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Approximate amenability and pseudo-amenability in Banach algebras

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Dedicated to Professor Anthony To-Ming Lau: Happy 80th birthday!

We survey the recent development of approximate amenability and pseudo-amenability in Banach algebras, concentrating on the relationship between the two notions. It has been conjectured that a Banach algebra with a bounded approximate identity is approximately amenable if and only if it is pseudo-amenable. So far we only have partial affirmative results concerning the conjecture. Some open problems are also posted.

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1. Amenability, approximate amenability and pseudo-amenability

A locally compact group G is amenable if there is a left invariant mean on $L^\infty(G)$, the space of all bounded Haar measurable functions on G. The notion was introduced by M. M. Day in 1950s in studying the Banach-Tarski paradox [8, 21]. It then became an essential concept in abstract harmonic analysis. B. E. Johnson discovered in 1970s that the amenability of a locally compact group G can characterize the cohomology property of its group algebra $L^1(G)$ [23]. This initiated the amenability theory in Banach algebras. Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. We recall that a mapping $D\colon \mathcal{A} \to X$ is a *derivation* if it is linear and satisfies the derivation identity

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A),$$

where $\cdot$ denotes the module action of $\mathcal{A}$ on X. For example, given any $\xi \in X$, the mapping $a \mapsto \mathrm{ad}_\xi(a) := a \cdot \xi - \xi \cdot a$ is a derivation. Such derivation is called

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an *inner* derivation. The derivation D is called *(boundedly) approximately inner* if there is a (resp. bounded) net $(\xi_\alpha) \subset X$ such that

$$D(a) = \lim_\alpha a \cdot \xi_\alpha - \xi_\alpha \cdot a \tag{1.1}$$

for all $a \in \mathcal{A}$ in the norm topology of X. If the convergence of (1.1) is in another topology τ of X, we will call D τ-approximately inner. (e.g. we can talk about weak* approximately inner derivations if X is a dual Banach space.)

In the sequel, the dual space of any Banach space X will be denoted by X^* and the second dual of it will be denoted by X^{**}. We use $\langle x, f\rangle$ to denote the evaluation $f(x)$ for $x \in X$ and $f \in X^*$.

Due to the investigation of F. Gourdeau in [20], we may define an amenable Banach algebra as follows.

Definition 1. The Banach algebra $\mathcal{A}$ is *amenable* if for each Banach $\mathcal{A}$-bimodule X, every continuous derivation D: $\mathcal{A} \to X$ is boundedly approximately inner.

This definition is equivalent to the original Johnson's definition in [23] which can be stated as follows: The Banach algebra $\mathcal{A}$ is amenable if every continuous derivation from $\mathcal{A}$ into the dual module X^* is inner for all Banach $\mathcal{A}$-bimodules X.

If $\mathcal{A}$ is a Banach algebra, then the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is naturally a Banach $\mathcal{A}$-bimodule with module actions given by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{A}).$$

A (bounded) net $(u_i) \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ is a (resp. bounded) *approximate diagonal* for $\mathcal{A}$ if it satisfies

$$\lim_i a \cdot u_i - u_i \cdot a = 0, \quad \lim_i \pi(u_i)a \to a \tag{1.2}$$

for all $a \in \mathcal{A}$ in the norm topology, where π: $\mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the product mapping defined by $\pi(a \otimes b) = ab$. Johnson showed in [24] that, instead of considering all Banach $\mathcal{A}$-bimodules, one only needs to consider $\mathcal{A}\widehat{\otimes}\mathcal{A}$ and determine whether there is a bounded approximate diagonal for $\mathcal{A}$ to reveal the amenability of $\mathcal{A}$. Precisely, he showed the following.

Theorem 2. *The Banach algebra $\mathcal{A}$ is amenable if and only if there is a bounded approximate diagonal for $\mathcal{A}$.*

