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Control theory on Wasserstein space: a new approach to optimality conditions

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This work is dedicated to the memory of Professor Roland Glowinski

We study the deterministic control problem in the Wasserstein space, following the recent works of Bonnet and Frankowska, but with a new approach. One of the major advantages of our approach is that it reconciles the closed loop and the open loop approaches, without the technicalities of the traditional feedback control methodology. It allows also to embed the control problem in the Wasserstein space into a control problem in a Hilbert space, similar to the lifting method introduced by P. L. Lions, used already in our previous works. The Hilbert space is different from that proposed by P. L. Lions, and it allows to recover the control problem in the Wasserstein space as a particular case.

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1. Introduction

In this paper, we give an alternative method for control problem in the Wasserstein space, following the papers of Bonnet, Frankowska and Rossi [4, 5, 6, 7]. We first study control problem where the state and control are in generic Hilbert spaces. We then embed the control problem in the Wasserstein space into a control problem in a Hilbert space, similar to the lifting method introduced by P. L. Lions, used already by Bensoussan-Yam [1]. The Hilbert space is different and allows to recover the control problem in the Wasserstein space as a particular case, which bypasses the evolution in the Wasserstein space. One of the major advantages of our approach is that it reconciles the closed loop and the open loop approaches, without the technicalities of the traditional feedback control methodology. We would like to emphasize that our results are appropriate for any generic Hilbert spaces, although the motivation of studying the control problem on Hilbert spaces is to study control problem in the Wasserstein space.

The first part of our results is to study the control problems in Hilbert spaces. We give a necessary condition in view of the maximum principle, and give a sufficient condition by showing the cost functional is convex. We derive from the optimality condition a forward-backward system defined in Hilbert spaces, the solution of which gives the optimal control. We give the solvability of the forward-backward system, and study the boundedness and continuity of the solution with respect to the initial time and initial state. Then, we study the regularity of the value function. By studying the growth and continuity conditions of the derivatives, we show that the value function is the unique solution of the Bellman equation. As a corollary, we show that the optimal control is of a feedback form. Our results extend the previous results in Bensoussan-Yam [1] to a more general case.

As an application of this model, we make a connection with the mean field type control problem, and show that the approach of Hilbert spaces simplifies the development greatly. In our formulation for the mean field type control problem, an admissible control (which we denote by $v_x(s)$) is a feedback with respect to the initial condition, but not with respect to the current state. To be more precise, the control is a function of time, and is indexed with respect to the initial condition (see Section 4 for details). The interest of our formulation $v_x(s)$ for an admissible control is that, the controlled dynamic is a physical state which we can observe with censors without knowing the initial state. Mathematically, the state $m(s)$ is a distribution. However, we cannot observe a probability, we can only compute it, so the initial probability is needed. This formulation leads to an open loop approach, and we shall show that, although it is an open loop approach, the optimal control is a feedback. We give an interpretation of all results for control problem in Hilbert spaces back to mean field type control problems. We also study the value function of mean field type control problem, and give a sensitivity relation as in Bonnet-Frankowska [6].

This article is organized as follows. In Section 2, we introduce the Wasserstein space and the derivatives of functionals. In Section 3, we give a formal presentation of the control problem in the Wasserstein space. In Section 4, we give another formulation of the control problem and introduce our approach. In Section 5, we study control problems in Hilbert spaces. We give a necessary condition and a sufficient condition in Theorems 5.5 and 5.8, respectively. We give the regularity of the corresponding forward-backward system in Lemmas 5.11 and 5.12. In Section 6, we study the regularity of the value function for control problems in Hilbert spaces, and give the solvability of the corresponding Bellman equation. In Section 7, we go back to mean field type control problems and interpret our results for control problem in Hilbert spaces as those for mean field type control problems. In Section 8, we study the value function and Bellman equation for the mean field type control problem.

2. Formalism

2.1. Wasserstein space

We consider the space $\mathcal{P}_2(\mathbb{R}^n)$ of probability measures on $\mathbb{R}^n$, with second moment, namely, $\int_{\mathbb{R}^n} |x|^2 dm(x) < \infty$, equipped with the 2-Wasserstein metric defined by

$$W_2(m, m') := \inf_{\pi \in \Gamma(m,m')} \sqrt{\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - x'|^2 \, \pi(dx, dx')},$$

where $\Gamma(m, m')$ denotes the set of joint probability measures with respective marginals m and m'. The infimum is attained, so we can find $\hat{X}_m, \hat{X}_{m'}$ in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is an atomless probability space, whose probability laws $\mathcal{L}X_m = m$, $\mathcal{L}X_{m'} = m'$, such that

$$W_2^2(m, m') = \mathbb{E} \left| \hat{X}_m - \hat{X}_{m'} \right|^2.$$

A family m_k converges to m in $\mathcal{P}_2(\mathbb{R}^n)$ if and only if it converges in the sense of the weak convergence and $W_2(m_k, \delta_0) \to W_2(m, \delta_0)$, see Villani [10] for details.

2.2. Functionals

Consider a functional F on $\mathcal{P}_2(\mathbb{R}^n)$. Continuity is clearly defined by the metric. For the concept of derivative in $\mathcal{P}_2(\mathbb{R}^n)$, we use the concept of functional derivative. The functional derivative of $F(m)$ at m is a function $\mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \ni (m, x) \mapsto \frac{dF}{d\nu}(m)(x)$ being continuous under the product topology, satisfying

$$\int_{\mathbb{R}^n} \left| \frac{dF}{d\nu}(m)(x) \right|^2 dm(x) \le c(m),$$

for some positive constant $c(m)$ depending locally on m, and

$$\text{(1)} \qquad \lim_{\epsilon \to 0} \frac{F(m + \epsilon(m' - m)) - F(m)}{\epsilon} = \int_{\mathbb{R}^n} \frac{dF}{d\nu}(m)(x)(dm'(x) - dm(x))$$

for any $m' \in \mathcal{P}_2(\mathbb{R}^n)$. Note that the definition (1) implies

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^n} \frac{dF}{d\nu}\big(m + \theta(m' - m)\big)(x)\,\big(dm'(x) - dm(x)\big)\,d\theta.$$

Of course $\frac{dF}{d\nu}(m)(x)$ is just a notation. We have not written $\frac{dF}{dm}(m)(x)$ to make the difference between the notation ν and the argument m. Also we prefer the notation $\frac{dF}{d\nu}(m)(x)$ to $\frac{\delta F}{\delta m}(m)(x)$ used in Carmona-Delarue [8], because there is no risk of confusion and it works pretty much like an ordinary Gâteaux derivative.

3. Formal presentation of the control problem

3.1. Evolution in the Wasserstein space

We consider a map $\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^n$, denoted by $g(x, m, v, s)$, where the argument v represents a control, with values in $\mathbb{R}^d$. Precise assumptions are not made explicit, but in m the function g has a functional derivative as defined above in (1). The evolution is defined by a feedback $v(x, s)$. A functional space for feedbacks could be

$$\|v\| = \sup_{x,s} \frac{|v(x,s)|}{1 + |x|} < \infty.$$

Unfortunately, it is not a Hilbert space. So, it is only a reference. The function g is continuous in all arguments and satisfies

$$|g(x, m, v, s)| \le c_g(1 + |x| + |v| + W_2(m, \delta_0)).$$

Next, $\frac{dg}{d\nu}(x, m, v, s)(\xi)$ is continuous in all arguments and

$$\left|\frac{dg}{d\nu}(x, m, v, s)(\xi)\right| \le c_g(1 + |x| + |v| + |\xi|).$$

Given a feedback $v(x, s)$, we solve the evolution equation in the Wasserstein space

$$\begin{cases} \dfrac{\partial}{\partial s} m(s)(x) + \mathrm{div}\,[g(x, m(s), v(x, s), s) m(s)(x)] = 0, \quad s > t, \\ m(t) = m. \end{cases} \tag{2}$$

This equation is called the Fokker-Planck equation associated to the drift vector g. If $\varphi(x)$ is a smooth test function, we can write (2) in a weak sense

$$\frac{d}{ds}\int_{\mathbb{R}^n}\varphi(x)dm(s)(x)=\int_{\mathbb{R}^n}D\varphi(x)\cdot g(x,m(s),v(x,s),s)dm(s)(x). \tag{3}$$

It is useful to consider the probability $m(s)$ as the state of a dynamical system, and (3) describes the evolution of the dynamics.

3.2. Cost functional

We introduce a function $\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^d\times\mathbb{R}^+\to\mathbb{R}$, denoted by $f(x,m,v,s)$, continuous in all arguments and satisfying

$$|f(x,m,v,s)|\leq c_f\left(1+|x|^2+|v|^2+W_2^2(m,\delta_0)\right).$$

Also, $\frac{df}{d\nu}(x,m,v,s)(\xi)$ is continuous in all arguments and

$$\left|\frac{df}{d\nu}(x,m,v,s)(\xi)\right|\leq c_f\left(1+|x|^2+|v|^2+|\xi|^2\right).$$

Similarly, a function $h(x,m):\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\to\mathbb{R}$, which is continuous and satisfies

$$|h(x,m)|\leq c_h(1+|x|^2+\mathcal{M}_2^2(m)),$$
$$\left|\frac{dh}{d\nu}(x,m)(\xi)\right|\leq c_h\left(1+|x|^2+|\xi|^2\right).$$

We then define the cost functional indexed by the parameters (m,t)

$$\begin{aligned}J_{mt}(v(\cdot))=&\int_t^T\int_{\mathbb{R}^n}f(\xi,m(s),v(\xi,s),s)dm(s)(\xi)ds\\&+\int_{\mathbb{R}^n}h(\xi,m(T))dm(T)(\xi).\end{aligned} \tag{4}$$

3.3. Necessary condition

We introduce the Lagrangian

$$L(x,m,v,s;q):=f(x,m,v,s)+q\cdot g(x,m,v,s),$$

and assume that there exists a unique $\hat{v}(x,m,s;q)$ satisfying

$$D_vL(x,m,\hat{v}(x,m,s;q),s;q)=0. \tag{5}$$

We next define the Hamiltonian

$$H(x,m,s;q):=L(x,m,\hat{v}(x,m,s;q),s;q),$$

and have

$$D_qH(x,m,s;q)=g(x,m,\hat{v}(x,m,s;q),s,q).$$

In Bensoussan-Frehse-Yam [2], it is formally proven that if $\hat{v}(x,s)$ is an optimal feedback, it can be characterized as follows. Denoting by $\hat{m}(s)$ the corresponding optimal state, we introduce the system of Fokker-Planck, Hamilton-Jacobi Bellman (HJB) equations

$$\begin{cases} \dfrac{\partial}{\partial s}\hat{m}(s)(x)+\text{div}\,[D_qH(x,\hat{m}(s),s;Du(x,s))\hat{m}(s)(x)]=0,\ \ s\in(t,T],\\ -\dfrac{\partial u}{\partial s}(x,s)=H(x,\hat{m}(s),s;Du(x,s))\\ \qquad\qquad\qquad +\displaystyle\int_{\mathbb{R}^n}\frac{dH}{d\nu}(\xi,\hat{m}(s),s,Du(\xi,s))(x)d\hat{m}(s)(\xi),\ \ s\in[t,T),\\ \hat{m}(t)=m,\ \ u(x,T)=h(x,\hat{m}(T))+\displaystyle\int_{\mathbb{R}^n}\frac{dh}{d\nu}(\xi,\hat{m}(T))(x)d\hat{m}(T)(\xi), \end{cases} \tag{6}$$

and we have the optimality condition

$$D_vL(x,\hat{m}(s),\hat{v}(x,s),s;Du(x,s))=0,\quad \text{a.e. } s\in[t,T],\ \text{a.s. } d\widehat{m}(s)(x),$$

which means

$$\hat{v}(x,s)=\hat{v}(x,\hat{m}(s),s;Du(x,s)). \tag{7}$$

4. Pontryagin maximum principle

4.1. Another formulation of the control problem

We can characterize the solution $m(s)$ of (2) as follows. Consider the dynamical system whose state is denoted by $x_{xmt}(s)$, solution of the McKean-Vlasov differential equation

$$\begin{cases} \dfrac{dx_{xmt}(s)}{ds}=g(x_{xmt}(s),m(s),v(x_{xmt}(s),s),s),\quad s>t,\\ x_{xmt}(t)=x, \end{cases} \tag{8}$$

with

$$m(s) := m_{mt}(s) = x_{\cdot mt}(s)\#m, \quad s \geq t,$$

where $x_{\cdot mt}(s)\#m$ represents the push forward of m by the map $x \mapsto x_{xmt}(s)$. The cost functional (4) can be written as

$$J_{mt}(v(\cdot)) = \int_t^T \int_{\mathbb{R}^n} f\left(x_{xmt}(s), m_{mt}(s), v(x_{xmt}(s), s), s\right) dm(x)ds + \int_{\mathbb{R}^n} h\left(x_{xmt}(T), m_{mt}(T)\right) dm(x).$$

The drawback of this formulation is that, we need assumptions on the feedback $v(x,s)$ to solve the differential equation (8). To obtain the Pontryagin maximum principle (PMP), we will proceed formally. The difficulty will be overcome with the specific open loop approach, without any loss.

