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Weighted composition operators preserving various Lipschitz constants

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Dedicated to Anthony To-Ming Lau on the occasion of his 80th birthday

Let $\mathrm{Lip}(X)$, $\mathrm{Lip}^b(X)$, $\mathrm{Lip}^{\mathrm{loc}}(X)$ and $\mathrm{Lip}^{\mathrm{pt}}(X)$ be the vector spaces of Lipschitz, bounded Lipschitz, locally Lipschitz and pointwise Lipschitz (real-valued) functions defined on a metric space (X, d_X), respectively. We show that if a weighted composition operator $Tf = h \cdot f \circ \varphi$ defines a bijection between such vector spaces preserving Lipschitz constants, local Lipschitz constants or pointwise Lipschitz constants, then $h = \pm 1/\alpha$ is a constant function for some scalar $\alpha > 0$ and φ is an α-dilation.

Let V be open connected and U be open, or both U, V are convex bodies, in normed linear spaces E, F, respectively. Let $Tf = h \cdot f \circ \varphi$ be a bijective weighed composition operator between the vector spaces $\mathrm{Lip}(U)$ and $\mathrm{Lip}(V)$, $\mathrm{Lip}^b(U)$ and $\mathrm{Lip}^b(V)$, $\mathrm{Lip}^{\mathrm{loc}}(U)$ and $\mathrm{Lip}^{\mathrm{loc}}(V)$, or $\mathrm{Lip}^{\mathrm{pt}}(U)$ and $\mathrm{Lip}^{\mathrm{pt}}(V)$, preserving the Lipschitz, locally Lipschitz, or pointwise Lipschitz constants, respectively. We show that there is a linear isometry $A : F \to E$, an $\alpha > 0$ and a vector $b \in E$ such that $\varphi(x) = \alpha Ax + b$, and h is a constant function assuming value $\pm 1/\alpha$. More concrete results are obtained for the special cases when $E = F = \mathbb{R}^n$, or when U, V are n-dimensional flat manifolds.

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1. Introduction

Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. Let $B(p, \epsilon)$ be the open ball in the metric space X centered at p of radius $\epsilon > 0$. We say

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that f is *Lipschitz* if the *Lipschitz constant*

$$L(f) = \sup_{x \neq y \in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)} < +\infty;$$

we say that f is *locally Lipschitz* if the *local Lipschitz constants*

$$L_p^{\mathrm{loc}}(f) = \lim_{\epsilon \to 0^+} \sup_{x \neq y \in B(p,\epsilon)} \frac{d_Y(f(x), f(y))}{d_X(x, y)} < +\infty, \quad \forall p \in X;$$

and we say that f is *pointwise Lipschitz* if the *pointwise Lipschitz constants*

$$L_p^{\mathrm{pt}}(f) = \limsup_{x \to p} \frac{d_Y(f(x), f(p))}{d_X(x, p)} < +\infty, \quad \forall p \in X.$$

Let $\mathrm{Lip}(X)$, $\mathrm{Lip}^b(X)$, $\mathrm{Lip}^{\mathrm{loc}}(X)$ and $\mathrm{Lip}^{\mathrm{pt}}(X)$ denote the (real) vector spaces of Lipschitz, bounded Lipschitz, locally Lipschitz, and pointwise Lipschitz (real-valued) functions on X into $Y = \mathbb{R}$, respectively. Clearly, $\mathrm{Lip}^b(X) \subseteq \mathrm{Lip}(X) \subseteq \mathrm{Lip}^{\mathrm{loc}}(X) \subseteq \mathrm{Lip}^{\mathrm{pt}}(X)$ and $0 \leq L^{\mathrm{pt}}(f) \leq L^{\mathrm{loc}} \leq L(f)$, in general, but all the inclusions and inequalities can be strict. However, for a bounded metric space X we have $\mathrm{Lip}(X) = \mathrm{Lip}^b(X)$. On the other hand, $\mathrm{Lip}(X) = \mathrm{Lip}(\overline{X})$ where $\overline{X}$ is the metric completion of X. See, e.g., [2, Examples 2.6 and 2.7], for more details.

We are interested in the question how the (resp. local, pointwise) Lipschitz constants determine the (resp. local, pointwise) Lipschitz function spaces. For example, if $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a bijective linear map preserving Lipschitz constants, namely,

$$L(Tf) = L(f), \quad \text{for every } f \in \mathrm{Lip}(X),$$

we ask if the underlying metric spaces X, Y are equivalent, and if T carries some good structure. The following results give us motivations.

Proposition 1.1. *(a)* (Weaver [11, Theorem D]; see also [12, Sections 2.6 and 2.7]) *Let X, Y be complete metric spaces of diameter at most 2, which cannot be written as a disjoint union of two nonempty subsets with distance ≥ 1. If $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a surjective linear isometry with respect to the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, then*

$$Tf = \alpha \cdot f \circ \varphi, \quad \forall f \in \mathrm{Lip}(X),$$

where $\varphi : Y \to X$ is a surjective isometry and α is a unimodular constant.

(b) (Wu [13], Araujo and Dubarbie [1, Corollary 6.1]) *Let X be a bounded complete metric space and Y be a compact metric space. If $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a linear bijection preserving zero products, that is,*

$$fg = 0 \quad \Longrightarrow \quad TfTg = 0,$$

then

$$Tf = h \cdot f \circ \varphi, \quad \forall f \in \mathrm{Lip}(X),$$

where φ is a bijective Lipschitz map from Y onto X with Lipschitz inverse φ^{-1}, and $h \in \mathrm{Lip}(Y)$ is nonvanishing.

See also, e.g., [2, 6, 10]. However, if the map T preserves only the Lipschitz constants without other assumption, T needs not carry a weighted composition operator form. For example, let Ψ be an arbitrary linear functional of the vector space $\mathrm{Lip}(X)$. Then the assignment $Tf(x) = f(x) + \Psi(f)$ defines a linear bijective map from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(X)$ preserving Lipschitz constants. But T does not assume a weighted composition form.

