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The convex decomposition of row-stochastic matrices

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We prove that every $m \times n$ row-stochastic (RS) matrix can be written as a convex combination of n^m many $\{0, 1\}$–RS matrices. In the special cases of 2×3 and 3×3 RS matrices, the proofs are given constructively. Algorithms for computing the convex decompositions of row-stochastic matrices are provided.

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1. Introduction

A *doubly stochastic* (DS, for short) matrix is a matrix of nonnegative real entries in which the sums of entries in every row and every column is 1. DS matrices have many applications in, for example, the theory of majorization, the assignment problems, discrete Markov chains, etc; see, e.g., [2, 3, 12, 15, 16]. Recall that a permutation matrix is a square $\{0, 1\}$–matrix with exactly one 1 per row and per column. Permutation matrices are DS matrices. Clearly, every convex combination of permutation matrices is again DS. The converse, named the Birkhoff-von Neumann theorem (BNT, for short), was proved by Birkhoff [1] and von Neumann [13], separately.

The BNT has been proved many times again in the literature with a number of different methods, some inductive, some constructive, and some existential, see [4, 5, 8, 9, 10, 14] for example. In 2016, Dufossé and Ucard [6] investigated the problem of decomposing a DS matrix as a convex sum of the minimum number of permutation matrices. They showed that the associated decision problem is strongly NP-complete. In 2018, Dufossé et al. [7] published some further notes on the Birkhoff-von Neumann decomposition of DS matrices.

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An $m \times n$ matrix $A = [a_{ij}]$ is said to be *row-stochastic* (RS, for short) if it is *nonnegative*, i.e., $a_{ij} \geq 0$ for all i, j, and each row sums up to 1, i.e.,

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, m.$$

Similarly, A is said to be *column-stochastic* (CS, for short) if it is nonnegative and each column sums up to 1, i.e.,

$$\sum_{i=1}^{m} a_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n.$$

The convex sets of all row-stochastic matrices and all column-stochastic matrices are denoted by $\mathrm{RSM}(m,n)$ and $\mathrm{CSM}(m,n)$, respectively. Moreover, a matrix A is said to be $\{0,1\}$*-row-stochastic* ($\{0,1\}$–RS, for short) if each row of A contains exactly one entry 1 and all others 0. In other words, for each i there exists a unique $k(i)$ such that $a_{i,k(i)} = 1$ and $a_{ij} = 0$ for all $j \neq k(i)$. It amounts to saying that $A = \left[\delta_{j,k(i)}\right]$. Clearly, there are exactly n^m many $\{0,1\}$–row-stochastic $m \times n$ matrices, denoted by $R_1, R_2, \ldots, R_{n^m}$. Similarly, we can define $\{0,1\}$*–column-stochastic* ($\{0,1\}$–CS, for short) matrices, and list all $\{0,1\}$–column-stochastic $m \times n$ matrices as $C_1, C_2, \ldots, C_{m^n}$.

It is clear that every convex combination of $\{0,1\}$–RS matrices is RS. Let

$$R(m,n) = \mathrm{conv}\{R_k : k = 1, 2, \ldots, n^m\},$$

which is the convex hull of $\{R_k : k = 1, 2, \ldots, n^m\}$. Then $\mathrm{RSM}(m,n) \supseteq R(m,n)$. An interesting question is whether the converse holds; namely, $\mathrm{RSM}(m,n) \subseteq R(m,n)$? For an affirmative answer we need to prove that for any $B \in \mathrm{RSM}(m,n)$, the equation

$$B = \sum_{k=1}^{n^m} x_k R_k, \tag{1}$$

has a nonnegative solution $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n^m} \end{bmatrix}^T$ such that $x_1 + x_2 + \cdots + x_{n^m} = 1$. This is equivalent to proving that for any $D \in \mathrm{CSM}(m,n)$, the equation

$$D = \sum_{j=1}^{m^n} y_j C_j, \tag{2}$$

has a nonnegative solution $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{m^n} \end{bmatrix}^T$ such that $y_1 + y_2 + \cdots + y_{m^n} = 1$.

In 2002, Li, et al. [11, Lemma 3.3] pointed out that the extreme points of the compact convex set $\mathrm{CSM}(m,n)$ constitute

$$\operatorname{ext}(\mathrm{CSM}(m,n)) = \{C_k : k = 1, 2, \ldots, m^n\}, \tag{3}$$

without a detail proof, but referring to an application of [11, Proposition 1.2]. Thus, the equation (2) has always the desired solutions $\mathbf{y}$, and so does the equation (1).

In Section 2, we provide a constructive proof of the solvability of (1) for the special, occasionally the most useful, cases of 2×3 and 3×3 RS matrices. In Section 3, an algorithm for finding all convex decompositions of a row-stochastic matrix and an efficient algorithm for computing a convex decomposition are presented.