4.2. Pontryagin maximum principle

We introduce $y_{xmt}(s)$ the solution of

$$\begin{cases} \dfrac{dy_{xmt}(s)}{ds} = D_q H(y_{xmt}(s), y_{\cdot mt}(s)\#m, s; q_{xmt}(s)), \quad s > t, \\ y_{xmt}(t) = x, \end{cases} \tag{9}$$

with

$$q_{xmt}(s) := Du(y_{xmt}(s), s).$$

Using the HJB equation (6), we can derive a backward differential equation for $q_{xmt}(s)$. We omit the calculations, very similar to those to obtain the PMP from Bellman equation in standard stochastic control. We can write

$$\begin{cases} -\dfrac{dq_{xmt}(s)}{ds} = D_x H(y_{xmt}(s), y_{\cdot mt}(s)\#m, s; q_{xmt}(s)) \\ \qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dH}{d\nu}(y_{\xi mt}(s), y_{\cdot mt}(s)\#m, s; q_{\xi mt}(s))(y_{xmt}(s)) dm(\xi), \\ q_{xmt}(T) = D_x h(y_{xmt}(T), y_{\cdot mt}(T)\#m) \\ \qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}(y_{\xi mt}(T), y_{\cdot mt}(T)\#m)(y_{xmt}(T)) dm(\xi). \end{cases} \tag{10}$$

Going back to (7) we introduce

$$u_{xmt}(s) := \widehat{v}(y_{xmt}(s), s) = \widehat{v}(y_{xmt}(s), y_{\cdot mt}(s)\#m, s; q_{xmt}(s)).$$

This function satisfies (see (5))

$$D_v L(y_{xmt}(s), y_{\cdot mt}(s)\#m, u_{xmt}(s), s; q_{xmt}(s)) = 0. \tag{11}$$

We can write (9), (10), (11) as follows

(12)

$$\begin{cases} y_{xmt}(s) = x + \displaystyle\int_t^s g(y_{xmt}(r), y_{\cdot mt}(r)\#m, u_{xmt}(r), r)dr, \\ q_{xmt}(s) = D_x h(y_{xmt}(T), y_{\cdot mt}(T)\#m) \\ \qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}(y_{\xi mt}(T), y_{\cdot mt}(T)\#m)(y_{xmt}(T))dm(\xi) \\ \qquad + \displaystyle\int_s^T \Big[D_x f(y_{xmt}(r), y_{\cdot mt}(r)\#m, u_{xmt}(r), r) \\ \qquad\qquad + D_x g(y_{xmt}(r), y_{\cdot mt}(r)\#m, u_{xmt}(r), r) q_{xmt}(r) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^n} \Big(D_\xi \frac{df}{d\nu}(y_{\xi mt}(r), y_{\cdot mt}(r)\#m, u_{\xi mt}(r), r)(y_{xmt}(r)) \\ \qquad\qquad\quad + D_\xi \frac{dg}{d\nu}(y_{\xi mt}(r), y_{\cdot mt}(r)\#m, u_{\xi mt}(r), r)(y_{xmt}(r)) \\ \qquad\qquad\qquad q_{\xi mt}(r) \Big) dm(\xi) \Big] dr. \end{cases}$$

4.3. Open loop approach

Under the form (11)–(12), we see that the feedback does not appear explicitly. More precisely, there is a feedback with respect to the initial conditions, not with respect to the current state. This leads to the following open loop approach: the controls are not feedbacks, they are functions of time only, but they are indexed with respect to the initial conditions. In other words, they are of the form $v_{xmt}(s)$. So we introduce the following control problem: To a control $v_{xmt}(s)$, we associate the state

$$\begin{cases} \dfrac{dx_{xmt}(s)}{ds} = g(x_{xmt}(s), x_{\cdot mt}(s)\#m, v_{xmt}(s), s), \quad s > t, \\ x_{xmt}(t) = x, \end{cases} \tag{13}$$

and the cost functional

$$J_{mt}(v_{\cdot mt}(\cdot)) = \int_t^T \int_{\mathbb{R}^n} f(x_{xmt}(s), x_{\cdot mt}(s)\#m, v_{xmt}(s), s)dm(x)ds \\ + \int_{\mathbb{R}^n} h(x_{xmt}(T), x_{\cdot mt}(T)\#m)dm(x). \tag{14}$$

The control problem (13)–(14) is an open loop control problem, whose corresponding PMP is exactly the system (11)–(12). The optimal control is exactly $u_{xmt}(s)$, obtained by the optimal feedback (4.2).

4.4. The approach of Bonnet-Frankowska

We reformulate the approach of Bonnet-Frankowska [5] in our framework, without the constraints. The control is a pure open loop control $v(s)$. The state depends necessarily on the initial conditions, namely

$$\begin{cases} \dfrac{dx_{xmt}(s)}{ds} = g(x_{xmt}(s), x_{\cdot mt}(s)\#m, v(s), s), \quad s > t, \\ x_{xmt}(t) = x, \end{cases}$$

and the cost is

$$J_{mt}(v(\cdot)) = \int_t^T \int_{\mathbb{R}^n} f(x_{xmt}(s), x_{\cdot mt}(s)\#m, v(s), s)dm(x)ds \\ + \int_{\mathbb{R}^n} h(x_{xmt}(T), x_{\cdot mt}(T)\#m)dm(x).$$

To obtain a PMP, they introduce the following Hamiltonian

$$\mathcal{H}(\varpi, v, s) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} L(x, \mu, v, s; q)d\varpi(x, q),$$

where $\varpi \in \mathcal{P}_2(\mathbb{R}^{2n})$ and $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ is the marginal of ϖ on the first component, namely,

$$d\mu(x) = \int_{\mathbb{R}^n} d\varpi(x, q),$$

where the integration is on the argument q only. We can compute the functional derivative

$$\frac{d\mathcal{H}}{d\nu}(\varpi, v, s)(x, q) = f(x, \mu, v, s) + q \cdot g(x, \mu, v, s) + \int_{\mathbb{R}^n} \frac{df}{d\nu}(y, \mu, v, s)(x)d\mu(y)$$

$$+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} r\cdot\frac{dg}{d\nu}(y,\mu,v,s)(x)d\varpi(y,r),$$

and the gradient with respect to the pair (x,q)

$$(15)\qquad \begin{aligned} D_{x,q}\frac{d\mathcal{H}}{d\nu}(\varpi,v,s)(x,q) &= \begin{pmatrix} D_x\frac{d\mathcal{H}}{d\nu}(\varpi,v,s)(x,q) \\ D_q\frac{d\mathcal{H}}{d\nu}(\varpi,v,s)(x,q)\end{pmatrix}, \\ D_x\frac{d\mathcal{H}}{d\nu}(\varpi,v,s)(x,q) &= D_xf(x,\mu,v,s)+D_xg(x,\mu,v,s)q \\ &\quad + \int_{\mathbb{R}^n}[D_\xi\frac{df}{d\nu}(y,\mu,v,s)(x) \\ &\quad + D_\xi\frac{dg}{d\nu}(y,\mu,v,s)(x)r]d\varpi(y,r), \\ D_q\frac{d\mathcal{H}}{d\nu}(\varpi,v,s)(x,q) &= g(x,\mu,v,s). \end{aligned}$$

To the optimal control $u_{mt}(s)$ is associated an optimal flow $\hat{\varpi}_{mt}(s)$ and satisfies the optimality condition

$$\mathcal{H}(\widehat{\varpi}_{mt}(s),u_{mt}(s),s)=\inf_v \mathcal{H}(\widehat{\varpi}_{mt}(s),v,s),\quad \text{a.e. } s\in[t,T].$$

The optimal flow corresponds to the push forward of a dynamical system $(y_{xmt}(s),q_{xmt}(s))$ as follows

$$\widehat{\varpi}_{mt}(s)=\begin{pmatrix} y_{\cdot mt}(s) \\ q_{\cdot mt}(s)\end{pmatrix}\#m.$$

The evolution of the dynamical system $(y_{xmt}(s),q_{xmt}(s))$ is defined by

(16)

$$\begin{cases} \dfrac{d}{ds}\begin{pmatrix} y_{xmt}(s) \\ q_{xmt}(s)\end{pmatrix} = \mathcal{J}_{2n}D_{x,q}\dfrac{d\mathcal{H}}{d\nu}(\widehat{\varpi}_{mt}(s),u_{mt}(s),s)(y_{xmt}(s),q_{xmt}(s)), \\ y_{xmt}(t)=x, \\ q_{xmt}(T)=D_xh(y_{xmt}(T),y_{\cdot mt}(T)\#m) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi\frac{dh}{d\nu}(y_{zmt}(T),y_{\cdot mt}(T)\#m)(y_{xmt}(T))dm(z), \end{cases}$$

where the matrix $\mathcal{J}_{2n}$ is defined by

$$\mathcal{J}_{2n} = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix}.$$

If we write (16) explicitly using formula (15), we obtain the system (12), except that $u_{xmt}(s)$ is replaced with $u_{mt}(s)$.

The main advantage of the formulation (13)–(14) is that, although it is also an open loop approach (except for the initial condition), it leads to the optimal feedback. In addition, as we will see in the following sections, the problem (13)–(14) can be embedded into a control problem on a Hilbert space, which bypasses the evolution in the Wasserstein space.

5. Control problems on Hilbert spaces

The formulation of problem (13)–(14) inspires us to study control problems where the state and the control are defined in Hilbert spaces. We consider two Hilbert spaces H and U, whose general elements are denoted by X and V, respectively. The inner products are denoted by $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_U$, and the corresponding norms by $\|\cdot\|_H$ and $\|\cdot\|_U$, respectively. For any initial $(t, X) \in [0, T] \times H$, in this section, we consider the following control problem

(17)
$$(\mathscr{P}^{t,X}) \quad \begin{cases} \inf\limits_{V(\cdot)\in L^2([t,T];U)} \ J_{Xt}(V(\cdot)), \\ \text{s.t.} \quad X_{Xt}^V(s) = X + \displaystyle\int_t^s G\left(X_{Xt}^V(r), v(r), r\right) dr, \quad s \in [t, T], \end{cases}$$

where

$$J_{Xt}(V) := \int_t^T F\left(X_{Xt}^V(s), V(s), s\right) ds + F_T\left(X_{Xt}^V(T)\right),\ V \in L^2([t,T];U),$$

and

$$G : H \times U \times [0,T] \to H, \quad F : H \times U \times [0,T] \to \mathbb{R}, \quad F_T : H \to \mathbb{R}.$$

For notational convenience, we drop the subscript (t, X) in the state process and cost function of $(\mathscr{P}^{t,X})$ in this section when there is no ambiguity. We will make a connection between problem (13)–(14) and the control problem on Hilbert spaces $(\mathscr{P}^{t,X})$ in Section 7. We would like to emphasize that, although the motivation of studying the control problem on Hilbert spaces is to study problem (13)–(14), our results for $(\mathscr{P}^{t,X})$ are appropriate for any generic Hilbert spaces H and U.

5.1. Necessary condition for $(\mathscr{P}^{t,X})$

We state our assumptions in this subsection. For notational convenience, we use the same constant $L > 0$ for all the conditions below.

(A1) The map G satisfies that for any $(X, V, s) \in H \times U \times [0, T]$,

$$\|G(X, V, s)\|_H \le L(1 + \|X\|_H + \|V\|_U).$$

The Gâteaux derivatives of G along any directions $\tilde{X} \in H$ and $\tilde{V} \in U$ at (X, V, s) exist and are continuous in (X, V), and satisfy

$$\|D_X G(X, V, s)(\tilde{X})\|_H \le L\|\tilde{X}\|_H, \quad \|D_V G(X, V, s)(\tilde{V})\|_H \le L\|\tilde{V}\|_U.$$

(A2) The functional F satisfies for any $(X, V, s) \in H \times U \times [0, T]$,

$$|F(X, V, s)| \le L(1 + \|X\|_H^2 + \|V\|_U^2).$$

The functional $H \times U \ni (X, V) \mapsto F(X, V, s) \in \mathbb{R}$ is continuously differentiable, with the derivatives

$$\|D_X F(X, V, s)\|_H + \|D_V F(X, V, s)\|_U \le L(1 + \|X\|_H + \|V\|_U).$$

The functional F_T satisfies for any $X \in H$,

$$|F_T(X)| \le L(1 + \|X\|_H^2),$$

and is continuously differentiable, with derivative

$$\|D_X F_T(X)\|_H \le L(1 + \|X\|_H).$$

5.1.1. Regularity of X^V in V. For a control $V \in L^2([t, T]; U)$ for $(\mathscr{P}^{t,X})$, we first show that the corresponding state process X^V belongs to $L^2([t, T]; H)$.