In this paper, based on [7], we study the structure of a weighted composition operator $Tf = h \cdot f \circ \varphi$ between Lipschitz function spaces defined on metric spaces X and Y, which preserves Lipschitz constants. We will see that φ is an α-*dilation* for a positive constant α, i.e., $d_X(\varphi(u), \varphi(v)) = \alpha d_Y(u, v)$, $\forall u, v \in Y$, and h is a constant function assuming the value $\pm 1/\alpha$. In particular, if V is open connected and U is open, or both U, V are convex bodies, in normed linear spaces E, F, respectively, then $\varphi(x) = \alpha Ax + b$ for a linear isometry $A : F \to E$ and a vector $b \in E$. In other words,

$$Tf(x) = \pm\alpha^{-1} f(Ax + b).$$

When $E = F = \mathbb{R}^n$, we obtain similar results for those bijective weighted composition operators preserving local or pointwise Lipschitz constants. We also extend these results to the case when U, V are flat manifolds.

2. Weighted composition operators preserving Lipschitz constants

All vector spaces discussed in this paper are over the real number field $\mathbb{R}$.

In this section, we study Lipschitz constant preserving weighted composition operators between Lipschitz function spaces. We begin with some well known results. Here, by an α-*dilation* (resp. *isometry*) between metric spaces X, Y for $\alpha > 0$, we mean a map $\varphi : X \to Y$ such that $d_Y(\varphi(x), \varphi(y)) =$

$\alpha d_X(x,y)$ (resp. when $\alpha = 1$) for all $x, y \in X$. Moreover, a *convex body* in a normed linear space is a (not necessarily closed or bounded) convex set with nonempty interior.

Proposition 2.1. *(a)* (Mazur-Ulam theorem; see, e.g., [4]) *Every surjective isometry $T : M \to N$ between normed linear spaces is affine; namely,*

$$T(\lambda x + (1-\lambda)y) = \lambda Tx + (1-\lambda)Ty, \quad \text{for all } x, y \in M \text{ and } 0 \le \lambda \le 1.$$

(b) (Mankiewicz [9]) *Every isometry $\varphi : U \to V$ from an open connected set (resp. convex body) U in a normed linear space M onto an open set (resp. convex body) V in a normed linear space N has a unique isometric extension from M onto N.*

Recall that a metric space (X, d_X) is *quasi-convex* if there is a constant $C > 0$ such that for any $x, y \in X$ there is a continuous path in X joining x to y with length at most $Cd_X(x,y)$.

Proposition 2.2 ([5, Theorems 3.9 and 3.12])**.** *Let $\varphi : Y \to X$ be a map between metric spaces. Then φ is Lipschitz if and only if*

(a) *$f \circ \varphi \in \mathrm{Lip}(Y)$ for each $f \in \mathrm{Lip}(X)$; or*
(b) *$f \circ \varphi \in \mathrm{Lip}^b(Y)$ for each $f \in \mathrm{Lip}^b(X)$, provided that both X, Y are quasi-convex.*

Theorem 2.3. *Let (X, d_X) and (Y, d_Y) be (resp. quasi-convex) metric spaces. Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ (resp. $T : \mathrm{Lip}^b(X) \to \mathrm{Lip}^b(Y)$) be a bijective weighted composition operator $Tf = h \cdot f \circ \varphi$ preserving Lipschitz constants. Then φ is an α-dilation from Y onto a dense subset $X_0 = \varphi(Y)$ of X for $\alpha > 0$, and h is a constant function assuming either $1/\alpha$ or $-1/\alpha$. If Y is complete, then $\varphi(Y) = X$.*

Proof. We verify the case when T sends $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$. The other case follows similarly.

Since T preserves Lipschitz constants, so does its inverse; indeed,

$$L(g) = L(TT^{-1}g) = L(T^{-1}g), \quad \text{for all } g \text{ in } \mathrm{Lip}(Y).$$

Observe that

$$0 = L(1) = L(T1) = L(h) = \sup_{u \neq v} \frac{|h(u) - h(v)|}{d_Y(u,v)}.$$

Hence $h = \pm 1/\alpha$ is a nonzero constant function for a scalar $\alpha > 0$.

For any f in $\mathrm{Lip}(X)$, we have $f \circ \varphi = \pm\alpha Tf \in \mathrm{Lip}(Y)$. It follows from Proposition 2.2 that φ is Lipschitz. Since T is bijective, φ is one-to-one with dense range. Let $X_0 = \varphi(Y)$, which is a dense subset of the metric space X. Define a bijective linear map $S : \mathrm{Lip}(Y) \to \mathrm{Lip}(X_0)$ such that $Sg = T^{-1}g\,|_{X_0}$; in other words, $Sg = \pm\alpha g \circ \psi$, where ψ is the bijection from X_0 onto Y defined by the condition $\psi(x) = y$ whenever $x = \varphi(y)$. Then a similar argument shows that ψ is Lipschitz from X_0 onto Y. Let $\overline{\varphi} : \overline{Y} \to \overline{X}$ and $\overline{\psi} : \overline{X} \to \overline{Y}$ be the unique Lipschitz extensions of φ and ψ between the metric completions $\overline{X}, \overline{Y}$ of X, Y, respectively. For any $x \in \overline{X}$, let $x_n \in X_0$ such that $x_n \to x$, we have $\overline{\varphi}(\overline{\psi}(x)) = \lim_n \overline{\varphi}(\overline{\psi}(x_n)) = \lim_n \varphi(\psi(x_n)) = \lim_n x_n = x$. It amounts to saying that $\overline{\varphi} \circ \overline{\psi} = I_{\overline{X}}$. In a similar manner, we see that $\overline{\psi} \circ \overline{\varphi} = I_{\overline{Y}}$, and thus $\overline{\varphi}^{-1} = \overline{\psi}$ is also Lipschitz.