2. The decompositions of 2×3 and 3×3 RS matrices

2.1. The 2×3 case

All 2×3 $\{0,1\}$–RS matrices are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$R_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, R_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$R_7 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, R_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, R_9 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given any 2×3 RS matrix $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$, the aim is to find a nonnegative vector
$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{bmatrix}^T$ in $\mathbb{R}^9$ such that

$$B = \sum_{k=1}^{9} x_k R_k. \tag{4}$$

Noting that by summing up all matrix entries of two sides, we see that $x_1 + \cdots + x_9 = 1$. Therefore, the matrix equation (4) has nonnegative solutions if and only if B is a convex combination of the $\{0,1\}$–RS matrices.

Rewrite (4) as a linear system of 6 equations in 9 unknowns:

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix}
=
\begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{21} \\ b_{22} \\ b_{23} \end{pmatrix}. \tag{5}
$$

A nonnegative solution $\mathbf{x}$ to (5) is given concretely according to the following five case rule.

$$
\begin{cases}
(1)\ b_{13} + b_{23} > 1 \quad (\text{Case } 1), \\
(2)\ b_{13} + b_{23} \le 1 \begin{cases}
b_{11} \le b_{23} \begin{cases} b_{13} \le b_{21} & (\text{Case } 2.1), \\ b_{13} > b_{21} & (\text{Case } 2.2), \end{cases} \\
b_{11} > b_{23} \begin{cases} b_{13} \le b_{21} & (\text{Case } 2.3), \\ b_{13} > b_{21} & (\text{Case } 2.4). \end{cases}
\end{cases}
\end{cases}
$$

Case 1. $\mathbf{x} = (0, 0, b_{11}, 0, 0, b_{12}, b_{21}, b_{22}, b_{13} + b_{23} - 1)^T$.
Case 2.1. $\mathbf{x} = (0, 0, b_{11}, b_{21} - b_{13}, b_{22}, b_{23} - b_{11}, b_{13}, 0, 0)^T$.
Case 2.2. $\mathbf{x} = (0, 0, b_{11}, 0, 1 - b_{13} - b_{23}, b_{23} - b_{11}, b_{21}, b_{13} - b_{21}, 0)^T$.
Case 2.3. $\mathbf{x} = (0, b_{11} - b_{23}, b_{23}, b_{21} - b_{13}, 1 - b_{11} - b_{21}, 0, b_{13}, 0, 0)^T$.
Case 2.4. $\mathbf{x} = (0, b_{11} - b_{23}, b_{23}, 0, b_{12}, 0, b_{21}, b_{13} - b_{21}, 0)^T$.

In other words, every 2×3 RS matrix B can be written as a convex combination of *at most* $2 \times (3 - 1) + 1 = 5$ many $\{0, 1\}$–RS matrices. However, such a convex combination representation is not necessarily unique. For example, when

$$b_{11} \ge b_{21} \ge 0.5 \ge b_{22} \ge b_{12} \ge b_{13} \ge b_{23},$$

The system (5) has many solutions indeed. The following are four of them.

$$\begin{bmatrix} b_{11}-b_{22} \\ b_{22} \\ 0 \\ b_{12} \\ 0 \\ 0 \\ b_{13}-b_{23} \\ 0 \\ b_{23} \end{bmatrix}, \begin{bmatrix} b_{11}+b_{21}-1 \\ b_{22} \\ b_{23} \\ b_{12} \\ 0 \\ 0 \\ b_{13} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-b_{13}-b_{22} \\ b_{22}-b_{12} \\ 0 \\ 0 \\ b_{12} \\ 0 \\ b_{13}-b_{23} \\ 0 \\ b_{23} \end{bmatrix}, \begin{bmatrix} b_{11}-b_{22} \\ b_{22} \\ 0 \\ b_{12}-b_{23} \\ 0 \\ b_{23} \\ b_{13} \\ 0 \\ 0 \end{bmatrix}.$$

2.2. The 3×3 case

All 3×3 $\{0,1\}$–RS matrices are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\cdots$$
$$R_{25} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, R_{26} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, R_{27} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given any 3×3 RS matrix $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, the aim is to find a nonnegative vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_{27} \end{bmatrix}^T$ in $\mathbb{R}^{27}$ such that

$$B = \sum_{k=1}^{27} x_k R_k. \tag{6}$$

Noting that by summing up all matrix entries of two sides, we see that $x_1 + \cdots + x_{27} = 1$. Therefore, the matrix equation (6) has nonnegative solutions if and only if B is a convex combination of the $\{0,1\}$–RS matrices.

