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Planar biharmonic vector fields; potentials and traces

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For Roland with fond memories of over 45 years of collegiality and friendship

This paper describes eigenfunction approximation, and representations, of planar biharmonic vector fields with prescribed normal and tangential boundary data. These enable the characterization of the dependence of the solutions on the data including convergence in various norms. These problems that have been extensively studied using finite element algorithms; here various analytical results about the solutions are obtained. Spectral representations for the solutions of some scalar harmonic and biharmonic boundary value problems are first described and their dependence on boundary data is summarized. For biharmonic problems, the solutions are described using spaces with bases of DBS eigenfunctions in a manner similar to the use of harmonic Steklov eigenfunctions for solving harmonic boundary value problems. These methods are then used to obtain eigenfunction expansions of the scalar potential and stream function of a biharmonic field with given normal and tangential boundary traces. The potentials are C^∞ and bounded and criteria for the solutions to be in specific spaces of fields are found. Some convergence results are stated and bounds of some norms are found. With further conditions on the traces, orthogonal expansions for the vorticity and the boundary trace of the vorticity of the flow are found.

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1. Introduction

This paper provides eigenfunction expansions for the solutions of a classic biharmonic boundary value problem. These representations are used to obtain results about the dependence of the solutions on their boundary data and whether the solutions are in specific Hilbert spaces of vector fields. The paper was motivated by the large literature on the computational solution of these problems. In particular it obtains analytical results for some of the problems studied in the well-known review paper on finite element simulations by Glowinski and Pironneau [9].

The initial application was that of describing solutions of the Stokes equations for solenoidal vector fields on bounded planar domains with Lipschitz boundaries. However, related problems arise as subproblems of much larger problems arising in studies of fluid flows, magnetic fields and elasticity theory. The problem is treated as one of finding representations of biharmonic vector fields subject to the tangential and normal components of the field being given. In particular the formulae are used to describe bounds on the solutions and their dependence on the boundary conditions.

2. Definitions and notation

In this paper, the definitions of the paper Auchmuty [5] will generally be used, with some additions from [1] and [4] for boundary traces and biharmonic functions.

Cartesian coordinates $x = (x_1, x_2)$ will be used and Euclidean norms and inner products are denoted by $|.|$ and $x \cdot y$. $\partial\Omega := \overline{\Omega} \setminus \Omega$. Often the position vector x is omitted in formulae for functions and fields and equality should be interpreted as holding a.e. with respect to 2-dimensional Lebesgue measure $\mathrm{d}^2\mathrm{x} = \mathrm{dx}_1\, \mathrm{dx}_2$ on Ω.

A curve in the plane is said to be a simple Lipschitz loop if it is a closed, non-self-intersecting curve with at least two distinct points and a uniformly Lipschitz parametrization. Such loops will be compact and have finite, nonzero, length. Arc-length measure will be denoted $\mathrm{d}\sigma$ and our standard assumption is

Condition B1. *Ω is a bounded region in $\mathbb{R}^2$ with boundary $\partial\Omega$ the union of a finite number of disjoint simple Lipschitz loops $\{\Gamma_j : 0 \le j \le J\}$.*

The unit outward normal to $\partial\Omega$ is denoted ν and the unit tangent is denoted $\tau = (-\nu_2, \nu_1)$ They are defined σ *a.e.*. When $u \in W^{1,p}(\Omega)$, the boundary trace of u is a Borel function $\gamma(u) \in L^p(\partial\Omega, \mathrm{d}\sigma)$.

Our interest is in the dependence of solutions of biharmonic problems and the Stokes problem on boundary fluxes and velocities. The boundary flux of u is denoted $D_\nu u := \nabla u \cdot \nu$. As is shown in [1], boundary traces of functions may be uniquely characterized by boundary integrals with harmonic functions. We shall use $\mathcal{H}(\Omega)$, the space of H^1-harmonic functions on Ω, and $W_H^{1,\infty}(\Omega)$, the subspace of harmonic functions in $W^{1,\infty}(\Omega)$.

To define weak normal and tangential derivatives of functions on the boundary we require that the function u, its derivatives $D_j u$ and its Laplacian Δu all be in $L^1(\Omega)$. The *boundary flux* (or normal derivative) of a function u is a function $g \in L^1(\partial\Omega, \mathrm{d}\sigma)$ that satisfies

$$\int_\Omega [\nabla h \cdot \nabla u + h\,\Delta u]\ \mathrm{d}^2\mathrm{x} = \int_{\partial\Omega} \mathrm{h\,g\,d}\sigma \quad \text{for all} \quad \mathrm{h} \in \mathrm{W}^{1,\infty}_{\mathrm{H}}(\Omega).$$

When this holds, we write $D_\nu u := g$ on $\partial\Omega$. Similarly, one has $D_\tau u := g$ on $\partial\Omega$ when

$$\int_\Omega \left[h\,\Delta u + \nabla^\perp h \cdot \nabla^\perp u\right]\ \mathrm{d}^2\mathrm{x} = -\int_{\partial\Omega} \mathrm{h\,g\,d}\sigma \quad \text{for all} \quad \mathrm{h} \in \mathrm{W}^{1,\infty}_{\mathrm{H}}(\Omega),$$

and $\nabla^\perp u := (u_{,2}, -u_{,1})$ is the scalar curl operator.