Lemma 5.1. *Under Assumption (A1), for any $V \in L^2([t, T]; U)$, the controlled state X^V satisfies*

$$\sup_{t \le s \le T} \left\|X^V(s)\right\|_H \le C(L, T)\left(1 + \|X\|_H + \|V\|_{L^2([t,T];U)}\right), \tag{18}$$

where $C(L, T)$ is a constant depending only on (L, T).

Proof. From the equation in (17) and Assumption (A1), we have for $s \in [t, T]$,

$$\left\|X^V(s)\right\|_H^2 \le C(L,T)\left[1 + \|X\|_H^2 + \int_t^s \left(\left\|X^V(r)\right\|_H^2 + \|V(r)\|_U^2\right) dr\right].$$

From Grönwall's inequality, we have (18). □

For $V, \tilde{V} \in L^2([t,T];U)$, we define $\mathcal{D}_{\tilde{V}} X^V \in L^2([t,T];H)$ the solution of the following equation: for $s \in [t,T]$,

$$\begin{aligned} \mathcal{D}_{\tilde{V}} X^V(s) = \int_t^s \Big[&D_X G\left(X^V(r), V(r), r\right)\left(\mathcal{D}_{\tilde{V}} X^V(r)\right) \\ &+ D_V G\left(X^V(r), V(r), r\right)\left(\tilde{V}(r)\right)\Big] dr. \end{aligned} \tag{19}$$

We have the following estimate for $\mathcal{D}_{\tilde{V}} X^V$.

Lemma 5.2. *Under Assumption (A1), for $V, \tilde{V} \in L^2([t,T];U)$, there is a unique solution $\mathcal{D}_{\tilde{V}} X^V$ of Equation (19) such that*

$$\sup_{t \le s \le T} \left\|\mathcal{D}_{\tilde{V}} X^V(s)\right\|_H \le C(L,T)\left\|\tilde{V}\right\|_{L^2([t,T];U)}. \tag{20}$$

Proof. We only prove the estimate (20) here. From equation (19) and Assumption (A1), we have

$$\begin{aligned} \left\|\mathcal{D}_{\tilde{V}} X^V(s)\right\|_H^2 &\le C(T)\int_t^s \Big(\left\|D_X G\left(X^V(r), V(r), r\right)\left(\mathcal{D}_{\tilde{V}} X^V(r)\right)\right\|_H^2 \\ &\qquad + \left\|D_V G\left(X^V(r), V(r), r\right)\left(\tilde{V}(r)\right)\right\|_H^2\Big) dr \\ &\le C(L,T)\int_t^s \left(\left\|\mathcal{D}_{\tilde{V}} X^V(r)\right\|_H^2 + \left\|\tilde{V}(r)\right\|_U^2\right) dr. \end{aligned}$$

By applying Grönwall's inequality, we obtain (20). □

We set $V^\epsilon := V + \epsilon\tilde{V}$ for $\epsilon \in (0,1)$. It is obvious that $V^\epsilon \in L^2([t,T];U)$. We denote by X^ϵ the state process corresponding to the control V^ϵ and set $Y^\epsilon := \frac{1}{\epsilon}\left(X^\epsilon - X^V\right)$. We have the following estimate, whose proof is given in Appendix A.1.1.

Lemma 5.3. *Under Assumption (A1), let* $V, \tilde{V} \in L^2([t,T];U)$*, and* Y^ϵ *and* $\tilde{X}^{V,\tilde{V}}$ *be defined above, we have*

$$\lim_{\epsilon \to 0} \sup_{t \le s \le T} \left\| Y^\epsilon(s) - \mathcal{D}_{\tilde{V}} X^V(s) \right\|_H = 0. \tag{21}$$

That is, $\mathcal{D}_{\tilde{V}} X^V(s)$ *is is actually the directional derivative* $D_{\tilde{V}} X^V(s)$ *of* $X^V(s)$ *along the direction* $\tilde{V} \in L^2([t,T];U)$*.*

Lemmas 5.2 and 5.3 show that the mapping

$$L^2([t,T];U) \ni V(\cdot) \mapsto X^V(s) \in H$$

is Gâteaux differentiable. The directional derivative $D_{\tilde{V}} X^V$ evaluated at V can be expressed as a Fréchet derivative $D_V X^V$ acting on $\tilde{V}$, i.e. $D_V X^V(\tilde{V})$, see [3]. Here, the subscript $\tilde{V}$ in $D_{\tilde{V}} X^V$ means the direction and the V in $D_V X^V(\tilde{V})$ means the evaluating point.

5.1.2. Lagrangian and adjoint process. We introduce the Lagrangian $\mathcal{L} : H \times U \times H \times [0,T] \to \mathbb{R}$ as

$$\mathcal{L}(X, V, s; Q) := \Big(Q, G(X, V, s) \Big)_H + F(X, V, s). \tag{22}$$

Then, we have

$$D_Q \mathcal{L}(X, V, s; Q) = G(X, V, s).$$

From Assumptions (A1) and (A2), we know that $\mathcal{L}$ satisfies

$$|\mathcal{L}(X, V, s; Q)| \le C(L) \left(1 + \|X\|_H^2 + \|V\|_U^2 + \|Q\|_H^2 \right), \tag{23}$$

and the functional $H \times U \ni (X, V) \mapsto \mathcal{L}(X, V, s; Q) \in \mathbb{R}$ is continuously differentiable, with the derivatives

$$\begin{aligned} &\|D_X \mathcal{L}(X, V, s; Q)\|_H + \|D_V \mathcal{L}(X, V, s; Q)\|_U \\ &\le C(L)(1 + \|X\|_H + \|V\|_U + \|Q\|_H). \end{aligned} \tag{24}$$

Let $\hat{V}$ be an optimal control for $(\mathscr{P}^{t,X})$ and $\hat{X}$ be the corresponding controlled state. We define the adjoint process as, for $s \in [t,T]$,

$$\hat{Q}(s) = D_X F_T \left(\hat{X}(T) \right) + \int_s^T D_X \mathcal{L} \left(\hat{X}(r), \hat{V}(r), r; \hat{Q}(r) \right) dr. \tag{25}$$

We have the following estimate Equation (25).

Lemma 5.4. *Under Assumptions (A1) and (A2), $\hat{Q}$ defined in (25) satisfies*

$$\sup_{t\le s\le T}\left\|\hat{Q}(s)\right\|_H \le C(L,T)\left(1+\|X\|_H+\left\|\hat{V}\right\|_{L^2([t,T];U)}\right). \tag{26}$$

Proof. From (25), Assumption (A2) and (24), we have

$$\begin{aligned}\|\hat{Q}(s)\|_H^2 &\le C(T)\left(\left\|D_XF_T\big(\hat{X}(T)\big)\right\|_H^2+\int_s^T\left\|D_X\mathcal{L}\big(\hat{X}(r),\hat{V}(r),r;\hat{Q}(r)\big)\right\|_H^2dr\right)\\ &\le C(L,T)\Big(1+\|\hat{X}(T)\|_H^2+\int_s^T\|\hat{X}(r)\|_H^2+\|\hat{V}(r)\|_U^2+\|\hat{Q}(r)\|_H^2dr\Big).\end{aligned}$$

From Grönwall's inequality, we deduce that

$$\sup_{t\le s\le T}\left\|\hat{Q}(s)\right\|_H^2 \le C(L,T)\left(1+\sup_{t\le s\le T}\left\|\hat{X}(s)\right\|_H^2+\left\|\hat{V}\right\|^2_{L^2([t,T];U)}\right).$$

From Lemma 5.1, we obtain (26). □

5.1.3. Necessary condition.

Now we give the necessary condition.

Theorem 5.5. *Under Assumptions (A1) and (A2), let $\hat{V}$ be an optimal control for $(\mathscr{P}^{t,X})$, $\hat{X}$ be the corresponding controlled state, and $\hat{Q}$ be the corresponding adjoint. Then, we have the optimality condition*

$$D_V\mathcal{L}\left(\hat{X}(s),\hat{V}(s),s;\hat{Q}(s)\right)\overset{U}{=}0,\quad a.e.\ s\in[t,T].$$

Proof. For any $\tilde{V}\in L^2([t,T];U)$, we set $V^\epsilon:=\hat{V}+\epsilon\tilde{V}$ for $\epsilon\in(0,1)$. It is obvious that $V^\epsilon\in L^2([t,T];U)$. We denote by X^ϵ the controlled state corresponding to V^ϵ, and set $Y^\epsilon:=\frac{1}{\epsilon}(X^\epsilon-\hat{X})$. From Assumptions (A1) and (A2), we have

$$\begin{aligned}&\frac{1}{\epsilon}\left[J\left(V^\epsilon\right)-J(\hat{V})\right]\\ &=\int_t^T\int_0^1\left(D_XF\left(\hat{X}(s)+\lambda\epsilon Y^\epsilon(s),V^\epsilon(s),s\right),Y^\epsilon(s)\right)_H d\lambda ds\\ &\quad+\int_t^T\int_0^1\left(D_VF\left(\hat{X}(s),\hat{V}(s)+\lambda\epsilon\tilde{V}(s),s\right),\tilde{V}(s)\right)_U d\lambda ds\\ &\quad+\int_0^1\left(D_XF_T\left(\hat{X}(T)+\lambda\epsilon Y^\epsilon(T)\right),Y^\epsilon(T)\right)_H d\lambda.\end{aligned} \tag{27}$$

Let $D_{\tilde{V}} X^{\hat{V}}$ be the solution of Equation (19) corresponding to $(\hat{V}, \tilde{V})$. From Assumption (A2), Lemmas 5.2 and 5.3, and the dominated convergence theorem, we deduce that

$$\begin{aligned}\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[J\left(V^{\epsilon}\right) - J(\hat{V})\right] = \int_t^T \Big[& \left(D_X F\left(\hat{X}(s), \hat{V}(s), s\right), D_{\tilde{V}} X^{\hat{V}}(s)\right)_H \\ & + \left(D_V F\left(\hat{X}(s), \hat{V}(s), s\right), \tilde{V}(s)\right)_U \Big] ds \\ & + \left(D_X F_T\left(\hat{X}(T)\right), D_{\tilde{V}} X^{\hat{V}}(T)\right)_H .\end{aligned} \tag{28}$$

From (19), (25), Assumption (A1) and the definition of $\mathcal{L}$, we have for $s \in (t, T)$,

$$\begin{aligned}\frac{d}{ds}\left(\hat{Q}(s), D_{\tilde{V}} X^{\hat{V}}(s)\right)_H = & \left(\hat{Q}(s), D_V G\left(\hat{X}(s), \hat{V}(s), s\right)\left(\tilde{V}(s)\right)\right)_H \\ & - \left(D_X F\left(\hat{X}(s), \hat{V}(s), s\right), D_{\tilde{V}} X^{\hat{V}}(s)\right)_H .\end{aligned}$$

We integrate for s between t and T to obtain

$$\begin{aligned} & \left(D_X F_T\left(\hat{X}(T)\right), D_{\tilde{V}} X^{\hat{V}}(T)\right)_H \\ & \quad + \int_t^T \left(D_X F\left(\hat{X}(s), \hat{V}(s), s\right), D_{\tilde{V}} X^{\hat{V}}(s)\right)_H ds \\ & = \int_t^T \left(\hat{Q}(s), D_V G\left(\hat{X}(s), \hat{V}(s), s\right)\left(\tilde{V}(s)\right)\right)_H ds.\end{aligned} \tag{29}$$

Plugging (29) into (28), from Assumption (A1), we deduce that

$$\begin{aligned} & \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[J(V^{\epsilon}) - J(\hat{V})\right] \\ & = \int_t^T \Big[\left(\hat{Q}(s), D_V G\left(\hat{X}(s), \hat{V}(s), s\right)\left(\tilde{V}(s)\right)\right)_H \\ & \qquad + \left(D_V F\left(\hat{X}(s), \hat{V}(s), s\right), \tilde{V}(s)\right)_U \Big] ds \\ & = \int_t^T \left(D_V \mathcal{L}\left(\hat{X}(s), \hat{V}(s), s; \hat{Q}(s)\right), \tilde{V}(s)\right)_U ds.\end{aligned} \tag{30}$$

The necessary condition is then obtained. □

5.1.4. Necessary condition for constrained problem. We also consider the following constrained control problem

$$(\mathscr{P}_C^{t,X}) \quad \begin{cases} \inf\limits_{V\in L^2([t,T];U)} \quad J_{Xt}(V), \\ \text{s.t.} \begin{cases} X_{Xt}^V(s) = X + \displaystyle\int_t^s G\left(X_{Xt}^V(r), V(r), r\right) dr, \\ \Psi_i\left(X_{Xt}^V(T)\right) \leq 0, \quad 1 \leq i \leq N, \end{cases} \end{cases} \tag{31}$$

where

$$\Psi_i : H \to \mathbb{R}, \quad 1 \leq i \leq N.$$

We have the following neceaasry condition for constrained problem $(\mathscr{P}_C^{t,X})$.