Rewrite (6) as a linear system $M\mathbf{x} = \mathbf{b}$ of 9 equations in 27 unknowns with the coefficient matrix M equal

$$\begin{bmatrix}
1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1
\end{bmatrix},$$

and

$$\mathbf{b} = \mathrm{vec}(B^T) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{21} & b_{22} & b_{23} & b_{31} & b_{32} & b_{33} \end{bmatrix}^T.$$

For convenience, we rewrite $\mathbf{x}$ as a 9×3 matrix:

$$\mathbf{x} = \begin{array}{cc}
\begin{array}{ccc} \textcircled{1} & \textcircled{2} & \textcircled{3} \end{array} & \\
\begin{bmatrix}
x_1 & x_{10} & x_{19}\\
x_2 & x_{11} & x_{20}\\
x_3 & x_{12} & x_{21}\\
x_4 & x_{13} & x_{22}\\
x_5 & x_{14} & x_{23}\\
x_6 & x_{15} & x_{24}\\
x_7 & x_{16} & x_{25}\\
x_8 & x_{17} & x_{26}\\
x_9 & x_{18} & x_{27}
\end{bmatrix} &
\begin{array}{c} \textcircled{1}\\ \textcircled{2}\\ \textcircled{3}\\ \textcircled{4}\\ \textcircled{5}\\ \textcircled{6}\\ \textcircled{7}\\ \textcircled{8}\\ \textcircled{9} \end{array}
\end{array}$$

Clearly, the nonnegative numbers $x_1, \ldots, x_{27}$ satisfying (6) is equivalent to the validity of the following seven conditions.

- (a) The sums of the entries in each column of $\mathbf{x}$ are equal to b_{11}, b_{12} and b_{13}, respectively;
- (b) The sum of the entries in the rows ①, ② and ③ of $\mathbf{x}$ is equal to b_{21};
- (c) The sum of the entries in the rows ④, ⑤ and ⑥ of $\mathbf{x}$ is equal to b_{22};
- (d) The sum of the entries in the rows ⑦, ⑧ and ⑨ of $\mathbf{x}$ is equal to b_{23};
- (e) The sum of the entries in the rows ①, ④ and ⑦ of $\mathbf{x}$ is equal to b_{31};
- (f) The sum of the entries in the rows ②, ⑤ and ⑧ of $\mathbf{x}$ is equal to b_{32};
- (g) The sum of the entries in the rows ③, ⑥ and ⑨ of $\mathbf{x}$ is equal to b_{33}.

We construct such a nonnegative solution to (6) explicitly according to the following 12 case rule.

Case (1): $b_{13} + b_{33} > 1$:

(1.1) $b_{13}+b_{23}-b_{31}\geq 1$ and $b_{13}-b_{21}+b_{33}\leq 1$:

$$\mathbf{x}=\left\{\begin{array}{l} x_9=b_{11}, x_{23}=b_{22}, x_{20}=1-b_{13}+b_{21}-b_{33},\\ x_{21}=b_{13}+b_{33}-1, x_{26}=b_{13}+b_{23}-b_{31}-1,\\ x_{18}=b_{12}, x_{25}=b_{31}, x_i=0 \text{ otherwise.}\end{array}\right\}$$

(1.2) $b_{13}+b_{23}-b_{31}\geq 1$ and $b_{13}-b_{21}+b_{33}\geq 1$:

$$\mathbf{x}=\left\{\begin{array}{l} x_9=b_{11}, x_{21}=b_{21}, x_{23}=1-b_{13}-b_{23}+b_{31}+b_{32},\\ x_{18}=b_{12}, x_{26}=b_{13}+b_{23}-b_{31}-1,\\ x_{25}=b_{31}, x_{24}=b_{22}-b_{31}, x_i=0 \text{ otherwise.}\end{array}\right\}$$

(1.3) $b_{13}+b_{23}-b_{31}\leq 1$ and $b_{13}-b_{21}+b_{33}\leq 1$:

$$\mathbf{x}=\left\{\begin{array}{l} x_{12}=b_{21}-b_{13}+b_{31}, x_{20}=1-b_{33}-b_{23}+b_{11}, x_9=b_{11},\\ x_{17}=b_{23}-b_{11}-b_{31}, x_{21}=b_{13}-b_{11}+b_{23}+b_{33}-b_{31}-1,\\ x_{15}=b_{22}, x_{25}=b_{31}, x_i=0 \text{ otherwise.}\end{array}\right\}$$

(1.4) $b_{13}+b_{23}-b_{31}\leq 1$ and $b_{13}-b_{21}+b_{33}\geq 1$:

$$\mathbf{x}=\left\{\begin{array}{l} x_9=b_{11}, x_{23}=b_{32}, x_{15}=1-b_{13}-b_{23}+b_{31},\\ x_{21}=b_{21}, x_{25}=b_{31}, x_{18}=b_{23}-b_{11}-b_{31},\\ x_{24}=b_{33}-b_{21}+b_{13}-1, x_i=0 \text{ otherwise.}\end{array}\right\}$$