On the other hand, the fact

$$L(f) = L(Tf) = L(\pm\alpha^{-1} f \circ \varphi) \leq \alpha^{-1} L(f) L(\varphi), \quad \forall f \in \mathrm{Lip}(X),$$

implies that $L(\varphi) \geq \alpha$. Assume $L(\varphi) > \alpha$. Then there are p and q in Y, such that

$$d_X(\varphi(p), \varphi(q)) > \alpha d_Y(p, q).$$

Define $\tilde{f} : X \to \mathbb{R}$ by

$$\tilde{f}(x) = \min\{d_X(x, \varphi(p)), d_X(\varphi(p), \varphi(q))\}.$$

Then

$$\begin{aligned} |\tilde{f}(x) - \tilde{f}(y)| &\leq |d_X(x, \varphi(p)) - d_X(y, \varphi(p))| \\ &\leq d_X(x, y), \quad \text{for all } x,\ y \text{ in } X, \end{aligned}$$

and

$$\|\tilde{f}(\varphi(q)) - \tilde{f}(\varphi(p))\| = d_X(\varphi(q), \varphi(p)).$$

Hence, $\tilde{f} \in \mathrm{Lip}^b(X)$ with $L(\tilde{f}) = 1$. But

$$\begin{aligned} |T\tilde{f}(p) - T\tilde{f}(q)| &= |h(p)\tilde{f} \circ \varphi(p) - h(q)\tilde{f} \circ \varphi(q)| \\ &= \alpha^{-1} d_X(\varphi(p), \varphi(q)) \\ &> d_Y(p, q). \end{aligned}$$

This implies $L(T\tilde{f}) > 1$, which is a contradiction. Hence $L(\varphi) = \alpha$. Similarly, $L(\psi) = \alpha^{-1}$.

If $u \neq v$ in Y and $\varphi(u) = s$, $\varphi(v) = t$ in X, then

$$\alpha = L(\varphi) \geq \frac{d_X(\varphi(u), \varphi(v))}{d_Y(u, v)} = \frac{d_X(s, t)}{d_Y(\psi(s), \psi(t))} \geq \frac{1}{L(\psi)} = \alpha,$$

and thus

$$d_X(\varphi(u), \varphi(v)) = \alpha d_Y(u, v).$$

In other words, φ is an α-dilation.

Finally, if $Y = \overline{Y}$ is complete then $X_0 = \varphi(Y)$ is also complete, and thus $\varphi(Y) = \overline{X_0} = X$. □

Corollary 2.4. *Assume either that U, V are open sets and V is connected, or that both U, V are convex bodies, in normed linear spaces E, F, respectively. Let the weighed composition operator $Tf = h \cdot f \circ \varphi$ define a bijection from* Lip(U) *onto* Lip(V) *preserving Lipschitz constants. Then there is a linear isometry $A : F \to E$ with dense range, a scalar $\alpha > 0$ and a vector $b \in E$ such that $\varphi(y) = \alpha Ay + b$ for all $y \in V$, and h is a constant function assuming value $\pm\alpha^{-1}$. In other words,*

$$Tf(y) = \pm\alpha^{-1} f(\alpha Ay + b), \quad \textit{for all } f \in \mathrm{Lip}(U) \textit{ and for all } y \in V.$$

Proof. By Theorem 2.3, we know that $h = \pm\alpha^{-1}$ is a constant function for some $\alpha > 0$, and φ is an α-dilation from V onto a dense subset of U. Indeed, φ extends uniquely to an α-dilation $\overline{\varphi}$ from $\overline{V}$ onto $\overline{U}$, where $\overline{U}, \overline{V}$ are the closures of U, V in the Banach space completions $\overline{E}, \overline{F}$ of E, F, respectively.

If V is open and connected in F, then so is the interior of $\overline{V}$ in $\overline{F}$. It is plain that $\overline{\varphi}$ sends the interior of $\overline{V}$ in the Banach space $\overline{F}$ onto the interior of $\overline{U}$ in the Banach space $\overline{E}$. Then $\overline{\varphi}$ extends uniquely to an affine α-dilation from $\overline{F}$ onto $\overline{E}$ by Proposition 2.1. In the other case, U, V are both convex bodies in E, F, and thus so are $\overline{U}, \overline{V}$ in the Banach spaces $\overline{E}, \overline{F}$, respectively. It follows from Proposition 2.1 again that $\overline{\varphi}$ extends uniquely to an affine α-dilation from $\overline{F}$ onto $\overline{E}$. In both cases, φ extends to a unique affine α-dilation from F into E with dense range. The assertion follows. □

Corollary 2.5. *Let U, V be convex bodies in normed linear spaces E, F, respectively. Let the weighed composition operator $Tf = h \cdot f \circ \varphi$ define a bijection from* Lip$^b(U)$ *onto* Lip$^b(V)$ *preserving Lipschitz constants. Then there is a surjective linear isometry $A : F \to E$, a scalar $\alpha > 0$ and a vector $b \in E$ such that $\varphi(y) = \alpha Ay + b$ for all $y \in V$, and h is a constant function assuming value $\pm\alpha^{-1}$. In other words,*

$$Tf(y) = \pm\alpha^{-1} f(\alpha Ay + b), \quad \textit{for all } f \in \mathrm{Lip}(U) \textit{ and for all } y \in V.$$

Proof. Note that convex bodies in normed spaces are quasi-convex. The proof now goes the same way as in Corollary 2.4. □