Theorem 5.6. *Under Assumptions (A1) and (A2), suppose that Ψ_i satisfies conditions in (A2) as F_T for $1 \leq i \leq N$. Let $\hat{V}$ be an optimal control for $(\mathscr{P}_C^{t,X})$ and $\hat{X}$ be the corresponding controlled state. Then, there exist nontrivial Lagrange multipliers $(\lambda_0, \lambda_1, \ldots, \lambda_N) \in \{0,1\} \times \mathbb{R}_+^N$, such that*

$$D_V \mathcal{L}_{\lambda_0}\left(\hat{X}(s), \hat{V}(s), s; \hat{Q}_\lambda(s)\right) \overset{U}{=} 0, \quad a.e.\ s \in [t,T], \tag{32}$$

$$\lambda_i \Psi_i\left(\hat{X}(T)\right) = 0, \quad 1 \leq i \leq N, \tag{33}$$

where

$$\mathcal{L}_{\lambda_0}(X, V, s; Q) := \left(Q, G(X,V,s)\right)_H + \lambda_0 F(X,V,s), \tag{34}$$

and $\hat{Q}_\lambda$ is defined as

$$\begin{aligned} \hat{Q}_\lambda(s) =& \lambda_0 D_X F_T\left(\hat{X}(T)\right) + \sum_{i=1}^N \lambda_i D_X \Psi_i\left(\hat{X}(T)\right) \\ &+ \int_s^T D_X \mathcal{L}_{\lambda_0}\left(\hat{X}(r), \hat{V}(r), r; \hat{Q}_\lambda(r)\right) dr, \quad s \in [t,T]. \end{aligned} \tag{35}$$

Proof. We define for $V \in L^2([t,T];U)$ and $(\lambda_0, \ldots, \lambda_N) \in \mathbb{R}^{N+1}$,

$$J(V, \lambda_0, \ldots, \lambda_N) := \lambda_0 J(V) + \sum_{i=1}^N \lambda_i \Psi_i(X^V(T)).$$

Then, from Lagrange multiplier method, there exist non-trivial Lagrange multipliers $(\lambda_0, \lambda_1, \ldots, \lambda_N) \in \{0,1\} \times \mathbb{R}_+^N$, such that (33) hold and for any $\tilde{V} \in L^2([t,T];U)$, and

$$\frac{d}{d\epsilon} J\left(\hat{V} + \epsilon \tilde{V}, \lambda_0 \ldots, \lambda_N\right)\Big|_{\epsilon=0} = 0. \tag{36}$$

From (28), we have

$$\begin{aligned}
&\frac{d}{d\epsilon} J\left(\hat{V} + \epsilon \tilde{V}, \lambda_0 \ldots, \lambda_N\right)\Big|_{\epsilon=0} \\
=&\lambda_0 \int_t^T \Big[\Big(D_X F\left(\hat{X}(s), \hat{V}(s), s\right), D_{\tilde{V}} X^{\hat{V}}(s)\Big)_H \\
&\qquad + \Big(D_V F\left(\hat{X}(s), \hat{V}(s), s\right), \tilde{V}(s)\Big)_U\Big] ds \\
&+ \lambda_0 \Big(D_X F_T\left(\hat{X}(T)\right), D_{\tilde{V}} X^{\hat{V}}(T)\Big)_H \\
&+ \sum_{i=1}^N \lambda_i \Big(D_X \Psi_i\left(\hat{X}(T)\right), D_{\tilde{V}} X^{\hat{V}}(T)\Big)_H,
\end{aligned} \tag{37}$$

where $D_{\tilde{V}} X^{\hat{V}}$ is the solution of equation (19) corresponding to $(\hat{V}, \tilde{V})$. From (19), (34) and (35), we have for $s \in (t,T)$,

$$\begin{aligned}
&\frac{d}{ds}\Big(\hat{Q}_\lambda(s), D_{\tilde{V}} X^{\hat{V}}(s)\Big)_H \\
=& \Big(\hat{Q}_\lambda(s), D_V G\Big(\hat{X}(s), \hat{V}(s), s\Big)\Big(\tilde{V}(s)\Big)\Big)_H \\
&- \lambda_0 \Big(D_X F\Big(\hat{X}(s), \hat{V}(s), s\Big), \tilde{X}^{\tilde{V}}(s)\Big)_H.
\end{aligned}$$

We integrate for s between t and T to obtain

$$\begin{aligned}
&\lambda_0 \int_t^T \Big(D_X F(\hat{X}(s), \hat{V}(s), s), D_{\tilde{V}} X^{\hat{V}}(s)\Big)_H ds \\
&+ \lambda_0 \Big(D_X F_T\left(\hat{X}(T)\right), D_{\tilde{V}} X^{\hat{V}}(T)\Big)_H \\
&+ \sum_{i=1}^N \lambda_i \Big(D_X \Psi_i\left(\hat{X}(T)\right), D_{\tilde{V}} X^{\hat{V}}(T)\Big)_H \\
=& \int_t^T \Big(\hat{Q}_\lambda(s), D_V G\left(\hat{X}(s), \hat{V}(s), s\right)\left(\tilde{V}(s)\right)\Big)_H ds.
\end{aligned} \tag{38}$$

From (36)–(38), we obtain (32). □

5.2. Sufficient condition for ($\mathscr{P}^{t,X}$)

In this subsection, we give a sufficient condition for ($\mathscr{P}^{t,X}$) under the following additional assumptions.

(A3) The map G is linear in X and V. That is

$$G(X,V,s)=G_0(s)+\mathcal{G}_1(s)X+\mathcal{G}_2(s)V,\quad (X,V,s)\in H\times U\times[0,T],$$

where $\mathcal{G}_1(s)$ and $\mathcal{G}_2(s)$ are linear maps on H and U respectively for $s\in[0,T]$, and

$$\|G_0(s)\|_H\le L,\quad \|\mathcal{G}_1(s)\|_{L(H;H)}\le L,\quad \|\mathcal{G}_2(s)\|_{L(U;U)}\le L.$$

(A4) The maps

$$\begin{aligned}&H\times U\ni(X,V)\mapsto D_XF(X,V,s)\in H,\\ &H\ni X\mapsto D_VF(X,V,s)\in U,\\ &H\ni X\mapsto D_XF_T(X)\in H,\end{aligned}$$

are L-Lipschitz continuous for any $s\in[0,T]$. Moreover, there exists $\lambda\ge 1$ such that, for any $X\in H$, $V^1,V^2\in U$ and $s\in[0,T]$,

$$F(X,V^2,s)-F(X,V^1,s)\ge\Big(D_VF(X,V^1,s),V^2-V^1\Big)_U+\lambda\|V^2-V^1\|_U^2.$$

Remark 5.7. *The linearity in X in Assumption (A3) will be used later in the study of the regularity of the forward-backward equations* (47)–(48) *in Lemma 5.11, and the regularity of the value function in Section 6. Just to obtain the sufficient condition for* ($\mathscr{P}^{t,X}$) *(Theorem 5.8) and the well-posedness of forward-backward equations* (47)–(48) *in Lemma 5.11 (Theorem 5.9), we only need G to be linear in V, that is,*

$$G(X,V,s)=G_1(X,s)+\mathcal{G}_2(s)V,$$

where G_1 also satisfies Assumption (A1), and $\mathcal{G}_2(s)$ is a bounded linear map on U. For the convexity assumption in (A4), we refer to [3, Remark 3.1] for the relation between it and the displacement monotonicity condition.

We now give a sufficient condition for ($\mathscr{P}^{t,X}$). The proof follows Bensoussan and Yam [1, Theorem 2.1].

Theorem 5.8. *Under Assumptions (A2)–(A4), there exists a constant Λ depending only on (L,T), such that when $\lambda \geq \Lambda$, $(\mathscr{P}^{t,X})$ has a unique optimal control.*

Proof. From (30) and Assumption (A3), we have the differentiability if J: for any $V, \tilde{V} \in L^2([t,T];U)$,

$$\begin{aligned}
&\frac{d}{d\epsilon} J\left(V + \epsilon\tilde{V}\right)\Big|_{\epsilon=0} \\
&= \int_t^T \left(D_V\mathcal{L}\left(X^V(s), V(s), s; Q^V(s)\right), \tilde{V}(s)\right)_U ds \\
&= \int_t^T \left[\left(\mathcal{G}_2(s)\tilde{V}(s), Q^V(s)\right)_H + \left(D_V F\left(X^V(s), V(s), s\right), \tilde{V}(s)\right)_U\right] ds,
\end{aligned} \tag{39}$$

where X^V is the controlled state and Q^V is the adjoint process corresponding to control V. Now, we prove that J is strictly convex when λ is large enough. For $V_1, V_2 \in L^2([t,T];U)$ and $\theta \in [0,1]$, we can write

$$\begin{aligned}
J(\theta V_1 + (1-\theta)V_2) &= J(V_1 + (1-\theta)(V_2 - V_1)) \\
&= J(V_1) + \int_0^1 \frac{d}{d\epsilon} J\left(V_1 + \epsilon(1-\theta)(V_2 - V_1)\right) d\epsilon.
\end{aligned} \tag{40}$$

From (39), we have

$$\begin{aligned}
&\int_0^1 \frac{d}{d\epsilon} J(V_1 + \epsilon(1-\theta)(V_2 - V_1)) \\
&= (1-\theta)\int_0^1 \int_t^T \Big[\left(\mathcal{G}_2(s)(V_2(s) - V_1(s)), Q^{1,\theta,\epsilon}(s)\right)_H \\
&\qquad + \Big(D_V F\left(X^{1,\theta,\epsilon}(s), V_1(s) + \epsilon(1-\theta)(V_2(s) - V_1(s)), s\right), \\
&\qquad\quad V_2(s) - V_1(s)\Big)_U\Big] ds d\epsilon,
\end{aligned}$$

where we denote by

$$X^{1,\theta,\epsilon} := X^{V_1+\epsilon(1-\theta)(V_2-V_1)}, \quad Q^{1,\theta,\epsilon} := Q^{V_1+\epsilon(1-\theta)(V_2-V_1)}.$$

Similarly, we can also write

$$\begin{aligned}
J(\theta V_1 + (1-\theta)V_2) &= J(V_2 + \theta(V_1 - V_2)) \\
&= J(V_2) + \int_0^1 \frac{d}{d\epsilon} J\left(V_2 + \epsilon\theta(V_1 - V_2)\right) d\epsilon,
\end{aligned} \tag{41}$$

while

$$
\begin{aligned}
&\int_0^1 \frac{d}{d\epsilon} J(V_2+\epsilon\theta(V_1-V_2))\\
=&(1-\theta)\int_0^1\int_t^T \Big[\Big(\mathcal{G}_2(s)(V_1(s)-V_2(s)), Q^{2,\theta,\epsilon}(s)\Big)_H\\
&\qquad + \Big(D_V F\left(X^{2,\theta,\epsilon}(s), V_2(s)+\epsilon\theta(V_1(s)-V_2(s)), s\right),\\
&\qquad\qquad V_1(s)-V_2(s)\Big)_U\Big] ds d\epsilon,
\end{aligned}
$$

where we also denote by

$$
X^{2,\theta,\epsilon} := X^{V_2+\epsilon\theta(V_1-V_2)}, \quad Q^{2,\theta,\epsilon} := Q^{V_2+\epsilon\theta(V_1-V_2)}.
$$

Adding θ of (40) to $(1-\theta)$ of (41), we have

(42)

$$
\begin{aligned}
&J(\theta V_1+(1-\theta)V_2)-\theta J(V_1)-(1-\theta)J(V_2)\\
=&\theta(1-\theta)\int_0^1\int_t^T \Big[\Big(\mathcal{G}_2(s)(V_2(s)-V_1(s)), Q^{1,\theta,\epsilon}(s)-Q^{2,\theta,\epsilon}(s)\Big)_H\\
&\qquad + \Big(D_V F\left(X^{2,\theta,\epsilon}(s), V_2(s)+\epsilon\theta(V_1(s)-V_2(s)), s\right)\\
&\qquad - D_V F\left(X^{1,\theta,\epsilon}(s), V_1(s)+\epsilon(1-\theta)(V_2(s)-V_1(s)), s\right),\\
&\qquad\qquad V_1(s)-V_2(s)\Big)_U\Big] ds d\epsilon.
\end{aligned}
$$

Similar as Lemma 5.1, from Assumption (A3), we have the following estimates for $X^{2,\theta,\epsilon}(s)-X^{1,\theta,\epsilon}(s)$,

$$
\sup_{t\le s\le T}\left\|X^{2,\theta,\epsilon}(s)-X^{1,\theta,\epsilon}(s)\right\|_H \le C(L,T)(1-\epsilon)\|V_2-V_1\|_{L^2([t,T];U)}. \tag{43}
$$

Similar as Lemma 5.4, from Assumptions (A3)–(A4) and estimate (43), we have the following estimate for $Q^{1,\theta,\epsilon}(s)-Q^{2,\theta,\epsilon}(s)$,

$$
\sup_{t\le s\le T}\left\|Q^{1,\theta,\epsilon}(s)-Q^{2,\theta,\epsilon}(s)\right\|_H \le C(L,T)(1-\epsilon)\|V_2-V_1\|_{L^2([t,T];U)}. \tag{44}
$$