Case (2): $b_{13}+b_{33}\leq 1$:

(2.1) $b_{13}\leq b_{21}$, $b_{13}\leq b_{31}$ and $b_{13}+b_{23}+b_{33}\geq 1+b_{11}$:

$$\mathbf{x}=\left\{\begin{array}{l} x_9=b_{11}, x_{17}=b_{32}, x_{12}=b_{21}-b_{13},\\ x_{15}=b_{22}, x_{19}=b_{13}, x_{16}=b_{31}-b_{13},\\ x_{18}=b_{33}+b_{23}-b_{11}+b_{13}-1, x_i=0 \text{ otherwise.}\end{array}\right\}$$

(2.2) $b_{13}\leq b_{21}$, $b_{13}\leq b_{31}$ and $b_{13}+b_{23}+b_{33}<1+b_{11}$:

$$\mathbf{x}=\left\{\begin{array}{l} x_{12}=b_{21}-b_{13}, x_{14}=1-b_{13}-b_{23}-b_{33}, x_6=b_{11},\\ x_{16}=b_{31}-b_{13}, x_{15}=b_{13}-b_{11}-b_{21}+b_{33}, x_{19}=b_{13},\\ x_{17}=b_{23}-b_{31}+b_{13}, x_i=0 \text{ otherwise.}\end{array}\right\}$$

(2.3) $b_{13}\leq b_{21}$, $b_{13}>b_{31}$ and $b_{13}+b_{23}+b_{33}\geq 1+b_{11}$:

$$\mathbf{x}=\left\{\begin{array}{l} x_{12}=b_{21}-b_{13}, x_{17}=1-b_{13}-b_{33}, x_9=b_{11},\\ x_{20}=b_{13}-b_{31}, x_{18}=b_{13}-b_{11}+b_{23}+b_{33}-1,\\ x_{15}=b_{22}, x_{19}=b_{31}, x_i=0 \text{ otherwise.}\end{array}\right\}$$

(2.4) $b_{13} > b_{21}$, $b_{13} \le b_{31}$ and $b_{13} + b_{23} + b_{33} \ge 1 + b_{11}$:

$$\mathbf{x} = \left\{ \begin{array}{l} x_{16} = b_{31} - b_{13}, x_{15} = 1 - b_{23} - b_{13}, x_9 = b_{11}, \\ x_{22} = b_{13} - b_{21}, x_{18} = b_{13} - b_{11} + b_{23} + b_{33} - 1, \\ x_{17} = b_{32}, x_{19} = b_{21}, x_i = 0 \text{ otherwise.} \end{array} \right\}$$

(2.5) $b_{13} \le b_{21}$, $b_{13} > b_{31}$ and $b_{13} + b_{23} + b_{33} < 1 + b_{11}$:

$$\mathbf{x} = \left\{ \begin{array}{l} x_{12} = b_{21} - b_{13}, x_{14} = 1 - b_{13} + b_{11} - b_{23} - b_{33}, \\ x_{17} = b_{23} - b_{11}, x_{15} = b_{33} - b_{11} + b_{13} - b_{21}, x_{19} = b_{31}, \\ x_9 = b_{11}, x_{20} = b_{13} - b_{31}, x_i = 0 \text{ otherwise.} \end{array} \right\}$$

(2.6) $b_{13} > b_{21}$, $b_{13} > b_{31}$ and $b_{13} + b_{23} + b_{33} \ge 1 + b_{11}$:

$$\mathbf{x} = \left\{ \begin{array}{l} x_{15} = 1 - b_{13} - b_{23}, x_{18} = b_{23} - b_{11} + b_{13} + b_{33} - 1, \\ x_{17} = 1 - b_{13} - b_{33}, x_{19} = b_{21} + b_{31} - b_{13}, x_9 = b_{11}, \\ x_{20} = b_{13} - b_{31}, x_{22} = b_{13} - b_{21}, x_i = 0 \text{ otherwise.} \end{array} \right\}$$

(2.7) $b_{13} > b_{21}$, $b_{13} \le b_{31}$ and $b_{13} + b_{23} + b_{33} < 1 + b_{11}$:

$$\mathbf{x} = \left\{ \begin{array}{l} x_{14} = 1 - b_{13} - b_{23} - b_{33}, x_{15} = b_{33} - b_{11}, x_9 = b_{11}, \\ x_{17} = b_{23} - b_{11} - b_{31} + b_{13}, x_{16} = b_{31} - b_{13}, \\ x_{22} = b_{13} - b_{21}, x_{19} = b_{21}, x_i = 0 \text{ otherwise.} \end{array} \right\}$$

