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An alternated inertial algorithm with weak and linear convergence for solving monotone inclusions

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Dedicated to Professor Anthony To-Ming Lau on the occasion of his 80th birthday

Inertial-based methods have the drawback of not preserving the Fejér monotonicity of iterative sequences, which may result in slower convergence compared to their corresponding non-inertial versions. To overcome this issue, Mu and Peng [Stat. Optim. Inf. Comput. **3** (2015), 241–248; MR3393305] suggested an alternating inertial method that can recover the Fejér monotonicity of even subsequences. In this paper, we propose a modified version of the forward-backward algorithm with alternating inertial and relaxation effects to solve an inclusion problem in real Hilbert spaces. The weak and linear convergence of the presented algorithm is established under suitable and mild conditions on the involved operators and parameters. Furthermore, the Fejér monotonicity of even subsequences generated by the proposed algorithm with respect to the solution set is recovered. Finally, our tests on image restoration problems demonstrate the superiority of the proposed algorithm over some related results.

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1. Introduction

The *Monotone Inclusion Problem* (shortly, MIP) is a fundamental problem in mathematics and operational research that arises in many fields, including

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physics, engineering, and economics. Solving the MIP is essential for numerous practical applications, such as image processing, signal processing, and machine learning; see, e.g., [1, 2, 3]. Recall that the MIP is described as finding a point in the intersection of two or more sets, and each of them is defined as the solution set of a monotone operator. In this paper, our goal is to solve the zero-point problem of the sum of two monotone operators. That is, we want to find the solution to the following mathematical problem

$$\text{find } x^* \in \mathcal{H} \text{ such that } 0 \in (A+B)x^*, \tag{1}$$

where $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, single-valued operator $A : \mathcal{H} \to \mathcal{H}$ is monotone, and multi-valued operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone. The solution set to problem (1) is denoted by Ω throughout the paper.

Monotone operators play a crucial role in optimization and variational analysis because they have special properties that make them suitable for solving a wide range of problems. Many optimization problems involve minimizing a function subject to constraints. The constraints are often expressed in terms of a set, and the optimization problem is to find a point in the intersection of these sets. Monotone operators arise naturally in this context because the solution sets of many optimization problems can be expressed as the zero points of monotone operators. The importance of monotone operators is not limited to optimization problems. Indeed, they also have been used in various areas, such as partial differential equations, image processing, and signal processing.

To solve problem (1) efficiently, various numerical methods have been developed. One approach is the operator splitting method, which decomposes the original problem into simpler sub-problems that can then be solved in turn. This type of algorithms has gained significant attention in recent years due to its efficiency in solving large-scale problems. It has been applied to various areas, such as compressed sensing, inverse problems, and convex optimization. The forward-backward (FB) splitting algorithm (see [4, 5]) is a widely-used method for solving optimization problems that can be decomposed into the sum of two convex functions. This algorithm consists of two phases: a forward phase that computes a gradient step of one of the convex functions, and a backward phase that computes a proximal operator of the other convex function. By alternating between these two phases, the algorithm produces an iterative sequence that converges to a solution of the original optimization problem. To improve the convergence condition and computational speed of the FB algorithm, Tseng [6] proposed an improved version of the FB algorithm, namely the forward-backward-forward (FBF) algorithm, also known as the Tseng splitting method. The FBF algorithm is known for its simplicity, efficiency, and versatility, and has been applied to a wide range of problems in various fields.

The convergence speed of algorithms is a key concern for scholars. It is known that the inertial method (cf. [7]), as an important technique for accelerating algorithm's convergence speed, was widely used by scholars. The basic idea of the inertial method is that the value of the current iterative sequence is jointly determined by the combination of two or more previous values. Over the past few decades, researchers proposed a large number of inertial-based algorithms to solve optimization problems, such as variational inequalities, equilibrium problems, inclusion problems, and splitting problems; see, e.g., [8, 9, 10, 11] and the references therein. On the other hand, looking back to the projection and contraction (PC) algorithm introduced by He [12] for solving variational inequality problems, it aims to improve the convergence conditions and computational efficiency of the extragradient method. Now, the PC algorithm is also used to solve monotone inclusion problems. Indeed, Gibali et al. [13] proposed a modified FB algorithm based on the inertia method, the FB algorithm, and the PC algorithm to solve the MIP. Under appropriate conditions, they proved the weak convergence of the suggested algorithm. Subsequently, Thong et al. [14] also introduced an improved FBF algorithm with inertial and relaxation effects to solve the MIP and established the weak and linear convergence of the algorithm.

It should be noted that the common drawback of the algorithms in references [13, 14] is that the generated iterative sequence does not enjoy Fejér monotonicity, which to some extent affects the computational efficiency of these algorithms. To overcome this, Mu and Peng [15] introduced an improved version of the inertial method, namely the alternated inertial method. This method maintains the original values of the sequence at even terms and uses the inertial method to update the values of the sequence at odd terms. The advantage of the alternated inertial method is that it can recover the Fejér monotonicity of even sub-sequences. Recently, Shehu et al. [16] proposed a modified FBF algorithm with alternating inertial terms to find solutions of the MIP. Under suitable conditions, they also established the weak and linear convergence of the proposed algorithm, and verified the computational efficiency of the method through experiments on optimal control problems. In recent years, the alternated inertial method was extended to solve other optimization problems; see, e.g., [17, 18, 19, 20, 21, 22, 23] and the references therein.

Inspired and motivated by the above results, our goal in this paper is to explore the theoretical analysis and practical applications of an improved PC algorithm for solving the monotone inclusion problem. The rest of the paper is organized as follows. In Section 2, we provide some definitions and lemmas that are needed subsequently. In Section 3, we introduce an improved PC algorithm with alternating inertial terms and relaxation effects to discover solutions to the MIP and analyze the weak convergence and linear convergence of the suggested algorithm under certain conditions of operators and parameters. In Section 4, the proposed algorithm in comparison with some pertinent algorithms is tested on image processing problems. Finally, we conclude the paper in Section 5, the last section.

2. Preliminaries

Throughout the paper, let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence of the sequence $\{u_n\}$ to x as $n \to \infty$ is indicated by $u_n \rightharpoonup x$, while the strong convergence of the sequence $\{u_n\}$ to x as $n \to \infty$ is represented by $u_n \to x$.

The following definitions are common and can be found in any books and articles on convex analysis or operator theory; see, e.g., [24].

Definition 2.1. Let $A : \mathcal{H} \to \mathcal{H}$ denote a single-valued operator and $B : \mathcal{H} \to 2^{\mathcal{H}}$ a multi-valued operator.