From (44) and Assumption (A3), we have

$$
\begin{aligned}
&\left|\int_0^1\int_t^T \Big(\mathcal{G}_2(s)(V_2(s)-V_1(s)),Q^{1,\theta,\epsilon}(s)-Q^{2,\theta,\epsilon}(s)\Big)_H ds d\epsilon\right| \\
\leq & C(L,T)\|V_2-V_1\|^2_{L^2([t,T];U)}.
\end{aligned} \tag{45}
$$

From (43) and Assumption (A4), we have

(46)

$$
\begin{aligned}
&\int_0^1\int_t^T \Big(D_V F\left(X^{1,\theta,\epsilon}(s),V_1(s)+\epsilon(1-\theta)(V_2(s)-V_1(s)),s\right) \\
&\quad - D_V F\left(X^{2,\theta,\epsilon}(s),V_2(s)+\epsilon\theta(V_1(s)-V_2(s)),s\right),\ V_2(s)-V_1(s)\Big)_U ds d\epsilon \\
\leq & \int_0^1\int_t^T \Big(D_V F\left(X^{2,\theta,\epsilon}(s),V_1(s)+\epsilon(1-\theta)(V_2(s)-V_1(s)),s\right) \\
&\quad - D_V F\left(X^{2,\theta,\epsilon}(s),V_2(s)+\epsilon\theta(V_1(s)-V_2(s)),s\right),\ V_2(s)-V_1(s)\Big)_U ds d\epsilon \\
&+\left|\int_0^1\int_t^T \Big(D_V F\left(X^{1,\theta,\epsilon}(s),V_1(s)+\epsilon(1-\theta)(V_2(s)-V_1(s)),s\right)\right. \\
&\qquad\qquad - D_V F\left(X^{2,\theta,\epsilon}(s),V_1(s)+\epsilon(1-\theta)(V_2(s)-V_1(s)),s\right), \\
&\qquad\qquad \left. V_2(s)-V_1(s)\Big)_U ds d\epsilon\right| \\
\leq & \big(-\lambda+C(L,T)\big)\|V_2-V_1\|^2_{L^2([t,T];U)}.
\end{aligned}
$$

Substituting (45) and (46) back to (42), we have

$$
\begin{aligned}
&J(\theta V_1+(1-\theta)V_2)-\theta J(V_1)-(1-\theta)J(V_2) \\
\leq & \theta(1-\theta)\big(-\lambda+C(L,T)\big)\|V_2-V_1\|^2_{L^2([t,T];U)}.
\end{aligned}
$$

Therefore, there exists a constant Λ depending only on (L,T), such that when $\lambda\geq\Lambda$, J is strictly convex. Next, we prove that $J(V)\to+\infty$ as $\|V\|_{L^2([t,T];U)}\to+\infty$. For any $V\in L^2([t,T];U)$, from Assumptions (A4) and (A2), we deduce that

$$
\begin{aligned}
J(V) &= \int_t^T F\left(X^V(s),V(s),s\right)ds + F_T\left(X^V(T)\right) \\
&\geq \int_t^T \Big[F(X^V(s),0,s)+\Big(D_V F(X^V(s),0,s),V(s)\Big)_U
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \lambda \|V(s)\|_U^2 \Big] ds + F_T\left(X^V(T)\right) \\
&\geq \int_t^T \Big[- L\left(1 + \left\|X^V(s)\right\|_H^2\right) - L\left(1 + \left\|X^V(s)\right\|_H\right) \|V(s)\|_U \\
&\qquad + \lambda \|V(s)\|_U^2 \Big] ds - L\left(1 + \left\|X^V(T)\right\|_H^2\right).
\end{aligned}
$$

From Lemma 5.1, we have

$$
J(V) \geq \big(- \lambda + C(L,T) \big) \|V\|^2_{L^2([t,T];U)} - C(L,T)\left(1 + \|X\|_H^2\right).
$$

So when λ is large enough such that $-\lambda + C(L,T) < 0$, we know that $J(V) \to +\infty$ as $\|V\|_{L^2([t,T];U)} \to +\infty$. This coercive property and the strict convexity imply that the functional J should possess a unique minimum. That is, $(\mathscr{P}^{t,X})$ has a unique optimal control. □

5.3. Forward-backward system for $(\mathscr{P}^{t,X})$

We derive from Theorem 5.5 the following forward-backward system for $(\hat{X}, \hat{V}, \hat{Q}) \in C([t,T];H) \times L^2([t,T];U) \times C([t,T];H)$: for $s \in [t,T]$,

$$
\begin{cases}
\hat{X}(s) = X + \displaystyle\int_t^s G\left(\hat{X}(r), \hat{V}(r), r\right) dr, \\
\hat{Q}(s) = D_X F_T\left(\hat{X}(T)\right) + \displaystyle\int_s^T D_X \mathcal{L}\left(\hat{X}(r), \hat{V}(r), r; \hat{Q}(r)\right) dr,
\end{cases}
\tag{47}
$$

with $\hat{V}$ satisfying the following optimal condition

$$
D_V \mathcal{L}\left(\hat{X}(s), \hat{V}(s), s; \hat{Q}(s)\right) \overset{U}{=} 0, \quad \text{a.e. } s \in [t,T]. \tag{48}
$$

As a consequence of Theorems 5.5 and 5.8, we have the following solvability of forward-backward equations (47)–(48). The proof is omitted here.

Theorem 5.9. *Under Assumptions (A1)–(A2), suppose that $(\mathscr{P}^{t,X})$ has a unique optimal control. Then, there is a unique solution $(\hat{X}, \hat{V}, \hat{Q}) \in C([t,T];H) \times L^2([t,T];U) \times C([t,T];H)$ of the forward-backward equations (47)–(48). As a corollary, under Assumptions (A2)–(A4), there exists a constant Λ depending only on (L,T), such that there is a unique solution of the forward-backward equations (47) & (48) when $\lambda \geq \Lambda$.*

From Assumption (A3), we know that for any $X, \tilde{X} \in H$, $V, \tilde{V} \in U$ and $s \in [0, T]$,

$$D_X G(X, V, s)\left(\tilde{X}\right) = \mathcal{G}_1(s)\tilde{X}, \quad D_V G(X, V, s)\left(\tilde{V}\right) = \mathcal{G}_2(s)\tilde{V}. \tag{49}$$

As a consequence of Assumptions (A3)–(A4), we know that the map $H \times U \times H \ni (X, V, Q) \mapsto D_X \mathcal{L}(X, V, s; Q) \in H$ is $C(L)$-Lipschitz continuous for any $s \in [0, T]$. And for any $X \in H$, $V^1, V^2 \in U$, $Q \in H$ and $s \in [0, T]$,

$$\begin{aligned} &\mathcal{L}\left(X, V^2, s; Q\right) - \mathcal{L}\left(X, V^1, s; Q\right) \\ &\geq \left(D_V \mathcal{L}(X, V^1, s; Q), V^2 - V^1\right)_U + \lambda \left\|V^2 - V^1\right\|_U^2. \end{aligned} \tag{50}$$

We define the feedback $\hat{V} : H \times [0, T] \times H \to U$ as

$$\hat{V}(X, s; Q) := \operatorname*{argmin}_{V \in U} \mathcal{L}(X, V, s; Q), \quad (X, s, Q) \in H \times [0, T] \times H. \tag{51}$$

From (50), we know $\hat{V}$ is well-defined. We define the Hamiltonian

$$\mathcal{H}(X, s; Q) := \mathcal{L}(X, \hat{V}(X, s; Q), s; Q), \quad (X, s, Q) \in H \times [0, T] \times H. \tag{52}$$

We have the following property for $\hat{V}$, whose proof is given in Appendix A.1.2.

Lemma 5.10. *Under Assumptions (A2)–(A4), $\hat{V}$ defined above satisfies, for any $X^1, X^2, Q^1, Q^2 \in H$ and $s \in [0, T]$,*

$$\begin{aligned} &\left\|\hat{V}(0, s; 0)\right\|_U \leq \frac{L}{2\lambda}, \\ &\left\|\hat{V}(X^2, s; Q^2) - \hat{V}(X^1, s; Q^1)\right\|_U \leq \frac{C(L)}{\lambda}\left(\left\|Q^2 - Q^1\right\|_H + \left\|X^2 - X^1\right\|_H\right). \end{aligned} \tag{53}$$

From the definition of $\mathcal{H}$ and Theorem 5.9, for any initial $(t, X) \in [0, T] \times H$, there is a unique solution $(Y_{Xt}, Q_{Xt}) \in C([t, T]; H) \times C([t, T]; H)$ of the forward-backward system: for $s \in [t, T]$,

$$\begin{cases} Y_{Xt}(s) = X + \displaystyle\int_t^s D_Q \mathcal{H}(Y_{Xt}(r), r; Q_{Xt}(r)) dr, \\ Q_{Xt}(s) = D_X F_T(Y_{Xt}(T)) + \displaystyle\int_s^T D_X \mathcal{H}(Y_{Xt}(r), r; Q_{Xt}(r)) dr. \end{cases} \tag{54}$$

We denote by

$$U_{Xt}(s) := \hat{V}\left(Y_{Xt}(s), s; Q_{Xt}(s)\right), \quad s \in [t,T], \tag{55}$$

then, $U_{Xt} \in L^2([t,T];U)$ is the unique optimal control of $(\mathscr{P}^{t,X})$ when λ is large enough. In the rest of this subsection, we study the regularity of (Y_{Xt}, U_{Xt}, Q_{Xt}) with respect to the initial (t,X). We first give the boundedness and continuity with respect to the initial $X \in H$. The following lemma is proved in Appendix A.1.3.

Lemma 5.11. *Under Assumptions (A2)–(A4), there exists a constant Λ depending only on (L,T), such that when $\lambda \geq \Lambda$, we have for any $X, X' \in H$,*

$$\|Y_{Xt}\|_{C([t,T];H)} + \|U_{Xt}\|_{C([t,T];U)} + \|Q_{Xt}\|_{C([t,T];H)} \leq C(L,T)(1+\|X\|_H), \tag{56}$$

$$\begin{aligned} &\|Y_{X't} - Y_{Xt}\|_{C([t,T];H)} + \|U_{X't} - U_{Xt}\|_{C([t,T];U)} + \|Q_{X't} - Q_{Xt}\|_{C([t,T];H)} \\ \leq & C(L,T)\|X' - X\|_H. \end{aligned} \tag{57}$$

We next give the continuity of (Y_{Xt}, U_{Xt}, Q_{Xt}) with respect to the initial time $t \in [0,T]$.

Lemma 5.12. *Under Assumptions (A2)–(A4), there exists a constant Λ depending only on (L,T), such that when $\lambda \geq \Lambda$, we have for any $X \in H$, $0 \leq t \leq t' \leq T$,*

$$\begin{aligned} &\|Y_{Xt'} - Y_{Xt}\|_{C([t',T];H)} + \|U_{Xt'} - U_{Xt}\|_{C^2([t',T];U)} \\ &\quad + \|Q_{Xt'} - Q_{Xt}\|_{C([t',T];H)} \leq C(L,T)(1+\|X\|_H)|t'-t|. \end{aligned} \tag{58}$$

Proof. From the uniqueness of solution of the forward-backward equations (47)–(48), we have for $s \in [t',T]$,

$$Y_{Xt}(s) = Y_{Y_{Xt}(t'),t'}(s), \quad U_{Xt}(s) = U_{Y_{Xt}(t'),t'}(s), \quad Q_{Xt}(s) = Q_{Y_{Xt}(t'),t'}(s).$$

Therefore, from Lemma 5.11, we have

$$\begin{aligned} &\|Y_{Xt'} - Y_{Xt}\|_{C([t',T];H)} + \|U_{Xt'} - U_{Xt}\|_{C^2([t',T];U)} + \|Q_{Xt'} - Q_{Xt}\|_{C([t',T];H)} \\ =& \left\|Y_{Xt'} - Y_{Y_{Xt}(t')t'}\right\|_{C([t',T];H)} + \left\|U_{Xt'} - U_{Y_{Xt}(t')t'}\right\|_{C^2([t',T];U)} \\ &+ \left\|Q_{Xt'} - Q_{Y_{Xt}(t')t'}\right\|_{C([t',T];H)} \\ \leq& C(L,T)\left\|Y_{Xt}(t') - X\right\|_H. \end{aligned}$$

From Cauchy's inequality and Lemmas 5.10 and 5.11, we deduce that

$$
\begin{aligned}
\left\|Y_{Xt}(t') - X\right\|_H^2 &= \left\| \int_t^{t'} G(Y_{Xt}(s), U_{Xt}(s), s) ds \right\|_H^2 \\
&\leq L|t'-t| \int_t^{t'} \left[1 + \|Y_{Xt}(s)\|_H^2 + \left\| \hat{V}(Y_{Xt}(s), s; Q_{Xt}(s)) \right\|_U^2 \right] ds \\
&\leq C(L,T) \left|t'-t\right| \int_t^{t'} \left[1 + \|Y_{Xt}(s)\|_H^2 + \|Q_{Xt}(s)\|_U^2 \right] ds \\
&\leq C(L,T) \left(1 + \|X\|_H^2\right) \left|t'-t\right|^2,
\end{aligned}
$$

from which we obtain (58). □

6. Value function and Bellman equation for $(\mathscr{P}^{t,X})$

In this section, we study the value function and Bellman equation for the control problem $(\mathscr{P}^{t,X})$. For $(t,X) \in [0,T] \times H$ and a control $V \in L^2([t,T];U)$, we denote by $X_{Xt}^V \in L^2([t,T];H)$ the corresponding controlled state, Q_{Xt}^V the corresponding adjoint process, and $J_{Xt}(V)$ the corresponding cost. We define the value function $\mathscr{V} : H \times [0,T] \to \mathbb{R}$ for Problem $(\mathscr{P}^{t,X})$ as

$$
\mathscr{V}(X,t) = \inf_{V \in L^2([t,T];U)} J_{Xt}(V). \tag{59}
$$

In view of Subsection 5.3, we have

$$
\mathscr{V}(X,t) = J_{Xt}(U_{Xt}) = \int_t^T F(Y_{Xt}(s), U_{Xt}(s), s) ds + F_T(Y_{Xt}(T)), \tag{60}
$$

where (Y_{Xt}, U_{Xt}, Q_{Xt}) are defined in (54)–(55) (when λ is large enough). Moreover, we need the following additional assumption on coefficients (G,F) in t.