When Ω is a bounded domain with Lipschitz boundary $\partial\Omega$, a non-zero function $s \in H^1(\Omega)$ is said to be a *harmonic Steklov eigenfunction* for Ω if there is a real number δ such that

$$\int_\Omega \nabla s \cdot \nabla v\, \mathrm{d}^2\mathrm{x} = \delta \int_{\partial\Omega} \mathrm{s\,v\,d}\sigma \quad \text{for all} \quad \mathrm{v} \in \mathrm{H}^1(\Omega). \tag{2.1}$$

For any domain Ω, the least eigenvalue is $\delta_0 = 0$ and the corresponding eigenfunctions are constants. All other eigenvalues are strictly positive. This eigenproblem has been studied since the 1950s by Fichera, Kuttler and Sigillito and many others. A recent analysis is in [7, chapter 3, section 3].

This analysis is based on the fact that there are orthogonal bases of spaces of harmonic functions, and trace spaces, generated by the harmonic Steklov eigenfunctions and their boundary traces. In particular the usual trace spaces $H^s(\partial\Omega)$ on functions on Lipschitz domains may be defined using the harmonic Steklov eigenfunctions under weaker requirements on the domain and with many useful properties.

In [3], and earlier, an algorithm for constructing successive harmonic Steklov eigenvalues and associated harmonic Steklov eigenfunctions is described and used to construct an orthogonal basis $\mathcal{S} := \{s_j : j \in \mathbb{N}_0\}$ of $L^2(\partial\Omega, \mathrm{d}\sigma)$ of $\mathcal{H}(\Omega)$ with respect to the ∂ inner product. When successive eigenvalues and eigenfunctions are obtained from this algorithm, an increasing, infinite sequence $\Lambda_{\mathcal{S}} := \{\delta_j : j \in \mathbb{N}_0\}$ of harmonic Steklov eigenvalues is found with $\delta_0 = 0 < \delta_1 \le \delta_2 \le \dots$ and $\delta_j \to \infty$ as $j \to \infty$. Choose associated normalized eigenfunctions so that

$$\int_{\partial\Omega} s_j\, s_k\ ds = 0 \quad \text{when } j \neq k, \quad \text{and} \quad \int_{\partial\Omega} s_j^2\ ds = 1 \quad \text{for all} \quad j, k.$$

Given $g \in L^2(\partial\Omega, \mathrm{d}\sigma)$, the *j-th Steklov coefficient* of g is

$$\hat{g}_j := \langle g, s_j \rangle_{\partial\Omega} := \int_{\partial\Omega} g\, s_j \, \mathrm{d}\sigma, \quad \text{and} \quad \int_{\partial\Omega} \mathrm{g}^2 \, \mathrm{d}\sigma \;=\; \sum_{\mathrm{j}=1}^{\infty} \hat{\mathrm{g}}_{\mathrm{j}}^2 \tag{2.2}$$

from Parseval's identity. The function $g_M(z) := \sum_{j=1}^{M} \hat{g}_j \; s_j((z)$ with $z \in \partial\Omega$ is called the *M-th Steklov approximation* of g and the sequence $\{g_M\}$ converges strongly to g from the Riesz-Fisher theorem.

More generally, for $s \geq 0$ define $H^s(\partial\Omega)$ to be the subspace of functions in $L^2(\partial\Omega, \mathrm{d}\sigma)$ with

$$\|g\|_{s,\partial\Omega}^2 := \sum_{j=1}^{\infty} (1+\delta_j)^{2s} \; \hat{g}_j^2 < \infty.$$

When Ω is the unit disk, the harmonic Steklov eigenvalues are the positive integers, and the corresponding eigenfunctions are multiples of $r^m \cos m\theta$ and $r^m \sin m\theta$. The above definitions agree with some standard definitions for Sobolev spaces on the unit circle.

The results to be described here involve functions that have L^2-Laplacians and boundary data. That is we will require $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ with boundary trace $\gamma(u)$ and flux $D_\nu u$, or tangential derivatives $D_\tau u$, in $L^2(\partial\Omega, \mathrm{d}\sigma)$. Define $H(\Delta, \Omega)$ to be the subspace of $H^1(\Omega)$ of functions with $\Delta u \in L^2(\Omega)$. It is a real Hilbert space with respect to the inner product

$$a_1(u, v) := \int_\Omega \left[\Delta u \, \Delta v + \nabla u \cdot \nabla v\right] \, \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} \mathrm{u}\,\mathrm{v}\, \mathrm{d}\sigma. \tag{2.3}$$

Let $H(\Delta, \partial\Omega)$ be the subspace that also has $D_\nu u \in L^2(\partial\Omega, \mathrm{d}\sigma)$ and the inner product

$$a_\partial(u, v) := \int_\Omega \Delta u \, \Delta v \, \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} \left[\mathrm{u}\,\mathrm{v} + \mathrm{D}_\nu \mathrm{u}\, \mathrm{D}_\nu \mathrm{v}\right] \, \mathrm{d}\sigma, \tag{2.4}$$

and $H_0(\Delta, \partial\Omega) := H(\Delta, \partial\Omega) \cap H_0^1(\Omega)$.