Example 2.6. (a) Let $Tf = h \cdot f \circ \varphi$ define a bijection $T : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$ preserving Lipschitz constants. It follows from Theorem 2.3 that φ is Lipschitz, and indeed an $L(\varphi)$-dilation of $\mathbb{R}^n$. Moreover, h is the constant function assuming either $1/L(\varphi)$ or $-1/L(\varphi)$. By Proposition 2.1, there is an $n \times n$ orthogonal matrix A such that $\varphi(x) = L(\varphi)Ax + \varphi(0)$. Let $\alpha = L(\varphi)$ and $b = \varphi(0)$, we have

$$Tf(x) = \pm\alpha^{-1} f(\alpha Ax + b) \quad \text{for all } x \in \mathbb{R}^n.$$

(b) Let $Tf = h \cdot f \circ \varphi$ define a bijection $T : \mathrm{Lip}([0,1]) \to \mathrm{Lip}([0,1])$ preserving Lipschitz constants. Applying Theorem 2.3, we see that φ is an $L(\varphi)$-dilation of $[0,1]$. This forces $L(\varphi) = 1$, and either $\varphi(x) = x$ or $\varphi(x) = 1 - x$. Consequently, T is given by

$$Tf(x) = \pm f(x) \quad \text{or} \quad Tf(x) = \pm f(1-x).$$

(c) More generally, let $Tf = h \cdot f \circ \varphi$ define a bijection $T : \mathrm{Lip}([a,b]) \to \mathrm{Lip}([c,d])$ preserving Lipschitz constants. Then $L(\varphi) = \frac{b-a}{d-c}$, $h = \pm\frac{d-c}{b-a}$, and either

$$\varphi(x) = \frac{b-a}{d-c}(x-c) + a \quad \text{or} \quad \varphi(x) = \frac{b-a}{d-c}(d-x) + a.$$

Consequently,

$$T(f) = \pm\frac{d-c}{b-a} f\left(\frac{b-a}{d-c}(x-c) + a\right),$$

or

$$T(f) = \pm\frac{d-c}{b-a} f\left(\frac{b-a}{d-c}(d-x) + a\right).$$

(d) Let $T : \mathrm{Lip}([0,1]^n) \to \mathrm{Lip}([0,1]^n)$ be a bijective weighted composition operator preserving Lipschitz constants. It follows from Corollary 2.4 and the structure of the symmetric group of the n-cube that

$$Tf(x) = \pm f(Px + b) \quad \text{for all } x \in [0,1]^n,$$

for an $n \times n$ signed permutation matrix P (in the sense that exactly one entry in each row and each column is ± 1, and elsewhere 0), and a vector $b \in \mathbb{R}^n$ in which all entries are either 0 or 1.

(e) There are similar versions for bounded Lipschitz functions in all above examples. The conclusions are identical.

3. Weighted composition operators preserving local/pointwise Lipschitz constants

We say that a weighted composition operator $Tf = h \cdot f \circ \varphi$ from $\mathrm{Lip}^{\mathrm{loc}}(X)$ into $\mathrm{Lip}^{\mathrm{loc}}(Y)$, or from $\mathrm{Lip}^{\mathrm{pt}}(X)$ into $\mathrm{Lip}^{\mathrm{pt}}(Y)$, preserves the local or pointwise Lipschitz constants (with respect to φ), if

$$L_p^{\mathrm{loc}}(Tf) = L_{\varphi(p)}^{\mathrm{loc}}(f) \quad \text{for all } f \in \mathrm{Lip}(X) \text{ and all } p \in Y,$$

or

$$L_p^{\mathrm{pt}}(Tf) = L_{\varphi(p)}^{\mathrm{pt}}(f) \quad \text{for all } f \in \mathrm{Lip}(X) \text{ and all } p \in Y.$$

In this section we consider weighted composition operators between locally or pointwise Lipschitz functions on Euclidean spaces preserving local or pointwise Lipschitz constants, respectively.

Proposition 3.1. *Let (X, d_X) and (Y, d_Y) be metric spaces without isolated point, and let $\varphi : Y \to X$ be a map. Suppose that*

$$f \circ \varphi \in \mathrm{Lip}^{\mathrm{loc}}(Y) \ (\textit{resp. } \mathrm{Lip}^{\mathrm{pt}}(Y)) \quad \textit{for all } f \in \mathrm{Lip}^{\mathrm{loc}}(X) \ (\textit{resp. } \mathrm{Lip}^{\mathrm{pt}}(X)).$$

Then φ is locally (resp. pointwise) Lipschitz from Y into X. If the composition map $f \mapsto f \circ \varphi$ preserves local (resp. pointwise) Lipschitz constants then $L_q^{\mathrm{loc}}(\varphi) = 1$ (resp. $L_q^{\mathrm{pt}}(\varphi) = 1$) for every $q \in Y$.

Proof. Let $q \in Y$, and let $f_q(x) = \min\{d_X(x, \varphi(q)), 1\}$. Then f_q is bounded and pointwise Lipschitz on X. In particular, $L_{\varphi(q)}^{\mathrm{pt}}(f_q) = 1$. Since q is not an isolated point in Y, we see that $d_X(\varphi(y), \varphi(q)) < 1$ eventually when $y \to q$. Hence,

$$\begin{aligned} L_q^{\mathrm{pt}}(f_q \circ \varphi) &= \limsup_{y \to q} \frac{|f_q(\varphi(y)) - f_q(\varphi(q))|}{d_Y(y, q)} \\ &= \limsup_{y \to q} \frac{\min\{d_X(\varphi(y), \varphi(q)), 1\}}{d_Y(y, q)} < +\infty. \end{aligned}$$

Thus,

$$L_q^{\mathrm{pt}}(\varphi) = \limsup_{y\to q} \frac{d_X(\varphi(y), \varphi(q))}{d_Y(y,q)} = L_q^{\mathrm{pt}}(f_q \circ \varphi) < +\infty.$$

Since this holds for every point q in Y, we see that φ is pointwise Lipschitz from Y into X. Moreover, $L_q^{\mathrm{pt}}(\varphi) = 1$ for all $q \in Y$ if the composition map $f \mapsto f \circ \varphi$ preserves pointwise Lipschitz constants.