(A5) The map $\mathcal{G}_2$ is independent of t, and for $(X,V) \in H \times U$ and $0 \leq t \leq t' \leq T$,

$$
\begin{aligned}
&\left\|G_0(t') - G_0(t)\right\|_H \leq L\left|t'-t\right|, \quad \left\|\mathcal{G}_1(t') - \mathcal{G}_1(t)\right\|_{L(H;H)} \leq L\left|t'-t\right|, \\
&\left|F(X,V,t') - F(X,V,t)\right| \leq L\left(1 + \|X\|_H^2 + \|V\|_U^2\right)\left|t'-t\right|, \\
&\left\|D_V F(X,V,t') - D_V F(X,V,t)\right\|_U \leq L\left(1 + \|X\|_H + \|V\|_U\right)\left|t'-t\right|.
\end{aligned}
$$

6.1. Regularity of $\mathscr{V}$

We have the following properties for the value function $\mathscr{V}$, whose proof is given in Appendix A.2.1.

Lemma 6.1. *Under Assumptions (A2)–(A5), there exists a constant Λ depending only on (L,T), such that when $\lambda \geq \Lambda$, the value function $\mathscr{V}$ is C^1 with the derivatives*

$$D_X \mathscr{V}(X,t) = Q_{Xt}(t), \tag{61}$$

$$\frac{\partial \mathscr{V}}{\partial t}(X,t) = -\mathcal{H}(X,t;Q_{Xt}(t)), \tag{62}$$

and satisfies the growth conditions

$$|\mathscr{V}(X,t)| \leq C(L,T)\left(1+\|X\|_H^2\right), \tag{63}$$

$$\begin{aligned} \|D_X \mathscr{V}(X,t)\|_H &\leq C(L,T)(1+\|X\|_H), \\ \left|\frac{\partial \mathscr{V}}{\partial t}(X,t)\right| &\leq C(L,T)\left(1+\|X\|_H^2\right), \end{aligned} \tag{64}$$

and the continuity condition

$$\begin{aligned} &\|D_X \mathscr{V}(X',t') - D_X \mathscr{V}(X,t)\|_H \\ \leq& C(L,T)\|X'-X\|_H + C(L,T)(1+\|X\|_H+\|X'\|_H)|t'-t|, \\ &\left|\frac{\partial \mathscr{V}}{\partial t}(X',t') - \frac{\partial \mathscr{V}}{\partial t}(X,t)\right| \\ \leq& C(L,T)\Big[(1+\|X\|_H+\|X'\|_H)\|X'-X\|_H \\ &\qquad + (1+\|X\|_H^2+\|X'\|_H^2)|t'-t|\Big]. \end{aligned} \tag{65}$$

As a consequence of Lemma 6.1, we have the following sensitivity relation between the maximum principle and the differential of the value function.

Corollary 6.2. *Under the assumptions in Lemma 6.1, for any $(t,X) \in [0,T]\times H$ and $s \in [t,T]$, the following relations hold:*

$$\begin{aligned} &D_X \mathscr{V}(Y_{Xt}(s),s) = Q_{Xt}(s), \\ &\frac{\partial \mathscr{V}}{\partial t}(Y_{Xt}(s),s) = -\mathcal{H}(Y_{Xt}(s),s;Q_{Xt}(s)). \end{aligned} \tag{66}$$

Proof. From the uniqueness of solution of the forward-backward equations (47)–(48), we have

$$U_{Y_{Xt}(s),s}(s) = U_{Xt}(s), \quad Q_{Y_{Xt}(s),s}(s) = Q_{Xt}(s), \quad s \in [t,T]. \tag{67}$$

From Lemma 6.1 and (67), we obtain (66). □

As a consequence of (61), we also have the following corollary.

Corollary 6.3. *For initial* $(t,X) \in [0,T] \times H$*, the solution of the control problem* $(\mathscr{P}^{t,X})$ *is a feedback.*

Proof. From Theorems 5.5 and 5.9, we know that the solution of the control problem $(\mathscr{P}^{t,X})$ is U_{Xt}. From (55), (61) and (66), we have for any $s \in [t,T]$,

$$U_{Xt}(s) = \hat{V}(Y_{Xt}(s), s; Q_{Xt}(s)) = \hat{V}\left(Y_{Xt}(s), s; D_X \mathscr{V}(Y_{Xt}(s), s)\right).$$

We define the map $\phi : H \times [t,T] \to U$ as

$$\phi(X,s) := \hat{V}(X, s; D_X \mathscr{V}(X,s)), \quad (X,s) \in H \times [t,T],$$

then, we know that $U_{Xt}(s) = \phi(Y_{Xt}(s), s)$, $s \in [t,T]$ is a feedback. □

6.2. Bellman equation

Now we introduce the Bellman equation:

$$\begin{cases} \dfrac{\partial \mathscr{V}}{\partial t}(X,t) + \mathcal{H}\Big(X, t; D_X \mathscr{V}(X,t)\Big) = 0, \quad (t,X) \in [0,T) \times H; \\ \mathscr{V}(X,T) = F_T(X), \quad X \in H. \end{cases} \tag{68}$$

We give the solvability of the Bellman equation (68).

Theorem 6.4. *Under Assumptions (A2)–(A5), there exists a constant* Λ *depending only on* (L,T)*, such that when* $\lambda \geq \Lambda$*, the value function* $\mathscr{V}$ *defined in* (59) *is the unique solution of the Bellman equation* (68) *(subject to* (63)–(65)*).*

Proof. From Lemma 6.1, we know that $\mathscr{V}$ is a solution of equation (68). Now we prove the uniqueness. Let $\mathscr{U}$ be a solution of equation (68) satisfying (63)–(65). For any initial $(t,X) \in [0,T] \times H$ and control $V \in L^2([t,T];U)$, we denote by $X_{Xt}^V \in C([0,T];H)$ the corresponding controlled state process. Then, the functional $\mathscr{U}(X_{Xt}^V(s),s)$ is differentiable and

(69)
$$
\begin{aligned}
&\frac{d}{ds}\mathscr{U}(X_{Xt}^V(s),s)\\
=&\frac{\partial \mathscr{U}}{\partial s}\left(X_{Xt}^V(s),s\right)+\left(D_X\mathscr{U}\left(X_{Xt}^V(s),s\right),G\left(X_{Xt}^V(s),V(s),s\right)\right)_H\\
=&-\mathcal{L}\Big(X_{Xt}^V(s),\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\right),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\Big)\\
&+\left(D_X\mathscr{U}\left(X_{Xt}^V(s),s\right),G\left(X_{Xt}^V(s),V(s),s\right)\right)_H\\
=&\Big(D_X\mathscr{U}\left(X_{Xt}^V(s),s\right),\mathcal{G}_2\Big(V(s)-\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\right)\Big)\Big)_H\\
&-F\Big(X_{Xt}^V(s),\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\right),s\Big).
\end{aligned}
$$

From the definition of $\hat{V}$, we have

$$
\begin{aligned}
&\Big(D_X\mathscr{U}\left(X_{Xt}^V(s),s\right),\mathcal{G}_2\Big(V(s)-\hat{V}\big(X_{Xt}^V(s),s;D_X\mathscr{U}(X_{Xt}^V(s),s)\big)\Big)\Big)_H\\
=&-\Big(D_VF\Big(X_{Xt}^V(s),\hat{V}\big(X_{Xt}^V(s),D_X\mathscr{U}(X_{Xt}^V(s),s),s\big),s\Big),\\
&\qquad V(s)-\hat{V}\big(X_{Xt}^V(s),s;D_X\mathscr{U}(X_{Xt}^V(s),s)\big)\Big)_U,\quad s\in(t,T).
\end{aligned}
$$

Substituting the last equality back to (69) and integration s over $[t,T]$, from Assumption (A4), we have

$$
\begin{aligned}
&J_{Xt}(V)-\mathscr{U}(X,t)\\
=&\int_t^T\Big[F\left(X_{Xt}^V(s),V(s),s\right)-F\Big(X_{Xt}^V(s),\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\right),s\Big)\\
&\qquad-\Big(D_VF\Big(X_{Xt}^V(s),\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\right),s\Big),\\
&\qquad\qquad V(s)-\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}(X_{Xt}^V(s),s)\right)\Big)_U\Big]ds\\
\geq&\lambda\int_t^T\left\|V(s)-\hat{V}\left(X_{Xt}^V(s),s;D_X\mathscr{U}\left(X_{Xt}^V(s),s\right)\right)\right\|_U^2ds.
\end{aligned}
$$

Therefore, we have

$$
\mathscr{U}(X,t)\leq J_{Xt}(V),\quad\forall\, V\in L^2([t,T];U).
$$

If we set $V = U_{Xt}$, we see that $\mathscr{U}(X,t) = J_{Xt}(U_{Xt})$. Therefore, $\mathscr{U}$ coincides with the value function. □

Remark 6.5. *Our results in Lemma 6.1 and Theorem 6.4 extend the previous results in Bensoussan and Yam [1, Theorem 2.1] where*

$$G(V) = V, \quad F(X,V) = \frac{\lambda}{2}\|V\|_U^2 + \mathcal{F}(X).$$

7. Application in mean field type control problems

In this section, we go back to problem (13)–(14). We apply our results in Section 5 to study the following McKean-Vlasov control problem for initial $(t,m) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^n)$ and $X \in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$:

$$(70) \qquad (\mathscr{P}^{t,m,X})\begin{cases} \inf_{v_{X.mt} \in L^2([t,T];L^2_m(\mathbb{R}^n;\mathbb{R}^d))} J_{Xmt}(v_{X.mt}), \\ \text{s.t. } X^v_{X_xmt}(s) = X_x + \int_t^s g\big(X^v_{X_xmt}(r), X^v_{X.mt}(r)\#m, v_{X_xmt}(r), r\big)dr, \end{cases}$$

where the cost functional is defined as

$$\begin{aligned} J_{Xmt}(v_{X.mt}) := &\int_t^T \int_{\mathbb{R}^n} f\left(X^v_{X_xmt}(s), X^v_{X.mt}(s)\#m, v_{X_xmt}(s), s\right) dm(x)ds \\ &+ \int_{\mathbb{R}^n} h\left(X^v_{X_xmt}(T), X^v_{X.mt}(T)\#m\right) dm(x). \end{aligned}$$

Here, $L^2_m(\mathbb{R}^n;\mathbb{R}^n)$ is the Hilbert space with respect to the inner product

$$(X,Y) := \int_{\mathbb{R}^n} X_x \cdot Y_x dm(x), \quad X,Y \in L^2_m(\mathbb{R}^n;\mathbb{R}^n).$$

For $m \in \mathcal{P}_2(\mathbb{R}^n)$ and $X \in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, we have $X\#m \in \mathcal{P}_2(\mathbb{R}^n)$, and

$$(71) \qquad \begin{aligned} &W_2(X\#m, \delta_0) = \|X\|_{L^2_m}, \\ &W_2(X\#m, X'\#m) \le \|X - X'\|_{L^2_m}, \quad X, X' \in L^2_m(\mathbb{R}^n;\mathbb{R}^n). \end{aligned}$$

We refer to Bonnet and Frankowska [6] for details. By letting X to be the identity function I, (70) introduce the following mean field type control problem

(72)

$$(\mathscr{P}^{t,m})\begin{cases}\inf_v \mathbb{E}\Big[\int_t^T f(\mathcal{X}_{mt}^v(s), \mathcal{L}\mathcal{X}_{mt}^v(s), v(s), s)ds + h(\mathcal{X}_{mt}^v(T), \mathcal{L}\mathcal{X}_{mt}^v(T))\Big],\\ \text{s.t. } \mathcal{X}_{mt}^v(s) = \eta + \int_t^s g(\mathcal{X}_{mt}^v(r), \mathcal{L}\mathcal{X}_{mt}^v(r), v(r), r)dr, \quad s \in [t,T],\end{cases}$$

where η is a random variable such that $\mathcal{L}\eta = m$, and a control $v(s),\ s \in [t,T]$, is a stochastic process adapted to the σ-algebra generated by η, such that $\mathbb{E}\int_t^T |v(s)|^2 ds < +\infty$. Necessarily, $v(s) = v_\eta(s)$ where $v.(\cdot)$ is deterministic and measurable. The mean field type control problem $(\mathscr{P}^{t,m})$ is equivalent to the control problem $(\mathscr{P}^{t,m,I})$, since $X_{I.mt}^v(s)\#m = \mathcal{L}\mathcal{X}_{mt}^v(s),\ s \in [t,T]$. For notational convenience, we drop the subscript (t,m,X) in the state process and cost functional for $(\mathscr{P}^{t,m,X})$ in this section when there is no ambiguity.