(i) A is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

(ii) A is called `monotone` if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}.$$

(iii) B is called `monotone` if

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}, u \in Bx, v \in By.$$

(iv) B is called μ-`strongly monotone` if there exists a number $\mu > 0$ such that

$$\langle u - v, x - y\rangle \geq \mu\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in Bx, v \in By.$$

(v) B is called `maximally monotone`, if it is monotone and if for any $(x,u) \in \mathcal{H} \times \mathcal{H}$, $\langle u-v, x-y\rangle \geq 0$ for every $(y,v) \in \text{Graph}(B)$ (the graph of operator B) implies that $u \in Bx$.

Definition 2.2. Let C be a nonempty subset of $\mathcal{H}$, and let $\{u_n\}$ be a sequence in $\mathcal{H}$. Then $\{u_n\}$ is Fejér monotone with respect to C if $\|u_{n+1}-u\| \leq \|u_n - u\|$, $\forall u \in C, n \in \mathbb{N}$.

Definition 2.3. Let $\{u_n\}$ be a sequence in $\mathcal{H}$. Then $\{u_n\}$ is said to converge R-linearly to p with rate $\rho \in [0,1)$ if there is a constant $c > 0$ such that $\|u_n - p\| \leq c\rho^n$, $\forall n \in \mathbb{N}$.

The following two lemmas play a crucial role in the weak convergence analysis of the algorithm presented in Section 3.

Lemma 2.1 ([25])**.** *Let $A : \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous and monotone and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone. Then $A + B$ is maximally monotone.*

Lemma 2.2 ([26])**.** *Let C be a nonempty set of $\mathcal{H}$ and $\{u_n\}$ be a sequence in $\mathcal{H}$. If $\lim_{n\to\infty} \|u_n - x\|$ exists for every $x \in C$, and every sequential weak cluster point of $\{u_n\}$ is in C, then $\{u_n\}$ converges weakly to a point in C.*

3. Main results

In this section, we propose an alternated inertial projection and contraction algorithm to solve the monotone inclusion problem. The proposed method can adaptively work without the prior information of the Lipschitz constant of the involved operator. Under some appropriate conditions, the weak and linear convergence of the proposed algorithm is proved. We first assume that the following conditions are satisfied for the convergence analysis of our algorithm:

(C1) the solution set of problem (1) is nonempty, i.e., $\Omega := (A+B)^{-1}(0) \neq \emptyset$;
(C2) operator $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and monotone and operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone;
(C3) let $\zeta_1 > 0$, $\chi \in (0,1]$, $\psi \in (0,1)$, $\alpha \in (0,2)$, and $\nu \in [0, \frac{2}{\alpha\chi} - 1)$. Choose $\{\xi_n\} \subset [1,\infty)$ such that $\sum_{n=0}^{\infty}(\xi_n - 1) < \infty$ and $\{\tau_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} \tau_n < \infty$.

We are now in the position to present Algorithm 3.1.

Remark 3.1. Note that the inertial parameter ν in (2) allows $\nu \geq 1$ when $\alpha\chi < 1$ (cf. $\nu \in [0, \frac{2}{\alpha\chi} - 1)$ in Condition (C3)).

Algorithm 3.1 An alternated inertial algorithm for solving monotone inclusions

Initialization: Give $\zeta_1 > 0$, $\chi \in (0,1]$, $\psi \in (0,1)$, $\alpha \in (0,2)$, and $\nu \in [0, \frac{2}{\alpha\chi} - 1)$. Choose $\{\xi_n\} \subset [1,\infty)$ such that $\sum_{n=0}^{\infty}(\xi_n - 1) < \infty$ and $\{\tau_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} \tau_n < \infty$. Select starting points $u_0, u_1 \in \mathcal{H}$ and set $n := 1$.

Iterative Steps: Given the iterates u_n, u_{n-1}, perform the following steps.

Step 1. Compute

$$s_n = \begin{cases} u_n & n = \text{ even;} \\ u_n + \nu\left(u_n - u_{n-1}\right) & n = \text{ odd.} \end{cases} \tag{2}$$

Step 2. Compute

$$p_n = \left(I + \zeta_n B\right)^{-1}\left(I - \zeta_n A\right) s_n.$$

If $p_n = s_n$, then stop and $p_n \in \Omega$. Otherwise, go to *Step 3.*

Step 3. Compute

$$t_n = s_n - \alpha\delta_n r_n,$$

where

$$r_n := s_n - p_n - \zeta_n\left(As_n - Ap_n\right), \quad \delta_n := \frac{\langle s_n - p_n, r_n\rangle}{\left\|r_n\right\|^2}. \tag{3}$$

Step 4. Compute

$$u_{n+1} = (1-\chi)s_n + \chi t_n.$$

Update ζ_{n+1} by

$$\zeta_{n+1} = \begin{cases} \min\left\{\dfrac{\psi\left\|s_n - p_n\right\|}{\left\|As_n - Ap_n\right\|}, \xi_n\zeta_n + \tau_n\right\} & \text{if } As_n - Ap_n \neq 0; \\ \xi_n\zeta_n + \tau_n & \text{otherwise.} \end{cases} \tag{4}$$

Set $n := n+1$ and go to *Step 1.*

Remark 3.2. If $p_n = s_n$, then, according to *Step 2* in Algorithm 3.1, we have

$$p_n = \left(I + \zeta_n B\right)^{-1}\left(I - \zeta_n A\right) p_n,$$

which is equivalent to

$$\left(I - \zeta_n A\right) p_n \in \left(I + \zeta_n B\right) p_n.$$

That is, $p_n \in (A+B)^{-1}(0)$. Thus the iterations of Algorithm 3.1 terminate when $p_n = s_n$.

The following lemmas are useful for the convergence analysis of Algorithm 3.1.

Lemma 3.1 ([23]). *Suppose that Condition (C3) holds. Then the sequence* $\{\zeta_n\}$ *generated by* (4) *is well defined and* $\lim_{n\to\infty}\zeta_n$ *exists.*

Lemma 3.2. *If* $p_n = s_n$ *or* $r_n = 0$ *in Algorithm 3.1, then* $p_n \in \Omega$.

Proof. By (4), one has

$$\begin{aligned}\|r_n\| &\geq \|s_n - p_n\| - \zeta_n\|As_n - Ap_n\| \\ &\geq \left(1 - \frac{\psi\zeta_n}{\zeta_{n+1}}\right)\|s_n - p_n\|.\end{aligned}$$

It can be easily proved that $\|r_n\| \leq \left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)\|s_n - p_n\|$. Then it follows that

$$\left(1 - \frac{\psi\zeta_n}{\zeta_{n+1}}\right)\|s_n - p_n\| \leq \|r_n\| \leq \left(1 + \frac{\psi\zeta_n}{\zeta_{n+1}}\right)\|s_n - p_n\|, \tag{5}$$

and thus $s_n = p_n$ if and only if $r_n = 0$. Hence, if $s_n = p_n$ or $r_n = 0$, then $p_n \in \Omega$ according to Remark 3.2. Thus, the proof of the lemma is finished. □

Lemma 3.3. *Assume that sequences* $\{u_{2n}\}$ *and* $\{p_{2n}\}$ *are formed by Algorithm 3.1. If* $\lim_{n\to\infty}\|u_{2n} - p_{2n}\| = 0$ *and* $\{u_{2n_k}\}$ *converges weakly to some* $p \in \mathcal{H}$, *then* $p \in \Omega$.