The case for local Lipschitzness is proved in [5, Lemma 3.15]. As an alternative proof, consider

$$\begin{aligned} L_{\varphi(q)}^{\mathrm{loc}}(f_q) &= \limsup_{y,z\to\varphi(q)} \frac{|f_q(y) - f_q(z)|}{d_X(y,z)} \\ &\le \limsup_{y,z\to\varphi(q)} \frac{|d_X(y,\varphi(q)) - d_X(z,\varphi(q))|}{d_X(y,z)} \le 1, \end{aligned}$$

while, in general,

$$L_{\varphi(q)}^{\mathrm{loc}}(f_q) \ge L_{\varphi(q)}^{\mathrm{pt}}(f_q) = 1.$$

Thus, $L_{\varphi(q)}^{\mathrm{loc}}(f_q) = 1$. In a similar fashion, we can verify that φ is locally Lipschitz, and $L_q^{\mathrm{loc}}(\varphi) = 1$ when the composition map $f \mapsto f \circ \varphi$ preserves local Lipschitz constants. □

Lemma 3.2. *Let $\varphi : [a,b] \to \mathbb{R}$ be a locally (resp. pointwise) Lipschitz function satisfying that*

$$L_p^{\mathrm{loc}}(\varphi) = \alpha \quad \textit{(resp. } L_p^{\mathrm{pt}}(\varphi) = \alpha\textit{)}, \quad \textit{for all } p \in [a,b].$$

If $\alpha = 0$ then $\varphi(x) = c$ for some fixed scalar c. In general, if φ is injective then $\varphi(x) = \alpha x + c$ or $\varphi(x) = -\alpha x + c$ on $[a,b]$ for some fixed scalar c.

Proof. Suppose

$$L_p^{\mathrm{loc}}(\varphi) = \limsup_{\substack{x,y\to p\\ x,y\in[a,b]}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} = \alpha \quad \text{for all } p \in [a,b].$$

Let $\epsilon > 0$. For any point $p \in [a,b]$, there is an open interval $B(p,\delta_p)$ centered at p with radius $\delta_p > 0$ such that

$$\frac{|\varphi(x) - \varphi(y)|}{|x-y|} < \alpha + \epsilon \quad \text{for all } x,y \in B(p,\delta_p) \cap [a,b].$$

By the Lebesgue's number lemma, there is a $\delta > 0$ such that for any partition $[a,b] = \bigcup_{i=1}^m [x_{i-1}, x_i]$ with all $|x_i - x_{i-1}| < \delta$, we have $x_{i-1}, x_i \in B(p_i, \delta_{p_i})$ for some $p_i \in [a,b]$, and thus

$$\sum_{i=1}^m |\varphi(x_i) - \varphi(x_{i-1})| < \sum_{i=1}^m (\alpha+\epsilon)|x_i - x_{i-1}| \\ =(\alpha+\epsilon)(b-a) < +\infty. \tag{3.1}$$

In particular, φ is of bounded variation on $[a,b]$. Consequently, φ is differentiable almost everywhere in $[a,b]$, and $|\varphi'(p)| = \alpha$ for almost all $p \in [a,b]$. If $\alpha = 0$ then φ assumes constant value c on $[a,b]$.

Suppose φ is injective now. Since φ is continuous and injective, it is monotone on $[a,b]$. Thus, $\varphi' = \alpha$ or $\varphi' = -\alpha$ almost everywhere on $[a,b]$. Hence, in the sense of Lebesgue integral,

$$\varphi(x) - \varphi(a) = \int_a^x \varphi'(t)\,dt = \pm\alpha(x-a).$$

Consequently, $\varphi(x) = \alpha x + c$ or $\varphi(x) = -\alpha x + c$ on $[a,b]$, where $c = \mp\alpha a + \varphi(a)$.

The other case when

$$L_p^{\mathrm{pt}}(\varphi) = \limsup_{x\to p} \frac{|\varphi(x) - \varphi(p)|}{|x-p|} = \alpha, \quad \text{for all } p \in [a,b],$$

follows similarly, except that in (3.1) we might break $|\varphi(x_i) - \varphi(x_{i-1})| \le |\varphi(x_i) - \varphi(p_i)| + |\varphi(p_i) - \varphi(x_{i-1})|$ where $x_{i-1}, x_i \in B(p_i; \delta_{p_i})$. □

The following example tells us that the injectivity condition is indispensable.

Example 3.3. Define $\varphi : [0,1] \to [\frac{1}{2}, 1]$ by

$$\varphi(x) = \begin{cases} x, & \frac{1}{2} \le x \le 1, \\ 1-x, & 0 \le x \le \frac{1}{2}. \end{cases}$$

Then $L_x^{\mathrm{loc}}(\varphi) = 1$ for all $x \in [0,1]$. But φ is not linear.

Lemma 3.4. *Let V be either an open connected set or a convex body in a normed linear space. Let h be locally or pointwise Lipschitz function on V with $L_q^{\mathrm{loc}}(h) = 0$ or $L_q^{\mathrm{pt}}(h) = 0$ for all $q \in V$. Then h is a constant function on V.*

Proof. Suppose $h \in \mathrm{Lip}^{\mathrm{pt}}(V)$ with $L^{\mathrm{pt}}_q(h) = 0$ for all $q \in V$. Let $(1-t)x+ty$ for $t \in [0,1]$ be a path lying in the interior of V. Define $f : [0,1] \to \mathbb{R}$ by $f(t) = h((1-t)x+ty)$. Then $f \in \mathrm{Lip}^{\mathrm{pt}}([0,1])$ with $L^{\mathrm{pt}}_t(f) = 0$ for all $t \in [0,1]$. By Lemma 3.2, f is a constant function. In other words, $h(x) = h(y)$. A connected argument shows that h is constant on V if V is open and connected. If V is a convex body, then we can see that h is constant on the interior of V. By continuity, h is constant on V. □

We also have the following well known results.