7.1. Embedding $(\mathscr{P}^{t,m,X})$ into Hilbert spaces

For any fixed $m \in \mathcal{P}_2(\mathbb{R}^n)$, we can make a connection bewteen the McKean-Vlasov control problem $(\mathscr{P}^{t,m,X})$ and the control problem on Hilbert spaces $(\mathscr{P}^{t,X})$ defined in (17) by setting $H := L_m^2(\mathbb{R}^n;\mathbb{R}^n)$, $U := L_m^2(\mathbb{R}^n;\mathbb{R}^d)$, and

$$G(X,V,s)|_x := g(X_x, X.\#m, V_x, s), \tag{73}$$

$$F(X,V,s) := \int_{\mathbb{R}^n} f(X_x, X.\#m, V_x, s)dm(x), \tag{74}$$

$$F_T(X) := \int_{\mathbb{R}^n} h(X_x, X.\#m)dm(x), \tag{75}$$

for $(X,V,s) \in L_m^2(\mathbb{R}^n;\mathbb{R}^n) \times L_m^2(\mathbb{R}^n;\mathbb{R}^d) \times [0,T]$ and $x \in \mathbb{R}^n$. We define the Lagrangian $L : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^d \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ as

$$L(x,m,v,s;q) := q \cdot g(x,v,m,s) + f(x,v,m,s), \tag{76}$$

and define $\mathcal{L} : L_m^2(\mathbb{R}^n;\mathbb{R}^n) \times L_m^2(\mathbb{R}^n;\mathbb{R}^d) \times [0,T] \times L_m^2(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ as

$$\mathcal{L}(X,V,s;Q) := \int_{\mathbb{R}^n} L(X_x, X\#m, V_x, s; Q_x)dm(x). \tag{77}$$

Then, we know that $(\mathcal{L}, G, F)$ satisfy (22). We have the following connection between the linear functional derivative in $\mathcal{P}_2(\mathbb{R}^n)$ and the Gâteaux derivative in $L_m^2(\mathbb{R}^n;\mathbb{R}^n)$, whose proof is given in Appendix A.3.1.

Lemma 7.1. *Let $f : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$ be a differentiable functional such that for any $\mu \in \mathcal{P}_2(\mathbb{R}^n)$, the map $\mathbb{R}^n \ni \xi \mapsto \frac{df}{d\nu}(\mu)(\xi) \in \mathbb{R}$ is differentiable with the derivative $D_\xi \frac{df}{d\nu}(\mu)(\xi)$ being continuous in (μ, ξ) and*

$$D_\xi \frac{df}{d\nu}(\mu)(\xi) \leq c(\mu)(1 + |\xi|), \quad (\mu, \xi) \in \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n.$$

For $m \in \mathcal{P}_2(\mathbb{R}^n)$, we define $F : L^2_m(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$ as

$$F(X) := f(X\#m), \quad X \in L^2_m(\mathbb{R}^n; \mathbb{R}^n).$$

Then, F is Gâteaux differentiable, and

$$D_X F(X)\Big|_x = D_\xi \frac{df}{d\nu}(X\#m)(X_x), \quad X \in L^2_m(\mathbb{R}^n; \mathbb{R}^n). \tag{78}$$

By letting X to be the identity function I, (78) becomes

$$D_X F(I)\Big|_x = D_x \frac{df}{d\nu}(m)(x),$$

which is identical to the L-derivative $\partial_m f(m)(x)$ in Carmona-Delarue [8]. We also refer to Bensoussan-Frehse-Yam [2] for further discussion.

7.2. Necessary condition for $(\mathscr{P}^{t,m,X})$

Our assumptions in this subsection are as follows. For notational convenience, we use the same constant $l > 0$ for all the conditions below.

(B1) The function g satisfies for $(x, m, v, s) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^d \times [0, T]$,

$$|g(x, m, v, s)| \leq l(1 + |x| + W_2(m, \delta_0) + |v|),$$

and is differentiable in (x, m, v), and the derivative $\frac{dg}{d\nu}(x, m, v, s)(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable. The derivatives $D_x g$, $D_\xi \frac{dg}{d\nu}$ and $D_v g$ are bounded by l and continuous in all variables.

(B2) The functions f and g satisfy for $(x, m, v, s) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^d \times [0, T]$,

$$|f(x, m, v, s)| \leq l\left(1 + |x|^2 + W_2^2(m, \delta_0) + |v|^2\right),$$
$$|h(x, m)| \leq l\left(1 + |x|^2 + W_2^2(m, \delta_0)\right).$$

The functional f is differentiable in (x, m, v) and $\frac{df}{d\nu}(x, m, v, s)(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is differentiable. The function h is differentiable in (x, m), and the derivatives $\frac{dh}{d\nu}(x, m)(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is differentiable. The derivatives $D_x f$, $D_\xi \frac{df}{d\nu}$, $D_v f$, $D_x h$ and $D_\xi \frac{dh}{d\nu}$ are continuous in all variables, and satisfy

$$\begin{aligned}
&|(D_x f, D_v f)(x, m, v, s)| \leq l(1 + |x| + W_2(m, \delta_0) + |v|),\\
&\left| D_\xi \frac{df}{d\nu}(x, m, v, s)(\xi) \right| \leq l(1 + |x| + W_2(m, \delta_0) + |v| + |\xi|),\\
&|D_x h(x, m)| \leq l(1 + |x| + W_2(m, \delta_0)),\\
&\left| D_\xi \frac{dh}{d\nu}(x, m)(\xi) \right| \leq l(1 + |x| + W_2(m, \delta_0) + |\xi|).
\end{aligned}$$

We now give the differentiability of maps G, F, F_T and $\mathcal{L}$ in $(X, V) \in L^2_m(\mathbb{R}^n; \mathbb{R}^n) \times L^2_m(\mathbb{R}^n; \mathbb{R}^d)$. The following lemma is proved in Appendix A.3.2.

Lemma 7.2. *Under Assumptions (B1)–(B2), G defined in (73) satisfies (A1); F defined in (74) and F_T defined in (75) satisfy (A2), with a constant $C(l)$ depending only on l. The functional $\mathcal{L}$ defined in (77) satisfies (23). For any $(s, Q) \in [0, T] \times L^2_m(\mathbb{R}^n; \mathbb{R}^n)$, the functional $L^2_m(\mathbb{R}^n; \mathbb{R}^n) \times L^2_m(\mathbb{R}^n; \mathbb{R}^d) \ni (X, V) \mapsto \mathcal{L}(X, V, s, Q) \in \mathbb{R}$ is continuously differentiable, with the derivatives satisfying (24). Moreover, for $s \in [0, T]$, $X, \tilde{X} \in L^2_m(\mathbb{R}^n; \mathbb{R}^n)$ and $V, \tilde{V} \in L^2_m(\mathbb{R}^n; \mathbb{R}^d)$, we have for $x \in \mathbb{R}^n$,*

(79)
$$\begin{aligned}
D_X G(X, V, s)(\tilde{X})\Big|_x &= (D_x g(X_x, X\#m, V_x, s))^* \tilde{X}_x\\
&\quad + \int_{\mathbb{R}^n} \left(D_\xi \frac{dg}{d\nu}(X_x, X\#m, V_x, s)(X_y) \right)^* \tilde{X}_y dm(y),
\end{aligned}$$

(80)
$$D_V G(X, V, s)(\tilde{V})\Big|_x = (D_v g(X_x, X\#m, V_x, s))^* \tilde{V}_x,$$

(81)
$$\begin{aligned}
D_X F(X, V, s)|_x &= D_x f(X_x, X\#m, V_x, s)\\
&\quad + \int_{\mathbb{R}^n} D_\xi \frac{df}{d\nu}(X_y, X\#m, V_y, s)(X_x) dm(y),
\end{aligned}$$

(82)
$$D_V F(X, V, s)|_x = D_v f(X_x, X\#m, V_x, s),$$

(83)

$$D_X F_T(X)|_x = D_x h(X_x, X\#m) + \int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}(X_y, X\#m)(X_x) dm(y),$$

and for $Q \in L^2_m(\mathbb{R}^n; \mathbb{R}^n)$*, we have for* $x \in \mathbb{R}^n$*,*

(84)

$$\begin{aligned} D_X \mathcal{L}(X, V, s; Q)|_x =& D_x g(X_x, X\#m, V_x, s) Q_x + D_x f(X_x, X\#m, V_x, s) \\ &+ \int_{\mathbb{R}^n} D_\xi \frac{dg}{d\nu}(X_y, X\#m, V_y, s)(X_x) Q_y dm(y) \\ &+ \int_{\mathbb{R}^n} D_\xi \frac{df}{d\nu}(X_y, X\#m, V_y, s)(X_x) dm(y), \end{aligned}$$

(85)

$$D_V \mathcal{L}(X, V, s; Q)|_x = D_v g(X_x, X\#m, V_x, s) Q_x + D_v f(X_x, X\#m, V_x, s).$$

Let $\hat{v}.(\cdot)$ be a optimal control for $(\mathscr{P}^{t,m,X})$ and $\hat{X}.(\cdot)$ be the corresponding state process. We define the adjoint process as

(86)

$$\begin{aligned} \hat{Q}_x(s) =& D_x h\left(\hat{X}_x(T), \hat{X}.(T)\#m\right) \\ &+ \int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}\left(\hat{X}_y(T), \hat{X}.(T)\#m\right)\left(\hat{X}_x(T)\right) dm(y) \\ &+ \int_s^T \Bigg[D_x L\left(\hat{X}_x(r), \hat{X}.(r)\#m, \hat{v}_x(r), r; \hat{Q}_x(r)\right) \\ &+ \int_{\mathbb{R}^n} D_\xi \frac{dL}{d\nu}\left(\hat{X}_y(r), \hat{X}.(r)\#m, \hat{v}_y(r), r; \hat{Q}_y(r)\right)\left(\hat{X}_x(r)\right) dm(y) \Bigg] dr. \end{aligned}$$

From Lemma 7.2, we know that $\hat{Q}.(\cdot)$ defined in (86) satisfies equation (25). The following necessary condition is then a direct consequence of Theorem 5.5 and Lemma 7.2.

Theorem 7.3. *Under Assumptions (B1)–(B2), let* $\hat{v}.(\cdot)$ *be an optimal control for* $(\mathscr{P}^{t,m,X})$*,* $\hat{X}.(\cdot)$ *be the corresponding controlled state process, and* $\hat{Q}.(\cdot)$ *be the corresponding adjoint process. Then, we have the optimality condition*

$$D_v L(\hat{X}_x(s), \hat{X}.(s)\#m, \hat{v}_x(s), s; \hat{Q}_x(s)) = 0, \ \ a.e.\ s \in [t, T], \ \ a.s.\ dm(x).$$

We also consider the following constrained McKean-Vlasov control problem

$$(\mathscr{P}_C^{t,m,X})\begin{cases} \inf\limits_{v_{X.mt}\in L^2([t,T];L^2_m(\mathbb{R}^n;\mathbb{R}^d))} J_{Xmt}(v_{X.mt}), \\ \text{s.t.} \begin{cases} X^v_{X_xmt}(s)=X_x+\displaystyle\int_t^s g(X^v_{X_xmt}(r),X^v_{X.mt}(r)\#m,v_{X_xmt}(r),r)dr, \\ \displaystyle\int_{\mathbb{R}^n} h_i\left(X^v_{X_xmt}(T),X^v_{X.mt}(T)\#m\right)dm(x)\leq 0,\ 1\leq i\leq N, \end{cases} \end{cases}$$

where

$$h_i:\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\to\mathbb{R},\quad 1\leq i\leq N.$$

By setting

$$\Psi_i(X):=\int_{\mathbb{R}^n} h_i(X_x,X.\#m)dm(x),\quad X\in L^2_m(\mathbb{R}^n;\mathbb{R}^n),\quad 1\leq i\leq N,$$

we can transform the constrained McKean-Vlasov control problem $(\mathscr{P}_C^{t,m,X})$ into a constrained control problem on Hilbert spaces $(\mathscr{P}_C^{t,X})$ defined in (31). As a consequence of Theorem 5.6, we have the following necessary condition for $(\mathscr{P}_C^{t,m,X})$.