Proof. Let $(v, u) \in \text{Graph}(A+B)$, i.e., $u \in (A+B)v$. By using the definition of p_{2n} and noting $s_{2n} = u_{2n}$, one obtains $(I - \zeta_{2n_k}A)u_{2n_k} \in (I + \zeta_{2n_k}B)\,p_{2n_k}$, which means that

$$\zeta_{2n_k}^{-1}\left(u_{2n_k} - p_{2n_k} - \zeta_{2n_k}Au_{2n_k}\right) \in Bp_{2n_k}.$$

Since operator B is maximally monotone, we deduce that

$$\left\langle u - Av - \zeta_{2n_k}^{-1}\left(u_{2n_k} - p_{2n_k} - \zeta_{2n_k}Au_{2n_k}\right), v - p_{2n_k}\right\rangle \geq 0.$$

This combining with the monotonicity of A finds that

$$\begin{aligned}\langle v - p_{2n_k}, u\rangle &\geq \left\langle v - p_{2n_k}, Av + \zeta_{2n_k}^{-1}\left(u_{2n_k} - p_{2n_k} - \zeta_{2n_k}Au_{2n_k}\right)\right\rangle \\ &= \langle v - p_{2n_k}, Av - Ap_{2n_k}\rangle + \langle v - p_{2n_k}, Ap_{2n_k} - Au_{2n_k}\rangle \\ &\quad + \left\langle v - p_{2n_k}, \zeta_{2n_k}^{-1}\left(u_{2n_k} - p_{2n_k}\right)\right\rangle \\ &\geq \langle v - p_{2n_k}, Ap_{2n_k} - Au_{2n_k}\rangle + \left\langle v - p_{2n_k}, \zeta_{2n_k}^{-1}\left(u_{2n_k} - p_{2n_k}\right)\right\rangle.\end{aligned}$$

We have $\lim_{k\to\infty}\|Ap_{2n_k}-Au_{2n_k}\|=0$ by means of $\lim_{n\to\infty}\|u_{2n}-p_{2n}\|=0$ and the fact that A is Lipschitz continuous. According to $\zeta_{2n_k}>0$, one has

$$\lim_{k\to\infty}\langle v-p_{2n_k},u\rangle=\langle v-p,u\rangle\geq 0,$$

which together with the maximal monotonicity of $(A+B)$ (cf. Lemma 2.1 and Condition (C2)) yields $0\in(A+B)p$, i.e., $p\in\Omega$. This completes the proof. □

Lemma 3.4. *Let $\{s_n\}$, $\{p_n\}$, and $\{t_n\}$ be three sequences created by Algorithm 3.1. Then*

$$\|t_n-p\|^2\leq\|s_n-p\|^2-\frac{2-\alpha}{\alpha}\|t_n-s_n\|^2,\quad\forall p\in\Omega,\tag{6}$$

and

$$\|s_n-p_n\|^2\leq\frac{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}{\left[(1-\frac{\psi\zeta_n}{\zeta_{n+1}})\alpha\right]^2}\|s_n-t_n\|^2.\tag{7}$$

Proof. Let $p\in\Omega$. According to the definition of t_n, one has

$$\|t_n-p\|^2=\|s_n-p\|^2-2\alpha\delta_n\langle s_n-p,r_n\rangle+\alpha^2\delta_n^2\|r_n\|^2.\tag{8}$$

From the definition of r_n, one sees that

$$\begin{aligned}\langle s_n-p,r_n\rangle&=\langle s_n-p_n,r_n\rangle+\langle p_n-p,r_n\rangle\\&=\langle s_n-p_n,r_n\rangle+\langle p_n-p,s_n-p_n-\zeta_n(As_n-Ap_n)\rangle.\end{aligned}\tag{9}$$

By using the definition of p_n, one obtains $(I-\zeta_nA)s_n\in(I+\zeta_nB)p_n$. Since B is maximally monotone, there exists $v_n\in Bp_n$ satisfying $(I-\zeta_nA)s_n=p_n+\zeta_nv_n$, which means that

$$v_n=\zeta_n^{-1}(s_n-p_n-\zeta_nAs_n).\tag{10}$$

Thanks to Lemma 2.1 and Condition (C2), we have that $(A+B)$ is maximally monotone. From $Ap_n+v_n\in(A+B)p_n$ and $0\in(A+B)p$, one infers that $\langle Ap_n+v_n,p_n-p\rangle\geq 0$. This together with (10) further implies that

$$\langle s_n-p_n-\zeta_n(As_n-Ap_n),p_n-p\rangle\geq 0.\tag{11}$$

Combining (8), (9), (11), and the definitions of δ_n and t_n, we have

$$\begin{aligned}\|t_n - p\|^2 &\leq \|s_n - p\|^2 - 2\alpha\delta_n \langle s_n - p_n, r_n\rangle + \alpha^2\delta_n^2 \|r_n\|^2 \\ &= \|s_n - p\|^2 - 2\alpha\delta_n^2 \|r_n\|^2 + \alpha^2\delta_n^2 \|r_n\|^2 \\ &= \|s_n - p\|^2 - \frac{2-\alpha}{\alpha} \|\alpha\delta_n r_n\|^2 \\ &= \|s_n - p\|^2 - \frac{2-\alpha}{\alpha} \|t_n - s_n\|^2, \quad \forall p \in \Omega.\end{aligned} \tag{12}$$

It follows from (4) that $\|As_n - Ap_n\| \leq \frac{\psi}{\zeta_{n+1}}\|s_n - p_n\|$, $\forall n \geq 1$, which combining with the definition of δ_n yields that

$$\begin{aligned}\delta_n\|r_n\|^2 = \langle r_n, s_n - p_n\rangle &\geq \|s_n - p_n\|^2 - \zeta_n \|As_n - Ap_n\| \|s_n - p_n\| \\ &\geq \left(1 - \frac{\psi\zeta_n}{\zeta_{n+1}}\right) \|s_n - p_n\|^2.\end{aligned} \tag{13}$$

Using (13) and $\|r_n\| \leq (1 + \frac{\psi\zeta_n}{\zeta_{n+1}})\|s_n - p_n\|$, one has

$$\delta_n^2\|r_n\|^2 \geq \left(1 - \frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2 \frac{\|s_n - p_n\|^4}{\|r_n\|^2} \geq \frac{\left(1 - \frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}{\left(1 + \frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}\|s_n - p_n\|^2. \tag{14}$$