3. The zD biharmonic boundary value problem

In the next section, properties of solutions of some planar Stokes problems are reduced to finding potentials that satisfying Laplacian and biharmonic boundary value problems. In particular results about the zero-Dirichlet Biharmonic (zDB) boundary value problem on a bounded Lipschitz planar domain Ω are used.

When $u \in L^2(\Omega), g \in L^1(\Omega)$, we say that $\Delta u = g$ on Ω provided

$$\int_\Omega u \, \Delta\varphi \, \mathrm{d}^2\mathrm{x} = \int_\Omega \mathrm{g} \, \varphi \, \mathrm{d}^2\mathrm{x} \quad \text{for all} \quad \varphi \in \mathrm{C}^2_\mathrm{c}(\Omega).$$

$u \in L^2(\Omega)$ is harmonic on Ω if this holds with $g \equiv 0$. Let $L^2_H(\Omega)$ denote the closed subspace of L^2-harmonic functions n Ω. A function $u \in H(\Delta, \Omega)$ is said to be (weakly) *biharmonic* on Ω provided

$$\int_\Omega \Delta u \, \Delta\varphi \, \mathrm{d}^2\mathrm{x} \;=\; 0 \quad \text{for all} \quad \varphi \in \mathrm{C}^2_\mathrm{c}(\Omega). \tag{3.1}$$

Let $\mathcal{B}(\Omega)$ be the space of all biharmonic functions in $H(\Delta, \partial\Omega)$, and $\mathcal{B}_0(\Omega) := \mathcal{B}(\Omega) \cap H^1_0(\Omega)$. This is a Hilbert space with respect to the inner product

$$b_\partial(u, v) := \int_\Omega \Delta u \, \Delta v \, \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} \mathrm{D}_\nu \mathrm{u} \, \mathrm{D}_\nu \mathrm{v} \, \mathrm{d}\sigma. \tag{3.2}$$

We would like to find biharmonic functions $\tilde{u} \in \mathcal{B}_0(\Omega)$ that satisfy boundary conditions

$$u = 0 \quad \text{and} \quad D_\nu u = g \quad \text{on } \partial\Omega \quad \text{with } g \in Z \tag{3.3}$$

where $Z \subset L^2(\partial\Omega, \mathrm{d}\sigma)$ is a class of functions on $\partial\Omega$ defined using the orthonormal basis $\mathcal{S}_K$ of $L^2(\partial\Omega, \mathrm{d}\sigma)$.

To describe the solutions of the zDB boundary value problem the specific orthogonal basis of $\mathcal{B}_0(\Omega)$ of Dirichlet Biharmonic Steklov (DBS) eigenfunctions constructed by the algorithm described in section 5 of [4] is used. The algorithm generates a maximal orthogonal set with respect to the b_∂ inner product, that provide representations for function in the space.

A function $b \in \mathcal{B}_0(\Omega)$ is a DBS eigenfunction provided there is a real number β, the DBS eigenvalue, such that

$$\int_\Omega \Delta b \, \Delta v \, \mathrm{d}^2\mathrm{x} = \beta \int_{\partial\Omega} \mathrm{D}_\nu \mathrm{b} \, \mathrm{D}_\nu \mathrm{v} \, \mathrm{d}\sigma \quad \text{for all } \mathrm{v} \in \mathrm{H}_0(\Delta, \partial\Omega). \tag{3.4}$$

Let $\Lambda_B := \{\beta_j : j \in \mathbb{N}\}$ be the increasing sequence of DBS eigenvalues repeated according to multiplicity and $\mathcal{S}_B := \{b_j : j \in \mathbb{N}\}$ be an associated sequence of DBS eigenfunctions normalized so that

$$\int_{\partial\Omega} D_\nu b_j\, D_\nu b_k\, d\sigma = 0 \quad \text{when } j \neq k, \quad \text{and} \quad \int_{\partial\Omega} (D_\nu b_j)^2\, d\sigma = 1 \quad \text{for all } j,k.$$

Then $$b_\partial(b_j, b_k) = (\beta_j + 1)\,\delta_{jk} \quad \text{for } j,k \in \mathbb{N}$$

Let $\tilde{b}_j := b_j/\sqrt{\beta_j + 1}$, so that $\tilde{\mathcal{S}}_B := \{\tilde{b}_j : j \in \mathbb{N}\}$ is orthonormal in $\mathcal{B}_0(\Omega)$ with respect to the b_∂ inner product. From theorem 5.3 in [4] this is an orthonormal basis.

Define $k_j := D_\nu b_j,\ h_j := (\Delta\, b_j)/\sqrt{\beta_j}$, then the sets $\mathcal{S}_K, \mathcal{S}_H$ of these functions are orthonormal bases of $L^2(\partial\Omega, d\sigma), \mathrm{L}^2_{\mathrm{H}}(\Omega)$ respectively. See section 6 of [4].