Proposition 3.5 (Rademacher/Stepanov Theorem; see, e.g., [3, 8])**.** *Let V be an open subset of $\mathbb{R}^n$. Every locally/pointwise Lipschitz function $\varphi(x) = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x))$ from V into $\mathbb{R}^n$ is differentiable on V almost everywhere. In particular, for every $v \neq 0$ in $\mathbb{R}$, the directional derivative $D_v\varphi_i(x) = v \cdot \nabla\varphi_i(x)$ exists for almost every x in V for $i = 1, 2, \cdots, n$.*

Theorem 3.6. *Let U and V be two open connected sets or two convex bodies in $\mathbb{R}^n$. Let $T : \mathrm{Lip}^{\mathrm{loc}}(U) \to \mathrm{Lip}^{\mathrm{loc}}(V)$ (resp. $T : \mathrm{Lip}^{\mathrm{pt}}(U) \to \mathrm{Lip}^{\mathrm{pt}}(V)$) be a bijective weighted composition operator defined by $Tf = h \cdot f \circ \varphi$ preserving local (resp. pointwise) Lipschitz constants. Then h assumes constant value $\pm\alpha^{-1}$ on V for some $\alpha > 0$, and φ assumes the form $\varphi(v) = \alpha Av + b$ for an $n \times n$ orthogonal matrix A and a point $b \in \mathbb{R}^n$.*

Proof. We present the proof for the case when U, V are open connected sets in $\mathbb{R}^2$, and $T : \mathrm{Lip}^{\mathrm{loc}}(U) \to \mathrm{Lip}^{\mathrm{loc}}(V)$ preserves local Lipschitz constants. The other cases follow similarly.

It is clear that φ is injective and $\varphi(V)$ is a dense subset of U. Since $h = T\mathbf{1}_U$ is locally Lipschitz with $L^{\mathrm{loc}}_{\varphi(p)}(h) = L^{\mathrm{loc}}_p(\mathbf{1}_U) = 0$ for all $p \in V$, we see that $h = \pm\alpha^{-1}$ is a constant function for some $\alpha > 0$ by Lemma 3.4. It then follows from Proposition 3.1 that both φ and φ^{-1} are locally Lipschitz.

We write $\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$. It follows from Proposition 3.5 that all partial derivatives $\frac{\partial\varphi_1}{\partial x}, \frac{\partial\varphi_1}{\partial y}, \frac{\partial\varphi_2}{\partial x}$ and $\frac{\partial\varphi_2}{\partial y}$ exist, and $L^{\mathrm{loc}}_{(x,y)}(\varphi_i) = \|\nabla\varphi_i(x,y)\|$ for $i = 1, 2$, for almost every point (x,y) in V. Let $I_1(x,y) = x$ and $I_2(x,y) = y$. Then I_1 and I_2 belong to $\mathrm{Lip}^{\mathrm{loc}}(V)$ with $L^{\mathrm{loc}}_{(x,y)}(I_1) = L^{\mathrm{loc}}_{(x,y)}(I_2) = 1$. It follows that

$$\alpha^{-1} L^{\mathrm{loc}}_{\varphi(x,y)}(\varphi_1) = L^{\mathrm{loc}}_{\varphi(x,y)}(h \cdot I_1 \circ \varphi) = L^{\mathrm{loc}}_{(x,y)}(I_1) = 1$$

for almost every point (x,y) in V. Thus

$$(\frac{\partial\varphi_1}{\partial x}(x,y))^2 + (\frac{\partial\varphi_1}{\partial y}(x,y))^2 = \alpha^2.$$

Dealing with I_2 instead, we also have

$$(\frac{\partial \varphi_2}{\partial x}(x,y))^2 + (\frac{\partial \varphi_2}{\partial y}(x,y))^2 = \alpha^2$$

for almost all (x,y) in V.

In general, for any f in $\mathrm{Lip}^{\mathrm{loc}}(U)$, by Proposition 3.5 we have

$$L^{\mathrm{loc}}_{(u,v)}(f) = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}_{|(u,v)}$$

for almost every (u,v) in U. Now let $f(x,y) = xy$ in $\mathrm{Lip}^{\mathrm{loc}}(U)$. We have $L^{\mathrm{loc}}_{(u,v)}(f) = \sqrt{v^2 + u^2}$ for all (u,v) in U. In particular,

$$L^{\mathrm{loc}}_{\varphi(x,y)}(f) = \sqrt{\varphi_1(x,y)^2 + \varphi_2(x,y)^2} \quad \text{for all } (x,y) \text{ in } U.$$

Hence

$$\begin{aligned}
\alpha\sqrt{\varphi_1^2+\varphi_2^2}\,|_{(x,y)} &= \alpha L^{\mathrm{loc}}_{\varphi(x,y)}(f) \\
&= \alpha L^{\mathrm{loc}}_{(x,y)}(Tf) = L^{\mathrm{loc}}_{(x,y)}(\varphi_1(x,y)\varphi_2(x,y)) \\
&= \sqrt{(\frac{\partial \varphi_1}{\partial x}\varphi_2 + \frac{\partial \varphi_2}{\partial x}\varphi_1)^2 + (\frac{\partial \varphi_1}{\partial y}\varphi_2 + \frac{\partial \varphi_2}{\partial y}\varphi_1)^2}\,|_{(x,y)} \\
&= \sqrt{\alpha^2(\varphi_1^2+\varphi_2^2) + 2(\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y}\frac{\partial \varphi_2}{\partial y})\varphi_1\varphi_2}\,|_{(x,y)}
\end{aligned}$$

for almost all (x,y) in V. This implies

$$\frac{\partial \varphi_1}{\partial x}(x,y)\frac{\partial \varphi_2}{\partial x}(x,y) + \frac{\partial \varphi_1}{\partial y}(x,y)\frac{\partial \varphi_2}{\partial y}(x,y) = 0$$

for almost all (x,y) in V. Therefore,

$$D[\varphi]_{|(x,y)} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} \end{pmatrix}_{|(x,y)}$$

exists, and $\alpha^{-1}D[\varphi]_{|(x,y)}$ is an orthogonal matrix for almost all (x,y) in V.