According to the definition of t_n and (14), one sees that

$$\|t_n - s_n\|^2 = \alpha^2\delta_n^2\|r_n\|^2 \geq \alpha^2\frac{\left(1 - \frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}{\left(1 + \frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}\|s_n - p_n\|^2.$$

Hence we obtain

$$\|s_n - p_n\|^2 \leq \frac{\left(1 + \frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}{\left[(1 - \frac{\psi\zeta_n}{\zeta_{n+1}})\alpha\right]^2}\|s_n - t_n\|^2.$$

With that, the proof of the lemma is concluded. □

Lemma 3.5. *Let sequence $\{u_n\}$ be generated by Algorithm 3.1 and Conditions (C1)–(C3) hold. Then the sequence $\{u_{2n}\}$ is Fejér monotone with respect to the solution set Ω of (1) and $\lim_{n\to\infty} \|u_{2n} - p\|$ exists, where $p \in \Omega$. Moreover,*

$$\lim_{n\to\infty} \|u_{2n} - p_{2n}\| = 0, \quad \lim_{n\to\infty} \|u_{2n+1} - u_{2n}\| = 0.$$

Proof. From the definition of u_{2n+2}, one sees that

$$\|t_{2n+1} - s_{2n+1}\|^2 = \frac{1}{\chi^2} \|u_{2n+2} - s_{2n+1}\|^2 . \tag{15}$$

It is known that the following inequality holds for any $x, y \in \mathcal{H}$ and $\nu \in [0, 1]$.

$$\|\nu x + (1-\nu)y\|^2 = \nu\|x\|^2 + (1-\nu)\|y\|^2 - \nu(1-\nu)\|x - y\|^2. \tag{16}$$

Combining (6), (15), and (16), we have

$$\begin{aligned}
&\|u_{2n+2} - p\|^2 \\
&= (1-\chi)\|s_{2n+1} - p\|^2 + \chi\|t_{2n+1} - p\|^2 \\
&\quad - \chi(1-\chi)\|s_{2n+1} - t_{2n+1}\|^2 \\
&\le (1-\chi)\|s_{2n+1} - p\|^2 + \chi\|s_{2n+1} - p\|^2 \\
&\quad - \chi\frac{2-\alpha}{\alpha}\|t_{2n+1} - s_{2n+1}\|^2 - \chi(1-\chi)\|s_{2n+1} - t_{2n+1}\|^2 \\
&= \|s_{2n+1} - p\|^2 - \left(\frac{2}{\alpha\chi} - 1\right)\|u_{2n+2} - s_{2n+1}\|^2 .
\end{aligned} \tag{17}$$

By using (17) (noting that $s_{2n} = u_{2n}$), one has

$$\begin{aligned}
\|u_{2n+1} - p\|^2 &\le \|s_{2n} - p\|^2 - \left(\frac{2}{\alpha\chi} - 1\right)\|u_{2n+1} - s_{2n}\|^2 \\
&= \|u_{2n} - p\|^2 - \left(\frac{2}{\alpha\chi} - 1\right)\|u_{2n+1} - u_{2n}\|^2 .
\end{aligned} \tag{18}$$

From (16), (18), and the definition of s_n, we obtain

$$\begin{aligned}
&\|s_{2n+1} - p\|^2 \\
&= (1+\nu)\|u_{2n+1} - p\|^2 - \nu\|u_{2n} - p\|^2 + \nu(1+\nu)\|u_{2n+1} - u_{2n}\|^2 \\
&\le (1+\nu)\left[\|u_{2n} - p\|^2 - \left(\frac{2}{\alpha\chi} - 1\right)\|u_{2n+1} - u_{2n}\|^2\right] \\
&\quad - \nu\|u_{2n} - p\|^2 + \nu(1+\nu)\|u_{2n+1} - u_{2n}\|^2 \\
&= \|u_{2n} - p\|^2 - (1+\nu)\left(\frac{2}{\alpha\chi} - 1 - \nu\right)\|u_{2n+1} - u_{2n}\|^2 .
\end{aligned} \tag{19}$$

Substituting (19) into (17), we have

$$\begin{aligned} \|u_{2n+2}-p\|^2 \le \|u_{2n}-p\|^2 - \left(\frac{2}{\alpha\chi}-1\right)\|u_{2n+2}-s_{2n+1}\|^2 \\ -(1+\nu)\left(\frac{2}{\alpha\chi}-1-\nu\right)\|u_{2n+1}-u_{2n}\|^2 . \end{aligned} \tag{20}$$

Since $\alpha \in (0,2)$, $\chi \in (0,1]$, and $\nu \in [0, \frac{2}{\alpha\chi}-1)$, it follows from (20) that

$$\|u_{2n+2}-p\| \le \|u_{2n}-p\|, \ \forall n \ge 1.$$

This implies that sequence $\{u_{2n}\}$ is Fejér monotone with respect to solution set Ω and sequences $\{\|u_{2n}-p\|\}$ and $\{u_{2n}\}$ are bounded. Furthermore, one obtains that $\lim_{n\to\infty}\|u_{2n}-p\|$ exists. Rearranging (20) and using the fact that $\{\|u_{2n}-p\|\}$ is bounded, we deduce that

$$\lim_{n\to\infty}\|u_{2n+1}-u_{2n}\| = 0. \tag{21}$$

It follows from the definition of r_{2n} and (4) that

$$\begin{aligned} \|r_{2n}\| &= \|s_{2n}-p_{2n}-\zeta_{2n}(As_{2n}-Ap_{2n})\| \\ &\le \|s_{2n}-p_{2n}\| + \zeta_{2n}\|As_{2n}-Ap_{2n}\| \\ &\le \left(1+\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}\right)\|s_{2n}-p_{2n}\|, \end{aligned}$$

which implies that

$$\frac{1}{\|r_{2n}\|} \ge \frac{1}{\left(1+\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}\right)\|s_{2n}-p_{2n}\|}. \tag{22}$$

By using the definition of r_{2n} and (4), one obtains

$$\begin{aligned} \langle s_{2n}-p_{2n}, r_{2n}\rangle &= \langle s_{2n}-p_{2n}, s_{2n}-p_{2n}-\zeta_{2n}(As_{2n}-Ap_{2n})\rangle \\ &= \|s_{2n}-p_{2n}\|^2 - \langle s_{2n}-p_{2n}, \zeta_{2n}(As_{2n}-Ap_{2n})\rangle \\ &\ge \|s_{2n}-p_{2n}\|^2 - \zeta_{2n}\|As_{2n}-Ap_{2n}\|\|s_{2n}-p_{2n}\| \\ &\ge \left(1-\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}\right)\|s_{2n}-p_{2n}\|^2 . \end{aligned} \tag{23}$$