Suppose $g \in L^2(\partial\Omega, d\sigma)$ has a DBS representation of the form

(3.5)
$$g(z) = \sum_{j=1}^{\infty} \hat{g}_j\ b_j(z) \text{ on } \partial\Omega \quad \text{with } \hat{g}_j := \langle D_\nu g, D_\nu b_j \rangle_{\partial\Omega} = \int_{\partial\Omega} D_\nu g\, D_\nu b_j\, d\sigma.$$

If only a finite number of the DBS coefficients $\hat{g}_j$ are non-zero, then the function

$$u(x) := E_B g(x) := \sum_{j=1}^{\infty} \hat{g}_j\ b_j(x) \quad \text{for } x \in \Omega \tag{3.6}$$

is a biharmonic function in $\mathcal{B}_0(\Omega)$ that provides a solution of (3.3) This is called the *zDB extension* of the boundary flux g. For general $g \in L^2(\partial\Omega, d\sigma)$, $E_B g$ need not be in $\mathcal{B}_0(\Omega)$. Define the M-th DBS approximation of g by

$$g_M(z) := \sum_{j=1}^{M} \hat{g}_j\, b_j(z) \quad \text{on } \partial\Omega.$$

Then each $u_M(x) := E_B\, g_M(x)$ has

(3.7)
$$\nabla^\perp u_M(x) = \sum_{j=1}^{M}\ \hat{g}_j\ \nabla^\perp b_j(x), \quad \Delta u_M(x) = \sum_{j=1}^{M} \sqrt{\beta_j}\ \hat{g}_j\ h_j(x) \quad \text{on } \Omega \text{ so}$$

$$\|D_\nu u_M\|^2_{2,\partial\Omega}\ = \sum_{j=1}^{M}\ \tilde{u}_j^2, \quad \text{and} \quad \int_\Omega |\Delta u_M|^2\ = \sum_{j=1}^{M}\ \beta_j \hat{g}_j^2. \tag{3.8}$$

from the orthogonality conditions. The boundary fluxes $D_\nu u_M$ converge to g in $L^2(\partial\Omega, \mathrm{d}\sigma)$. However, $E_B g \in \mathcal{B}_0(\Omega)$ if and only if the boundary flux g has a DBS representation with $\sum_{j=1}^{\infty} \beta_j \ \hat{g}_j^2 < \infty$. Since the β_j increase to ∞, there are $g \in L^2(\partial\Omega, \mathrm{d}\sigma)$ for which $\Delta(E_B g)$ is not L^2.

Note that each u_M can be written as a finite rank integral operator

(3.9)

$$u_M(x) := \int_{\partial\Omega} K_M(x,z)\, g(z)\, \mathrm{d}\sigma, \quad \text{with } \mathrm{K_M(x,z)} := \sum_{\mathrm{j=1}}^{\mathrm{M}} \mathrm{b_j(x)\, k_j(z)} \text{ on } \Omega \times \partial\Omega.$$

To quantify conditions on the flux, let $W^s(\partial\Omega)$ denote the subspace of functions in $L^2(\partial\Omega, \mathrm{d}\sigma)$ with $\sum_{j=1}^{\infty}(1+\beta_j)^s \ \hat{g}_j^2 < \infty$. For $s \geq 0$ these spaces constitute a scale of real Hilbert spaces with respect to the inner products

$$b_s(g_1, g_2) := \sum_{j=1}^{\infty} (1+\beta_j)^s \ \hat{g}_{1j} \ \hat{g}_{2j},$$

and the embedding $W^{s_1}(\partial\Omega) \subset W^{s_2}(\partial\Omega)$ is strict when $0 \leq s_1 < s_2$.

The following theorem summarizes the conditions on the boundary flux g for the solutions of the zDB boundary value problem to be in $\mathcal{B}_0(\Omega)$. It provides an explicit orthogonal representation in terms of the DBS eigenfunctions of the solution of the biharmonic boundary value problem (3.3) as well as its Laplacian ω and the boundary trace of ω, in terms of the boundary data g_τ.

Theorem 3.1. *When (B1) holds and* $g \in W^1(\partial\Omega)$, *then* $\tilde{u}(x) = \sum_{j=1}^{\infty} \hat{g}_j \ b_j(x)$ *is a solution of* (3.3) *with* $b_\partial(\tilde{u}, \tilde{u}) = b_1(g, g)$. *The function* $\omega := \Delta\tilde{u}$ *is in* $L^2(\Omega)$ *and harmonic with*

$$\omega(x) = \lim_{M\to\infty} \int_{\partial\Omega} K_M(x,z)\, \mathrm{g}_\tau(\mathrm{z})\, \mathrm{d}\sigma(\mathrm{z}) \ = \sum_{\mathrm{j=1}}^{\infty} \ \sqrt{\beta_{\mathrm{j}}}\, \hat{\mathrm{g}}_{\mathrm{j}} \ \mathrm{h_j(x)} \quad \textit{for } \mathrm{x} \in \Omega.$$

The spaces $W^1(\partial\Omega)$ *and* $\mathcal{B}_0(\Omega)$ *are isometrically isomorphic.*

Proof. Given $g \in W^1(\partial\Omega)$, the biharmonic extension of g is given by (3.5) and is a solution of (3.3). This solution is in $\mathcal{B}_0(\Omega)$ if and only if $\|\tilde{u}\|^2_{W^1(\partial\Omega)} = b_1(g,g)$ is finite. This implies that the spaces are isometric and the zDB extension E_B is continuous, 1-1 and onto as a map from $W^1(\partial\Omega)$ to $\mathcal{B}_0(\Omega)$. □

For ω to have an L^2-boundary trace, g must be in a smaller subspace of $L^2(\partial\Omega, \mathrm{d}\sigma)$.