(2.8) $b_{13} > b_{21}$, $b_{13} > b_{31}$ and $b_{13} + b_{23} + b_{33} < 1 + b_{11}$, i.e. $b_{23} < b_{11}$:

$$\mathbf{x} = \left\{ \begin{array}{l} x_{14} = 1 - b_{13} + b_{11} - b_{23} - b_{33}, x_{15} = b_{33} - b_{11}, \\ x_{19} = b_{21} + b_{31} - b_{13}, x_{17} = b_{23} - b_{11}, x_9 = b_{11}, \\ x_{20} = b_{13} - b_{31}, x_{22} = b_{13} - b_{21}, x_i = 0 \text{ otherwise.} \end{array} \right\}$$

This shows that (6) always has a nonnegative solution $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_{27} \end{bmatrix}^T$ in $\mathbb{R}^{27}$ satisfying $\sum_{k=1}^{27} x_k = 1$. In other words, every 3×3 row-stochastic matrix B can be written as a convex combination of *at most* $3 \times (3-1) + 1 = 7$ many $\{0, 1\}$–row-stochastic matrices.

3. Convex decomposition algorithms

3.1. Computing all possible convex decompositions of an RS matrix into $\{0,1\}$–RS matrices

Given an RS matrix $B = [b_{ij}] \in \mathbb{R}^{m \times n}$, we define a matrix M by

$$M = \sum_{k=1}^{m} \mathbf{e}_{m,k} \otimes \mathbf{1}_{n^{k-1}}^T \otimes I_n \otimes \mathbf{1}_{n^{m-k}}^T \in \mathbb{R}^{mn \times n^m}$$

where $\mathbf{e}_{m,k} \in \mathbb{R}^m$ whose entries are zero except $k$the element being one, $\mathbf{1}_p \in \mathbb{R}^p$ whose elements are all one, $I_n \in \mathbb{R}^{n \times n}$ is the $n \times n$ identity matrix, and $\otimes$ denotes the Kronecker product. For computing all possible convex decompositions of B, the simplex algorithm can be applied to solve for $\mathbf{x}$ in a system of linear equations and linear inequalities

$$\begin{cases} M\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}, \end{cases} \tag{7}$$

where $\mathbf{b} = \operatorname{vec}\left(B^T\right) \in \mathbb{R}^{mn}$ and $\mathbf{x} = \left(x_1, \dots, x_{n^m}\right)^T \in \mathbb{R}^{n^m}$.

Let $\mathbf{m}_k$ be the kth column of the matrix M. Let $R_k = \operatorname{vec}^{-T}(\mathbf{m}_k)$; in other words, $\mathbf{m}_k = \operatorname{vec}(R_k^T)$. Then we can consider R_k as the kth among all n^m many $\{0,1\}$–RS matrices. It is easy to see that the system (7) is equivalent to the convex decomposition equation

$$B = \sum_{k=1}^{n^m} x_k R_k, \quad x_k \geq 0. \tag{8}$$

For example, let

$$B = \begin{bmatrix} 0.92 & 0.08 \\ 0.69 & 0.31 \\ 0.57 & 0.43 \end{bmatrix} \in \operatorname{RSM}(3,2).$$

Then its convex decomposition equation is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_8 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 0.08 \\ 0.69 \\ 0.31 \\ 0.57 \\ 0.43 \end{bmatrix}, \tag{9}$$

where $x_1, x_2, \ldots, x_8$ are nonnegative. And the eight $\{0,1\}$–RS matrix are ordered as follows:

$$R_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, R_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$
$$R_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, R_6 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, R_7 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, R_8 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Now, summing up the equations $M\mathbf{x} = \mathbf{b}$ in (7) yields equation $m\mathbf{1}_{n^m}^T\mathbf{x} = m$; that is, $\mathbf{1}_{n^m}^T\mathbf{x} = 1$. Together with the fact that the solution set in (7) forms a polyhedron, it shows that the solution set of all coefficients $\mathbf{x} = (x_1, \ldots, x_{n^m})^T$ is the convex hull of some finitely many points (i.e. vertices of the solution set) in $\mathbb{R}^{n^m}$. We are going to find out all these vertices.

First, to find a solution of system (7), we consider the linear optimization problem **P1**:

$$\text{minimize} \quad \mathbf{1}_{mn}^T\mathbf{y}$$

$$\text{subject to} \begin{cases} M\mathbf{x} + \mathbf{y} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{cases}$$

Obviously, $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}$ is a basic feasible solution of **P1**. If the minimum is reached (i.e. cost is zero), then the system (7) has a solution. In this case, the basic feasible solution of **P1** is $\begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{0} \end{bmatrix}$, and $\tilde{\mathbf{x}}$ is a solution of system (7).

