) \cap I$ have support $\{x_1, x_2, \ldots, x_k\} \subseteq G \setminus E$. Then $v = \sum_{i=1}^{k} v(x_i) 1_{x_i}$ so that

$$v \cdot T_1 = \sum_{i=1}^{k} v(x_i) 1_{x_i} \cdot T_1 = \sum_{i=1}^{k} v(x_i) \alpha_i \phi_{x_i}$$

for some finite collection of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

Finally since E is a set of spectral synthesis, there exists a sequence $\{u_n\} \subset I$ with $u_n \in c_{00}(G)$ such that $\lim_{n\to\infty} \|u_n - u\|_{A(G)} = 0$. If $w \in I$ with $\|w\|_{A(G)} \leq 1$, then

$$
\begin{aligned}
| \langle u \cdot T_1, w \rangle - \langle u_n \cdot T_1, w \rangle | &= | \langle T_1, wu \rangle - \langle T_1, wu_n \rangle | \\
&= | \langle T_1, w(u - u_n) \rangle | \\
&\leq \|T_1\|_{I^*} \|u - u_n\|_{A(G)}.
\end{aligned}
$$

It follows that $\lim_{n\to\infty} \|u_n \cdot T_1 - u \cdot T_1\|_{I^*} = 0$. However, $u_n \cdot T_1 \in \operatorname{span}\{\phi_x \mid x \in G \setminus E\}$ and hence we have that $T = u \cdot T_1 \in \overline{\operatorname{span}\{\phi_x \mid x \in G \setminus E\}}^{-\|\cdot\|_{I^*}}$. We get that

$$
UCB(I) \subseteq \overline{\operatorname{span}\{\phi_x \mid x \in G \setminus E\}}^{-\|\cdot\|_{I^*}}. \qquad \square
$$

Since the empty set is a set of Spectral synthesis for all of $A(G)$, $A_{cb}(G)$ and $A_M(G)$ the next result follows immediately.

5.2.

Corollary. *Let G be a locally compact group. Then $\mathcal{Q}_{cb}^{\delta}(G) \subseteq UCB(A_{cb}(G))$ and $\mathcal{Q}_M^{\delta}(G) \subseteq UCB(A_M(G))$. Moreover, if G is discrete, then $UCB(A_{cb}(G)) = \mathcal{Q}_{cb}^{\delta}(G) = \mathcal{Q}_{cb}^{\lambda}(G)$ and $UCB(A_M(G)) = \mathcal{Q}_M^{\delta}(G) = \mathcal{Q}_M^{\lambda}(G)$.*

5.3.

Remark. If G is discrete, then $UCB(A_{cb}(G)) = \mathcal{Q}_{cb}^{\lambda}(G) = \mathcal{Q}_{cb}^{\delta}(G)$ and $UCB(A_M(G)) = \mathcal{Q}_M^{\lambda}(G) = \mathcal{Q}_M^{\delta}(G)$. It follows that

$$
UCB(A_{cb}(G))^* = B_{cb}^{\lambda}(G)
$$

and

$$
UCB(A_M(G))^* = B_M^{\lambda}(G)
$$

respectively. Moreover, the normal algebra structure of both $B_{cb}^{\lambda}(G)$ and $B_M^{\lambda}(G)$ agrees with the multiplication on $UCB(A_{cb}(G))^*$ and $UCB(A_M(G))^*$ respectively that is inherited from being the dual of a topologically introverted subspace.

If I is a closed ideal in $A_{cb}(G)$, and if G is discrete and $E = Z(I)$ is a set of spectral synthesis, then we have seen that

$$
\overline{\operatorname{span}\{\phi_x \mid x \in G \setminus E\}}^{-\|\cdot\|_{I^*}} = UCB(I).
$$

In this case,

$$UCB(I)^* = \overline{I}^{-w^*} \subseteq B_{cb}^\lambda(G).$$

Similarly, if $I \subset A_M(G)$, then

$$UCB(I)^* = \overline{I}^{-w^*} \subseteq B_M^\lambda(G)$$

and if $I \subset A(G)$, then

$$UCB(I)^* = \overline{I}^{-w^*} \subseteq B^\lambda(G).$$

5.4.

Proposition. *Let $\mathcal{A}(G)$ denote any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a non-zero closed ideal in $\mathcal{A}(G)$. If I has a bounded approximate identity, then $WAP(I) \subseteq UCB(I)$.*

Proof. Assume that I has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \Omega}$. Let $T \in WAP(I)$. Then for $u \in I$,

$$\lim_{\alpha \in \Omega} \langle u_\alpha \cdot T, u \rangle = \lim_{\alpha \in \Omega} \langle T, u_\alpha u \rangle = \langle T, u \rangle.$$

That is $\{u_\alpha \cdot T\}_{\alpha \in \Omega}$ converges in the weak-* topology on I^* to T. But since $T \in WAP(I)$ and since $\{u_\alpha\}_{\alpha \in \Omega}$ is bounded, there exists a subnet $\{u_{\alpha_\beta}\}_{\beta \in \Omega_1}$ such that $\{u_{\alpha_\beta} \cdot T\}_{\beta \in \Omega_1}$ converges weakly in I^* to some $T_0 \in I^*$. But this means that $T = T_0$ so T is in the weak closure of $\{u \cdot T | u \in I\}$ and hence in the weak closure of $UCB(I)$. But $UCB(I)$ is weakly closed so $T \in UCB(I)$. $\qquad\square$

5.5.