Amenability in Banach algebras turns out to be a significant concept in the Banach algebra theory. It reflects various profound features of important Banach algebras. For example, a C^*-algebra is amenable if and only if it is nuclear (due to A. Conne and U. Haagerup, see [4, 28]), the weighted group algebra $L^1(G, \omega)$ is amenable if and only if the group G is amenable and the weight ω is diagonally bounded (due to N. Grønbæk under condition $\omega(e) = 1$ [22], and to F. Ghahramani, R.J. Loy and Y. Zhang for the general [15]), the amenability of a von Neumann algebra (which natuarally involves only ultraweak continuous derivations) is equivalent to the injectivity of the algebra, and is also equivalent to the property P, semidiscreteness and approximate finite dimensionality of the algebra (due to the tremendous investigations of A. Connes, J. Schwartz, J. Tomiyama, M.D. Choi, E.G. Effros, E.C. Lance, B.E. Johnson, Kadison, and Ringorose; we refer to [26] for attributions of these investigations). On the other hand, amenability for a Banach algebra is a very restrictive condition. For example, amenable algebras only form a small class in the family of semigroup algebras ([6]), and the Fourier algebra $A(G)$ on a locally compact group G is amenable if and only if G has a commutative subgroup of a finite index (due to A. T.-M. Lau, R. J. Loy and G. A. Willis [25], and B.E. Forrest and V. Runde [11]). Dropping the boundedness conditions in Definition 1 and Theorem 2, we have two natural ways to generalize the amenability in Banach algebras. We give them as follows.

Definition 3. We call the Banach algebra $\mathcal{A}$ *approximately amenable* if for each Banach $\mathcal{A}$-bimodule X, every continuous derivation D: $\mathcal{A} \to X$ is approximately inner (meaning that the identity (1.1) holds in the norm topology without requiring that (ξ_i) is bounded).

Remark. Approximately amenable Banach algebras were introduced in [12]. The original definition was the following: The Banach algebra $\mathcal{A}$ is approximately amenable if for each Banach $\mathcal{A}$-bimodule X, every continuous derivation from $\mathcal{A}$ into X^*, the dual of X, is approximately inner. The Banach algebra defined in Definition 3 was called *approximately contractible* in [12]. However, It was shown in [14] that the above two notions are the same (see Theorem 7 below).

Definition 4. We call the Banach algebra $\mathcal{A}$ *pseudo-amenable* if there exists an approximate diagonal for $\mathcal{A}$ (i.e. there is a net $(u_i) \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$, unnecessarily being bounded, such that the convergence identities in (1.2) hold.

Example 5. The following are notable examples of approximately amenable Banach algebras.

1. The semigroup algebras $\ell^1(\mathbb{N}_\wedge)$ and $\ell^1(\mathbb{N}_\vee)$ are approximately amenable but not amenable, where $\mathbb{N}_\wedge$ and $\mathbb{N}_\vee$ are the semigroups of integers under products $m \wedge n = \min\{m, n\}$ and $m \vee n = \max\{m, n\}$ respectively [14]. In fact, let $\mathcal{I}$ be any totally ordered set taken as a semigroup under the product $\vee$ (or $\wedge$). Then $\ell^1(\mathcal{I})$ is approximately amenable [3, Theorem 6.1].
2. If G is an amenable locally compact group containing an open abelian subgroup H, then the Fourier algebra $A(G)$ is approximately amenable [17]. It is not amenable if H is not of a finite index.
3. Let $\{\mathcal{A}_i : i \in I\}$ be a collection of amenable unital Banach algebras. The c_0 direct sum
$$(\oplus_{i\in I} \mathcal{A}_i)_{c_0}$$
is approximately amenable ([12]). It is not amenable unless the set $\{M(\mathcal{A}_i) : i \in I\}$ of the amenability constants of the collection is bounded.

Approximate amenability and pseudo-amenability are different notions. This is affirmed by the examples below.

Example 6. Notable pseudo-amenable Banach algebras include the following.

1. Let $\mathbb{F}_2$ be the the free group on two generators. Then the Fourier algebra $A(\mathbb{F}_2)$ is pseudo-amenable but not approximately amenable ([3]). In fact, if G is a locally compact group that contains an open abelian subgroup and if $A(G)$ has an approximate identity, then the Fourier algebra $A(G)$ is pseudo-amenable [17].
2. Let G be an amenable locally amenable SIN group. Then every nontrivial Segal algebra $S(G)$ on G is pseudo-amenable [18] but not approximately amenable [1].
3. Let $\{\mathcal{A}_i : i \in I\}$ be a collection of amenable Banach algebras. Then the c_0 and the ℓ^p ($p \geq 1$) direct sums
$$(\oplus_{i\in I} \mathcal{A}_i)_{c_0} \qquad \text{and} \qquad (\oplus_{i\in I} \mathcal{A}_i)_p$$
are pseudo-amenable. The latter is never approximately amenable if the collection is infinite ([18], [7]).

