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Some tools to study sets of spectral synthesis for the Fourier algebra A(G) of a LCA group G and application

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This paper is dedicated to my old friend Professor Anthony To-Ming Lau on the occasion of his 80^{th} birthday

Let G be a locally compact Abelian group and $A(G)$ its Fourier Algebra. In this paper, in the setting of the Fourier algebra $A(G)$ of G, we present some tools to study the sets of synthesis and Ditkin sets and, to illustrate the use of them, we give a large number applications of the tools introduced.

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1. Introduction

Let G be a locally compact abelian (= LCA) group and $\widehat{G}$ its dual group. The Fourier algebra $A(G)$ of G is the isometric image of the group algebra $L^1(\widehat{G})$ of $\widehat{G}$ under the Fourier transform. The algebra $A(G)$ is one of the most studied Banach algebras of harmonic analysis; the reader can find ample information about this algebra, among other places, in the classical book of Rudin [Ru]. The algebra $A(G)$ is a commutative, semisimple, regular and Tauberian Banach algebra with a bounded approximate identity. The structure space (or Gelfand spectrum) $\Phi_{A(G)}$ of $A(G)$, via evaluation functionals, is G; and, the Gelfand transform of $A(G)$ is just the natural inclusion $j : A(G) \to C_0(G)$. We assume that the group G is not discrete. The nondefined terms used in this section will be defined in the next section.

In this paper we present some tools to study the set of synthesis and Ditkin sets in the setting of the Fourier algebra $A(G)$ of a LCA group G. We also present many applications of these tools to illustrate the use of them. Our aim in these applications is to get as much information as we can about sets of synthesis and Ditkin sets in order to understand better where the difficulties lie in the study of these sets. The great majority of the results presented in the paper are new and they appear for the first time here.

To explain the content of the paper we need some notation and terminology. To every nonempty closed subset E of G, the following two ideals are usually associated.

$$k(E) = \{u \in A(G) : u = 0 \text{ on } E\};$$

and,

$$j(E) = \{u \in A(G) : \text{The support of } u \text{ is compact and disjoint from } E\}.$$

The ideal $k(E)$ is the largest closed ideal of $A(G)$ with hull E; and the ideal $J(E) = \overline{j(E)}$ is the smallest closed ideal of $A(G)$ with hull E. When these two closed ideals coincide, the set E is said to be a set of synthesis. If the following stronger condition,

$$\text{for each } \; u \in k(E), \; u \in \overline{uj(E)}$$

holds, the set E is said to be a Ditkin set. We also recall that a closed subset H of G is said to be a Helson set if the restriction homomorphism $R : A(G) \to C_0(H), R(u) = u \mid H$, is surjective. For basic results about sets of synthesis and Helson sets, our reference is Rudin's book [Ru, Chapter 7 and Chapter 5].

Two long-standing open problems of the subject are the following.

Union Problem 1. This problem asks whether the union of two sets of synthesis is a set of synthesis?

S-set-D-set Problem 1. This problem asks whether every set of synthesis is a Ditkin set?

These are not the only open problems of the subject (=sets of spectral synthesis); as we shall see below, there are many other important unsolved problems in this area.

Before continuing with the introduction, we present a short history of the subject. This subject is an important chapter of Harmonic Analysis with many applications in many parts of mathematics, from the ideal theory of Banach algebras, the number theory, the spectral theory of the bounded linear operators to the differential equations, among many others. In the books [Be], [Gr-McGe] the reader can find ample information about the set of spectral synthesis, Ditkin and Helson sets; and, in the books [Be] and [Ka-Sa] some notes about the historical origin of the subject. In the books [Me] and [Be, Sections 2.3-2.5], and also in the earlier works of Y. Meyer, the reader can find some information about the connections of this subject to the number theory, in particular to Pisot numbers and Cantor set type sets.

Some authors (see e.g. [Be, Section 1.5] and [Ka-Sa, Capter IX]) take the origin of the subject to the beginning of the wave equation in 1750's. We shall not go that far; we shall start with Wiener's celebrated paper [Wi] (1932). In this paper Wiener, among many other important results, proves that, for the group algebra $A = L^1(\mathbb{R})$, the ideal

$$\{f \in L^1(\mathbb{R}) : \text{The support of } \hat{f} \text{ is compact}\}$$

is dense in the algebra $L^1(\mathbb{R})$. In today's terminology this is equivalent to saying that the empty set is a set of synthesis for the algebra $L^1(\mathbb{R})$. A few years after Wiener's paper, in 1939 Ditkin [Di] proved that any closed set $E \subseteq \mathbb{R}$ whose boundary does not contain any nonempty perfect set is a set of synthesis for the algebra $L^1(\mathbb{R})$. Such a set is in fact a Ditkin set and this result is at the origin of the "Ditkin set" notion. Until Laurent Schwartz's paper [SC] it was not known whether every closed subset of the Euclidean group $\mathbb{R}^n$ is a set of synthesis or not for the group algebra $L^1(\mathbb{R}^n)$. In 1948 Schwartz proved that, for $n \geq 3$, the unit sphere $S = \{x \in \mathbb{R}^n :\| x \|= 1\}$ of $\mathbb{R}^n$ fails to be a set of synthesis [SC]. In contrast with this result, in 1958 Herz proved that the unit circle $S = \{x \in \mathbb{R}^2 :\| x \|= 1\}$ is a set of synthesis for the algebra $L^1(\mathbb{R}^2)$ [Her1], a situation evoking the Banach-Tarski paradox. At that time an important open problem was whether Schwartz's result is special to the Euclidean group $\mathbb{R}^n$ or it is universally valid for any noncompact LCA group G. In his celebrated paper [Ma] P. Malliavin proved that every noncompact LCA group $\widehat{G}$ contains a closed set that fails to be a set of synthesis for the group algebra $L^1(G)$. In 1967, opening a new avenue to the subject, N. Varopoulos, using the projective tensor products, gave a completely different proof of Malliavin's Theorem [Va1]. Around that time another important problem was whether every Helson set is a set of uniqueness. In his land marking paper [Kö] Th. Körner proved that a Helson set in the unit circle group $\mathbb{T}$ need not be a set of uniqueness nor a set of synthesis. Around that time yet another important problem, solved in the affirmative by Drury [Dr] and Varopoulos [Va2], was whether the union of two Helson sets is a Helson set. After the 1970s most of the works in this area are about constructing/discovering sets (mostly in $\mathbb{R}^n$ or $\mathbb{T}$) that are sets of synthesis or Helson sets. See e.g. [Her2], [Do1], [Do2], [Mü1], [Mü2]. After the 1990s the works in this area are slowed down and eventually became sporadic.

In a paper like this one, which is not a review article, it was not possible to make justice to all of the researchers in this area; the author asks for the understanding of the contributors of the subject. In the books [Be], [Gr-McGe] and [Me] the readers can find many more references to the works that have appeared before 1980.

To explain the content of the paper we need some more notation and terminology. For $\varphi \in A(G)^*$, by $Sp(\varphi)$ we denote the support (or spectrum) of the functional φ. For $u \in A(G)$ and $\varphi \in A(G)^*$, by $u \cdot \varphi$ we denote the functional defined on $A(G)$ by

$$< u \cdot \varphi, v >=< \varphi, vu > .$$

We note here that

$$Sp(u \cdot \varphi) \subseteq Sp(\varphi) \cap Supp(u).$$

To every closed subset E of G, we associate the following closed subset of E.

$$\sigma(E) = \overline{\cup_{u,\varphi} Sp(u \cdot \varphi)},$$

where the union is taken over all $u \in k(E)$ and all φ in $J(E)^{\perp}$. The set $\sigma(E)$ is a perfect subset of the boundary of E and, by the very definition of it, $\sigma(E) = \emptyset$ iff E is a set of synthesis.

If the set E is not a Ditkin set then there is a $u \in k(E)$ such that $u \notin \overline{uj(E)}$. So, there is a $\varphi \in A(G)^*$ such that $< u, \varphi > \neq 0$ and $\sigma(u \cdot \varphi) \subseteq E$. Similar to the set $\sigma(E)$, to the set E we associate another perfect set, denoted as $\sigma_\natural(E)$ and defined as follows,

$$\sigma_\sharp(E) = \overline{\cup_{u,\varphi} Sp(u \cdot \varphi)},$$

where the union is taken over all $u \in k(E)$ and all φ in $A(G)^*$ such that $Sp(u \cdot \varphi) \subseteq E$. By the very definition of it, $\sigma_\sharp(E) = \emptyset$ iff the set E is a Ditkin set. It is easy to see that, for $u \in k(E)$ and $\varphi \in A(G)^*$, $Sp(u \cdot \varphi) \subseteq \partial E$ whenever $Sp(u \cdot \varphi) \subseteq E$. Hence

$$\sigma(E) \subseteq \sigma_\sharp(E) \subseteq \partial E.$$

For two closed ideals I and J of $A(G)$, IJ is the closed ideal generated by the set

$$\{uv : u \in I \text{ and } v \in J\}.$$

Here we note that, for any closed set E, $J(E)J(E) = J(E)$ since the ideal $J(E)$ is the smallest closed ideal in $A(G)$ with hull E. To the set E we also associate the following closed ideal

$$I = \{u \in A(G) : k(E)u \subseteq J(E)\}.$$

This ideal, by its very definition, is the largest closed ideal in $A(G)$ for which the equality $k(E)I = J(E)$ holds. The hull of this ideal is $\sigma(E)$ [Ka-Ül-1]. Now on this ideal will be denoted as $I(\sigma(E))$. We shall also use throughout the paper the following notion, introduced here.

Definition 1.1. A subset D of G will be called a LS-set if it is closed, nonempty and $\sigma(D) = D$.

A notable example to the LS-sets is the unit sphere of $\mathbb{R}^n$ for $n \geq 3$ [Ka-Ül-2]. The letters LS stand for Laurent Schwartz because of this example. As we shall see, the LS-sets abundantly exist and they are everywhere. Concerning the synthesis problem, the LS-sets are worst possible.

After having introduced these notions, we can now explain the content of the paper.

After a preliminary section, in Section 3 we prove the following three theorems and some consequences of them. Below by a "closed set" we always mean a closed subset of G.

Theorem A. *Let D and E be two closed sets such that $D \subseteq E$. Then the equality $k(E)J(D) = J(E)$ holds iff $\sigma(E) \subseteq D$.*

Theorem B. *Let D and E be two closed sets and $F = D \cup E$. Then $\sigma(F) \subseteq D$ iff $\sigma(E) \subseteq D$.*

Theorem C. *Let D and E be two closed sets and $F = D \cup E$. Then $\sigma_\sharp(F) \subseteq D$ iff $\sigma_\sharp(E) \subseteq D$.*

These three theorems together with the sets $\sigma(E)$, $\sigma_\sharp(E)$, Lemma 6.6 below and LS-sets are our main **tools** to study the sets of synthesis and Ditkin sets. The set $\sigma(E)$ has been introduced in the paper [Ül1]; this set and the ideal $I(\sigma(E))$ are extensively used in the papers [Ka-Ül-1] and [Ka-Ül-2] to study the sets of synthesis. The set $\sigma_\sharp(E)$, the notion of LS-set, Lemma 6.6 and the three theorems above are new and these theorems are some of the main results of the paper. These theorems, to some extent, transform the synthesis problems into problems of topology.

In Section 4 we present a few general results about the problem of how to make a given set into set of synthesis or Ditkin sets. Two of these results are the following.

1. Let D and E be two closed sets such that the set $L = D \cap \overline{E \setminus D}$ is a Ditkin set. Then

(i) The set $\overline{E \setminus D}$ is a set of synthesis iff $\sigma(E) \subseteq D$.
(ii) The set $F = D \cup E$ is a set of synthesis iff $\sigma(E) \subseteq D$ and the set D is a set of synthesis.

2. Let E be a closed set and (D_n) a sequence of Ditkin sets such that the set $F = E \cup (\cup_{n\geq 1} D_n)$ is closed. Suppose that $\sigma_\sharp(E) \subseteq \cup_{n\geq 1} D_n$. Then the set F is a Ditkin set.

In Section 5 we present a series of results related to LS-sets. Among them we cite here the following results.

3. Let S be a LS-set in G. Then, for any proper closed subset E of S, the set $D = \overline{S \setminus E}$ is also a LS-set.

This result says that, concerning synthesibility, the most of the closed subsets of S are as bad as the set S itself.

In [Re, Theorem 2] Reiter proves the following theorem.

Theorem (Reiter)**.** *Let E be a closed subset of $G = \mathbb{R}^n$ for $n \geq 3$, S the unit sphere of $\mathbb{R}^n$ and $F = E \cup S$. If the set F is a set of synthesis then $S \subseteq E$.*

The following result, which is a far reaching generalization of this theorem of Reiter, shows that in this theorem, the hypothesis that the set F is a set of synthesis is not necessary; it is enough to assume that $\overline{F \setminus \sigma(F)} = F$. This latter hypothesis is satisfied, for instance, if F is the closure of its interior or there is a set of synthesis in between $\sigma(F)$ and F.