Combining the definition of u_{2n+1} and δ_{2n}, (22), and (23), we have

$$\begin{aligned}\|u_{2n+1}-s_{2n}\| &= \|\chi\,(t_{2n}-s_{2n})\| = \chi\alpha\delta_{2n}\,\|r_{2n}\| \\ &= \chi\alpha\frac{\langle s_{2n}-p_{2n}, r_{2n}\rangle}{\|r_{2n}\|} \geq \chi\alpha\left(\frac{1-\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}}{1+\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}}\right)\|s_{2n}-p_{2n}\|\,. \end{aligned} \tag{24}$$

From Lemma 3.1 and $\psi \in (0,1)$, one can check that

$$\lim_{n\to\infty}\frac{1-\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}}{1+\frac{\psi\zeta_{2n}}{\zeta_{2n+1}}} = \frac{1-\psi}{1+\psi} > 0.$$

By using (21) and (24) (noting that $s_{2n}=u_{2n}$), we deduce that

$$\lim_{n\to\infty}\|u_{2n}-p_{2n}\| = 0.$$

Therefore, the lemma has been demonstrated. □

We can now proceed to prove the weak convergence of Algorithm 3.1.

Theorem 3.1 (Weak convergence). *Let $\{u_n\}$ be any sequence generated by Algorithm 3.1 and Conditions (C1)–(C3) hold. Then $\{u_n\}$ converges weakly to an element $p \in \Omega$.*

Proof. Lemma 3.5 implies that $\{u_{2n}\}$ is bounded, and thus it has weakly convergent subsequences. Let $z \in \mathcal{H}$ denote the weak limit of a subsequence $\{u_{2n_k}\}$ of $\{u_{2n}\}$. Combining this with the fact that $\lim_{n\to\infty}\|u_{2n}-p_{2n}\|=0$ and Lemma 3.3, we obtain $z \in \Omega$. Thus, by Lemma 3.5, $\lim_{n\to\infty}\|u_{2n}-p\|$ exists for all $p \in \Omega$. Using Lemma 2.2, we can show that the whole sequence $\{u_{2n}\}$ converges weakly to a point in Ω. We now prove that this weak limit is unique. Suppose that $\{u_{2n}\}$ converges weakly to both p and q in Ω. Then

$$\begin{aligned}\|p-q\|^2 &= \langle p, p-q\rangle - \langle q, p-q\rangle \\ &= \lim_{n\to\infty}\langle u_{2n}, p-q\rangle - \lim_{n\to\infty}\langle u_{2n}, p-q\rangle \\ &= \lim_{n\to\infty}\langle u_{2n}-u_{2n}, p-q\rangle = 0.\end{aligned}$$

Hence, the weak limit p is unique. By definition, we obtain that

$$\lim_{n\to\infty}\langle u_{2n}-p, x\rangle = 0$$

for all $x \in \mathcal{H}$. Recalling that $\lim_{n\to\infty} \|u_{2n+1} - u_{2n}\| = 0$ in Lemma 3.5, we have for all $x \in \mathcal{H}$,

$$\begin{aligned}
|\langle u_{2n+1} - p, x\rangle| &= |\langle u_{2n+1} - p + u_{2n} - u_{2n}, x\rangle| \\
&\leq |\langle u_{2n} - p, x\rangle| + |\langle u_{2n+1} - u_{2n}, x\rangle| \\
&\leq |\langle u_{2n} - p, x\rangle| + \|u_{2n+1} - u_{2n}\| \, \|x\| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

This implies that $\{u_{2n+1}\}$ also converges weakly to p. Thus we conclude that the sequence $\{u_n\}$ converges weakly to a point $p \in \Omega$. The proof is completed. □

With that, we are now going to prove the linear convergence of Algorithm 3.1 under the condition that operator B is strongly monotone.

Theorem 3.2 (Linear convergence)**.** *Let any sequence $\{u_n\}$ be created by Algorithm 3.1. Assume that Conditions (C1), (C2)′, and (C3) hold.*

(C2)′ The operator $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and monotone, and the operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ is σ-strongly monotone.

Then $\{u_n\}$ converges to the unique solution p of the problem (1) with an R-linear rate.

Proof. It follows from $p_n = (I + \zeta_n B)^{-1} (I - \zeta_n A)s_n$ that $(I - \zeta_n A)\, s_n \in (I + \zeta_n B)\, p_n$. Thus

$$\zeta_n^{-1}(s_n - p_n - \zeta_n A s_n) \in B p_n. \tag{25}$$

Let $p \in \Omega$, i.e., $p \in (A+B)^{-1}(0)$, which implies that $-Ap \in Bp$. Combining this with (25) and the fact that B is strongly monotone with constant σ, we deduce that

$$\langle s_n - p_n - \zeta_n A s_n + \zeta_n A p, p_n - p\rangle \geq \sigma\zeta_n \|p_n - p\|^2 .$$

By using the monotonicity of A, one has

$$\begin{aligned}
&\langle s_n - p_n - \zeta_n (A s_n - A p_n), p_n - p\rangle \\
&\geq \sigma\zeta_n \|p_n - p\|^2 + \zeta_n \langle A p_n - Ap, p_n - p\rangle \\
&\geq \sigma\zeta_n \|p_n - p\|^2 .
\end{aligned} \tag{26}$$

Combining (8), (9), (12), and (26), we obtain

$$\|t_n - p\|^2 \leq \|s_n - p\|^2 - \frac{2-\alpha}{\alpha} \|t_n - s_n\|^2 - 2\sigma\alpha\delta_n\zeta_n \|p_n - p\|^2 . \tag{27}$$

This together with (16) and the definition of u_{n+1} implies that

$$\begin{aligned}
&\|u_{n+1}-p\|^2\\
&\leq (1-\chi)\|s_n-p\|^2+\chi\|t_n-p\|^2\\
&\leq (1-\chi)\|s_n-p\|^2+\chi\|s_n-p\|^2-\chi\frac{2-\alpha}{\alpha}\|t_n-s_n\|^2\\
&\quad -2\chi\sigma\alpha\delta_n\zeta_n\|p_n-p\|^2.
\end{aligned}$$

That is,

$$\|u_{n+1}-p\|^2\leq\|s_n-p\|^2-\chi\frac{2-\alpha}{\alpha}\|t_n-s_n\|^2-2\chi\sigma\alpha\delta_n\zeta_n\|p_n-p\|^2. \tag{28}$$

From the definition of δ_n, (5), and (13), we have

$$\delta_n=\frac{\langle s_n-p_n,r_n\rangle}{\|r_n\|^2}\geq\frac{\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)\|s_n-p_n\|^2}{\|r_n\|^2}\geq\frac{\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}. \tag{29}$$

Combining (7), (28), and (29), we can present that

$$\begin{aligned}
\|u_{n+1}-p\|^2\leq\|s_n-p\|^2&-\chi\frac{2-\alpha}{\alpha}\frac{\left[\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)\alpha\right]^2}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}\|s_n-p_n\|^2\\
&-2\chi\sigma\alpha\zeta_n\frac{\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}\|p_n-p\|^2.
\end{aligned} \tag{30}$$