We don't know whether or not the approximate amenability is a stronger notion than the pseudo-amenability. The following result characterizing approximate amenability reveals some connection between the two notions.

Theorem 7 ([14]). *For a Banach algebra $\mathcal{A}$, the following are equivalent.*

1. *$\mathcal{A}$ is approximately amenable;*
2. *For each Banach $\mathcal{A}$-bimodule X, every continuous derivation D from $\mathcal{A}$ into the dual module X^* is approximately inner;*
3. *For each Banach $\mathcal{A}$-bimodule X, every continuous derivation D from $\mathcal{A}$ into X^* is weak* approximately inner;*
4. *the unitization $\mathcal{A}^{\#}$ of $\mathcal{A}$ is pseudo-amenable.*
5. *the unitization $\mathcal{A}^{\#}$ of $\mathcal{A}$ is approximately amenable.*

2. A conjecture

A Banach $\mathcal{A}$-bimodule X is *neo-unital* if it satisfies

$$X = \mathcal{A}X\mathcal{A} := \{axb : a, b \in \mathcal{A}, x \in X\}.$$

Theorem 8 ([12]). *Suppose that the Banach algebra $\mathcal{A}$ has a bounded approximate identity. Then the following are equivalent.*

1. *$\mathcal{A}$ is approximately amenable;*
2. *For each neo-unital Banach $\mathcal{A}$-bimodule X, each continuous derivation $D\colon \mathcal{A} \to X^*$ is approximately inner.*

Although approximate amenability and pseudo-amenability are different notions for a Banach algebra, they are most likely equivalent if the Banach algebra has a bounded approximate identity. This is indeed a conjecture in the area. A proof to this conjecture was attempted in [18]. But there was a gap in the proof which assumed an unjustified affirmative answer to the following question.

Question. If $\mathcal{A}$ has a b.a.i., is the condition stated in Theorem 7(3) equivalent to the following?

(3') For each right neo-unital Banach $\mathcal{A}$-bimodule X (i.e. $X = X\mathcal{A}$), every continuous derivation $D\colon \mathcal{A} \to X^*$ is weak* approximately inner.

So far we do not have an answer to the question. Theorem 8 seems supporting an affirmative answer to it. To better understand the problem, here we pin some observations as follows. First, when a one-sided module action is trivial, we have the following factor.

Lemma 9. *Let $\mathcal{A}$ be a Banach algebra with a right (or left) approximate identity (e_α), and let X be a Banach $\mathcal{A}$-bimodule. Suppose that $D\colon \mathcal{A} \to X$ is* *a continuous derivation such that $D(\mathcal{A}) \subset Y$, where Y is an $\mathcal{A}$-submodule of X and the right (resp. left) $\mathcal{A}$-module action restricting to Y is trivial. Then D is approximately inner. If in addition X is a dual Banach $\mathcal{A}$-bimodule, $(D(e_\alpha))$ is bounded (in particular, this is the case when (e_α) is bounded) and Y is weak* closed, then D is inner.*

Proof. Simply take $\eta_\alpha = D(e_\alpha)$. Then

$$D(a) = \lim_\alpha a\eta_\alpha - \eta_\alpha a \quad (a \in \mathcal{A}).$$

If X is a dual Banach $\mathcal{A}$-bimodule, $(D(e_\alpha))$ is bounded and Y is weak* closed, we take a weak* cluster point η of (η_α). Then $D = \mathrm{ad}_\eta$. $\square$

Let X be any Banach $\mathcal{A}$-bimodule. Then X^* is a left $\mathcal{A}^{**}$-module, where $\mathcal{A}^{**}$ is equipped with the first Arens product. The left module action of $\mathcal{A}^{**}$ on X^* is defined by

$$\langle x, u \cdot f\rangle = \langle fx, u\rangle \quad (x \in X, f \in X^*, u \in \mathcal{A}^{**}),$$

where $fx \in \mathcal{A}^*$ is given by

$$\langle a, fx\rangle = \langle xa, f\rangle \quad (a \in \mathcal{A}).$$

The action extends the natural left module action of $\mathcal{A}$ on X^*, and the mapping $u \mapsto u \cdot f$: $\mathcal{A}^{**} \to X^*$ is indeed weak*-weak* continuous for each $f \in X^*$.