4. Let E be a closed subset of G, S a LS-set and $F = E \cup S$. Suppose that $\overline{F \setminus \sigma(F)} = F$. Then $S \subseteq E$ and $\overline{E \setminus S} = E$

5. For any closed subset E of G, we have $\overline{E \setminus \sigma(E)} = E$ iff for any LS-set $S \subseteq E$ we have $\overline{E \setminus S} = E$.

Section 5 also contains several other results related to LS-sets and the set $\sigma(E)$.

In Section 6 we present a few results related to Helson sets. Two of these are the following. Before stating these results we need to introduce the following notion.

Let E be a closed set and x a point in its boundary ∂E. We shall say that "E is locally Helson at x" if for some neighbourhood V of x, the set $H = \overline{V} \cap \partial E$ is a Helson set. With this notion, we prove the following result.

6. Let E be a closed set, $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u\cdot\varphi) \subseteq E$. Suppose that $u\cdot\varphi \in k(\partial E)^\perp$. Then no point x of ∂E at which E is locally Helson can be in the set $D = Sp(u \cdot \varphi)$.

The second main result of Section 6 is the following one.

7. Let (H_n) be a sequence of Helson sets and $E = \cup_{n\geq 1} H_n$. Then E is a hereditarily set of synthesis (i.e. every closed subset of E is a set of synthesis) iff it is a hereditarily Ditkin iff it does not contain any LS-set.

In Section 7 we listed a series of results, remarks, and open problems with comments.

The paper is essentially self-contained; except a few results taken from the papers [Ka-Ül-2] and [Ül2], as far as we know the subject, all the other results and tools presented in this paper are new. The main ingredients of the proofs are the standard results of the subject and the Baire Theorem.

2. Preliminaries and notation

Our notation and terminology are standard. For any Banach space X, by X^* we denote its dual space. We always consider X as naturally embedded into its bidual X^{**}. For $x \in X$ and $f \in X^*$, by $< x, f >$, and also by $< f, x >$, we denote the natural duality between X and X^*. For any subspace Y of X, by $Y^\perp$ we denote the annihilator of Y in X^*. The weak-star topology of X^* is $\sigma(X^*, X)$; the weak topology of X^* is $\sigma(X^*, X^{**})$.

Fourier Algebra $A(G)$ of a LCA group G. Let G be a locally compact abelian group and $A(G)$ its Fourier algebra. This is a commutative semisimple, regular, Tauberian Banach function algebra. The term Tauberian means that the ideal $A_c(G) = \{u \in A(G) : Supp(u) \text{ is compact}\}$ is dense in $A(G)$. This is equivalent to saying that the empty set is set of synthesis. This is also equivalent to saying that every proper closed ideal of $A(G)$ is contained in a regular maximal ideal of $A(G)$. The structure space $\Phi_{A(G)}$ of the algebra A(G) is $\{\rho_x : x \in G\}$, where ρ_x is the evaluation functional at x defined by $< \rho_x, u >= u(x)$. The space $\Phi_{A(G)}$ is homeomorphic to G and we usually do not distinguish them.

The algebra $A(G)$ and its maximal ideals have bounded approximate identities. For this algebra, the closed scattered subsets (the empty set included), in particular, the finite subsets of G, are Ditkin sets. The dual space of $A(G)$ is denoted as $PM(G)$ and each element of $PM(G)$ is called a pseudomeasure on G. Our main reference about the algebra $A(G)$ is Rudin's classical book [Ru].

***A^** as an *A* module.** Let A be a commutative Banach Algebra.

For $u \in A$ and $\varphi \in A^*$, by $u \cdot \varphi$ we denote the functional on A, defined, for $v \in A$, by:

$$< u \cdot \varphi, v >=< \varphi, vu >$$

It is immediate to see that $\| u \cdot \varphi \| \leq \| u \| \cdot \| \varphi \|$ so that $u \cdot \varphi$ is in A^*.

For the action $A \times A^* \to A^*$, defined by $(u, \varphi) \mapsto u \cdot \varphi$, A^* is an A-module.

For the algebra $A = A(G)$, this is the only module action on $A(G)^*$ that will be used in this paper.

Our Banach algebra terminology is that of the book [Da, Chapter 3].

The spectrum (or support) of a functional φ in A^*. Let A be a commutative semisimple, regular and Tauberian Banach algebra. For $\varphi \in A^*$, the spectrum (or the support) $Sp(\varphi)$ of φ is defined in several equivalent ways. Below we state three of them. The set $Sp(\varphi)$ is a closed subset of Φ_A and is defined as follows. For more on this notion, see [Ka, Chapter 5] and [Re-St, Chapter 7].

$$\text{For } \gamma \in \Phi_A, \gamma \in Sp(\varphi) \text{ iff, for any } u \in A, \quad u \cdot \varphi = 0 \text{ implies that } u(\gamma) = 0. \tag{2.1}$$

$$\text{For } \gamma \in \Phi_A, \gamma \in Sp(\varphi) \text{ iff, for each neighbourhood } V \text{ of } \gamma, \text{ there is an } u \in A \text{ such that } Supp(u) \subseteq V \text{ and } < u, \varphi > \neq 0. \tag{2.2}$$

$$\text{For } \gamma \in \Phi_A, \gamma \in Sp(\varphi) \text{ iff there is a net } (u_\alpha) \text{ in } A \text{ such that } u_\alpha \cdot \varphi \to \gamma \text{ in the weak-star topology of } A^*. \tag{2.3}$$

We note here that, since the algebra A is supposed to be Tauberian, for $\varphi \in A^*, Sp(\varphi) = \varnothing$ iff $\varphi = 0$. In the special case where $A = C_0(\Omega)$ for a locally compact space Ω, for $\mu \in M(\Omega) = C_0(\Omega)^*$, by (2.2), the set $Sp(\mu)$ is just the support of the measure μ, the complement of the largest open subset of Ω on which μ vanishes.

The properties of the spectrum that we shall need are:

$$Sp(\varphi) = \varnothing \quad \text{iff} \quad \varphi = 0. \tag{2.4}$$

$$\text{For any } u \in A \text{ and any } \varphi \in A^*, \quad Sp(u \cdot \varphi) \subseteq Sp(\varphi) \cap Supp(u). \tag{2.5}$$

$$\text{For any closed subset } E \text{ of } \Phi_A, \quad Sp(\varphi) \subseteq E \text{ iff } \varphi \in J(E)^\perp. \tag{2.6}$$

$$\text{If } E \text{ is a closed subset of } \Phi_A \text{ and if } (\varphi_i)_{i \in I} \text{ is a weak-star convergent net in } A^* \text{ converging to some } \varphi \in A^*, \text{ the inclusions } Sp(\varphi_i) \subseteq E \text{ for all } i \in I \text{ imply that } Sp(\varphi) \subseteq E \text{ too.} \tag{2.7}$$

The product IJ of two closed ideals I and J. Let I and J be two closed ideals of $A(G)$. By IJ we denote the closed ideal of $A(G)$ generated by the set $\{uv : u \in I \,\text{and}\, v \in J\}$. This is the closure of the set

$$\{\sum_{i=1}^{n} u_i v_i : u_i \in I \text{ and } v_i \in J\}.$$

If $J = I$, instead of IJ we will write I^2.

We shall also need some topological notions. Below X is a Hausdorff topological space and all the sets considered are subsets of X.

Locally Finite Family of Sets. A family of sets $(E_i)_{\in I}$ is said to be "locally finite" if, for every $x \in X$, there is a neighbourhood of x that meets only finitely many of E_i. In this case

$$\overline{\cup_{i \in I} E_i} = \cup_{i \in I} \overline{E_i}$$

In particular, the union of a locally finite family of closed sets is closed [En, p. 16].

Relative Interior. Let $D \subseteq E$ be two nonempty sets. Then the interior of the set D as a subset of E is denoted as $Int_E(D)$. A point x of E is in the set $Int_E(D)$ iff there is a neighbourhood V of x such that $V \cap E \subseteq D$. If $D \subseteq E_1 \subseteq E_2$ then, obviously,

$$Int_{E_2}(D) \subseteq Int_{E_1}(D)$$

Now let $D \subseteq E$ be two closed sets. It is important to notice that

$$\overline{E \setminus D} = E \setminus Int_E(D).$$

Relative boundary. Let $D \subseteq E$ be two closed sets. The boundary of D as a subset of E is the set

$$\partial_E(D) = D \cap \overline{E \setminus D}.$$

We note that, if $D \subseteq E_1 \subseteq E_2$ then

$$\partial_{E_1}(D) \subseteq \partial_{E_2}(D).$$

Baire Theorem. We will use several times the following form of the Baire Theorem. Let Ω be a locally compact space, E and F_n closed subsets of Ω such that $E \subseteq \cup_{n\geq 1} F_n$. Suppose that, for each $n \geq 1$, $\overline{E \setminus F_n} = E$. Then $E = \emptyset$.

The other notions and notation will be introduced as needed.

3. Some tools to study sets of synthesis and Ditkin sets

In this section we present the proofs of the three theorems stated in the introduction, which are our main tools to study sets of synthesis, and consider a few consequences of them.

By a "closed set", as above, we always mean a closed subset of G, which we consider as identified with the structure space of the Banach algebra $A(G)$. By E we denote a generic nonempty closed subset of G. When we work with a sequence of Ditkin sets (D_n) we never assume that the union of them is closed.

We recall that, in the introduction, to every closed set E we have associated the following closed set

$$\sigma(E) = \overline{\cup_{u,\varphi} Sp(u \cdot \varphi)},$$

where the union is taken over all $u \in k(E)$ and $\varphi \in J(E)^{\perp}$; and the following closed ideal

$$I(\sigma(E)) = \{u \in A(G) : k(E)u \subseteq J(E)\}.$$

By the very definition of the set $\sigma(E)$, the set E is a set of synthesis iff $\sigma(E) = \emptyset$. The set $\sigma(E)$ is the smallest closed subset of E for which this equivalence holds. It is immediate to see that $\sigma(E) = \emptyset$ iff there is a Ditkin set F such that $\sigma(E) \subseteq F \subseteq E$.
Here we recall that the empty set is a Ditkin set for the algebra $A(G)$. As proved in [Ka-Ül-1], the hull of the ideal $I(\sigma(E))$ is the set $\sigma(E)$ so that

$$J(\sigma(E)) \subseteq I(\sigma(E)) \subseteq k(\sigma(E)).$$

So, since $J(E)J(E) = J(E)$ and $J(E) \subseteq k(E)$, we have

$$k(E)J(\sigma(E)) = J(E).$$

The ideal $I(\sigma(E))$ is the largest closed ideal I of $A(G)$ for which the equality

$$k(E)I = J(E)$$

holds. For instance, from this equality we see that

$$k(E)^2 = J(E) \text{ iff } k(E) \subseteq I(\sigma(E)).$$

An important result about the set $\sigma(E)$ is the following theorem.

Theorem 3.1. *Let D and E be two arbitrary closed sets. Then*

(i) $k(E)J(D) \subseteq J(E)$ *iff* $\sigma(E) \subseteq D$.
(ii) *If* $k(E)J(D) = k(E)$ *then* $D \subseteq E$ *and* $\overline{\sigma(E) \setminus D} = \sigma(E)$.
(iii) *If* $D \subseteq E$ *then* $k(E)J(D) = J(E)$ *iff* $\sigma(E) \subseteq D$.

Proof. (i) Suppose first that $k(E)J(D) \subseteq J(E)$. Then, for $u \in k(E), v \in J(D)$ and $\varphi \in J(E)^{\perp}$, $Sp(vu \cdot \varphi) = \emptyset$. So, this being true for each $v \in J(D)$, $Sp(u \cdot \varphi) \subseteq D$. This shows that $\sigma(E) \subseteq D$. Conversely, suppose that $\sigma(E) \subseteq D$. Then $J(D) \subseteq J(\sigma(E))$. Hence $k(E)J(D) \subseteq k(E)J(\sigma(E))$. Since $k(E)J(\sigma(E)) = J(E)$, we see that $k(E)J(D) \subseteq J(E)$.

(ii) Suppose that $k(E)J(D) = k(E)$. Then $k(E) \subseteq J(D)$ so that $D \subseteq E$. To prove that $\overline{\sigma(E) \setminus D} = \sigma(E)$, we first note that, for $u \in J(D)$ and any $\varphi \in A(G)^*$, since $J(D) = \overline{j(D)}$, $Sp(u \cdot \varphi) \subseteq \overline{Sp(\varphi) \setminus D}$. So, for $u \in k(E), v \in J(D)$ and $\varphi \in J(E)^{\perp}$, we have

$$Sp(uv \cdot \varphi) = Sp(v \cdot u \cdot \varphi) \subseteq \overline{\sigma(E) \setminus D}.$$

Since $k(E)J(D) = k(E)$, from the preceding line we conclude that $\overline{\sigma(E) \setminus D} = \sigma(E)$.