Summing up the equations in **P1**, we have equation $m\mathbf{1}_{n^m}^T\mathbf{x} + \mathbf{1}_{mn}^T\mathbf{y} = m$. Replacing the cost function $\mathbf{1}_{mn}^T\mathbf{y}$ in **P1** with $\left(m - m\mathbf{1}_{n^m}^T\mathbf{x}\right)$, we have an equivalent problem **P2**:

$$\text{maximize} \quad m\mathbf{1}_{n^m}^T\mathbf{x} - m$$

$$\text{subject to} \begin{cases} M\mathbf{x} + \mathbf{y} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{cases}$$

Suppose $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}$ is a basic feasible solution of **P2** with cost $-m$. When the cost reaches the maximal value 0, the basic feasible solution is in the form of $\begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{0} \end{bmatrix}$, and $\tilde{\mathbf{x}}$ is a solution of system (7).

Let $[d]$ denote the set $\{1, 2, \ldots, d\}$ for simplicity. The simplex algorithm for solving **P2** are as follows.

Step 1: Construct the initial tableau

$$\begin{array}{|c|c|} \hline H & \mathbf{h} \\ \hline \mathbf{c}^T & \alpha \\ \hline \end{array} \quad \text{where } H = M,\ \mathbf{h} = \mathbf{b},\ \mathbf{c}^T = m\mathbf{1}_{n^m}^T, \quad \text{and } \alpha = m.$$

Step 2: Find a column s with a positive element in the bottom row.

If there is no positive element in the bottom row, then the cost reaches maximum 0.

Step 3: Determine the pivot row r where

$$\frac{h_r}{H_{rs}} = \min_{i \in [mn]} \left\{ \frac{h_i}{H_{is}} : H_{is} > 0 \right\}.$$

Step 4: Update the tableau by Gauss elimination on pivot H_{rs}.

Step 5: Go to step 2.

After calculating with the simplex algorithm, the resulting tableau for solving (9) is

$$\tag{10} \begin{array}{|cccccccc|c|} \hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0.43 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0.08 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 & -2 & 0.18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00 \\ \hline \end{array}.$$

Consequently, we obtain a solution $\mathbf{x}^{(1)} = \begin{bmatrix} 0.18 & 0.43 & 0.31 & 0 & 0.08 & 0 & 0 & 0 \end{bmatrix}^T$ of the system (8), which is a vertex of the polyhedral solution set. The tableau also shows that columns $4, 6, 7, 8$ are inactive, so the vertex $\mathbf{x}^{(1)}$ may have at most four adjacent vertices.

Next, we can compute the pivot and update the tableau by Gauss elimination to achieve an adjacent vertex. For the column $s = 4$ of tableau (10), the pivot row $r = 1$ is determined by

$$\frac{h_r}{H_{rs}} = \min_{i \in [mn]} \left\{ \frac{h_i}{H_{is}} : H_{is} > 0 \right\}.$$

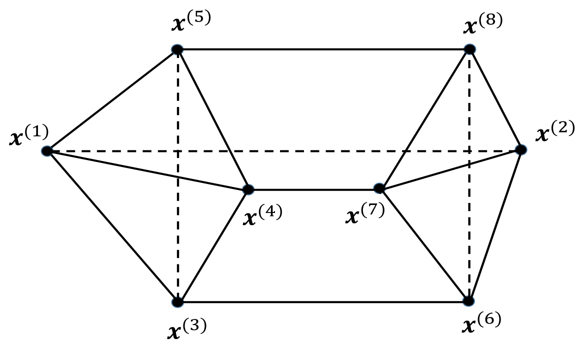

Figure 1: The adjacent relation of vertices $\{\mathbf{x}^{(k)}\}_{k=1}^{8}$.

After updating with Gauss elimination, the tableau becomes

$$\begin{array}{|cccccccc|c|}
\hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0.31 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00 \\
0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0.12 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0.08 \\
1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0.49 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00 \\
\hline
\end{array}. \tag{11}$$

Hence, the solution $\mathbf{x}^{(2)} = \begin{bmatrix} 0.49 & 0.12 & 0 & 0.31 & 0.08 & 0 & 0 & 0 \end{bmatrix}^T$ is an adjacent vertex of $\mathbf{x}^{(1)}$. Applying the same routine on column $6, 7, 8$ of tableau (10), we obtain the other three adjacent vertices:

$$\begin{aligned}
\mathbf{x}^{(3)} &= \begin{bmatrix} 0.26 & 0.35 & 0.31 & 0 & 0 & 0.08 & 0 & 0 \end{bmatrix}^T, \\
\mathbf{x}^{(4)} &= \begin{bmatrix} 0.26 & 0.43 & 0.23 & 0 & 0 & 0 & 0.08 & 0 \end{bmatrix}^T, \\
\mathbf{x}^{(5)} &= \begin{bmatrix} 0.34 & 0.35 & 0.23 & 0 & 0 & 0 & 0 & 0.08 \end{bmatrix}^T.
\end{aligned}$$