Now let $(e_\alpha) \subset \mathcal{A}$ be a bounded approximate identity and E a weak* cluster point of (e_α) in $\mathcal{A}^{**}$. Suppose that D: $\mathcal{A} \to X^*$ is a continuous derivation. We can write

$$X^* = EX^* \oplus (id - E)X^*, \quad \text{and} \quad D = D_1 + D_2, \tag{2.1}$$

where $EX^* = \{E \cdot f : f \in X^*\}$, $(id - E)X^* = \{f - E \cdot f : f \in X^*\}$, $D_1 = E \cdot D$ and $D_2 = D - E \cdot D$. The mappings D_1 and D_2 are derivations from $\mathcal{A}$ into EX^* and $(id-E)X^*$ respectively. From Lemma 9, we see that D_2 is inner since $(id - E)X^* = (X\mathcal{A})^\perp$ is a weak* closed submodule of X^* and the left $\mathcal{A}$-module action on $(id - E)X^*$ is trivial. On the other hand, $EX^* = (X\mathcal{A})^*$. If the condition (3') holds, D_1 is weak* approximately inner as a derivation into $(X\mathcal{A})^*$. However, this does not mean that D_1 is weak* approximately inner as a derivation into X^*. So we cannot conclude that D: $\mathcal{A} \to X^*$ is weak* approximately inner.

We now have only a partial solution to the above conjecture which we will present in the remainder of the section.

Proposition 10. *The following statements are equivalent.*

1. *the Banach algebra $\mathcal{A}$ is approximately amenable and has a bounded approximate identity;*
2. *The Banach algebra $\mathcal{A}$ is pseudo-amenable with an approximate diagonal $(m_\mu)_{\mu\in\Gamma} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $(\pi(m_\mu))_{\mu\in\Gamma}$ is bounded.*

Proof. The proof of (1) ⇒ (2) was already given in [18].

Suppose that (2) holds. Then $(\pi(m_\mu))_{\mu\in\Gamma}$ is a bounded approximate identity for $\mathcal{A}$. Without loss of generality we may assume

$$\text{wk}^*\lim_\mu \pi(m_\mu) = E \in \mathcal{A}^{**}.$$

Clearly $Ea = aE = a$ for all $a \in \mathcal{A}$. Let X be any Banach $\mathcal{A}$-bimodule and $D\colon \mathcal{A} \to X^*$ be a continuous derivation. We show that the condition Theorem 7(3) holds. Indeed the decompositions formulated in (2.1) hold. From the discussion in the sentence after Lemma 9, we only need to show that D_1 defined by $D_1(a) = E \cdot D(a)\colon \mathcal{A} \to X^*$ $(a \in \mathcal{A})$ is weak* approximately inner.

Let $\Phi\colon \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ be the bounded linear mapping determined by $\Phi(a \otimes b) = a \cdot D(b)$ and let $\xi_\mu = \Phi(m_\mu)$. Then

$$\pi(m_\mu) \cdot D(a) = (a \cdot \xi_\mu - \xi_\mu \cdot a) - \Phi(am_\mu - m_\mu a) \quad (a \in \mathfrak{A}).$$

As we have known, $\pi(m_\mu) \cdot D(a) \to E \cdot D(a)$ in the weak* topology of X^* for each $a \in \mathcal{A}$. On the other hand,

$$\|\Phi(am_\mu - m_\nu a)\| \le \|D\|\|am_\mu - m_\mu a\| \to 0$$

for $a \in \mathcal{A}$. We therefore have

$$D_1(a) = E \cdot D(a) = \text{wk}^*\lim_\mu (a \cdot \xi_\mu - \xi_\mu \cdot a) \quad (a \in \mathcal{A}).$$

This shows that $D_1\colon \mathcal{A} \to X^*$ is weak* approximately inner. Thus $D = D_1 + D_2$ is weak* approximately inner. □

Indeed, the proof still works if we drop the boundedness condition on $(\pi(m_\mu))_{\mu\in\Gamma}$ but assume it is weak* convergent in $\mathcal{A}^{**}$. So we have the following result.

Proposition 11. *If $\mathcal{A}$ has an approximate diagonal $(m_\mu)_{\mu\in\Gamma} \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $(\pi(m_\mu))_{\mu\in\Gamma}$ is weak* convergent in $\mathcal{A}^{**}$, then $\mathcal{A}$ is approximately amenable.*

The following is an important special case of Proposition 10.