(iii) This assertion follows from the first assertion and the equality

$$k(E)J(\sigma(E)) = J(E). \qquad \square$$

For any Ditkin subset D of E, as one can see easily, $k(E) = k(E)J(D)$. By the preceding theorem, this implies that $\overline{\sigma(E) \setminus D} = \sigma(E)$. In particular, by Baire Theorem, $\sigma(E) = \emptyset$ if there is a sequence of Ditkin sets (D_n) such that $\sigma(E) \subseteq \cup_{n \geq 1} D_n \subseteq E$.

As a first application of this theorem we present the following result. Before this we recall that, as proved by Helson [He] (see also [Sa1]), given any two closed distinct ideals I and J with the same hull, there is a closed ideal K strictly in between I and J. In the next example, in the case where $J = J(E)$ and $I = I(E)$, we give a very simple proof of this result.

Example 3.2. Suppose that the set E is not a set of synthesis so that $J(E) \neq k(E)$ and $\sigma(E) \neq \emptyset$. Let $D \subseteq \sigma(E)$ be a closed set such that $\overline{\sigma(E) \setminus D} \neq \sigma(E)$. Set $K = k(E)J(D)$. Then, by Theorem 3.1 (i), $K \neq J(E)$;

and, by Theorem 3.1 (ii), $K \neq k(E)$. Since the hull of K is E, the ideal K is strictly in between $J(E)$ and $k(E)$. Varying D we obtain infinitely many closed ideals in between $J(E)$ and $k(E)$.

The most important results of this paper are the preceding theorem and the following two theorems, which have many applications and can be used in many different ways. We shall see some of their applications in the subsequent sections. In the theorems below D and E are arbitrary closed sets.

Remark 3.3. The proofs of the next two theorems use the following simple result.

Let D be a closed set and $\varphi \in A(G)^*$ be a functional such that, for all $v \in J(D)$, $Sp(v \cdot \varphi) \subseteq D$. Then, in this case, $Sp(\varphi) \subseteq D$.

To see that this claim is true, for a contradiction, suppose that there is an x in $Sp(\varphi)$ that is not in D. Then there is a relatively compact neighbourhood V of x such that $V \cap D = \emptyset$. Let $v \in A(G)$ be a function such that $v(x) = 1$ and $Supp(v) \subseteq V$. Then, since $v \in j(D)$, by hypothesis, $Sp(v \cdot \varphi) \subseteq D$. Since $v(x) = 1$ and $x \in Sp(\varphi)$, $x \in Sp(v \cdot \varphi)$. This contradicts the hypothesis that, for all $v \in J(D)$, $Sp(v \cdot \varphi) \subseteq D$. So the above claim is justified.

Theorem 3.4. *Let D and E be two closed sets and $F = D \cup E$. Then*

$$\sigma(F) \subseteq D \quad \textit{iff} \quad \sigma(E) \subseteq D.$$

Proof. Suppose first that $\sigma(F) \subseteq D$. Let us see that then $k(E)J(D) \subseteq J(E)$. To see this inclusion, let $u \in k(E)$, $v \in J(D)$ and $\varphi \in J(E)^{\perp}$. Then, since $k(E)J(D) \subseteq k(E)k(D)$ and $k(E)k(D) \subseteq k(E \cup D) = k(F)$, and since $J(E)^{\perp} \subseteq J(F)^{\perp}$, so $vu \in k(F)$ and $\varphi \in J(F)^{\perp}$, we see that

$$Sp(vu \cdot \varphi) \subseteq \sigma(F) \subseteq D.$$

Thus, for all $v \in J(D)$, $Sp(vu \cdot \varphi) \subseteq D$. This, by the above remark, implies that $Sp(u \cdot \varphi) \subseteq D$. This inclusion in turn implies that, for each $v \in J(D)$, $Sp(vu \cdot \varphi) = \emptyset$. So $vu \cdot \varphi = 0$. As the algebra $A(G)$ has an approximate identity, $< vu, \varphi >= 0$. This proves that φ vanishes on the ideal $k(E)J(D)$. This being true for each $\varphi \in J(E)^{\perp}$, we conclude that $k(E)J(D) \subseteq J(E)$. Hence, by Theorem 3.1 (i), $\sigma(E) \subseteq D$.

Conversely, now suppose that $\sigma(E) \subseteq D$. So, for all $u \in k(E)$ and $\varphi \in J(E)^{\perp}$, $Sp(u \cdot \varphi) \subseteq D$. Take $w \in k(F)$ and $\psi \in J(F)^{\perp}$. Since for

$v \in J(D)$,

$$Sp(v \cdot \psi) \subseteq \overline{F \setminus D}$$

and

$$\overline{F \setminus D} \subseteq \overline{E \setminus D} \subseteq E,$$

we see that $v \cdot \psi \in J(E)^{\perp}$. Hence, since $k(F) \subseteq k(E)$, we have $w \in k(E)$ so that $Sp(wv \cdot \psi) \subseteq \sigma(E)$. By hypothesis, $\sigma(E) \subseteq D$. Thus, for all $v \in J(D)$,

$$Sp(vw \cdot \psi) \subseteq \; D.$$

This, by Remark 3.3, implies that $Sp(w \cdot \psi) \subseteq D$. This inclusion in turn implies that

$$< w \cdot \psi, v >=< \psi, vw >= 0.$$

This shows that ψ vanishes on the ideal $k(F)J(D)$. This being true for each $\psi \in J(F)^{\perp}$, we conclude that $k(F)J(D) \subseteq J(F)$. Hence, by Theorem 3.1 (i), $\sigma(F) \subseteq D$. This finishes the proof. □

We note that, in this theorem, the roles of D and E are symmetric. This theorem, as we shall see in the next sections, has many applications. Here in the next example we present some simple applications of this theorem.

Examples 3.5. 1. Let $D \subseteq E$ be two closed sets. Suppose that D is a set of synthesis. Writing E as

$$E = \overline{E \setminus D} \cup D,$$

and using the hypothesis that D is a set of synthesis, so $\sigma(D) = \emptyset$, by Theorem 3.4, we get that $\sigma(E) \subseteq \overline{E \setminus D}$. This shows that, for any set of synthesis D contained in E, $\sigma(E) \cap Int_E(D) = \emptyset$. Thus, E is a set of synthesis iff $E = \cup_D Int_E(D)$, where the union is taken over all subsets D of E that are sets of synthesis. Equivalently, E is a set of synthesis iff, given any $x \in E$, there are a neighbourhood V of x and a set of synthesis $D \subseteq E$ such that $V \cap E \subseteq D$.

2. Let $D \subseteq E$ be two closed sets. Suppose that the set E is a set of synthesis. Then in this case $\sigma(D) \subseteq \partial_E(D)$. If D is open in E then $\partial_E(D) = \emptyset$, and D is a set of synthesis.

3. If D is an arbitrary closed subset of E, writing E as

$$E = [\overline{E \setminus D} \cup \sigma(E)] \cup D$$

and remembering that $\sigma(D) \subseteq D$, we see that

$$\sigma(D) \subseteq \sigma(E) \cup \partial_E(D).$$

4. Let U be an closed-open subset of G. For instance U maybe an open subgroup of G or the union of the finitely many cosets of it. Then, for any closed set E, the set $D = E \cap U$ is closed in G and open in E so that $\overline{E \setminus D} = E \setminus D$. Hence, if E is a set of synthesis, as noted above, so is D. In particular, if H is an open subgroup of G then, for any set of synthesis E, the set $D = E \cap H$ is a set of synthesis.

The next theorem is the analogue of Theorem 3.4 for Ditkin sets. Before stating this theorem, we recall the definition of the set $\sigma_\sharp(E)$.

If the set E is not a Ditkin set then there is a $u \in k(E)$ such that $u \notin \overline{uj(E)}$. Hence, there is a $\varphi \in A(G)^*$ such that $< u, \varphi > \neq 0$ and $Sp(u \cdot \varphi) \subseteq E$. Below we take the closure of the union of all such sets $Sp(u \cdot \varphi)$. Precisely,

$$\sigma_\sharp(E) = \overline{\cup_{u,\varphi} Sp(u \cdot \varphi)},$$

where the union is taken over all $u \in k(E)$ and for all $\varphi \in A(G)^*$ such that $Sp(u \cdot \varphi) \subseteq E$. Thus, for $u \in k(E)$ and $\varphi \in A(G)^*$,

$$Sp(u \cdot \varphi) \subseteq \sigma_\sharp(E) \quad \text{iff} \quad Sp(u \cdot \varphi) \subseteq E.$$

In the preceding line, if E is a Ditkin set, since then $u \in \overline{uk(E)}$, the inclusion $Sp(u \cdot \varphi) \subseteq E$ implies that $Sp(u \cdot \varphi) = \emptyset$ so that $\sigma_\sharp(E) = \emptyset$. Conversely, if $\sigma_\sharp(E) = \emptyset$ then, for $u \in k(E)$ and $\varphi \in A(G)^*$, the inclusion $Sp(u \cdot \varphi) \subseteq E$ implies that $Sp(u \cdot \varphi) = \emptyset$, so $< u, \varphi >= 0$, and the set E is a Ditkin set. Hence, E is a Ditkin set iff $\sigma_\sharp(E) = \emptyset$.

Theorem 3.6. *Let D and E be two closed sets and $F = D \cup E$. Then $\sigma_\sharp(F) \subseteq D$ iff $\sigma_\sharp(E) \subseteq D$*

Proof. Suppose first that $\sigma_\sharp(F) \subseteq D$. This means that whenever for a $w \in k(F)$ and $\varphi \in A(G)^*$ we have $Sp(w \cdot \varphi) \subseteq F$ then we also have $Sp(w \cdot \varphi) \subseteq D$.

To show that then $\sigma_\sharp(E) \subseteq D$, we start with a $u \in k(E)$ and a $\varphi \in A(G)^*$ such that $Sp(u \cdot \varphi) \subseteq E$. Then, since for $v \in J(D)$, the product vu is in $k(F)$, and since $Sp(vu \cdot \varphi) \subseteq E \subseteq F$, by hypothesis, $Sp(vu \cdot \varphi) \subseteq D$. This being true for each $v \in J(D)$, by Remark 3.3, $Sp(u \cdot \varphi) \subseteq D$. Hence $\sigma_\sharp(E) \subseteq D$.

To prove the reverse implication, suppose that $\sigma_\sharp(E) \subseteq D$. This means that, for $u \in k(E)$ and $\varphi \in A(G)^*$, whenever $Sp(u \cdot \varphi) \subseteq E$, we have

$Sp(u \cdot \varphi) \subseteq D$. Now to prove that $\sigma_\sharp(F) \subseteq D$, we start with a $w \in k(F)$ and a $\varphi \in A(G)^*$ such that $Sp(w \cdot \varphi) \subseteq F$. Then, for $v \in J(D)$,

$$Sp(vw \cdot \varphi) \subseteq \overline{F \setminus D} \subseteq E.$$

Hence, by hypothesis, $Sp(vw \cdot \varphi) \subseteq D$. This being true for each $v \in J(D)$, by Remark 3.3 again, $Sp(w \cdot \varphi) \subseteq D$. This implies that $\sigma_\sharp(E) \subseteq D$. This completes the proof. □

Next, as a first application of the preceding theorem, we give the following corollary. This result will permit us to produce new Ditkin sets from the known ones.

Corollary 3.7. *Let E be a Ditkin set and D a closed subset of E. Suppose that the relative boundary of D, $\partial_E D = D \cap \overline{E \setminus D}$, is a Ditkin set. Then the set $F = \overline{E \setminus D}$ is also a Ditkin set.*

Proof. Since $E = F \cup D$ and since E is a Ditkin set, so $\sigma_\sharp(E) = \emptyset$, by the preceding theorem, $\sigma_\sharp(F) \subseteq D$. Since $\sigma_\sharp(F) \subseteq F$ too, $\sigma_\sharp(F) \subseteq \partial_E(D) \subseteq F$. Since the set $\partial_E(D)$ is a Ditkin set, we conclude that the set F is a Ditkin set. □

In the example below by a "triangle" we mean a triangle together with its interior.

Example 3.8. Let $G = \mathbb{R}^2$ and E be the closed unit disk in $\mathbb{R}^2$. Let also D_1,..., D_n be pairwise disjoint triangles situated in the interior of E. Let $D = \cup_{k=1}^n D_k$. Then, since the union of finitely many Ditkin sets is a Ditkin set, the relative boundary of D, which is the union of the boundaries of the sets D_k's, is a Ditkin set. So the set $F = \overline{E \setminus \cup_{k=1}^n D_k}$, which is a kind of lace (= dentelle in French), is a Ditkin set.