Continuing this process, we can apply this routine on column $6, 7, 8$ of tableau (11) for seeking adjacent vertices of $\mathbf{x}^{(2)}$. The computing result shows that, beside $\mathbf{x}^{(1)}$, the other three adjacent vertices of $\mathbf{x}^{(2)}$ are obtained as follow:

$$\begin{aligned}
\mathbf{x}^{(6)} &= \begin{bmatrix} 0.57 & 0.04 & 0 & 0.31 & 0 & 0.08 & 0 & 0 \end{bmatrix}^T, \\
\mathbf{x}^{(7)} &= \begin{bmatrix} 0.49 & 0.20 & 0 & 0.23 & 0 & 0 & 0.08 & 0 \end{bmatrix}^T,
\end{aligned}$$

$$\mathbf{x}^{(8)} = \begin{bmatrix} 0.57 & 0.12 & 0 & 0.23 & 0 & 0 & 0 & 0.08 \end{bmatrix}^T.$$

Finally, these eight points $\left\{\mathbf{x}^{(k)}\right\}_{k=1}^{8}$ are all vertices of the polyhedral solution set. The adjacent relation of them is illustrated in Figure 1. Note that every convex combination of $\left\{\mathbf{x}^{(k)}\right\}_{k=1}^{8}$ is the coefficients of a convex decomposition of the RS matrix B. In this sense, we have solved *all* possible convex decompositions of the RS matrix B into $\{0,1\}$–RS matrices.

3.2. Convex decomposition algorithm for finding one convex decomposition

On the occasion of requiring only one convex decomposition of an RS matrix into $\{0,1\}$–RS matrices, here we provide an efficient algorithm. Given

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \in \mathrm{RSM}(m,n).$$

Step 1: Set $B^{(0)} = \left[b_{ij}^{(0)}\right] = B$ and determine the index $l_i^{(0)} = \min\{j \in [n] \mid b_{ij}^{(0)} > 0\}$. Then $b_{i,l_i^{(0)}}^{(0)}$ is the first nonzero entry of the i–th row of $B^{(0)}$. Define $R_1 = \left[h_{ij}^{(1)}\right]$ where

$$h_{ij}^{(1)} = \begin{cases} 1, & j = l_i^{(0)} \\ 0, & j \neq l_i^{(0)} \end{cases},$$

and set

$$x_1 = \min\left\{b_{i,l_i^{(0)}}^{(0)} \mid i \in [m]\right\} = b_{i_1,l_{i_1}^{(0)}}^{(0)}.$$

Clearly, $0 < x_1 \leq 1$. If $x_1 = 1$, then $B^{(0)}$ is an $\{0,1\}$–RS matrix R_1, and $B = B^{(0)} = R_1$. In this case, the convex decomposition is obtained and the algorithm terminates. Otherwise, set

$$B^{(1)} = \left[b_{ij}^{(1)}\right] := \frac{1}{1-x_1}(B^{(0)} - x_1 R_1).$$

and go to Step 2.

Step 2: Determine the index $l_i^{(1)} = \min\left\{j \in [n] \mid b_{ij}^{(1)} > 0\right\}$. Then $b_{i,l_i^{(1)}}^{(1)}$ is the first nonzero entry of i–th row of $B^{(1)}$. Define $R_2 = \left[h_{ij}^{(2)}\right]$ where

$$h_{ij}^{(2)} = \begin{cases} 1, & j = l_i^{(0)} \\ 0, & j \neq l_i^{(0)} \end{cases},$$

and set

$$x_2 = \min\left\{b_{i,l_i^{(1)}}^{(1)} \mid i \in [m]\right\} = b_{i_2,l_{i_2}^{(1)}}^{(1)}.$$

Clearly, $0 < x_2 \leq 1$. If $x_2 = 1$, then $B^{(1)}$ is an $\{0,1\}$–RS matrix R_2, and $B = x_1R_1 + (1 - x_1)B^{(1)}$. In this case, the convex decomposition is obtained and the algorithm terminates. Otherwise, set

$$B^{(2)} = \left[b_{ij}^{(2)}\right] := \frac{1}{1 - x_2}(B^{(1)} - x_2R_2).$$

and go to next step.

If $B^{(s)}, R_s$ and x_s are defined with $0 < x_s < 1$ and

$$B^{(s)} = \left[b_{ij}^{(s)}\right] := \frac{1}{1 - x_s}(B^{(s-1)} - x_sR_s) \text{ for } s = 1, 2, \ldots, k - 1,$$

go to Step k.