Remark. It follows immediately from the definitions that for any commutative Banach algebra $\mathcal{A}$, we have that $AP(\mathcal{A}) \subseteq WAP(\mathcal{A})$. Hence if $\mathcal{A}(G)$ denotes any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$ and if I is a non-zero closed ideal in $\mathcal{A}(G)$ with a bounded approximate identity, then $AP(I) \subseteq UCB(I)$.

5.6.

Theorem. *Let $\mathcal{A}(G)$ denote any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$.*

   i) *Let I be a non-zero closed ideal in $\mathcal{A}(G)$ with $UCB(I) \subseteq WAP(I)$, then G is discrete.*

   ii) *Let I be a non-zero closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$. Then $\phi_{x_{|_I}} \in AP(I)$. In particular, if G is discrete and $Z(I)$ is a set of spectral synthesis, then $UCB(I) \subseteq AP(I)$.*

  iii) *If G is discrete, $Z(I)$ is a set of spectral synthesis, and I has a bounded approximate identity, then $UCB(I) = AP(I)$.*

Proof.

   i) This is [8, Corollary 4.19].

   ii) Let $\{u_\alpha\}_{\alpha \in \Omega}$ be a net in I such that $\|u_\alpha\|_{\mathcal{A}(G)} \leq 1$, we find a subnet $\{u_{\alpha_\beta}\}_{\beta \in \Omega_1}$ such that $\lim_{\beta \in \Omega_1} u_{\alpha_\beta}(x) = c \in \mathbb{C}$. It follows that if $u \in I$ with $\|u\|_{\mathcal{A}(G)} \leq 1$, then

$$
\begin{aligned}
| \langle c\phi_{x_{|_I}}, u \rangle - \langle u_{\alpha_\beta} \cdot \phi_{x_{|_I}}, u \rangle | &= | \langle \phi_{x_{|_I}}, cu \rangle - \langle \phi_{x_{|_I}}, u_{\alpha_\beta} u \rangle | \\
&= | cu(x) - u_{\alpha_\beta}(x)u(x) | \\
&\leq |c - u_{\alpha_\beta}(x) |
\end{aligned}
$$

    This shows that $\lim_{\beta \in \Omega_1} \|u_{\alpha_\beta} \cdot \phi_{x_{|_I}} - c\phi_{x_{|_I}}\|_{I^*} = 0$. Hence $\phi_{x_{|_I}} \in AP(I)$.

    The fact that $UCB(I) \subseteq AP(I)$ follows from Lemma 5.1

  iii) This follows from ii) and Remark 5.5. $\square$

5.7.

Corollary. *Let $\mathcal{A}(G)$ denote either of the algebras $A_{cb}(G)$ or $A_M(G)$. Then $Q_M^\delta(G) \subset AP(\mathcal{A}(G))$ and $Q_{cb}^\delta(G) \subset AP(\mathcal{A}(G))$. In particular, G is discrete if and only if $UCB(\mathcal{A}(G)) \subseteq AP(\mathcal{A}(G))$. Moreover, if $\mathcal{A}(G)$ has a bounded approximate identity, then G is discrete if and only if $UCB(\mathcal{A}(G)) = AP(\mathcal{A}(G))$.*

5.8.

Proposition. *Let G be a locally compact group. Then*

   i) $\mathcal{Q}_{cb}^\lambda(G) \subseteq UCB(A_{cb}(G))$

   ii) $\mathcal{Q}_{cb}^\lambda(G) \subseteq WAP(A_{cb}(G))$

iii) $\mathcal{Q}^\lambda_{cb}(G) = UCB(A_{cb}(G))$ *if and only if* G *is discrete.*
iv) $\mathcal{Q}^\lambda_M(G) \subseteq UCB(A_M G).$
v) $\mathcal{Q}^\lambda_{cb}(G) \subseteq WAP(A_M(G)).$
vi) $\mathcal{Q}^\lambda_M(G) = UCB(A_M G)$ *if and only if* G *is discrete.*

Proof.

i) Since $C_{00}(G)$ is dense in $L^1(G)$ and hence in $\mathcal{Q}^\lambda_{cb}(G)$, it is enough to show that every $f \in C_{00}(G)$ is in $UCB(A_{cb}(G))$.
Assume that $f \in C_{00}(G)$ with $supp(f) = K$. Then there exits a $u \in A_{cb}(G)$ such that $u(x) = 1$ for each $x \in K$. In particular $u \cdot f = uf = f$. Hence $f \in UCB(A_{cb}(G))$.