We can repeat this example in any dimension with other geometrical figures that are known to be Ditkin sets to obtain new Ditkin sets. In $G = \mathbb{R}^n$, it is known that the polyhedral sets and the star-shaped bodies are Ditkin sets [Ru, Section 7.5].

The results stated in Examples 3.5 remain valid if we replace the term "set of synthesis" by "Ditkin set". We shall not repeat them. In the following remarks we display a couple of useful results about the sets $\sigma(E)$ and $\sigma_\sharp(E)$

Remarks 3.9. Let D and E be two closed sets.

1. Suppose that $\sigma(E) \subseteq D \subseteq E$. Then

$$\sigma(E) \subseteq \sigma_\sharp(D).$$

Taking into account the inclusion $k(E) \subseteq k(D)$, this follows from the definition of the set $\sigma(E)$ and $\sigma_\sharp(F)$. This inclusion shows once more that if there is a Ditkin set in between $\sigma(E)$ and E then E is a set of synthesis.

2. Suppose that $\sigma_\sharp(E) \subseteq D \subseteq E$. Then

$$\sigma_\sharp(E) \subseteq \sigma_\sharp(D).$$

To see that this inclusion holds, let $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u \cdot \varphi) \subseteq E$. Then, since $\sigma_\sharp(E) \subseteq D$, $Sp(u \cdot \varphi) \subseteq D$. As $k(E) \subseteq k(D)$, so $u \in k(D)$, by the definition of the set $\sigma_\sharp(D)$, we see that $Sp(u \cdot \varphi) \subseteq \sigma_\sharp(D)$. Hence $\sigma_\sharp(E) \subseteq \sigma_\sharp(D)$.

This result shows in particular that $\sigma_\sharp(E) \subseteq \partial D$ so that $\sigma_\sharp(E) \subseteq \partial D \cap \partial E$. This also shows that (take $D = \partial E$) if the boundary of the set E is a Ditkin set then so is the set E, a well-known result. Below we shall see a much better result.

Next we state a condition implying that a set E for which the given condition holds is a Ditkin set. To our knowledge, the hypothesis of the next result is weaker than the hypothesis of any known result with the same conclusion.

Theorem 3.10. *Let E be a closed set. Suppose that for some sequence of Ditkin sets $(D_n)_{n\geq 1}$ we have $\sigma_\sharp(E) \subseteq \cup_{n\geq 1} D_n \subseteq E$. Then E is a Ditkin set.*

Proof. Since E is a Ditkin set iff $\sigma_\sharp(E) = \emptyset$, we are going to prove that, under the given hypothesis, $\sigma_\sharp(E) = \emptyset$. To this end, let $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u \cdot \varphi) \subseteq E$. Hence, by the definition of the set $\sigma_\sharp(E)$, $Sp(u \cdot \varphi) \subseteq \sigma_\sharp(E)$. So

$$Sp(u \cdot \varphi) \subseteq \cup_{n\geq 1} D_n.$$

Fix an $n \geq 1$. Since $u \in k(E)$, $k(E) \subseteq k(D_n)$ and D_n is a Ditkin set, there is a sequence (v_k) in $j(D_n)$ such that $\| u - uv_k \| \to 0$, as $k \to \infty$. Since $Supp(v_k) \cap D_n = \emptyset$,

$$Sp(v_k u \cdot \varphi) \subseteq Sp(u \cdot \varphi) \setminus D_n.$$

Hence, taking the limit on k, we get that

$$Sp(u \cdot \varphi) \subseteq \overline{Sp(u \cdot \varphi) \setminus D_n}.$$

So,

$$Sp(u \cdot \varphi) = \overline{Sp(u \cdot \varphi) \setminus D_n}.$$

The set $Sp(u \cdot \varphi)$, considered as a topological space, is locally compact; and the preceding lines show that the set $O_n = Sp(u \cdot \varphi) \setminus D_n$ is dense in the space $Sp(u \cdot \varphi)$. The set O_n is open in the space $Sp(u \cdot \varphi)$ so that, by Baire Theorem, the intersection of the sets O_n is also dense in the space $Sp(u \cdot \varphi)$. As $Sp(u \cdot \varphi) \subseteq \cup_{n \geq 1} D_n$, this is possible only if $Sp(u \cdot \varphi) = \emptyset$. This, by the definition of the set $\sigma_\sharp(E)$, implies that this set is empty, so E is a Ditkin set. □

In the above proof, after the equality $Sp(u \cdot \varphi) = \overline{Sp(u \cdot \varphi) \setminus D_n}$, the conclusion was clear from Baire Theorem. As we use this kind of argument very often below, for the sake of the reader, we have included the full details of the proof.

4. First uses of the tools introduced

In this section, to illustrate the use of the tools introduced, we present some applications of the theorems proved in the preceding section.

Most of the results of this section are about the following question.

Question: If a given set E is not a set of synthesis, can we make it into a set of synthesis (Ditkin set) by adding (or subtracting) a small set to it (or from it)?

So the problem is similar to the closure problem in topology. If a set in a topological space is not closed, we can make it into a closed set by adding to the set its accumulation points. In the synthesis problem the subset of E that hinder the set E from being a set of synthesis (Ditkin set) is the set $\sigma(E)$ ($\sigma_\sharp(E)$). So, to make E into a set of synthesis (Ditkin set) either we should cut off the set $\sigma(E)$ ($\sigma_\sharp(E)$) or we should coat it by a better set. Below we present a couple of results to illustrate these possibilities. Before this, we present a result that shows that the set $\sigma(E)$ ($\sigma_\sharp(E)$) (if the set E is not a LS-set) is in general much smaller than the boundary of the set E.

Lemma 4.1. *Let $D \subseteq E$ be two closed sets. Suppose that D is a set of synthesis (Ditkin set). Set $F = \overline{E \setminus D} \cap \partial E$. Then $\sigma(E) \subseteq \sigma_\sharp(F)$ $\quad (\sigma_\sharp(E) \subseteq \sigma_\sharp(F))$.*

Proof. Since the set D is a set of synthesis (Ditkin set), as seen in Examples 3.5, $\sigma(E) \subseteq \overline{E \setminus D}$ $\quad (\sigma_\sharp(E) \subseteq \overline{E \setminus D})$. So $\sigma(E) \subseteq F$ $\quad (\sigma_\sharp(E) \subseteq F)$. Hence, as seen in Remarks 3.9, $\sigma(E) \subseteq \sigma_\sharp(F)$ $\quad (\sigma_\sharp(E) \subseteq \sigma_\sharp(F))$. □

So, in the preceding lemma, larger the set D is, smaller is the set $\sigma(E)$ ($\sigma_\sharp(E)$).

The next result tells us about what to do to make the set E into a set of synthesis (Ditkin set).

Lemma 4.2. *Let E be a closed set.*

(i) Suppose that, for some closed set D, the set $F = E \cup D$ is a set of synthesis (Ditkin set). Then $\sigma(E) \subseteq D \quad (\sigma_\sharp(E) \subseteq D)$.

(ii) Suppose that D is a Ditkin set. Then the set $F = E \cup D$ is a set of synthesis (Ditkin set) iff $\sigma(E) \subseteq D \quad (\sigma_\sharp(E) \subseteq D)$.

Proof. (i) Since the set F is a set of synthesis (Ditkin set), $\sigma(F) = \emptyset$ $(\sigma_\sharp(F) = \emptyset)$. So $\sigma(F) \subseteq D \ \ (\sigma_\sharp(F) \subseteq D)$. Hence, by Theorem 3.4 (Theorem 3.6), $\sigma(E) \subseteq D \ \ (\sigma_\sharp(E) \subseteq D)$.

(ii) To see that assertion (ii) holds, we recall once more that a closed set K is a set of synthesis (Ditkin set) iff there is a Ditkin set H such that $\sigma(K) \subseteq H \subseteq K$ $(\sigma_\sharp(K) \subseteq H \subseteq K)$. We have seen this result above (see Remarks 3.9); it also follows from Theorem 3.1 (ii) and from Theorem 3.10, respectively. Assertion (ii) follows from (i) and the result just recalled. □

In the next result we make a "surgical operation" on the set E to make it into a set of synthesis. Below D and E are two arbitrary closed sets, and $L = D \cap \overline{E \setminus D}$. The set L is the "stitching path". The suture should be good for the surgical operation to be successful. Note that L is just the relative boundary of the set $D \cap E$ in E.

Theorem 4.3. *Let D, E be two closed sets and $L = D \cap \overline{(E \setminus D)}$. Suppose that the set L is a Ditkin set. Then*

(i) The set $\overline{E \setminus D}$ is a set of synthesis iff $\sigma(E) \subseteq D$.

(ii) The set $F = E \cup D$ is a set of synthesis iff $\sigma(E) \subseteq D$ and the set D is a set of synthesis.

Proof. (i) Suppose first that the set $\overline{E \setminus D}$ is a set of synthesis so that $\sigma(\overline{E \setminus D}) = \emptyset$. Writing E as

$$E = \overline{E \setminus D} \cup (D \cap E)$$

and applying Theorem 3.4, we see that $\sigma(E) \subseteq D$.

Conversely, suppose that $\sigma(E) \subseteq D$. Then, by Theorem 3.4 again, $\sigma(\overline{E \setminus D}) \subseteq D$ so that $\sigma(\overline{E \setminus D}) \subseteq L \subseteq \overline{E \setminus D}$. Since the set L is a Ditkin set, we get that $\sigma(\overline{E \setminus D}) = \emptyset$. This proves that the set $\overline{E \setminus D}$ is a set of synthesis.

(ii) Suppose first that the set $F = E \cup D$ is a set of synthesis. Then, since $\sigma(F) = \emptyset$, $\sigma(F) \subseteq D$. So, by Theorem 3.4, $\sigma(E) \subseteq D$. Now, writing F as

$$F = \overline{E \setminus D} \cup D$$

and using again Theorem 3.4, we see that $\sigma(D) \subseteq \overline{E \setminus D}$. Since $\sigma(D) \subseteq D$ too, we see that $\sigma(D) \subseteq L \subseteq D$. Since L is a Ditkin set, we conclude that D is a set of synthesis.

Conversely, suppose that $\sigma(E) \subseteq D$ and the set D is a set of synthesis. Then, since $\sigma(D) \subseteq \overline{E \setminus D}$, by Theorem 3.4, $\sigma(F) \subseteq \overline{E \setminus D}$. Since, by hypothesis, $\sigma(E) \subseteq D$, by Theorem 3.4 again, $\sigma(F) \subseteq D$. Hence $\sigma(F) \subseteq L \subseteq F$. Since L is a Ditkin set, we conclude that the set F is a set of synthesis. □

The first part of this theorem says that if we cut off from the set E the set $\sigma(E)$ by a set D containing $\sigma(E)$ then, if the suture is good, the remaining closed set $\overline{E \setminus D}$ is a set of synthesis. The second part of the theorem says that if we transplant to the set E a set of synthesis D containing the set $\sigma(E)$, if the stitching is good, then the set $E \cup D$ becomes a set of synthesis. In this Theorem, instead of assuming that "L is a Ditkin set" if we assume that "there is a sequence of Ditkin sets (D_n) such that $L \subseteq \cup_{n \geq 1} D_n \subseteq D \cap E$" then the theorem remains valid.

For the proof of the results below we need the following lemma, which is of independent interest and an illustration of the use of the tools introduced.

Lemma 4.4. *Let $D \subseteq E$ be two closed sets.*

(i) Suppose that $\sigma(D) \cap \sigma(E) = \emptyset$. Then $k(E) \subseteq J(D)$.

(ii) Suppose that $k(E) \subseteq J(D)$. Then $\sigma(E) \subseteq \overline{E \setminus D}$ and $\sigma(D) \subseteq \overline{E \setminus D}$.

(iii) Suppose that $\sigma(E) \cap D = \emptyset$. Then $k(E) = k(E)J(D) = k(E)k(D)$.

Proof. (i) To prove the inclusion $k(E) \subseteq J(D)$, we are going to prove that each $\varphi \in J(D)^{\perp}$ vanishes on $k(E)$. So we take a u in $k(E)$ and a φ in $J(D)^{\perp}$. Since $k(E) \subseteq k(D)$, $Sp(u \cdot \varphi) \subseteq \sigma(D)$. Since $J(D)^{\perp} \subseteq J(E)^{\perp}$, $Sp(u \cdot \varphi) \subseteq \sigma(E)$ too. Since $\sigma(D) \cap \sigma(E) = \emptyset$, we see that $Sp(u \cdot \varphi) = \emptyset$. Hence $< u, \varphi >= 0$, and $k(F) \subseteq J(D)$.

(ii) We first prove the inclusion $\sigma(E) \subseteq \overline{E \setminus D}$. To prove this we note that, since $k(E) \subseteq J(D)$,

$$k(E)J(\overline{E \setminus D}) \subseteq J(D)J(\overline{E \setminus D}) = J(E)).$$

From this, by Theorem 3.1 (i), we conclude that $\sigma(E) \subseteq \overline{E \setminus D}$.