Let $\beta:=\min\left\{\frac{\chi\alpha(2-\alpha)}{2}\frac{(1-\psi)^2}{(1+\psi)^2},\chi\sigma\alpha\zeta\frac{1-\psi}{(1+\psi)^2}\right\}$, where $\zeta:=\lim_{n\to\infty}\zeta_n$. Note that $\beta\in(0,\frac{1}{2})$. Then we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\chi\frac{2-\alpha}{\alpha}\frac{\left[\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)\alpha\right]^2}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}=\chi\alpha(2-\alpha)\frac{(1-\psi)^2}{(1+\psi)^2}\geq 2\beta,\\
&\lim_{n\to\infty}\chi\sigma\alpha\zeta_n\frac{\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2}=\chi\sigma\alpha\zeta\frac{1-\psi}{(1+\psi)^2}\geq\beta.
\end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ such that

$$\chi \frac{2-\alpha}{\alpha} \frac{\left[\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)\alpha\right]^2}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2} \geq 2\beta, \quad \chi\sigma\alpha\zeta_n \frac{\left(1-\frac{\psi\zeta_n}{\zeta_{n+1}}\right)}{\left(1+\frac{\psi\zeta_n}{\zeta_{n+1}}\right)^2} \geq \beta, \quad \forall n \geq N.$$

From (30), we have

$$\begin{aligned} \|u_{n+1}-p\|^2 &\leq \|s_n-p\|^2 - 2\beta\|p_n-s_n\|^2 - 2\beta\|p_n-p\|^2 \\ &\leq \rho\|s_n-p\|^2, \quad \forall n \geq N, \end{aligned} \tag{31}$$

where $\rho := 1-\beta \in (0,1)$. By using (31) (noting that $s_{2n} = u_{2n}$), one has

$$\|u_{2n+1}-p\|^2 \leq \rho\|u_{2n}-p\|^2, \tag{32}$$

and

$$\|u_{2n+2}-p\|^2 \leq \rho\|s_{2n+1}-p\|^2. \tag{33}$$

Note that (27) can reduced to (6). Thus by the same arguments in (15)–(19), we also have

$$\|s_{2n+1}-p\|^2 \leq \|u_{2n}-p\|^2, \quad \forall n \geq 1. \tag{34}$$

It follows from (33) and (34) that

$$\begin{aligned} \|u_{2n+2}-p\|^2 &\leq \rho\|u_{2n}-p\|^2 \leq \rho^2\|u_{2n-2}-p\|^2 \\ &\leq \cdots \leq \rho^n\|u_2-p\|^2, \quad \forall n \geq 1, \end{aligned}$$

which implies that

$$\|u_{2n}-p\|^2 \leq \frac{\|u_2-p\|^2}{\rho}\rho^n, \quad \forall n \geq 1. \tag{35}$$

Combining (32) and (35), we have

$$\begin{aligned} \|u_{2n+1}-p\|^2 &\leq \rho\|u_{2n}-p\|^2 \\ &\leq \|u_{2n}-p\|^2 \leq \frac{\|u_2-p\|^2}{\rho}\rho^n, \ \forall n \geq 1. \end{aligned} \tag{36}$$

Therefore we conclude that $\{u_n\}$ converges R-Linearly to p in virtue of (35) and (36). This finishes the proof of the theorem. □

4. Numerical experiments

In this section, we apply Algorithm 3.1 to solve the image processing problem and compare it with the methods (see Appendix A) in the literature [13, 14, 16]. All codes were executed in MATLAB 2018a on a personal computer with RAM 8.00 GB.

Example 4.1. (**Image Restoration Problem**) The image restoration problem refers to the task of improving the quality of a degraded or corrupted image. The degradation process can be caused by various factors such as noise, blur, loss of detail, and other distortions. The objective of image restoration is to recover the original image as much as possible, by removing these degradation effects. Overall, the image restoration problem is an important field of research in computer vision and image processing, with applications in a wide range of areas including medical imaging, astronomy, and surveillance.

Formulated Model The problem of image restoration can be formulated as the following model:

$$\mathbf{C}\mathbf{x} = \mathbf{b} + \mathbf{v}, \tag{37}$$

where $\mathbf{C} \in \mathbb{R}^{m\times k}$ is a convolution matrix, $\mathbf{x} \in \mathbb{R}^k$ is the original image data, $\mathbf{b} \in \mathbb{R}^m$ is the degraded image data, and $\mathbf{v} \in \mathbb{R}^m$ is the noise vector. This problem can be approached as a constraint optimization problem with the goal of minimizing the function $f(\mathbf{x}) = \|\mathbf{C}\mathbf{x} - \mathbf{b}\|^2$, subject to the constraint that $\mathbf{x} \in C$. This model can be transformed into a split feasibility problem by defining C as a box in $\mathbb{R}^k$ and Q as either $\{\mathbf{b}\}$ if no noise is added (i.e. $\mathbf{v} = \mathbf{0}$), or as a set $Q = \{\mathbf{y} \in \mathbb{R}^m \mid \|\mathbf{y} - (\mathbf{b} + \mathbf{v})\| \le \varepsilon\}$ for a small enough $\varepsilon > 0$.

Grayscale Image Degradation In this experiment, we picked four grayscale images with a size of 515×512 as our test matters. As is well known, the range of each pixel value in a grayscale image is from 0 to 1. This means that the range of C is $[0, 1]$, that is, $0 \le \mathbf{x}_{i,j} \le 1$ for each $1 \le i \le 512$ and $1 \le j \le 512$. Two stages of degradation are applied to the original image to create the degraded image: initially, a 9×9 Gaussian blur with a standard deviation of 2 is applied to the original image, and then zero-mean Gaussian white noise with a standard deviation of 10^{-4} is added.

Evaluation Indicators To evaluate the quality of the reconstructed image compared to the original image, we use the Signal-to-Noise Ratio

(SNR) in decibels and the Structural Similarity Index (SSIM). The SNR is calculated as follows:

$$\text{SNR} := 20 \log_{10} \frac{\|\mathbf{x}\|}{\|\tilde{\mathbf{x}} - \mathbf{x}\|},$$

where $\mathbf{x}$ is an original image and $\tilde{\mathbf{x}}$ is a restored image. The calculation of SSIM directly calls the function "`ssimval=ssim(x̃,x)`" in MATLAB. It is well known that higher values of SNR and SSIM indicate better reconstructions.

Algorithm Parameter Settings The parameters of the proposed Algorithm 3.1 and the algorithms presented in [13, 14, 16] are set as follows.