Theorem 3.2. *When (B1) holds, then the boundary trace $\gamma(\omega) \in L^2(\partial\Omega, \mathrm{d}\sigma)$ if and only if $g \in W^2(\partial\Omega)$.*

Proof. Taking limits in (3.8) and use the fact that $\gamma(h_j) = \sqrt{\beta_j}\, k_j$ on $\partial\Omega$ to see that

$$\gamma(\omega)(z) = -\sum_{j=1}^{\infty} \beta_j\, \hat{g}_j\, k_j(z) \quad \text{so} \quad \|\gamma(\omega)\|^2_{2,\partial\Omega} = \sum_{j=1}^{\infty} \beta_j^2\, \hat{g}_j^2. \tag{3.10}$$

This is finite and the limit exists in $L^2(\partial\Omega, \mathrm{d}\sigma)$ if and only if $g \in W^2(\partial\Omega)$. □

4. Biharmonic planar vector fields

In both planar elasticity and fluid mechanics, the solution of boundary value problems are often reduced to solving various problems with zero boundary conditions together with homogenous problems (Lu = 0) on a domain with non-zero boundary conditions. The paper of Glowinski and Pirroneau was concerned with the finite element algorithms for planar biharmonic problems related to Stokes' flows. Here this problem is reformulated as the analytical problem of determining the dependence of a planar biharmonic vector field $\mathbf{v}$ on the boundary traces v_ν, v_τ.

To do this, Hilbert spaces with inner products that include boundary integrals and Sobolev traces will be used. In particular the definitions of spaces such as $H^s(\partial\Omega)$ will be the spectral definitions of [1]. All vector fields are assumed to involve L^1_{loc}, Borel measurable representatives on their domains.

Here $H(\mathrm{div}, \Omega)$ is the subspace of L^2-vector fields on Ω with $\mathrm{div}\,\mathbf{v}, |\mathbf{v}|$ in $L^2(\Omega; \mathbb{R}^2)$ **and** the normal trace $v_\nu := \mathbf{v} \cdot \nu \in \mathbf{L^2}(\boldsymbol{\partial\Omega}, \mathrm{d}\sigma)$. It is a Hilbert space with respect to the inner product.

$$b_{\mathrm{div}}(v, w) := \int_\Omega [\mathrm{div}\,\mathbf{v}\,\mathrm{div}\,\mathbf{w} + \mathbf{v} \cdot \mathbf{w}]\; \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} \mathrm{v}_\nu\, \mathrm{w}_\nu\, \mathrm{d}\sigma. \tag{4.1}$$

$H(\mathrm{curl}, \Omega)$ is the subspace where $v_\tau = \mathbf{v} \cdot \tau$ on $\partial\Omega$ **and** $b_{\mathrm{curl}}(v, v)$ is finite where

$$b_{\mathrm{curl}}(v, w) := \int_\Omega [\mathrm{curl}\,\mathbf{v}\,\mathrm{curl}\,\mathbf{w} + \mathbf{v} \cdot \mathbf{w}]\; \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} \mathrm{v}_\tau\, \mathrm{w}_\tau\, \mathrm{d}\sigma. \tag{4.2}$$

Then $H_{DC}(\Omega) := H(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega)$ is the Hilbert subspace with the inner product

$$
(4.3) \qquad \langle \mathbf{v},\, \mathbf{w} \rangle_{\mathbf{DC}} := \int_\Omega [\mathrm{div}\, \mathbf{v}\, \mathrm{div}\, \mathbf{w} + \mathrm{curl}\, \mathbf{v}\, \mathrm{curl}\, \mathbf{w} + \mathbf{v} \cdot \mathbf{w}]\ \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} [\mathrm{v}_\nu \mathrm{w}_\nu + \mathrm{v}_\tau \mathrm{w}_\tau]\ \mathrm{d}\sigma
$$

A vector field $\mathbf{v} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ is said to be *solenoidal* provided a weak form of the equation $\mathrm{div}\, \mathbf{v} = \mathbf{0}$ on $\mathbf{\Omega}$ holds. That is

$$
(4.4) \qquad \int_\Omega \mathbf{v} \cdot \nabla\varphi\, \mathrm{d}^2\mathrm{x} = 0 \quad \text{for all} \quad \varphi \in \mathbf{C}^{\mathbf{1}}_{\mathbf{c}}(\mathbf{\Omega}).
$$

$V(\Omega)$ will denote the subspace of solenoidal vector fields in $H_{DC}(\Omega)$ and $V_0(\Omega)$ is the space of *no flux* fields with $v_\nu \equiv 0$ on $\partial\Omega$.