To prove the inclusion $\sigma(D) \subseteq \overline{E \setminus D}$, let $u \in k(D)$ and $\varphi \in J(D)^{\perp}$. Then, since $k(E) \subseteq J(D)$, $\varphi \in k(E)^{\perp}$. So, since, in the weak-star topology of the space $A(G)^*$,

$$k(E) = \overline{Span(E)}^{w^*},$$

for some net (φ_α) in $Span(E)$, $\varphi_\alpha \to \varphi$ in the weak-star topology of $A(G)^*$. Since φ_α is in the space $Span(E)$, we see that the functionals $u \cdot \varphi_\alpha$ are in the space $Span(E \setminus D)$. So $Sp(u \cdot \varphi) \subseteq \overline{E \setminus D}$. This proves that $\sigma(D) \subseteq \overline{E \setminus D}$. Actually, we have proved more: $u \cdot \varphi$ is in the space $k(\overline{E \setminus D})^{\perp}$.

(iii) Since $k(E)J(D) \subseteq k(E)k(D) \subseteq k(E)$, it is enough to prove that $k(E) \subseteq k(E)J(D)$. To prove this inclusion, let $\varphi \in A(G)^*$ be a functional that vanishes on $k(E)J(D)$. Then, for each $u \in k(E)$, $Sp(u \cdot \varphi) \subseteq D$. Since $J(E) = J(E)J(E)$, so $J(E) \subseteq k(E)J(D)$, $\varphi \in J(E)^{\perp}$. Hence $Sp(u \cdot \varphi) \subseteq \sigma(E)$. Thus, $Sp(u \cdot \varphi) \subseteq \sigma(E) \cap D$. So, by hypothesis, $Sp(u \cdot \varphi) = \emptyset$, so that $< u, \varphi >= 0$. This proves that any $\varphi \in A(G)^*$ that vanishes on $k(E)J(D)$ also vanishes on $k(E)$ so that $k(E) = k(E)J(D)$. □

Next we present a general result involving the union of countably many Ditkin sets. We recall that each one-point set in G is a Ditkin set so that any closed separable subset E of G can be written as in the proposition. So, one should not expect that such a set E is or should be a set of synthesis.

Proposition 4.5. *Let (D_n) be a sequence of Ditkin sets and $E = \overline{\cup_{n\geq 1} D_n}$. Then, for any closed set $D \subseteq \cup_{n\geq 1} D_n$, we have*

(i) $k(E) = k(E)J(D)$.

(ii) $\sigma(E) \subseteq \overline{E \setminus D}$.

(iii) $\overline{\sigma(E) \setminus D} = \sigma(E)$.

Proof. (i) Let $\varphi \in A(G)^*$ be a functional that vanishes on $k(E)J(D)$. Then, for each $u \in k(E)$, $u \cdot \varphi$ is in the space $J(D)^{\perp}$ so that

$$Sp(u \cdot \varphi) \subseteq D \subseteq \cup_{n\geq 1} D_n.$$

From this, as in the proof of Theorem 3.10, we deduce that $Sp(u \cdot \varphi) = \emptyset$. Since the algebra $A(G)$ has an approximate identity, from $Sp(u \cdot \varphi) = \emptyset$ we get that $< u, \varphi >= 0$. This proves that φ vanishes on $k(E)$ so that $k(E) = k(E)J(D)$.

(ii) The equality $k(E) = k(E)J(D)$ implies that $k(E) \subseteq J(D)$. This inclusion, by Lemma 4.4 above, implies that $\sigma(E) \subseteq \overline{E \setminus D}$.

(iii) This assertion follows from Theorem 3.1(ii). □

From this proposition we see once more that if $\sigma(E) \subseteq \cup_{n\geq 1} D_n \subseteq E$ then the set E is a set of synthesis. Note that, from (i), we also have $k(E) = k(E)k(D)$.

In the next result we shall see that if a closed set E is not a Ditkin set then by adding some sets to it we can make it into a Ditkin set. Note that in the next theorem the hypothesis is about the set E but the conclusion is about the set F. Note also that if $\partial E \subseteq \cup_{n\geq 1} D_n$ then the main hypothesis is satisfied.

Theorem 4.6. *Let E be a closed set and (D_n) a sequence of Ditkin sets such that the set $F = E \cup (\cup_{n\geq 1} D_n)$ is closed. Suppose that $\sigma_\sharp(E) \subseteq \cup_{n\geq 1} D_n$. Then the set F is a Ditkin set.*

Proof. We are going to show that, under the given hypothesis,

$$\sigma_\sharp(F) \subseteq \sigma_\sharp(E).$$

To prove this, we start with a $u \in k(F)$ and $\varphi \in A(G)^*$ such that $Sp(u\cdot\varphi) \subseteq F$. Since $k(F) \subseteq k(D_n)$ for each n, and D_n is a Ditkin set,

$$Sp(u \cdot \varphi) \subseteq \overline{Sp(u \cdot \varphi) \setminus D_n}.$$

So

$$Sp(u \cdot \varphi) = \overline{Sp(u \cdot \varphi) \setminus D_n}.$$

Hence, by Baire Theorem,

$$Sp(u \cdot \varphi) = \overline{Sp(u \cdot \varphi) \setminus \cup_{n\geq 1} D_n}.$$

Since $F = E \cup (\cup_{n\geq 1} D_n)$, it follows that $Sp(u \cdot \varphi) \subseteq E$. Since $k(F) \subseteq k(E)$, so $u \in k(E)$, from the inclusion $Sp(u \cdot \varphi) \subseteq E$, by the definition of the set $\sigma_\sharp(E)$, we conclude that

$$Sp(u \cdot \varphi) \subseteq \sigma_\sharp(E).$$

Hence,

$$\sigma_\sharp(F) \subseteq \sigma_\sharp(E).$$

Since, by hypothesis, $\sigma_\sharp(E) \subseteq \cup_{n\geq 1} D_n$, so $\sigma_\sharp(F) \subseteq \cup_{n\geq 1} D_n$, and since each D_n is a Ditkin set, by Theorem 3.10, F is a Ditkin set. □

5. About the sets that fail to be sets of synthesis

To understand better the sets of synthesis one should understand as much as one can why a given closed set fails to be a set of synthesis, and one should collect as much information as one can about the sets that fail to be set of synthesis. This is what we want to do in this section.

In this section the LS-sets play an important role. We recall that a set D is said to be a LS-set if it is nonempty, closed and $\sigma(D) = D$. As noted in the introduction, the unit sphere of the Euclidean group $G = \mathbb{R}^n$, for $n \geq 3$, is such a set [Ka-Ül-2]. In this section we shall prove that

(1) Almost every closed subset E of G contains a LS-set.
(2) Every substantial closed subset of a LS-set is a LS-set.

Since the closed scattered sets in G are set of synthesis, one should not expect that all the closed subsets of a LS-set are LS-sets.

(3) For a LS-set S and a closed set E, for the set $F = E \cup S$, we can have $\overline{F \setminus \sigma(F)} = F$ only if $S \subseteq E$ and $\overline{E \setminus S} = E$.

This result is a far reaching generalization of Reiter's Theorem mentioned in the introduction, which is Theorem 2 in [Re].

Before starting with the results of this section, we recall that for the equality $k(E)^2 = k(E)$ to hold the set E need not be a set of synthesis. As proved in [Ül2], for any Helson set H in G, the equality $k(H)^2 = k(H)$ holds. There are many other sets E that fail to be a set of synthesis but for them the equality $k(E)^2 = k(E)$ holds.

Proposition 5.1. *Let E be a closed sets that fails to be a set of synthesis but such that $k(E)^2 = k(E)$. Then the set $S = \sigma(E)$ is a LS-set.*

Proof. The proof uses Theorem 3.1 twice. Since $k(E)^2 = k(E)$, for any closed subset D of E, the equality $k(E) = k(E)k(D)$ also holds. In particular

$$k(E) = k(E)k(S).$$

Hence, since $S = \sigma(E)$, by Theorem 3.1,

$$k(E)J(\sigma(S)) = k(E)k(S)J(\sigma(S)) = k(E)J(S) = J(E).$$

The equality $k(E)J(\sigma(S)) = J(E)$, by Theorem 3.1, implies that $S \subseteq \sigma(S)$. Hence $\sigma(S) = S$. □

This result shows that any closed set E that contains a Helson set H failing to be a set of synthesis contains a LS-set. So, except the rare sets that are hereditarily set of synthesis (such a set is called a "set of spectral resolution" (Malliavin)), practically every nonempty closed set E in G contains a LS-set.

Another case where we have $\sigma(\sigma(E)) = \sigma(E)$ is the following.

Proposition 5.2. *Suppose that, for a closed set F in between $\sigma(E)$ and E, the equality $k(E) = k(E)k(F)$ holds. Then $\sigma(\sigma(E)) = \sigma(E)$ and $\sigma(E) \subseteq \sigma(F)$.*

Proof. Let $u \in k(E)$, $v \in k(F)$ and $\varphi \in J(E)^{\perp}$. Since $\sigma(E) \subseteq F$, $Sp(u\cdot\varphi) \subseteq F$. So $u\cdot\varphi \in J(F)^{\perp}$ and $Sp(vu\varphi) \subseteq \sigma(F)$. Since $k(E) = k(E)k(F)$, from the inclusion $Sp(vu\varphi) \subseteq \sigma(F)$ we conclude that $\sigma(E) \subseteq \sigma(F)$. Since $Sp(u\cdot\varphi) \subseteq \sigma(E)$ and $k(F) \subseteq k(\sigma(E))$, so $v \in k(\sigma(E))$, we have $\sigma(vu \cdot \varphi) \subseteq \sigma(\sigma(E))$. Whence, since $k(E) = k(E)k(F)$, we conclude that $\sigma(\sigma(E)) = \sigma(E)$. □

This proposition also shows that if F is a set of synthesis and it is in between $\sigma(E)$ and E then E is a set of synthesis iff $k(E) = k(E)k(F)$ (see Theorem 3.1). We note that if E is the union of two sets of synthesis E_i, $(i = 1, 2)$ then $\sigma(E) \subseteq E_i \subseteq E$ $(i = 1, 2)$ so that there are sets of synthesis in between $\sigma(E)$ and E.

The closed sets E that fail to be a set of synthesis can be split into two categories.

(1) The sets E for which $\overline{E \setminus \sigma(E)} \neq E$.
(2) The sets E for which $\overline{E \setminus \sigma(E)} = E$.

In general, the sets in category (1) are easier to deal with; they fail badly to be sets of synthesis. The sets in category (2) are more subtle and we know very little about them. One of our concerns in this section is to get some information about these sets. More precisely, we are looking for the answer of the **problem** that asks when does, for a closed E, the equality $\overline{E \setminus \sigma(E)} = E$ hold? We shall present below various results about this problem.

Before the following results, we recall that if $S \subseteq E$ are two closed sets then $\overline{E \setminus S} = E \setminus Int_E(S)$ so that $\overline{E \setminus S} = E$ iff $Int_E(S) = \emptyset$. So, if the set E is such that $\overline{Int(E)} = E$ then, since $\sigma(E) \subseteq \partial E$, we have $\overline{E \setminus \sigma(E)} = E$. This observation shows that for the equality $\overline{E \setminus \sigma(E)} = E$ to hold the set E need not be a set of synthesis. Actually, for many classes of sets failing to be set of synthesis, this equality hold. For instance, if there is a set of synthesis F such that $\sigma(E) \subseteq F \subseteq E$ then, one can see as an application of Theorem 3.4, that $\overline{E \setminus \sigma(E)} = E$. In fact, as the next result shows, this holds under much weaker hypothesis.

Lemma 5.3. *Let E be a closed set. Suppose that, for some sequence of sets of synthesis (F_n), we have $\sigma(E) \subseteq \cup_{\geq 1} F_n \subseteq E$. Then $\overline{E \setminus \sigma(E)} = E$.*

Proof. Let $D_n = F_n \cap \sigma(E)$ so that $\cup_{n\geq 1} D_n = \sigma(E)$. By Example 3.5 (1),

$$\sigma(E) \subseteq \overline{E \setminus F_n} \subseteq \overline{E \setminus D_n}.$$

Since $D_n \subseteq \sigma(E)$, we see that $\overline{E \setminus D_n} = E$. Hence, by Baire Theorem, $\overline{E \setminus \sigma(E)} = E$. □

This result shows in particular that if E is a union of countably many sets of synthesis then $\overline{E \setminus \sigma(E)} = E$.

The next result shows that if $\overline{E \setminus \sigma(E)} \neq E$ then $\sigma(E)$, so E, contains a LS-set.