For any two points (x_1, y_1) and (x_2, y_2) in an open ball B contained in V, let $C : [0, 1] \to V$ be any smooth curve in B joining (x_1, y_1) to (x_2, y_2). Using Lebesgue integration, we have

$$\begin{aligned}\|\varphi(x_1, y_1) - \varphi(x_2, y_2)\| &\leq \int_0^1 \|\frac{d}{dt}(\varphi(C(t)))\| \, dt \\ &= \int_0^1 \|D[\varphi(C(t))]\frac{d}{dt}C(t)\| \, dt = \alpha \int_0^1 \|\frac{d}{dt}C(t)\| \, dt.\end{aligned}$$

Hence, taking infimum over all such smooth curves C, we have

$$\|\varphi(x_1, y_1) - \varphi(x_2, y_2)\| \leq \alpha\|(x_1, y_1) - (x_2, y_2)\|.$$

In particular, the injective map φ is continuous on the open set $V \subset \mathbb{R}^n$. By the invariance of domain, $U_0 = \varphi(V)$ is an open dense subset of the open set $U \subset \mathbb{R}^n$. It is clear that $T^{-1}g \mid_{U_0} = \alpha g \circ \varphi^{-1}$ preserves local Lipschitz constants. By a similar argument, we have $\alpha D[\varphi^{-1}]_{|q}$ exists as an orthogonal matrix for almost every point $q \in U_0$. Hence, the reverse inequality

$$\|\varphi(x_1, y_1) - \varphi(x_2, y_2)\| \geq \alpha\|(x_1, y_1) - (x_2, y_2)\|$$

also holds. In other words, φ is an α-dilation from B onto $\varphi(B)$ for any open ball B contained in V. It follows from Proposition 2.1 that $\varphi|_B$ can be uniquely extended to an α-dilation from $\mathbb{R}^n$ onto itself. By a connectedness argument, we see that φ can be extended uniquely to the same α-dilation of $\mathbb{R}^n$ onto itself. In other words, $\varphi(v) = \alpha Av + b$ for an $n \times n$ orthogonal matrix A and a vector $b \in \mathbb{R}^n$, as asserted. □

Example 3.7. Let $T : \mathrm{Lip}^{\mathrm{loc}}([0, 1]^n) \to \mathrm{Lip}^{\mathrm{loc}}([0, 1]^n)$ be a bijective weighted composition operator preserving local Lipschitz constants. Then

$$Tf = \pm f(Px + b),$$

where P is an $n \times n$ signed permutation matrix and $b \in \mathbb{R}^n$ has entries either 0 or 1.

Example 3.8. Let $T : \mathrm{Lip}^{\mathrm{loc}}([a, b]) \to \mathrm{Lip}^{\mathrm{loc}}([c, d])$ defined by $Tf = h \cdot f \circ \varphi$ be a bijection and preserve local Lipschitz constants. Then $h = \frac{d-c}{b-a}$ or $h = -\frac{d-c}{b-a}$ and $\varphi(x) = \frac{b-a}{d-c}(x - c) + a$ or $\varphi(x) = \frac{b-a}{d-c}(d - x) + a$.

4. Local/pointwise Lipschitz constant preservers of flat manifolds

In this section we consider locally (resp. pointwise) Lipschitz functions defined on flat manifolds. A *flat manifold* of dimension n is a set M with a family of injective mappings, called *charts*, $\phi_\alpha : U_\alpha \subseteq \mathbb{R}^n \to \phi_\alpha(U_\alpha) \subseteq M$ of open connected sets U_α containing 0 such that:

(1) $M \subseteq \cup_\alpha \phi_\alpha(U_\alpha)$;
(2) For any pair α, β with $W = \phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$, the transition map $\phi_\alpha^{-1} \circ \phi_\beta|_{\phi_\beta^{-1}(W)}$ is a diffeomorphism from $\phi_\beta^{-1}(W)$ onto $\phi_\alpha^{-1}(W)$, and $D[\phi_\alpha^{-1} \circ \phi_\beta]$ is orthogonal matrix-valued everywhere;
(3) The family $\{(U_\alpha, \phi_\alpha)\}_\alpha$ is maximal with respect to the conditions (1) and (2).

For example, lines, circles, planes, spheres and the Möbius strip are all flat manifolds. Note that a flat manifold becomes a metric space when it is equipped with the geodesic distance between points.

Definition 4.1. Let M be a flat manifold of dimension n. A function $f : M \to \mathbb{R}$ is *locally* (resp. *pointwise*) *Lipschitz* if for all p in M, there is a chart $\phi_p : U \subseteq \mathbb{R}^n \to \phi_p(U) \subseteq M$ with $\phi_p(0) = p$ such that $f \circ \phi_p : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is locally (resp. pointwise) Lipschitz at 0. Moreover, we define the *local* (resp. *pointwise*) *Lipschitz constant* of f at p by $L_p^{\mathrm{loc}}(f) = L_0^{\mathrm{loc}}(f \circ \phi_p)$ (resp. $L_p^{\mathrm{pt}}(f) = L_0^{\mathrm{pt}}(f \circ \phi_p)$).