ii) Let $\{u_\alpha\}_{\alpha \in \Omega}$ be a net in $A_{cb}(G)$ with $\|u_\alpha\|_{A_{cb}(G)} \leq 1$ for each $\alpha \in \Omega$. Since

$$\{u_\alpha\}_{\alpha \in \Omega} \subseteq B^\lambda_{cb}(G) = \mathcal{Q}^\lambda_{cb}(G)^*,$$

by passing to a subnet if necessary we can assume that $\{u_\alpha\}_{\alpha \in \Omega}$ converges in the $\sigma(B^\lambda_{cb}(G), \mathcal{Q}^\lambda_{cb}(G))$ topology to some $u \in B^\lambda_{cb}(G)$.
Now consider $m \in A_{cb}(G)^{**}$. Let $v = m_{|_{\mathcal{Q}^\lambda_{cb}(G)}}$. If $f \in \mathcal{Q}^\lambda_{cb}(G)$, then

$$
\begin{aligned}
\lim_{\alpha \in \Omega} \langle m, u_\alpha \cdot f \rangle &= \lim_{\alpha \in \Omega} \langle v, u_\alpha \cdot f \rangle \\
&= \lim_{\alpha \in \Omega} \int_G v(x) u_\alpha(x) f(x) dx \\
&= \lim_{\alpha \in \Omega} \langle u_\alpha, v \cdot f \rangle \\
&= \langle u, v \cdot f \rangle \\
&= \langle v, u \cdot f \rangle \\
&= \langle m, u \cdot f \rangle.
\end{aligned}
$$

That is $\{u_\alpha \cdot f\}_{\alpha \in \Omega}$ converges weakly to $u \cdot f$. This shows that $f \in WAP(A_{cb}(G))$.

iii) Assume that $\mathcal{Q}^\lambda_{cb}(G) = UCB(A_{cb}(G))$. Then by ii) above we have that $UCB(A_{cb}(G)) \subseteq WAP(A_{cb}(G))$. In particular, Theorem 5.6 i) shows that G is discrete.
The converse is Corollary 5.2.
The proofs of iv), v) and vi) are similar to those of the corresponding statements about $A_{cb}(G)$. $\square$

5.9.

Remark. If $\mathcal{A}(G)$ is any of $A(G)$, $A_M(G)$ or $A_{cb}(G)$, then the question of when $\mathcal{A}(G) = WAP(\mathcal{A}(G))$ is equivalent to the question of identifying

groups G such that $\mathcal{A}(G)$ is Arens regular. For G amenable, it is known that G must be finite. In general, while it is strongly suspected that $\mathcal{A}(G) = WAP(\mathcal{A}(G))$ if and only if G is finite, surprisingly, even for $A(G)$, this is still not known to be true. However, we have recently shown that if $\mathcal{A}(G) = WAP(\mathcal{A}(G))$, then G must be discrete [8, Theorem 5.1].

Granirer has shown that $UCB(A(G)) = VN(G)$ if and only if G is compact. This motivates us asking: If $\mathcal{A}(G)$ is either $A_M(G)$ or $A_{cb}(G)$, is $UCB(\mathcal{A}(G)) = \mathcal{A}(G)^*$ if and only if G is compact. While we strongly expect the answer to this question to be yes, we are unable to show this in general. However, we can show that if $UCB(\mathcal{A}(G)) = \mathcal{A}(G)^*$, then G has a compact open subgroup, and every amenable subgroup is compact.

5.10.

Theorem. *Let $\mathcal{A}(G)$ be either $A_M(G)$ or $A_{cb}(G)$. If $UCB(\mathcal{A}(G)) = \mathcal{A}(G)^*$, then G has a compact open subgroup, and every closed amenable subgroup is compact. In particular, if G is almost connected, then G is compact.*

Proof. Assume that H is a closed amenable subgroup. Then $H \in Ext_{\mathcal{A}(G)}$. It follows from Proposition 4.13 that $R^*(UCB(A(H)) = UCB(\mathcal{A}(G)) \cap VN_H(G)$. However, if we assume that $UCB(\mathcal{A}(G)) = \mathcal{A}(G)^*$, we have that $R^*(UCB(A(H)) = VN_H(G)$. This in turn means that $UCB(A(H)) = VN(H)$. It then follows from Granirer's result that H is compact.

To see that G has an open subgroup we note that if the connected component G_0 is not amenable, then G_0 contains the free group $\mathbb{F}_2$ on two generators. In particular, G_0 contains an infinite non-compact abelian group which is impossible from our argument above. As a consequence it must be the case that G_0 is compact. From here we note that G also has an open almost connected subgroup G_1. As G_1/G_0 is compact, G_1 must also be compact. $\square$

Acknowledgments

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