A vector field $\mathbf{v} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ is said to be *irrotational* provided

$$
(4.5) \qquad \int_\Omega \mathbf{v} \cdot \nabla^\perp\varphi\, \mathrm{d}^2\mathrm{x} = 0 \quad \text{for all} \quad \varphi \in \mathrm{C}^1_{\mathrm{c}}(\Omega).
$$

This is a weak form of the equation $\mathrm{curl}\, \mathbf{v} = \mathbf{0}$ on $\mathbf{\Omega}$.

A vector field is said to be a *harmonic* on Ω if it is both solenoidal and irrotational on Ω. The subspace of $H_{DC}(\Omega)$ of harmonic vector fields is denoted $\mathcal{H}(\Omega, \mathbb{R}^2)$.

A vector field $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \mathbf{\Omega})$ is said to be *biharmonic* on Ω when it is solenoidal, and $\mathrm{curl}^{\,2}\, \mathbf{v} \in \mathbf{L^2}(\mathbf{\Omega}, \mathbb{R}^\mathbf{2})$ is irrotational.

The time independent Stokes system on Ω is the system of 3 scalar equations for 3 unknown functions v_1, v_2, p on Ω

$$
(4.6) \qquad \mu\, \mathrm{curl}^{\,2}\, \mathbf{v} = \mathbf{f} - \nabla\mathrm{p}, \quad \text{and} \quad \mathrm{div}\, \mathbf{v} = \mathbf{0} \quad \text{on } \mathbf{\Omega}.
$$

Here $\mathbf{f}$ is given data, $\mu > 0$ and the usual problem is to find solutions with given boundary traces $\mathrm{g}_\nu, \mathrm{g}_\tau$ where

$$
(4.7) \qquad \mathbf{v} \cdot \nu = \mathrm{g}_\nu, \qquad \mathbf{v} \cdot \tau = \mathrm{g}_\tau \quad \text{on } \partial\Omega.
$$

Solutions of the Stokes system are biharmonic fields whenever the source term $\mathbf{f} \in \mathbf{L^2}(\mathbf{\Omega}, \mathbb{R}^\mathbf{2})$ is irrotational and $p \in H^1(\Omega)$.

When $\mathrm{g}_\nu \equiv \mathrm{g}_\tau \equiv 0$ on $\partial\Omega$ the analysis of this system has been studied in many papers and texts. This paper concerns the dependence of the solutions on the boundary data. Our simplest requirement is

Condition B2. $\mathbf{v} \in \mathrm{V}(\Omega)$, $\mathrm{g}_\nu, \mathrm{g}_\tau \in \mathrm{L}^2(\partial\Omega, \mathrm{d}\sigma)$ *and* $\int_{\partial\Omega} \mathrm{g}_\nu \, \mathrm{d}\sigma = 0$.

In particular we find Steklov eigenfunction representations of a scalar potential and stream function of these biharmonic fields. These potentials are smooth functions but may have singular gradients near the boundary so further conditions may be needed for specific integrability properties of the fields on Ω. Thus conditions on the boundary data for the fields to have $|\mathbf{v}| \in \mathbf{L^2}(\boldsymbol{\Omega}; \mathbb{R}^2)$ or L^2-vorticity will first be described.

5. Harmonic component of a solenoidal field

The problem of finding the harmonic (or gradient) component of a solenoidal vector field has been well-studied, but the author is not aware of previous descriptions of the expressions for the solution in terms of the harmonic Steklov eigenfunctions of the domain.

Let $H^1_m(\Omega)$ be the subspace of $H^1(\Omega)$ of functions that have mean value 0. That is $\int_\Omega \varphi \, \mathrm{d}^2\mathrm{x} = 0$. Let $\tilde{\varphi} \in H^1_m(\Omega)$ be a minimizer of the functional $\mathcal{E} : H^1_m(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}(\varphi) := \int_\Omega |\nabla\varphi|^2 \, \mathrm{d}^2\mathrm{x} \; - \; 2 \int_{\partial\Omega} \mathrm{g}_\nu \, \varphi \, \mathrm{d}\sigma. \tag{5.1}$$

A family of minimizers that differ by constants exist under the above conditions from standard arguments. Let $\tilde{\varphi}$ be such a minimizer then $\tilde{\mathbf{V}} := -\nabla\tilde{\varphi}$ is a harmonic field on Ω with $D_\nu \tilde{\varphi} = \mathrm{g}_\nu$. This field $\tilde{\mathbf{V}}$ is called the **gradient component of the flow** and has the following properties.