Lemma 4.2. *Let M be a flat manifold. A function $f : M \to \mathbb{R}$ is locally (resp. pointwise) Lipschitz at p with respect to a chart (ϕ_p, U) is equivalent to the same property with respect to another chart (ψ_p, V) at p. Indeed, we have $L_0^{\mathrm{loc}}(f \circ \phi_p) = L_0^{\mathrm{loc}}(f \circ \psi_p)$ (resp. $L_0^{\mathrm{pt}}(f \circ \phi_p) = L_0^{\mathrm{pt}}(f \circ \psi_p)$).*

Proof. Let $W = \phi_p(U) \cap \psi_p(V)$. Observe that

$$L_0^{\mathrm{pt}}(f \circ \phi_p) = L_0^{\mathrm{pt}}(f \circ \phi_p\,|_{\phi_p^{-1}(W)}) \quad \text{and} \quad L_0^{\mathrm{pt}}(f \circ \psi_p) = L_0^{\mathrm{pt}}(f \circ \psi_p\,|_{\psi_p^{-1}(W)}).$$

It follows

$$\begin{aligned} L_0^{\mathrm{pt}}(f \circ \phi_p) &= L_0^{\mathrm{pt}}(f \circ \psi_p \circ \psi_p^{-1} \circ \phi_p\,|_{\phi_p^{-1}(W)}) \\ &\leq L_0^{\mathrm{pt}}(f \circ \psi_p) \cdot L_0^{\mathrm{pt}}(\psi_p^{-1} \circ \phi_p\,|_{\phi_p^{-1}(W)}) = L_0^{\mathrm{pt}}(f \circ \psi_p) \end{aligned}$$

and

$$L_0^{\mathrm{pt}}(f \circ \psi_p) = L_0^{\mathrm{pt}}(f \circ \phi_p \circ \phi_p^{-1} \circ \psi_p\,|_{\psi_p^{-1}(W)})$$

$$\leq L_0^{\mathrm{pt}}(f\circ\phi_p)\cdot L_0^{\mathrm{pt}}(\phi_p^{-1}\circ\psi_p\,|_{\psi_p^{-1}(W)}) = L_0^{\mathrm{pt}}(f\circ\phi_p).$$

Hence $L_0^{\mathrm{pt}}(f\circ\phi_p) = L_0^{\mathrm{pt}}(f\circ\psi_p)$.

The case for the local Lipschitz constants is similar. □

Theorem 4.3. *Let M, N be two n-dimensional flat manifolds. Let $\sigma : N \to M$ such that the composition operator $Tf = f\circ\sigma$ defines a bijective linear map $T : \mathrm{Lip}^{\mathrm{pt}}(M) \to \mathrm{Lip}^{\mathrm{pt}}(N)$ satisfying that $L_x^{\mathrm{pt}}(Tf) = L_{\sigma(x)}^{\mathrm{pt}}(f)$ (resp. $T : \mathrm{Lip}^{\mathrm{loc}}(M) \to \mathrm{Lip}^{\mathrm{loc}}(N)$ satisfying that $L_x^{\mathrm{loc}}(Tf) = L_{\sigma(x)}^{\mathrm{loc}}(f)$) for all x in N. Then σ is a local isometry in the sense that for any point $p\in N$, and any chart $\phi : U \to M$ of $\sigma(p)$ and $\psi : V \to N$ of p such that $\sigma(\psi(V)) \subseteq \phi(U)$, the induced map $\phi^{-1}\circ\sigma\circ\psi : V \to U$ is an isometry.*

Proof. Let p be in N, equipped with charts $\phi : U \to M$ and $\psi : V \to N$ such that $\psi(0) = p$, $\phi(0) = \sigma(p)$ and $\sigma(\psi(V)) \subseteq \phi(U)$. Note that both U, V are open and connected in $\mathbb{R}^n$. The composition map $T(g\circ\phi^{-1})\circ\psi = g\circ(\phi^{-1}\circ\sigma\circ\psi)$ defines a bijection from $\mathrm{Lip}^{\mathrm{pt}}(U)$ onto $\mathrm{Lip}^{\mathrm{pt}}(V)$ preserving the pointwise Lipschitz constants. It follows from Theorem 3.6 that $\phi^{-1}\circ\sigma\circ\psi$ extends to an isometry from $\mathbb{R}^n$ onto $\mathbb{R}^n$.

The case for local Lipschitz functions is similar. □

Example 4.4. Let S^2 be the unit sphere in $\mathbb{R}^3$. Let $T : \mathrm{Lip}^{\mathrm{pt}}(S^2) \to \mathrm{Lip}^{\mathrm{pt}}(S^2)$ be a bijection such that $Tf = f\circ\sigma$, and $L_p^{\mathrm{pt}}(Tf) = L_{\sigma(p)}^{\mathrm{pt}}(f)$ for all $p\in S^2$. By Theorem 4.3, σ is a local isometry, and thus a surjective isometry with respect to the geodesic metric on S^2.

Example 4.5. Let $0 < r < R$ and

$$S^1\times S^1 = \{((R+r\cos\theta)\cos\phi, (R+r\cos\theta)\sin\phi, r\sin\theta)\in\mathbb{R}^3 \colon 0\leq\theta\leq 2\pi,\ 0\leq\phi\leq 2\pi\ \}$$

be a 2-dimensional torus. Let $T : \mathrm{Lip}^{\mathrm{pt}}(S^1\times S^1) \to \mathrm{Lip}^{\mathrm{pt}}(S^1\times S^1)$ be a bijection such that $Tf = f\circ\sigma$ and $L_p^{\mathrm{pt}}(Tf) = L_{\sigma(p)}^{\mathrm{pt}}(f)$ for all $p\in S^1\times S^1$. It follows from Theorem 4.3 that σ is a local isometry, and thus it is a surjective isometry of $S^1\times S^1$ in the geodesic metric.

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