Corollary 12. *Suppose that the Banach algebra $\mathcal{A}$ has a central bounded approximate identity. Then it is approximately amenable if and only if it is pseudo-amenable.*

Proof. We only need to show the sufficiency. Suppose that (e_α) is a central bounded approximate identity of $\mathcal{A}$, and suppose that (m_μ) is an approximate diagonal for $\mathcal{A}$. Consider $(m_{\alpha,\mu})$, where $m_{\alpha,\mu} = e_\alpha m_\mu$. We have

$$\|am_{\alpha,\mu} - m_{\alpha,\mu}a\| = \|e_\alpha\|\|am_\mu - m_\mu a\|, \quad \pi(m_{\alpha,\mu}) = e_\alpha \pi(m_\mu).$$

Therefore, by a standard way, one may extract a subnet (n_ν) from $(m_{\alpha,\mu})$ such that (n_ν) is still an approximate diagonal for $\mathcal{A}$ and such that $(\pi(n_\nu))$ is bounded. So Proposition 10 applies. □

Within the author's knowledge, all notable applications of [18, Proposition 3.2] so far are for commutative Banach algebras. They are valid since we have the following simple consequence of Corollary 12.

Corollary 13. *Let $\mathcal{A}$ be a commutative Banach algebra with a bounded approximate identity. Then $\mathcal{A}$ is approximately amenable if and only if it is pseudo-amenable.*

We notice that the monograph [28] also contains some investigation concerning the conjecture discussed in this section.

3. Open questions

We post some open problems in this section to conclude this note.

First, the conjecture discussed in the previous section can be simply stated as the following problem.

Problem 1. If $\mathcal{A}$ has a bounded approximate identity and is pseudo-amenable, must it be approximately amenable?

Given two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is also a Banach algebra with the product defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd \quad (a, c \in \mathcal{A},\ b, d \in \mathcal{B}).$$

It has been shown in [13] that, for approximately amenable Banach algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\widehat{\otimes}\mathcal{B}$ may not be approximately amenable even both $\mathcal{A}$ and $\mathcal{B}$ are unital. Since approximate amenability for a unital Banach algebra is the same as pseudo-amenability, this implies further that the tensor product of unital pseudo-amenable Banach algebras may not be pseudo-amenable. Conditions on $\mathcal{A}$ and $\mathcal{B}$ under which $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately (pseudo) amenable have been investigated in [12, 2, 19].

Problem 2. For Banach algebras $\mathcal{A}$ and $\mathcal{B}$, how to characterize approximate amenability and pseudo-amenability of the tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$?

In particular, we are interested in the special case where $\mathcal{A} = \mathcal{B} = \ell^1(\mathbb{N}_\wedge)$.

Problem 3. Is $\ell^1(\mathbb{N}_\wedge \times \mathbb{N}_\wedge) = \ell^1(\mathbb{N}_\wedge)\widehat{\otimes}\ell^1(\mathbb{N}_\wedge)$ approximately (or pseudo-) amenable?

We note that, by Corollary 13, the approximate amenability and the pseudo-amenability of $\ell^1(\mathbb{N}_\wedge \times \mathbb{N}_\wedge)$ are the same, since $\mathbb{N}_\wedge \times \mathbb{N}_\wedge$ is abelian and $\ell^1(\mathbb{N}_\wedge \times \mathbb{N}_\wedge)$ has a bounded approximate identity.

We refer to [27, 5, 9, 10] for the investigation of approximate and pseudo-amenability in general semigroup algebras.

It is known that, for any weight function $\omega(x)$ on a locally compact group G, the weighted group algebra $L^1(G, \omega)$ is amenable if and only if it is isomorphic to the group algebra $L^1(G)$ and G is amenable [22, 15]. While the group algebra $L^1(G)$ is approximately amenable and pseudo-amenable if and only if it is amenable [12, 18].

Problem 4. How to characterize the approximate amenability and pseudo-amenability of the weighted group algebra $L^1(G, \omega)$? Is there a weight ω on some G such that $L^1(G, \omega)$ is approximately or pseudo-amenable but is not amenable?

We have partial answers to the above question in [16].

Problem 5. How to characterize the approximate amenability and pseudo-amenability of a C*-algebra?

In fact, there is no published investigation so far on the generalized amenability of a C*-algebra at all.

We refer to [29] for more open questions in generalized amenability of Banach algebras.

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