Lemma 5.4. *Suppose that* $\overline{E \setminus \sigma(E)} \neq E$. *Then the set* $S = \overline{Int_E(\sigma(E))}$ *is a LS-set.*

Proof. We first note that, since the set $\sigma(E)$ is a closed set and $S = \overline{Int_E(\sigma(E))}$, $\overline{E \setminus S} = \overline{E \setminus \sigma(E)}$. This being noted, we write the set E as

$$E = [\overline{E \setminus S} \cup \sigma(S)] \cup S.$$

Theorem 3.4 implies that

$$\sigma(E) \subseteq \overline{E \setminus S} \cup \sigma(S).$$

This in turn, taking the equality $\overline{E \setminus S} = \overline{E \setminus \sigma(E)}$ into account, implies that

$$\sigma(S) \cup \overline{E \setminus S} = E.$$

From the last equality we deduce that $Int_E(S) \subseteq \sigma(S)$. So $\overline{Int_E(S)} \subseteq \sigma(S)$. Since $S = \overline{Int_E(\sigma(E))}$ is the closure of an open set, $\overline{Int_E(S)} = S$, and so $\sigma(S) = S$. □

The next theorem completes the preceding result.

Theorem 5.5. *Let* E *be a closed set. Then* $\overline{E \setminus \sigma(E)} = E$ *iff, for any LS-set* $S \subseteq E$, $\overline{E \setminus S} = E$.

Proof. By the preceding lemma, if $\overline{E \setminus \sigma(E)} \neq E$ then the set $S = \overline{Int_E(\sigma(E))}$ is a LS-set and $\overline{E \setminus S} \neq E$.

Conversely, suppose that $\overline{E \setminus \sigma(E)} = E$. Let $S \subseteq E$ be a LS-set. Writing E as

$$E = [\overline{E \setminus S} \cup \sigma(E)] \cup S$$

and applying Theorem 3.4, we get that

$$S \subseteq \overline{E \setminus S} \cup \sigma(E).$$

So

$$E = \overline{E \setminus S} \cup \sigma(E).$$

It follows

$$E \setminus \sigma(E) \subseteq \overline{E \setminus S}.$$

Since, by hypothesis, $\overline{E \setminus \sigma(E)} = E$, we conclude that $\overline{E \setminus S} = E$. □

From the above lemma we also deduce easily that the following result holds. So we can safely omit the proof.

Theorem 5.6. *Let E be a closed set. Then E contains a LS-set iff E contains a closed set F such that $\overline{F \setminus \sigma(F)} \neq F$.*

The next result shows that most of the subsets of a LS-set are also LS-sets.

Theorem 5.7. *Let S be a LS-set and D a closed proper subset of it. Then the set $E = \overline{S \setminus D}$ is also a LS-set.*

Proof. If $\overline{S \setminus D} = S$ then there is nothing to prove. So we suppose that $\overline{S \setminus D} \neq S$. Then we write S as

$$S = E \cup (D \cup \sigma(E)).$$

Using the hypothesis that S is a LS-set, from Theorem 3.4, we see that

$$S \subseteq D \cup \sigma(E)).$$

Hence,

$$S = D \cup \sigma(E).$$

So

$$S \setminus D \subseteq \sigma(E).$$

Hence, since the set $\sigma(E)$ is closed,

$$E \subseteq \sigma(E).$$

From this we conclude that $\sigma(E) = E$ so that E is a LS-set □

This result tells us that Laurent Schwartz could not have chosen a set better than the unit sphere of $\mathbb{R}^n$, $(n \geq 3)$ to prove his theorem.

The next result shows that the sets that are not much larger than a LS-set, concerning synthesibility, are as bad as the LS-sets.

Theorem 5.8. *Let E be a closed set and S a LS-set contained in E. Suppose that $\overline{E \setminus S} \neq E$. Then neither the boundary of E nor any closed set in between S and E is a set of synthesis.*

Proof. Let us first see that, for any closed set F containing a LS-set D, $Int_F(D) \subseteq \sigma(F)$. To see this it is enough to write F as follows

$$F = [\overline{F \setminus D} \cup \sigma(F)] \cup D$$

and apply Theorem 3.4 to get that $D \subseteq [\overline{F \setminus D} \cup \sigma(F)]$. From this inclusion, the equality $\overline{F \setminus D} \cup \sigma(F) = F$ and then the inclusion $Int_F(D) \subseteq \sigma(F)$ follow.

Now, for a contradiction, suppose that there is a set of synthesis F such that $S \subseteq F \subseteq E$. Then, by the above observation,

$$Int_E(S) \subseteq Int_F(S) \subseteq \sigma(F).$$

Since F is a set of synthesis, so $\sigma(F) = \emptyset$, and $Int_E(S) \neq \emptyset$, we have a contradiction. So there is no set of synthesis in between S and E.

Since $Int_E(S) \subseteq \sigma(E)$ and $\sigma(E) \subseteq \partial E$, if the boundary of E were a set of synthesis, for the set $D = Int_E(S)$, which is a LS-set, we would have $\overline{\partial E \setminus D} = \partial E$. But then we would have $\overline{E \setminus D} = E$. Since $\overline{E \setminus D} = \overline{E \setminus S}$, this is not the case. From this we conclude that the boundary of E is not a set of synthesis. □

Here we note that, if $\overline{E \setminus \sigma(E)} \neq E$ then, the preceding theorem together with Lemma 5.4 show once more that no closed set in between $\sigma(E)$ and E is a set of synthesis.

The next result is a far reaching generalization of Reiter's Theorem stated in the Introduction.

Theorem 5.9. *Let E be a closed set, S a LS-set and $F = E \cup S$. Suppose that $\overline{F \setminus \sigma(F)} = F$. Then $S \subseteq E$ and $\overline{E \setminus S} = E$*

Proof. The proof is very similar to the proof of Theorem 5.5 above; for the sake of the reader we repeat it. We write F as

$$F = [\overline{F \setminus S} \cup \sigma(F)] \cup S.$$

Since $\sigma(S) = S$, by Theorem 3.4,

$$F = \overline{F \setminus S} \cup \sigma(F).$$

So,

$$F \setminus \sigma(F) \subseteq \overline{F \setminus S} \subseteq \overline{E \setminus S}.$$

Since $\overline{F \setminus \sigma(F)} = F$, we conclude that

$$\overline{E \setminus S} = F.$$

It follows that $S \subseteq E$ and $\overline{E \setminus S} = E$. □

The last result of this section is a general result of independent interest; it will be used in the last section.

Theorem 5.10. *Let E be an arbitrary closed set and $F = \overline{E \setminus \sigma(E)}$. Then $\overline{F \setminus \sigma(F)} = F$.*

Proof. If $\overline{E \setminus \sigma(E)} = E$ then there is nothing to prove. So we suppose that $F \neq E$. Writing E as

$$E = \overline{E \setminus \sigma(E)} \cup \sigma(E)$$

and applying Theorem 3.4 we get the following.

$$\sigma(F) \subseteq \sigma(E).$$

Since

$$\sigma(F) \subseteq F \quad \text{and} \quad F = \overline{E \setminus \sigma(E)} = E \setminus Int_E \sigma(E),$$

we see that

$$\sigma(F) \subseteq \sigma(E) \cap (E \setminus Int_E(S)) \subseteq \sigma(E) \setminus Int_E(\sigma(E)).$$

So

$$\sigma(F) \subseteq \partial_E(\sigma(E)).$$

Hence

$$F \setminus \partial_E(\sigma(E)) = (E \setminus Int_E(\sigma(E))) \setminus \partial_E(\sigma(E)) = E \setminus \sigma(E).$$

So, since $\overline{E \setminus \sigma(E)} = F$, we have

$$\overline{F \setminus \partial_E(\sigma(E))} = F.$$

Since

$$\sigma(F) \subseteq \partial_E(\sigma(E)),$$

we see that

$$\overline{F \setminus \sigma(F)} = F.$$ □

The reader can find more results about the LS-sets and the set $\sigma(E)$ in Section 7.

6. Around Helson sets

In this section we consider two classes of sets that are closely related to Helson sets and study the synthesibility of them. These are sets of the form

(1) $E = \cup_{n\geq 1} H_n$, where each H_n is an Helson set.

And,

(2) The closed sets that are locally Helson sets.

We note that, since the closed subsets of Helson sets are Helson sets, any closed set E that is a subset of a union $\cup_{n\geq 1} H_n$ of Helson sets H_n can be written as $E = \cup_{n\geq 1} E \cap H_n$. So, E is a union of countably many Helson sets. Since there exist compact countable sets in G that are not Helson sets [Ru, p.117 and Section 6.8], a set E as in (1) need not be a Helson set.

We shall first consider the locally Helson sets. We start by recalling the definition of these sets.

Definition 6.1. Let E be a closed set and x a point in its boundary. We shall say that "E is locally Helson at x" if there is a neighbourhood V of x such the set $H = \overline{V} \cap \partial E$ is an Helson set. If E is locally Helson at every x in its boundary then the set E will be said to be a locally Helson set.

A locally Helson set need not be a Helson set. Here are some examples.

Examples 6.2. 1. Let $G = \mathbb{R}$ and, for each integer $n \geq 1$, let $H_n \subseteq [n, n+1]$ be a Helson set. If necessary, adding to H_n the points n and $n+1$, we can and do assume that these points are in H_n. Let $E = \cup_{n\geq 1} H_n$. The set E is closed since the family $(H_n)_{n\geq 1}$ is locally finite. The set E is not a Helson set. For this we recall that a Helson set in $\mathbb{R}$ does not contain an arbitrary long arithmetic progression [Ru, p. 117 and Section 6.8]. However the set E is a locally Helson set. Indeed, since every $x \in E$ has a neighbourhood that meets at most two of the sets H_n and since the union of two Helson sets is a Helson set, we can easily see that the set E is a locally Helson set.

2. Let $(H_\alpha)_{\alpha\in I}$ be a locally finite family of Helson sets in G. We suppose that the set I is infinite. Then the set $E = \cup_{\alpha\in I} H_\alpha$ is a locally Helson set but, as in the preceding example, in general is not a Helson set.

3. Let E be a closed boundary set (i.e. the interior of E is empty) in G and $H \subseteq E$ a Helson set such that $\overline{E \setminus H} \neq E$. Let x be a point in E that is not in $\overline{E \setminus H}$. Then the point x has a closed neighbourhood V such that $V \cap E \subseteq H$ so that the set $V \cap E$ is a Helson set. This shows that the set E is locally Helson at every point x in $E \setminus \overline{E \setminus H}$. So, if $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \cdots$,

are Helson sets, $E = \cup_{n\geq 1} H_n$ and $F = \cap_{n\geq 1}\overline{E \setminus H_n}$ then E is locally Helson set at each $x \notin F$. So, if E is also locally Helson at each $x \in F$ then the set E is a locally Helson set.

Let $j : A(G) \to C_0(G)$ be the natural injection. Since j is a continuous linear operator, $j^*(C_0(G))^*$ is a subspace of the space $A(G)^*$. Identifying $M(G)$ with $j^*(C_0(G))^*$ we can and do consider $M(G)$ as a subspace of $A(G)^*$.

Since, for a Helson set H, the restriction homomorphism $R : A(G) \to C_0(H)$, $R(u) = u \mid H$, is surjective and $ker(R) = k(H)$, we have $k(H)^\perp \subseteq M(G)$. In [Sa2] Saeki, extending Körner's result cited above, proved that every nondiscrete locally compact Abelian group G contains a Helson set that fails to be a set of synthesis for the algebra $A(G)$. Hence, for a closed subset E of G, the inclusion $k(E)^\perp \subseteq M(G)$ does not imply that E is a set of synthesis.

We shall need the following result proved in [Ül2].

Theorem 6.3. *Let E be a closed set, $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u \cdot \varphi) \subseteq E$. Suppose that the functional $u \cdot \varphi$ is in the subspace $M(G)$ of $A(G)^*$. Then $u \cdot \varphi = 0$.*

Let again E be a closed set, $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u \cdot \varphi) \subseteq E$. Then, as one can see easily, $Sp(u \cdot \varphi) \subseteq \partial E$. One of the main results of this section is the following lemma. This lemma gives us some information about the question where the set $Sp(u \cdot \varphi)$ lies in the boundary of E.