- Choose $\zeta_1 = 1$, $\psi = 0.8$, $\alpha = 1.5$, $\nu = 0.3$, $\chi = 0.9$, $\xi_n = 1 + \frac{1}{(n+1)^2}$, and $\tau_n = \frac{1}{n+1}$ for the proposed Algorithm 3.1.
- Set $\nu_n = 0.3$, $\alpha = 1.5$, and $\zeta_n = \frac{0.3}{L}$ (where $L = \|\mathbf{C}^*\mathbf{C}\|$) for the Algorithm 1 introduced by Gibali et al. [13] (shortly, GTV Alg. 1). Take $\zeta_1 = 1$, $\psi = 0.8$, $\nu = 0.3$, $\chi = 0.4$, and $\tau_n = \frac{1}{n+1}$ for the Algorithm 1 presented by Thong et al. [14] (shortly, TCPDL Alg. 1). Pick $\zeta_1 = 1$, $\nu = 0.1$, $\psi = 0.8$, and $\chi = 0.9$ for the Algorithm 1 suggested by Shehu et al. [16] (shortly, SLDY Alg. 1).

Restoration Results Now we can use these algorithms to solve the image restoration problem. The starting points for all algorithms are $\mathbf{u_0} = \mathbf{u_1} = \mathbf{b}$, and the iteration process stops after 200 iterations. The original test images, the degraded images, and the images recovered by our Algorithm 3.1 are displayed in Figures 1, 2, 3, and 4, respectively. The variation of SNR and SSIM values with the number of iterations for all algorithms on the four test images is illustrated in Figures 5 and 6, respectively. Finally, the SNR and SSIM values of all algorithms after 200 iterations for the four test images are stated in Table 1.

Table 1: Numerical results for all algorithms under different images

Algorithms	Cameraman		Lena		Mandril		Pirate	
	SNR	SSIM	SNR	SSIM	SNR	SSIM	SNR	SSIM
Our Alg. 3.1	29.5488	0.9570	27.5188	0.9032	22.2740	0.8506	23.7240	0.8556
SLDY Alg. 1	28.5557	0.9485	27.0257	0.8958	21.6515	0.8251	23.3439	0.8435
TCPDL Alg. 1	27.7342	0.9406	26.5654	0.8885	21.0840	0.7977	23.0044	0.8319
GTV Alg. 1	27.4197	0.9373	26.3897	0.8856	20.8604	0.7857	22.8753	0.8273

Remark 4.1. It is intuitively evident from Figures 1, 2, 3, and 4 that the Algorithm 3.1 proposed in this paper can effectively restore the original im-

Figure 1: The original Cameraman image, the degraded image, and the image recovered by our Algorithm 3.1.

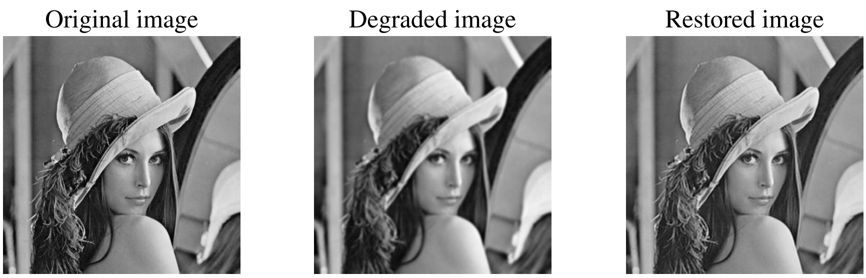


Figure 2: The original Lena image, the degraded image, and the image recovered by our Algorithm 3.1.

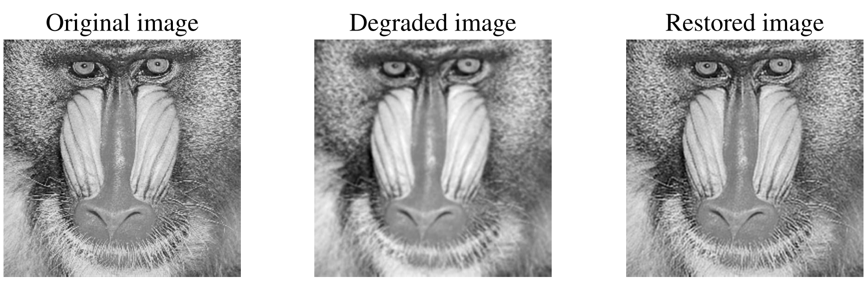
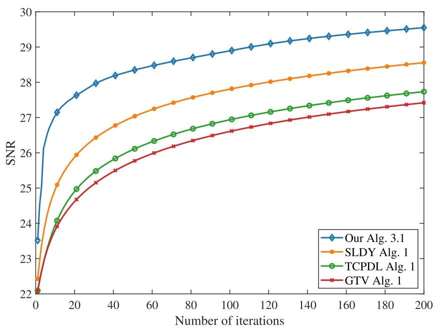


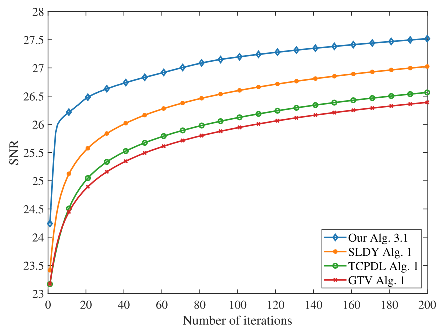
Figure 3: The original Mandril image, the degraded image, and the image recovered by our Algorithm 3.1.

age from the degraded image. According to numerical results, the suggested Algorithm 3.1 has higher SNR and SSIM values than the algorithms in references [13, 14, 16] under the same tests (cf. Figures 5 and 6, and Table 1),

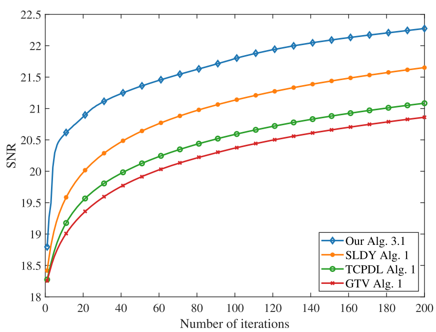
Figure 4: The original Pirate image, the degraded image, and the image recovered by our Algorithm 3.1.