Theorem 5.1. *Assume that Ω is a bounded domain with a Lipschitz boundary $\partial\Omega$ and $\mathrm{g}_\nu \in \mathrm{L}^2(\partial\Omega, \mathrm{d}\sigma)$ satisfies $\int_{\partial\Omega} \mathrm{g}_\nu \, \mathrm{d}\sigma = 0$. There is a unique minimizer $\tilde{\varphi}$ of $\mathcal{E}$ on $H^1_m(\Omega)$. It is a harmonic function on Ω and $\tilde{\varphi}$ has the harmonic Steklov representation*

$$\tilde{\varphi}(x) = \sum_{j=1}^\infty \frac{\hat{g}_j}{\delta_j} \, s_j(x) \; = \; \lim_{M \to \infty} \int_{\partial\Omega} N_M(x, z) \, \mathrm{g}_\nu(\mathrm{z}) \, \mathrm{d}\sigma \quad \textit{for } \mathrm{x} \in \Omega \tag{5.2}$$

with $N_M(x, z) \; := \; \sum_{j=1}^M \delta_j^{-1} s_j(x) \, s_j(z)$. When $\mathrm{g}_\nu \in \mathrm{H}^{\mathrm{s}}(\partial\Omega)$, then $\tilde{\varphi} \in \mathcal{H}^{s+3/2}(\Omega)$ and $\tilde{\mathbf{V}} := -\nabla\tilde{\varphi}$ has

$$\|\tilde{\mathbf{V}}\|_2^2 \; = \; \sum_{j=1}^\infty \delta_j^{-1} \, \hat{\mathrm{g}}_{\mathrm{j}}^2 \; \le \; \delta_1^{-1} \|\mathrm{g}_\nu\|^2_{2,\partial\Omega} \tag{5.3}$$

Proof. The functional $\mathcal{E}$ is continuous, strictly convex and coercive on $H^1_m(\Omega)$ so there is a unique minimizer of $\mathcal{E}$ that satisfies the extremality equation

$$\int_\Omega \nabla\tilde{\varphi}\cdot\nabla\xi\,\mathrm{d}^2\mathrm{x} = \int_{\partial\Omega} \mathrm{g}_\nu\,\xi\,\mathrm{d}\sigma \quad \text{for all} \quad \xi \in \mathrm{H}^1_\mathrm{m}(\Omega).$$

This minimizer is a weak solution of the system

$$\Delta\tilde{\varphi} \;=\; 0 \quad \text{on } \Omega, \quad \text{and} \quad D_\nu\tilde{\varphi} = \mathrm{g}_\nu \quad \text{on } \partial\Omega.$$

Thus $\tilde{\varphi}$ is harmonic on Ω. Substituting s_j for ξ yields that

$$\int_\Omega \nabla\tilde{\varphi}\cdot\nabla s_j\,\mathrm{d}^2\mathrm{x} \;=\; \delta_\mathrm{j}\;\hat{\mathrm{g}}_\mathrm{j} \quad \text{when } \mathrm{j}\geq 1$$

and $\mathrm{g}_\nu = \mathrm{g}$ here. Thus the expansion (5.2) holds. The series converges from the Riesz-Fisher theorem as $\mathcal{S}$ is an orthogonal basis of the subspace of harmonic H^1 functions of $H^1_m(\Omega)$.

The analysis of these Neumann harmonic boundary value problems in [1] shows that, when the Neumann data is in $H^s(\partial\Omega)$, the solution is in $\mathcal{H}^{s+3/2}(\Omega)$. Taking gradients and using properties of the harmonic Steklov eigenfunctions and Parseval's identity yields the first part of (5.3). Since the δ_j are increasing the last inequality holds. □

6. Biharmonic component of solenoidal fields in $V_0(\Omega)$

There are a number of different decompositions of planar vector fields on bounded domains that are called Helmholtz decompositions and involve a gradient and a curl. Usually L^2-orthogonal decompositions are studied, as in [5]. Here a biharmonic stream function that provides a field orthogonal to the harmonic gradient field described in the preceding section will be found.

Given a biharmonic field $\mathbf{w}$ and the potential $\tilde{\varphi}$ one has that $\mathbf{w} := \mathbf{v}-\nabla\tilde{\varphi}$ is in $V_0(\Omega)$ with $\mathbf{w}\cdot\tau = \mathrm{g}_\tau - \mathrm{D}_\tau\tilde{\varphi}$ on $\partial\Omega$. Now a biharmonic stream function $\tilde{\psi}$ will be found with the property that $\mathbf{w} = \nabla^\perp\tilde{\psi}$, so that

$$\mathbf{v} = \nabla^\perp\tilde{\psi} + \nabla\tilde{\varphi} \quad \text{with} \quad \tilde{\psi}\in \mathrm{V}_0(\Omega),\ \tilde{\varphi}\in \mathrm{H}^1_\mathrm{m}(\Omega). \tag{6.1}$$

Consider the problem of solving the zDB boundary value problem of Section 3 for a function $\tilde{\psi} \in \mathcal{B}_0(\Omega)$ satisfying (3.3) with $g \in W^s(\partial\Omega)$. The spectral solution of the problem, from (3.6), has the form $\tilde{\psi}(x) = \lim_{M\to\infty}\psi_M(x)$ with $\psi_M(x) = \sum_{j=1}^M \hat{g}_j\; b_j(x)$. Each ψ_M is biharmonic, so