Lemma 6.4. *Let E be a closed set, $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u \cdot \varphi) \subseteq E$. Set $D = Sp(u \cdot \varphi)$. Suppose that the functional $u \cdot \varphi$ is in the space $k(\partial E)^\perp$. Then no point $x \in \partial E$ at which E is locally Helson can be in the set D.*

Proof. Let $x \in \partial E$ be a point at which E is locally Helson so that there is a neighbourhood V of x such that the set $H = \overline{V} \cap \partial E$ is a Helson set. Let $v \in A(G)$ be a function such that $v(x) = 1$ and $Supp(v) \subseteq V$. Since $u \cdot \varphi \in k(\partial E)^\perp$, there is a net (φ_α) in the space $Span(\partial E)$ that converges in the weak-star topology of $A(G)^*$ to $u \cdot \varphi$. Then, since

$$Sp(v \cdot \varphi_\alpha) \subseteq Sp(\varphi_\alpha) \cap Supp(v) \subseteq H,$$

and since the functionals $Sp(v \cdot \varphi_\alpha)$ are finitely supported, so $v \cdot \varphi_\alpha \in k(H)^\perp$, the functional $vu \cdot \varphi$ is the space $k(H)^\perp$. Since $k(H)^\perp \subseteq M(G)$, by the preceding theorem, $vu \cdot \varphi = 0$. Since $v(x) = 1$, x cannot be in D (see (2.1) in Section 2). □

Let $LH(E) = \{x \in \partial E : E$ is locally Helson at $x\}$. This set is open in ∂E. The preceding lemma shows that, if ∂E is a set of synthesis then $\sigma_\sharp(E) \cap LH(E) = \emptyset$ so that $\sigma_\sharp(E) \subseteq \partial E \setminus LH(E)$.

As a consequence of this lemma, we have the following result. We can safely omit the proof of this theorem; it follows easily from the preceding lemma.

Theorem 6.5. *Let E be a closed set that is locally Helson. Suppose that the boundary of E is a set of synthesis. Then E is a Ditkin set.*

From this theorem we see that any Helson set of synthesis is a hereditary Ditkin set, a known result [La-Ül].

In the rest of this section our concern will be the synthesis problem for a closed E that is union of countably many Helson sets (H_n). If each of the sets H_n is a set of synthesis then, since a Helson set of synthesis is a Ditkin set, the set E is a Ditkin set. For this reason we do not assume that the sets (H_n) are sets of synthesis. Our problem is to find out when the set E is a set of synthesis.

For $u \in A(G)$, we set $Z_u = \{x \in G : u(x) = 0\}$. This is the zero set of u. Note that

$$Supp(u) = \overline{G \setminus Z_u} = G \setminus Int(Z_u).$$

We shall use the set Z_u frequently below.

• From this point on until the end of the paper we assume that the group G is metric and $\sigma - compact$. In such a group, every open set U can be written as

$$U = \cup_{n \geq 1} K_n$$

with, for each $n \geq 1$, K_n is compact and $K_n \subseteq Int K_{n+1}$. Now let $D \neq G$ be a nonempty closed set and $U = G \setminus D$. Write U as above and, for each $n \geq 1$, choose a function $u_n \in A(G)$ such that $u_n = 1$ on K_n and $Supp(u_n) \subseteq K_{n+1}$. Set

$$w = \sum_{n=1}^{\infty} \frac{u_n}{2^n(1+ \parallel u_n \parallel)}.$$

Then $Z_w = D$ and $w \in J(D)$, see [Be, p. 67, Theorem 1.4.3].

• Thus, given any proper closed set $D \subseteq G$, there is a function $w \in J(D)$ such that $Z_w = D$.

Since $Z_w = D$, as one can see readily,

$$w \in \overline{wJ(D)}, \quad \text{and} \quad J(D) = \overline{wA(G)}.$$

Now let E be a closed set. Let $w \in J(\sigma(E))$ be such that $Z_w = \sigma(E)$. Then, since

$$J(\sigma(E)) = \overline{wA(G)},$$

we have (see Theorem 3.1)

$$\overline{k(E)w} = J(E).$$

This implies that the following lemma holds.

Lemma 6.6. *Let E be a closed set and $w \in J(\sigma(E))$ be such that $Z_w = \sigma(E)$. Then, for any $\varphi \in J(E)^\perp$, the functional $w \cdot \varphi$ is in the space $k(E)^\perp$.*

This lemma also is one of our tools to study the set of synthesis.

• This lemma in particular shows that if E is a Helson set then, for each $\varphi \in J(E)^\perp$, the functional $w \cdot \varphi$ is a measure.

To illustrate the use of this lemma, we consider the following situation.

Let E be a closed set, $u \in k(E)$ and $\varphi \in J(E)^\perp$. Set $D = Sp(u \cdot \varphi)$. Then the functional $u \cdot \varphi$ is in the space $J(D)^\perp$. Now let $w \in J(\sigma(D))$ be such that $Z_w = \sigma(D)$. Then, by the preceding lemma, the functional $wu \cdot \varphi$ is in the space $k(D)^\perp$. So, if the set D is a Helson set, by Theorem 6.3, $wu \cdot \varphi = 0$. On the other hand, using the fact that $Z_w = \sigma(D)$, as one can see easily,

$$Sp(wu \cdot \varphi) = \overline{D \setminus \sigma(D)}.$$

Hence, (under the hypothesis that D is a Helson set), since $wu \cdot \varphi = 0$,

$$\overline{D \setminus \sigma(D)} = \emptyset.$$

From this we see that $D = \sigma(D)$ so that D is a LS-set. Thus, in the case where $D = Sp(u \cdot \varphi)$ is a Helson set, the set D is either empty or it is a LS-set. We shall see below that the same conclusion holds in the case where D is the union of a sequence of Helson sets. The above observation also shows that a Helson is a set of synthesis iff it does not contain a LS-set.

For the proof of the next theorem we need the following result [Ül2].

Theorem 6.7. *Let H be a Helson set. Then, for each $u \in k(H)$, $u \in \overline{uk(H)}$.*

This theorem shows that, for any Helson set H, the ideal $k(H)$ has approximate units. The theorem below is the second main result of this section. Before this theorem, for the sake of clarity, we present the essential of the argument in a separate lemma. Below we use more than once the following result in the next remark.

Remark 6.8. Let E be a closed set, $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u\cdot\varphi) \subseteq E$. Suppose that, for some closed set $F \subseteq E$, $u\cdot\varphi \in k(F)^{\perp}$. Then, for any Helson set $H \subseteq F$, $u \cdot \varphi \in k(\overline{F \setminus H})^{\perp}$. Indeed, since $u \cdot \varphi \in k(F)^{\perp}$, there is a net (φ_α) in $Span(F)$ such that $\varphi_\alpha \to u\cdot\varphi$ in the weak-star topology of $A(G)^*$. For any $v \in k(H)$, the functional $v\cdot\varphi_\alpha$ is in the space $Span(F\setminus H)$. It follows that $vu \cdot \varphi \in k(\overline{F \setminus H})^{\perp}$. Since $u \in \overline{uk(H)}$, using this, we get that $u \cdot \varphi \in k(\overline{F \setminus H})^{\perp}$.

Lemma 6.9. *Let $E = \cup_{n\geq 1} H_n$ be a closed set that is the union of countably many Helson sets. Let $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $Sp(u \cdot \varphi) \subseteq E$. Then $u \cdot \varphi = 0$ iff the functional $u \cdot \varphi$ is in the space $k(D)^{\perp}$, where $D = Sp(u \cdot \varphi)$.*

Proof. Suppose that the functional $u \cdot \varphi$ is in the space $k(D)^{\perp}$. We write D as $D = \cup_{n\geq 1} D \cap H_n$ so that $D = \cup_{n\geq 1} D_n$, where $D_n = H_n \cap D$. So D is the union of countably many Helson sets D_n. Fix an $n \geq 1$. Since $k(E) \subseteq k(D_n)$ and $u \in k(E)$, u is in the ideal $k(D_n)$. Hence, by Theorem 6.7, $u \in \overline{uk(D_n)}$. On the other hand, since $u \cdot \varphi$ is in the space $k(D)^{\perp}$, for each $v \in k(D_n)$, the functional $vu \cdot \varphi$ is in the space $k(\overline{D \setminus D_n})^{\perp}$. From this we get that

$$Sp(vu \cdot \varphi) \subseteq \overline{D \setminus D_n}.$$

Using the fact that $u \in \overline{uk(D_n)}$, we conclude that

$$Sp(u \cdot \varphi) \subseteq \overline{D \setminus D_n}.$$

Since $D = Sp(u \cdot \varphi)$, we see that $D = \overline{D \setminus D_n}$. Hence, by Baire Theorem, $D = \emptyset$, and $u \cdot \varphi = 0$. The reverse implication is trivial. □

Now we can prove the next result.

Theorem 6.10. *Let E be a closed. Suppose that E is the union of a sequence of Helson sets H_n. Then the following assertions are equivalent.*

(i) The set E is a set of synthesis.

(ii) The set $\sigma(E)$ is a set of synthesis.

(iii) For $u \in k(E)$ and $\varphi \in J(E)^{\perp}$, the functional $u \cdot \varphi$ is in the space $k(\sigma(E))^{\perp}$.

(iv) The set $\sigma(E)$ does not contain any LS-set.

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear. To prove the implication $(iii) \Rightarrow (iv)$ it is enough to prove that (iii) implies that $\sigma(E) = \emptyset$. To prove this, take $u \in k(E)$ and $\varphi \in J(E)^{\perp}$. Then, by hypothesis, the functional $u \cdot \varphi$ is in the space $k(\sigma(E))^{\perp}$. Next we write, as above, $\sigma(E)$ as a union of countably many Helson sets:

$$\sigma(E) = \cup_{n\geq 1} D_n, \quad \text{where} \quad D_n = \sigma(E) \cap H_n.$$

Since $u \in \overline{uk(D_n)}$, as in the proof of the preceding lemma, we get that

$$Sp(u \cdot \varphi) \subseteq \overline{\sigma(E) \setminus D_n}.$$

Hence, by definition of the set $\sigma(E)$, $\sigma(E) = \overline{\sigma(E) \setminus D_n}$. This, by Baire Theorem, implies that $\sigma(E) = \emptyset$. So the set $\sigma(E)$ does not contain any LS-set.

To prove the implication $(iv) \Rightarrow (i)$, suppose that the set $\sigma(E)$ does not contain a LS-set. We want to prove that then $\sigma(E) = \emptyset$. To see this, we take a $u \in k(E)$ and a $\varphi \in J(E)^{\perp}$. Set $D = Sp(u \cdot \varphi)$. Then $u \cdot \varphi \in J(D)^{\perp}$. Now let $w \in J(\sigma(D))$ be such that $Z_w = \sigma(D)$. Then, by Lemma 6.6, the functional $wu \cdot \varphi$ is in the space $k(D)^{\perp}$. Since $\sigma(D) = Z_w$,

$$Sp(wu \cdot \varphi) = \overline{D \setminus \sigma(D)}.$$

Since the set $\sigma(E)$, so the set D, does not contain any LS-set, by Lemma 5.4, $\overline{D \setminus \sigma(D)} = D$ so that $Sp(wu \cdot \varphi) = D$. Since the functional $wu \cdot \varphi$ is in the space $k(D)^{\perp}$, we are in the hypothesis of Lemma 6.8. Hence $wu \cdot \varphi = 0$. Since

$$Sp(wu \cdot \varphi) = \overline{D \setminus \sigma(D)} = D,$$

we conclude that $D = \emptyset$. Hence $u \cdot \varphi = 0$, so $< u, \varphi >= 0$, and E is a set of synthesis. □

Exactly the same proof shows that the set E is a Ditkin set iff $\sigma_\sharp(E)$ does not contain any LS-set iff $\sigma_\sharp(E)$ is a set of synthesis (see item A/9 in the next section). We shall not repeat the proof.

The final result of this section is the following theorem.

Theorem 6.11. *Let E be a closed that is the union of a sequence of Helson sets H_n. Then the following three assertions are equivalent.*

(i) E is hereditary Ditkin.

(ii) E is hereditary set of synthesis.

(iii) E does not contain any LS-set.

Proof. The only implication that needs a proof is the implication $(iii) \Rightarrow (ii)$). To prove this implication, let F be a nonempty closed subset of E. Then we can write F as a union of countably many Helson sets. Since F does not contain any LS-set, by the note preceding this theorem, the set F is a Ditkin set so that the set E is hereditary Ditkin. □

7. Questions, results, remarks and comments

In this final sections we have collected a certain number of results, remarks, notes, and some problems that we were not able to solve; the interested readers may wish to work on them. For the sake of the readers we have added some comments to the questions posed.

In this section, as in the previous sections, E is a nonempty closed subset of G.

A. Concerning the synthesis problem, the most critical part of the set E is the set $\sigma(E)$. Any information about this set is valuable. Here are a few questions about this set.

1. In Lemma 4.2 we have seen that if D is a Ditkin set and $\sigma(E) \subseteq D$ then the set $F = E \cup D$ is a set of synthesis. Does this result hold if D is a set of synthesis? This is the most difficult and most important question stated here.

2. It is not difficult to see that

$$\overline{E \setminus \sigma(E)} = \overline{E \setminus \sigma(\sigma(E))}.$$

In spite of this, we do not know whether, for each closed set E, $\sigma(E) = \sigma(\sigma(E))$. As seen in Proposition 5.1, this is the case if $k(E)^2 = k(E)$. The other possibility is the case where $\overline{\sigma(E) \setminus \sigma(\sigma(E))} = \sigma(E)$. We do not know whether there is a a set E that fails to be a set of synthesis but for which this equality holds. In this connection, see the item 9 below.