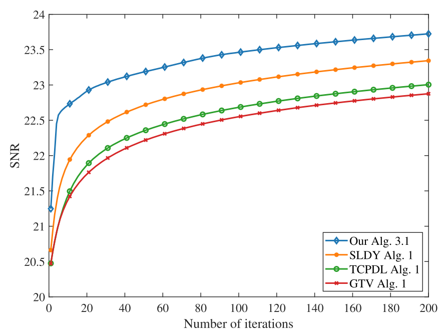
(a) Cameraman image



(b) Lena image



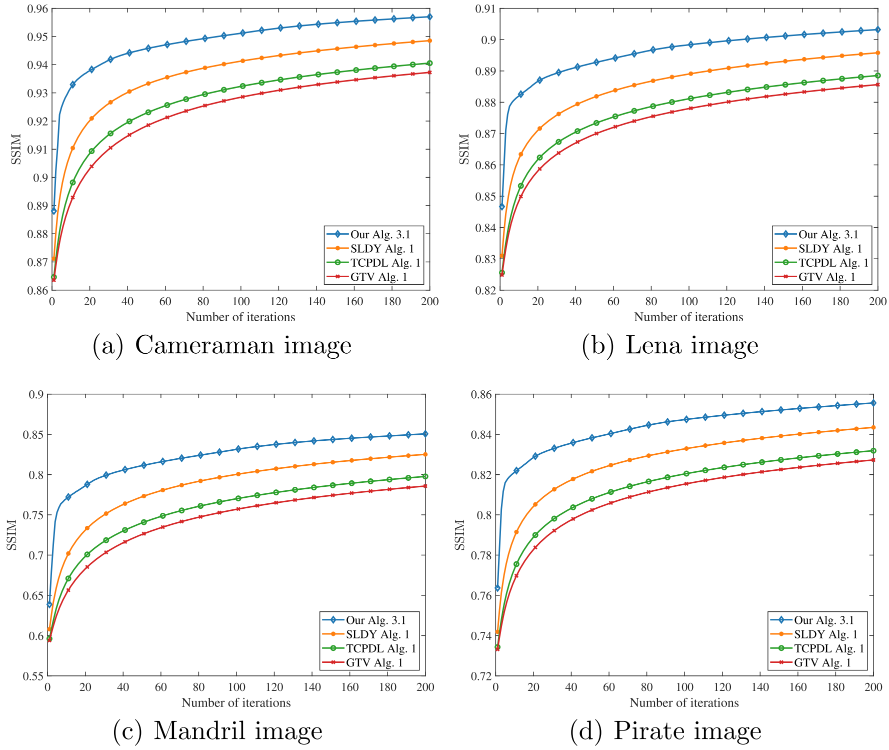
(c) Mandril image



(d) Pirate image

Figure 5: The variation of SNR for all algorithms with four images.

which means that our algorithm performs better. Therefore, the solution scheme introduced in this paper can provide a reference for solving image processing problems and develop new ideas for addressing other monotone



(a) Cameraman image (b) Lena image

(c) Mandril image (d) Pirate image

Figure 6: The variation of SSIM for all algorithms with four images.

inclusion problems.

5. Conclusions

In this paper, a novel iterative scheme was introduced based on the alternating inertial method, the forward-backward algorithm, the projection and contraction algorithm, and the relaxation method to solve the monotone inclusion problem. The weak convergence of the suggested algorithm was proved under the assumption that the involved single-valued operator is Lipschitz continuous and monotone, and the involved multi-valued operator is maximally monotone in real Hilbert spaces. Furthermore, the R-linear convergence of the proposed algorithm was established under the condition that the multi-valued operator is strongly monotone. Finally, our algorithm is applied to image restoration problems and it exhibits better performance than related results. The proposed algorithm improves and generalizes many known results in the literature.

Appendix A. Algorithms for comparison in numerical experiments

In the appendix, we state the three comparison algorithms in the literature [13, 14, 16] referred to in this paper. The symbol Ω in following algorithms denotes the solution set of problem (1).

Algorithm Gibai et al's Algorithm 1 [13] (GTV Alg. 1)

Initialization: Take $\alpha \in (1,2)$, $\{\nu_n\} \subset [0,1)$, and $\{\zeta_n\} \subset \left(0, \frac{1}{L}\right)$. Let $u_0, u_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Given the iterates u_n, u_{n-1}, perform the following steps.
Step 1. Compute $s_n = u_n + \nu_n (u_n - u_{n-1})$.
Step 2. Compute $p_n = (I + \zeta_n B)^{-1} (I - \zeta_n A) s_n$. If $p_n = s_n$ then stop and $p_n \in \Omega$.
Step 3. Compute $u_{n+1} = s_n - \alpha \delta_n r_n$, where

$$r_n := s_n - p_n - \zeta_n (As_n - Ap_n), \quad \delta_n := \frac{\langle s_n - p_n, r_n \rangle}{\|r_n\|^2}.$$

Set $n := n+1$ and go to *Step 1.*

Algorithm Thong et al.'s Algorithm 1 [14] (TCPDL Alg. 1)

Initialization: Let $\zeta_1 > 0$, $\chi \in (0, \frac{1}{2})$, $\psi \in (0,1)$, and $\nu \in [0,1]$. Let $\{\tau_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$. Select $u_0, u_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Given the iterates u_n, u_{n-1}, perform the following steps.
Step 1. Compute $s_n = u_n + \nu (u_n - u_{n-1})$.
Step 2. Compute $p_n = (I + \zeta_n B)^{-1} (I - \zeta_n A) s_n$. If $p_n = s_n$ then stop and $p_n \in \Omega$.
Step 3. Compute $u_{n+1} = (1-\chi)u_n + \chi (p_n - \zeta_n(Ap_n - As_n))$. Update ζ_{n+1} by

$$\zeta_{n+1} = \begin{cases} \min\left\{ \dfrac{\psi \|s_n - p_n\|}{\|As_n - Ap_n\|}, \zeta_n + \tau_n \right\} & \text{if } As_n - Ap_n \neq 0; \\ \zeta_n + \tau_n & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to *Step 1.*

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Algorithm Shehu et al.'s Algorithm 1 [16] (SLDY Alg. 1)

Initialization: Let $\zeta_1 > 0$, $\chi \in (0,1]$, $\psi \in (0,1)$, and $\nu \in [0, \frac{1-\psi}{1+\psi})$. Let $u_0, u_1 \in \mathcal{H}$ and set $n := 1$.

Iterative Steps: Given the iterates u_n, u_{n-1}, perform the following steps.

Step 1. Compute

$$s_n = \begin{cases} u_n, & n = \text{ even;} \\ u_n + \nu\left(u_n - u_{n-1}\right), & n = \text{ odd.} \end{cases}$$

Step 2. Compute $p_n = \left(I + \zeta_n B\right)^{-1}\left(I - \zeta_n A\right)s_n$. If $p_n = s_n$ then stop and $p_n \in \Omega$.

Step 3. Compute $u_{n+1} = (1-\chi)s_n + \chi\left(p_n - \zeta_n(Ap_n - As_n)\right)$. Update ζ_{n+1} by

$$\zeta_{n+1} = \begin{cases} \min\left\{\dfrac{\psi\left\|s_n - p_n\right\|}{\left\|As_n - Ap_n\right\|}, \zeta_n\right\} & \text{if } As_n - Ap_n \neq 0; \\ \zeta_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to *Step 1*.

Conflict of interest

The authors declare that they have no conflict of interest.

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