$\mathbf{w_M}(\mathbf{x}) := \nabla^\perp \psi_\mathbf{M}(\mathbf{x})$ is in $V_0(\Omega)$ and is a biharmonic field with $\mathbf{w_M} \cdot \nu = \mathbf{0}$ on $\partial\Omega$. The tangential trace is

$$(\mathbf{w_M} \cdot \tau)(\mathrm{z}) \;=\; \sum_{\mathbf{j=1}}^{\mathbf{M}} \hat{\mathrm{g}}_\mathrm{j}\ \mathrm{k_j(z)} \quad \text{for } \mathrm{z} \in \partial\boldsymbol{\Omega}, \quad \text{and also} \tag{6.2}$$

$$\operatorname{curl} \mathbf{w_M}(\mathrm{x}) \;=\; -\Delta\psi_\mathrm{M}(\mathrm{x}) \;=\; -\sum_{\mathrm{j=1}}^{\mathrm{M}} \sqrt{\beta_\mathrm{j}}\ \hat{\mathrm{g}}_\mathrm{j}\ \mathrm{h_j(x)} \quad \text{on } \Omega. \tag{6.3}$$

The following theorem provides a criterion on the boundary data $\mathrm{g}_\nu, \mathrm{g}_\tau$ of a solenoidal field to have a biharmonic extension to Ω. Moreover there is an explicit spectral representation of the stream function for this biharmonic field. That is there are explicit formulae, in terms of harmonic Steklov and DBS eigenfunctions, for this biharmonic field.

Theorem 6.1. *Suppose (B1)–(B2) hold,* $g_1 := \mathrm{g}_\nu - \mathrm{D}_\tau \in \mathrm{W}^1(\partial\Omega)$ *and* $\tilde{\varphi} \in \mathcal{B}_0(\Omega)$ *as above. Then the biharmonic function* $\tilde{\psi}$ *is bounded and* C^∞ *on* Ω. $\mathbf{w} = \nabla^\perp\tilde{\psi}$ *is in* $H(curl, \Omega)$ *and there is a constant* C *depending only on* Ω *such that*

(6.4)

$$b_{\mathrm{curl}}(\mathbf{w},\, \mathbf{w}) \;=\; \int_{\boldsymbol{\Omega}} \left[|\mathbf{w}|^\mathbf{2} + |\operatorname{curl} \mathbf{w}|^\mathbf{2}\right]\, \mathrm{d}^2\mathrm{x} + \int_{\partial\Omega} |\mathbf{w}|^\mathbf{2}\, \mathrm{d}\sigma \leq \mathrm{C}\ \mathrm{b_1(g_1, g_1)}$$

Proof. From the formulae for ψ_M, $\mathbf{w_M}$ above and the orthogonality properties, one sees that

$$b_\partial(\psi_M, \psi_M) \;=\; \sum_{j=1}^{M} (\beta_j + 1)\ \hat{g}_j^2 \;\leq\; b_1(g_1, g_1)$$

There is a constant C that depends only on Ω such that

$$\int_\Omega |\operatorname{curl} \mathbf{w_M}|^\mathbf{2}\, \mathrm{d}^2\mathrm{x} \;+ \int_{\partial\Omega} |\mathbf{w_M}|^\mathbf{2}\, \mathrm{d}\sigma \;\geq\; \mathrm{C} \int_\Omega |\mathbf{w_M}|^\mathbf{2}\, \mathrm{d}^2\mathrm{x}$$

from the analysis in [6, section 8]. Thus (6.4) holds for each M, so the limit as $M \to \infty$ exists and the associated $\tilde{\psi}$ yields a $\mathbf{w} := \nabla^\perp\tilde{\psi} \in \mathbf{H}(\mathrm{curl}, \boldsymbol{\Omega})$. This $\tilde{\psi}$ is C^∞ and is bounded as its Laplacian is L^2 and Ω is a bounded planar domain. Moreover the sequence of fields $\{\mathbf{w_M}\}$ converges to $\mathbf{w}$ in $H(\mathrm{curl}, \Omega)$ as $M \to \infty$. □

These properties now imply the following representation theorem for such biharmonic fields.

Corollary 6.1. *Suppose (B1)–(B2) hold,* $\tilde{\varphi}$ *is as in Theorem 5.1 and* $\mathrm{g}_\tau, \mathrm{D}_\tau\tilde{\varphi} \in \mathrm{W}^1(\partial\Omega)$*. If* $\mathbf{v}$ *is a biharmonic field on* Ω*, then there is a function* $\tilde{\psi} \in \mathcal{B}_0(\Omega)$ *such that* (4.7) *and* (6.1) *hold.*

For Stokes problems, $\mathbf{v}$ is the fluid velocity so this theorem says that both under the conditions of the corollary both the velocity and the vorticity are L^2. From (6.3) one has that the boundary trace of the M-th approximation of the vorticity is

$$\omega_M(z) := -\sum_{j=1}^{M} \beta_j \ \hat{g}_j \ k_j(z)$$

Then the 2-norm of this trace has $\|\omega_M\|^2_{2,\partial\Omega} = \sum_{j=1}^{M} \beta_j^2 \ \hat{g}_j^2$ so the boundary trace of the vorticity $\omega := -\Delta\tilde{\psi}$ will have finite 2-norm only when the function g_1 in Theorem 6.1 is in $W^2(\partial\Omega)$. It would be of interest to investigate the behavior of this trace when it is not an L^2 function.

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