3. For a Helson set H, the set H is a set of synthesis iff the set $\sigma(H)$ is a set of synthesis. This is also true for any set E for which $k(E)^2 = k(E)$.

4. If $k(E)^2 = k(E)$ and there is a set of synthesis in between $\sigma(E)$ and E then E is a set of synthesis. As stated in question 1 above, we do not know whether, for an arbitrary closed set E, existence of a set of synthesis in between $\sigma(E)$ and E implies that E is a set of synthesis. Note that if E is the union of two sets of synthesis E_1 and E_2 then $\sigma(E) \subseteq E_i \subseteq E$ for $i = 1, 2$.

5. Suppose that

$$\overline{E \setminus \sigma(E)} \neq E$$

and set $F = \overline{E \setminus \sigma(E)}$. In Theorem 5.10 we have seen that $\overline{F \setminus \sigma(F)} = F$. What can we say about synthesibility of the set F?

6. Let E be a closed set and $D = \cup_{F \subseteq E} Int_E(F)$, where the union is taken on all sets of synthesis F contained in E. One can prove easily that if $\overline{D} = E$ then $\overline{E \setminus \sigma(E)} = E$. Is the converse of this result true?

7. The only information about the ideal $J(E)$ that we have is its definition ($J(E) = \overline{j(E)}$). Here is a result that distinguishes the elements of $J(E)$ among those of $k(E)$. It is easy to see the following: Let $u \in k(E)$. Then $u \in J(E)$ iff there is a sequence $(u_n)_{n\geq 1}$ in $k(E)$ such that $Supp(u_n) \cap \sigma(E) = \emptyset$ and $\| u_n - u \| \to 0$.

8. This remark is about Ditkin sets. Let E be a closed set that is not a Ditkin set. Let $w \in J(\sigma_\sharp(E))$ be such that $Z_w = \sigma_\sharp(E)$. Then, as one can see easily, for $u \in k(E)$, the product uw is in the space $\overline{uJ(E)}$. So, if $u \in \overline{uJ(\sigma_\sharp(E))}$ then $u \in \overline{uJ(E)}$. From this we see that the ideal $J(\sigma_\sharp(E)$ cannot have approximate units unless the set E is a Ditkin set.

9. Continuing with the preceding remark, suppose that the set $D = \sigma_\sharp(E)$ is the union of countably many Helson sets D_n. This is the case, for instance, if the boundary of E is contained in the union of countably many Helson sets. Let $u \in k(E)$ and $\varphi \in A(G)^*$ be such that $\sigma(u \cdot \varphi) \subseteq E$. Then $u \cdot \varphi \in J(D)^\perp$. Now choose a $w \in J(\sigma(D))$ such that $Z_w = \sigma(D)$. Then, by Lemma 6.6, $wu \cdot \varphi \in k(D)^\perp$. So, by Remark 6.8, for each n, $wu \cdot \varphi \in k(\overline{D \setminus H_n})^\perp$. On the other hand, since $Z_w = \sigma(D)$,

$$Sp(u \cdot \varphi) \setminus \sigma(D) \subseteq Sp(wu \cdot \varphi).$$

Hence,

$$Sp(u \cdot \varphi) \setminus \sigma(D) \subseteq \overline{D \setminus H_n}.$$

This implies that $D \setminus \sigma(D) \subseteq \overline{D \setminus H_n}$. So

$$\overline{D \setminus \sigma(D)} \subseteq \overline{D \setminus H_n}.$$

Now suppose that $\overline{D \setminus \sigma(D)} = D$ so that, for each n, $D = \overline{D \setminus H_n}$. Hence, by Baire Theorem, $D = \emptyset$, and E is a Ditkin set. This proves that if the boundary of E is contained in the union of countably many Helson sets and $\overline{D \setminus \sigma(D)} = D$, in particular, if the D is a set of synthesis, then $D = \emptyset$. This applies in particular to any closed set E that is the union of countably many Helson sets and shows that such a set E is a Ditkin set iff the set $D = \sigma_\sharp(E)$ is a set of synthesis.

10. Let F be a closed set and O an open set such that $\sigma(F) \subseteq O$. Set $E = F \setminus O$. It is easy to see that $k(F) \subseteq J(E)$ so that $\sigma(E) \cap O = \emptyset$ (see Lemma 4.4). Is the set E a set of synthesis?

11. Let $G = \mathbb{R}^n$ and S be the unit sphere of $\mathbb{R}^n$. As mentioned in the introduction, for $n = 2$, S is a set of synthesis whereas, for $n \geq 3$, S is not a set of synthesis; a situation quite similar to Banach-Tarski Paradox. This observation led us to ask the question: Does, for ≥ 3, the nonamenability of

the discrete version of the rotation group SO_n have anything to do with the nonsythesibility of S? In this connection, see the paper [Ka-Ül-2].

12. We single out here a result proved during the proof of Theorem 5.8: For any closed set E and any LS-set S contained in E, the inclusion $\overline{Int_E(S)} \subseteq \sigma(E)$ holds so that $\overline{E \setminus \sigma(E)} \subseteq \overline{E \setminus S}$. This result shows that if $\overline{E \setminus \sigma(E)} = E$, in particular, if E is a set of synthesis, then one cannot write E as a union of countably many LS-sets. In the reverse direction, it is also true that a LS-set cannot be written as a union of countably many sets of synthesis.

13. The following two results are also about LS-sets. The first result follows from the preceding result; the second can be proved in a similar way. Let S be a LS-set and E a closed set.

(i) Suppose that $S \subseteq E$. Then $\overline{E \setminus S} = E$ iff $\sigma(E) \cap Int_E(S) = \emptyset$.
(ii) Suppose that $E \subseteq S$. Then $\overline{S \setminus E} = S$ iff $\sigma(E) \cap Int_S(E) = \emptyset$.

14. **A way of producing LS-sets**. Here we indicate a way, closely related to "sets of uniqueness problem", of producing LS-sets. Let $X \subseteq A(G)^*$ be a (not necessarily closed) nontrivial $A(G)$-module. i.e. $X \neq \{0\}$. For instance, for any $\varphi \in A(G)^* \setminus \{0\}$, the space $Y = \{u \cdot \varphi : u \in A(G)\}$ is such an $A(G)$-module. Let $\varphi \in X$, $\varphi \neq \{0\}$, and set $E = Sp(\varphi)$. Suppose that $k(E)^\perp \cap X = \{0\}$. Then $\sigma(E) = E$. Indeed, let $w \in J(\sigma(E))$ be such that $Z_w = \sigma(E)$. Then, since $\varphi \in J(E)^\perp$, by Lemma 6.6, $w \cdot \varphi \in k(E)^\perp$. Since X is an $A(G)$-module, $w \cdot \varphi \in X$ too. Since $k(E)^\perp \cap X = \{0\}$, $w \cdot \varphi = \{0\}$. This implies that $\sigma(E) = E$ so that E is a LS-set.

As a special case of the above remark, let now $X = PF(G)$ be the space of the pseudo functions on G. This is an $A(G)$-module. We recall that, for any Helson set H, $k(H)^\perp \cap PF(G) = \{0\}$, see [Ül3] for more general results. Hence, if for some $\varphi \in PF(G) \setminus \{0\}$, the set $E = Sp(\varphi)$ is an Helson set then E is a LS-set. Since not every Helson set is a set of uniqueness ([Kö], [Sa2]), there are Helson sets H such that, $J(H)^\perp \cap PF(G) \neq \{0\}$. So, there are nontrival functionals φ in $PF(G)$ for which the sets $E = Sp(\varphi)$ are LS-sets.

15. Let E be a closed. Suppose that E does not contain any LS-set. Is then E a set of synthesis? As seen above, the answer is "yes" in the case where E is the union of countably many Helson sets.

B. Let F be another closed such that $E \subseteq F$.

1. While working on synthesis problem, the following problem arises very often.

When does the inclusion $k(F) \subseteq J(E)$ hold?

It is easy to see the following.

Suppose that for a sequence of Ditkin set (D_n) we have

$$\sigma(E) \cap \sigma(F) \subseteq \cup_{n \geq 1} D_n \subseteq F.$$

Then $k(F) \subseteq J(E)$. Can we replace in this result the hypothesis that "D_n is a Ditkin set" by "D_n is a set of synthesis"?

2. When do have $k(F) = k(F)k(E)$? If E is an Helson set or Ditkin set this is the case. Is there any other case (not involving Helson sets, Ditkin sets or the union of them) where this equality hold? For a related result, see Lemma 4.4 and item 4 below.

3. If $k(F) = k(F)J(E)$ then $k(F) \subseteq J(E)$. When is the converse true?

4. We recall that a commutative Banach algebra B is said to have approximate units if for each $u \in B$, $u \in \overline{uB}$.

This being recalled, we write the set F in the preceding item, for some (not uniquely determined) closed set D, as $F = D \cup E$. Let $B = k(E)/J(F)$. If the Banach algebra B has approximate units then $k(F) = k(F)k(E)$ and so $k(F) = k(D)k(E)$. Does this result have a kind of converse?

The following also holds for any closed set E: The equality $k(E)^2 = k(E)$ holds iff the algebra $B = k(E)/k(E)^2$ has approximate units.

Another interesting result related to the above ones is this. Let D and E be two sets of synthesis and $F = D \cup E$. Suppose that the Banach algebra $B = k(E)/J(F)$ or the Banach algebra $B = k(D)/J(F)$ has approximate units. Then the set F is a set of synthesis.

5. If the ideal $k(E)$ has approximate units then, whatever the set F containing E is, the equality $k(F) = k(F)k(E)$ holds. Is the converse true?

C. The following results and questions are related to Helson sets.

1. Let H be a Helson set. Suppose that, for any LS-set $D \subseteq H$, $\overline{H \setminus D} = H$. Does this imply that H is a set of synthesis?

2. Let H be a Helson set or a boundary set that is locally Helson. Then it easy to see that $\sigma_\sharp(H) = \sigma(H)$. Is this true for a set E for which $k(E)^2 = k(E)$? This is equivalent to S-set-D-set Problem.

Let E be the union of countably many Helson sets (H_n).

3. We know that a Helson set is a Ditkin set iff it is a set of synthesis. Most probably this is also true for the set E but we were not able to prove it. This is closely related to S-set-D-set problem.

4. We do not know either whether the ideal $k(E)$ has approximate units or at least $k(E)^2 = k(E)$.

5. We do not know either whether $\overline{E \setminus \sigma(E)} = E$.

D. A General Result. Let B be a commutative semisimple, regular Banach algebra such that the empty set is a Ditkin set for B. Then one can prove that the following two assertions are equivalent.

(i) Every closed subset of Φ_B is a Ditkin set.
(ii) Every closed subset of Φ_B is a set of synthesis.

This result shows that for a Banach algebra such as B, to prove that there is a closed set in Φ_B that fails to be a set of synthesis it is enough to prove that there is a closed set in Φ_B that fails to be Ditkin set. Since, for any Ditkin subset E of Φ_B, the ideal $k(E)$ has approximate units, it is enough to prove that there is a closed set E in Φ_B such that the ideal $k(E)$ does not have approximate units. As a special case of this observation, we consider the following case.

Let $G = \mathbb{R}$ and E a Cantor set (i.e. a compact, perfect and totally disconnected set) in G. Let $A(E) = A(G)/k(E)$ be the quotient algebra. Then the algebra $A = A(E)\hat{\otimes}A(E)$ satisfies the hypotheses imposed above on the algebra B. To prove that there is a closed subset $F \subseteq \Phi_A$ that fails to be a set of synthesis, it is enough to prove that, for some closed subset $F \subseteq \Phi_A$, the ideal $k(F)$ does not have approximate units. This problem looks to be an easier problem than the problem of proving that there is a closed subset $F \subseteq \Phi_A$ that fails to be a set of synthesis (a result proved by Varopoulos).

E. Closely related to the preceding results, we consider the following problem. Let E be a closed subset of G that is a hereditarily set of synthesis. Is then E hereditarily Ditkin? Let $B = A(G)/k(E)$ then the algebra B is semisimple, regular and the empty set is a Ditkin for B. The structure space of B is E so that every closed subset of Φ_B is a set of synthesis for B, see [Ka, Theorem 5.2.7]. So, by the preceding result, every closed subset of Φ_B is a Ditkin set for the algebra B. The problem is whether this is true for the algebra $A(G)$?

Finally, we note that all the results presented in this paper in the setting of the Fourier algebra $A(G)$ of a LCA group G are also valid for any abstract Banach algebra having the relevant properties of $A(G)$. In particular, they are valid for the Eymard's Fourier algebra A(G) of an amenable (noncommutative) locally compact group G. The reader can find ample information about this algebra in the monograph [Ka-La].

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