Step k: Determine the index $l_i^{(k-1)} = \min\left\{j \in [n] \mid b_{ij}^{(k-1)} > 0\right\}$. Then $b_{i,l_i^{(k-1)}}^{(k-1)}$ is the first nonzero entry of i–th row of $B^{(k-1)}$. Define $R_k = \left[h_{ij}^{(k)}\right]$ where

$$h_{ij}^{(k)} = \begin{cases} 1, & j = l_i^{(k-1)} \\ 0, & j \neq l_i^{(k-1)} \end{cases}$$

and set

$$x_k = \min\left\{b_{i,l_i^{(k-1)}}^{(k-1)} \mid i \in [m]\right\} = b_{i_k,l_{i_k}^{(k-1)}}^{(k-1)}.$$

Clearly, $0 < x_k \leq 1$. If $x_k = 1$, then $B^{(k-1)}$ is an $\{0,1\}$–RS matrix R_k, and

$$B = x_1R_1 + (1 - x_1)x_2R_2 + \cdots + (1 - x_1)(1 - x_2)\cdots(1 - x_{k-1})B^{(k-1)}.$$

Otherwise, set

$$B^{(k)} = \left[b_{ij}^{(k)}\right] := \frac{1}{1 - x_k}(B^{(k-1)} - x_kR_k),$$

and go to Step $k+1$.

The following theorem shows that this algorithm must be terminated in *at most* $m(n-1)+1$ many steps, and in each step at most mn comparisons and m subtractions are performed.

Theorem 3.1. *Let B be an $m \times n$ RS matrix. It can be written as a convex combination of at most $m(n-1)+1$ many $\{0,1\}$–RS matrices.*

Proof. If B is an $m \times n$ RS matrix but not a $\{0,1\}$–RS matrix, then x_1 defined at Step 1 in the above algorithm satisfies $0 < x_1 < 1$.

Now, we claim that the matrix $B^{(s)}$ generated at each Step s is an RS matrix. To prove this, let $r_i(X) = \sum_{j=1}^{n} x_{ij}$ for any $m \times n$ matrix $X = [x_{ij}]$. Then from the formula $B^{(s)} = \frac{1}{1-x_s}(B^{(s-1)} - x_s R_s)$ we see that

$$r_i\left(B^{(s)}\right) = \frac{1}{1-x_s}\left(r_i\left(B^{(s-1)}\right) - x_s r_i\left(R^{(s)}\right)\right) = \frac{1}{1-x_s}(r_i\left(B^{(s-1)}\right) - x_s).$$

Thus, if $B^{(s-1)}$ is an RS matrix, so is $B^{(s)}$. Since $B^{(0)} = B$ is an RS matrix, we conclude that $B^{(s)}$ is an RS matrix for $s \geq 1$.

Clearly, $b^{(1)}_{i_1,l^{(0)}_{i_1}} = 0$. If $b^{(0)}_{i,j} = 0$, then we have $j \neq l^{(0)}_{i_1}$, and so $b^{(1)}_{i,j} = 0$ since $h^{(1)}_{i,j} = 0$. This shows that $\text{zero}(B^{(0)}) < \text{zero}(B^{(1)})$ where $\text{zero}(A)$ denotes the number of the zero entries in a matrix A. Thus,

$$\text{zero}(B^{(0)}) < \text{zero}(B^{(1)}) < \text{zero}(B^{(2)}) < \cdots \leq mn - m$$

provided that $B^{(0)}, B^{(1)}, \ldots$ are defined. Thus, the sequence $B^{(0)}, B^{(1)}, \ldots$ must be a finite sequence. In other words, there exists a positive integer k such that $0 < x_s < 1$ for $s = 1, 2, \ldots, k-1$, and $x_k = 1$. In this case, $k - 1 \leq \text{zero}(B^{(k-1)}) \leq mn - m = m(n-1)$ since $B^{(k-1)}$ is a $\{0,1\}$–RS matrix. Hence, $k \leq m(n-1)+1$. □

The following example demonstrates the operation of the algorithm.

Example 3.2.

$$\begin{aligned} B &= \begin{bmatrix} \mathbf{0.6} & 0.1 & 0.3 \\ \mathbf{0.2} & 0.4 & 0.4 \\ \mathbf{0.3} & 0.2 & 0.5 \end{bmatrix} \\ &= 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{0.4} & 0.1 & 0.3 \\ 0 & \mathbf{0.4} & 0.4 \\ \mathbf{0.1} & 0.2 & 0.5 \end{bmatrix} \end{aligned}$$

$$
= 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 0.1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{0.3} & 0.1 & 0.3 \\ 0 & \mathbf{0.3} & 0.4 \\ 0 & \mathbf{0.2} & 0.5 \end{bmatrix}
$$

$$
\vdots
$$

$$
\begin{aligned}
&= 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 0.1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 0.2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&\quad + 0.1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 0.3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

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