Theorem 7.4. *Under Assumptions (B1)–(B2), suppose that h_i satisfies conditions in (B2) as h for $1\leq i\leq N$. Let $\hat{v}.(\cdot)$ be an optimal control for $(\mathscr{P}_C^{t,X})$ and $\hat{X}.(\cdot)$ be the corresponding controlled state process. Then, there exist non-trivial Lagrange multipliers $(\lambda_0,\lambda_1,\ldots,\lambda_N)\in\{0,1\}\times\mathbb{R}^N_+$, such that*

$$D_vL_{\lambda_0}\left(\hat{X}_x(s),\hat{X}.(s)\#m,\hat{v}_x(s),s;\hat{Q}_{\lambda,x}(s)\right)=0,\quad a.e.\ s\in[t,T],\ a.s.\ dm(x),$$
$$\lambda_ih_i\left(\hat{X}_x(T),\hat{X}.(T)\#m\right)=0,\quad a.s.\ dm(x),\quad 1\leq i\leq N,$$

where

$$\begin{aligned} L_{\lambda_0}(x,m,v,s;q)&:=q\cdot g(x,m,v,s)+\lambda_0f(x,v,m,s),\\ &(x,m,v,s;q)\in\mathbb{R}^n\times\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^d\times[0,T]\times\mathbb{R}^n, \end{aligned}\tag{87}$$

and $\hat{Q}_{\lambda,x}(\cdot)$ *is defined as, for* $s \in [t,T]$,

(88)

$$\begin{aligned}
\hat{Q}_{\lambda,x}(s) =& \lambda_0 D_x h\left(\hat{X}_x(T), \hat{X}_\cdot(T)\#m\right) \\
&+ \lambda_0 \int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}\left(\hat{X}_y(T), \hat{X}_\cdot(T)\#m\right)\left(\hat{X}_x(T)\right) dm(y) \\
&+ \sum_{i=1}^N \lambda_i \Big[D_x h_i\left(\hat{X}_x(T), \hat{X}_\cdot(T)\#m\right) \\
&\qquad + \int_{\mathbb{R}^n} D_\xi \frac{dh_i}{d\nu}\left(\hat{X}_y(T), \hat{X}_\cdot(T)\#m\right)\left(\hat{X}_x(T)\right) dm(y) \Big] \\
&+ \int_s^T \Big[D_x L_{\lambda_0}\left(\hat{X}_x(r), \hat{X}_\cdot(r)\#m, \hat{v}_x(r), r; \hat{Q}_{\lambda,x}(r)\right) \\
&+ \int_{\mathbb{R}^n} D_\xi \frac{dL_{\lambda_0}}{d\nu}\left(\hat{X}_y(r), \hat{X}_\cdot(r)\#m, \hat{v}_y(r), r; \hat{Q}_{\lambda,y}(r)\right)\left(\hat{X}_x(r)\right) dm(y) \Big] dr.
\end{aligned}$$

7.3. Recovering the results of Bonnet-Frankowska-Rossi

Bonnet-Rossi [7] study the case

$$\begin{aligned}
&g(x,m,v,s) := g^1(m)(x,s) + v(x), \quad f(x,m,v,s) := f^1(m,v), \\
&h(x,m) := h^1(m), \quad (x,m,v,s) \in \mathbb{R}^d \times \mathcal{P}_c(\mathbb{R}^d) \times U \times [0,T],
\end{aligned} \tag{89}$$

where the set of admissible controls is $\mathcal{U} := L^1([0,T];U)$, U is a non-empty and closed subset of $\left\{v \in C^1(\mathbb{R}^d, \mathbb{R}^d) \text{ s.t. } \|v(\cdot)\|_{C^1(\mathbb{R}^d)} \le L_U\right\}$, and $\mathcal{P}_c(\mathbb{R}^d)$ is the set of distributions that have compact support. Then, the Pontryagin maximization condition (26) in [7, Theorem 5] is a result of our Theorem 7.3, except that their initial distribution has compact support, while our initial distribution belongs to $\mathcal{P}_2(\mathbb{R}^n)$. Bonnet-Frankowska [5] study constrained McKean-Vlasov control problem when

$$\begin{aligned}
&g(x,m,v,s) := g^2(m,v,s)(x), \quad f(x,m,v,s) := f^2(m,v,s), \\
&h(x,m) := h^2(m), \quad h_i(x,m) := h_i^2(m), \quad 1 \le i \le N, \\
&\qquad\qquad (x,m,v,s) \in \mathbb{R}^d \times \mathcal{P}_c(\mathbb{R}^d) \times U \times [0,T],
\end{aligned} \tag{90}$$

and the set of admissible controls is $\mathcal{U} := \{s \mapsto v(s) \in U \text{ s.t. } v(\cdot) \text{ is measure}\}$, and U is a compact metric space. Then, from our Theorem 7.4, we can obtain the necessary result [5, Theorem 4.9] for constrained problem. We can see that, in case (89), the controls are closed-loop and should be continuously differentiable in x; and in case (90), the controls are open-loop. Inspired by their works Bonnet-Rossi [7] and Bonnet-Frankowska [5], we try to extend their approach, so that we allow our model by including for general $(\mathscr{P}^{t,m,X})$ and $(\mathscr{P}_C^{t,m,X})$ the admissible control set that contains both open-loop and closed-loop. An admissible control only need to be L_m^2-integrable in x, and need not to be continuous. Another advantage is that, our coefficients are of more general forms. The cost function is of quadratical growth, which includes the common linear quadratic cases. Moreover, we only need the distribution to be in $\mathcal{P}_2(\mathbb{R}^n)$.

7.4. Sufficient condition for $(\mathscr{P}^{t,m,X})$

We give a sufficient condition for $(\mathscr{P}^{t,m,X})$ under the following additional assumptions.

(B3) The function g is linear. That is,

$$g(x,m,v,s) = g_0(s) + g_1(s)x + g_2(s)\int_{\mathbb{R}^n} y dm(y) + g_3(s)v,$$

with $g_0(s) \in \mathbb{R}^n$, $g_1(s) \in \mathbb{R}^{n\times n}$, $g_2(s) \in \mathbb{R}^{n\times n}$ and $g_3(s) \in \mathbb{R}^{n\times d}$ being bounded by l.

(B4) The functions $D_x f$, $D_\xi \frac{df}{d\nu}$, $D_v f$, $D_x h$ and $D_\xi \frac{dh}{d\nu}$ are l-Lipschitz continuous. Moreover, there exists $\lambda > 0$ such that

$$f(x,m,v',s) - f(x,m,v,s) \geq D_v f(x,m,v,s)\cdot(v'-v) + \lambda|v'-v|^2.$$

From Assumption (B4), for any $X \in L_m^2(\mathbb{R}^n;\mathbb{R}^n)$, $V^1, V^2 \in L_m^2(\mathbb{R}^n;\mathbb{R}^d)$ and $s \in [0,T]$, we have

$$\begin{aligned}
&F(X,V^2,s) - F(X,V^1,s)\\
&= \int_{\mathbb{R}^n} \left[f(X_x, X\#m, V_x^2, s) - f(X_x, X\#m, V_x^1, s)\right] dm(x)\\
&\geq \int_{\mathbb{R}^n} \left[D_v f(X_x, X\#m, V_x^1, s)\cdot(V_x^2 - V_x^1) + \lambda|V_x^2 - V_x^1|^2\right] dm(x)\\
&= \int_{\mathbb{R}^n} D_V F(X,V^1,s)\big|_x \cdot (V_x^2 - V_x^1) dm(x) + \lambda\|V^2 - V^1\|_U^2.
\end{aligned}$$

Then, we know that F satisfies the convexity conditions in (A4). From Lemma 7.2, we know that (G, F, F_T) defined in (73)–(75) satisfy conditions (A3)–(A4), with a constant $C(l)$ depending only on l. As a consequence of Theorem 5.8, we have the following solvability of $(\mathscr{P}^{t,m,X})$.

Theorem 7.5. *Under Assumptions (B2)–(B4), there exists a constant* Λ *depending only on* (l, T)*, such that when* $\lambda \geq \Lambda$*, the cost function* J_{Xmt} *is strictly convex, and* $(\mathscr{P}^{t,m,X})$ *has a unique optimal control.*

7.5. Forward-backward system for $(\mathscr{P}^{t,m,X})$

We derive from Theorem 7.3 the following forward-backward system for $(\hat{X}.(\cdot), \hat{v}.(\cdot), \hat{Q}.(\cdot))$ in spaces $C([t,T]; L^2_m(\mathbb{R}^n;\mathbb{R}^n)) \times L^2([t,T]; L^2_m(\mathbb{R}^n;\mathbb{R}^d)) \times C([t,T]; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$: for $(s,x) \in [t,T] \times \mathbb{R}^n$,

$$\begin{cases} \hat{X}_x(s) = X_x + \displaystyle\int_t^s g\left(\hat{X}_x(r), \hat{X}.(r)\#m, \hat{v}_x(r), r\right) dr, \\ \hat{Q}_x(s) = D_x h\left(\hat{X}_x(T), \hat{X}.(T)\#m\right) \\ \qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}\left(\hat{X}_y(T), \hat{X}.(T)\#m\right)\left(\hat{X}_x(T)\right) dm(y) \\ \qquad + \displaystyle\int_s^T \Big[D_x L\left(\hat{X}_x(r), \hat{X}.(r)\#m, \hat{v}_x(r), r; \hat{Q}_x(r)\right) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dL}{d\nu}\left(\hat{X}_y(r), \hat{X}.(r)\#m, \hat{v}_y(r), r; \hat{Q}_y(r)\right) \\ \qquad\qquad\qquad \left(\hat{X}_x(r)\right) dm(y)\Big] dr, \end{cases} \tag{91}$$

with $\hat{v}.(\cdot)$ satisfying the optimal condition

$$D_v L\left(\hat{X}_x(s), \hat{X}.(s)\#m, \hat{v}_x(s), s; \hat{Q}_x(s)\right) = 0, \text{ a.e. } s \in [t,T], \text{ a.s. } dm(x). \tag{92}$$

As a consequence of Theorem 5.9, we have the following solvability result for forward-backward equations (91)–(92). We refer to Bensoussan-Tai-Yam [3] and Ciampa-Rossi [9] for further discussion on these forward-backward equations.

Theorem 7.6. *Under Assumptions (B2)–(B4), there exists a constant* Λ *depending on* (l, T)*, such that there is a unique solution* $(\hat{X}.(\cdot), \hat{v}.(\cdot), \hat{Q}.(\cdot)) \in C([t,T]; H) \times L^2([t,T]; U) \times C([t,T]; H)$ *of the forward-backward equations (91)–(92) when* $\lambda \geq \Lambda$*.*

We define the feedback $\hat{v} : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ as

$$\hat{v}(x,m,s;q) := \operatorname*{argmin}_{v \in \mathbb{R}^d} L(x,m,v,s;q).$$

From Assumptions (B3)–(B4), we know $\hat{v}$ is well-defined. We define the Hamiltonian such that for $(x, m, s, q) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times [0, T] \times \mathbb{R}^n$,

$$H(x, m, s; q) := L(x, m, \hat{v}(x, m, s; q), s; q), \tag{93}$$

and define $\hat{V} : L^2_m(\mathbb{R}^n; \mathbb{R}^n) \times [0, T] \times L^2_m(\mathbb{R}^n; \mathbb{R}^n) \to L^2_m(\mathbb{R}^n; \mathbb{R}^d)$ as

$$\hat{V}(X, s; Q)\Big|_x := \hat{v}(X_x, X\#m, s; Q_x), \quad x \in \mathbb{R}^n. \tag{94}$$

From (77) and (94), we know that

$$\hat{V}(X, s; Q) = \operatorname*{argmin}_{V \in L^2_m(\mathbb{R}^n;\mathbb{R}^d)} \mathcal{L}(X, V, s; Q),$$

which concides with the definition in (51) in Section 5. We define $\mathcal{H}$: $L^2_m(\mathbb{R}^n; \mathbb{R}^n) \times [0, T] \times L^2_m(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$ as

$$\mathcal{H}(X, s; Q) := \int_{\mathbb{R}^n} H(X_x, X\#m, s; Q_x) dm(x).$$

From (77), (93) and (94), we know that

$$\mathcal{H}(X, s; Q) = \mathcal{L}(X, \hat{V}(X, s; Q), s; Q),$$

which concides with the definition in (52) in Section 5. From above definitions and Theorem 7.6, for any initial $(t, X) \in [0, T] \times L^2_m(\mathbb{R}^n; \mathbb{R}^n)$, there is a unique solution $(Y_{X.mt}(\cdot), Q_{X.mt}(\cdot)) \in C([t, T]; L^2_m(\mathbb{R}^n; \mathbb{R}^n)) \times C([t, T]; L^2_m(\mathbb{R}^n; \mathbb{R}^n))$ of the forward-backward system

(95)
$$\begin{cases} Y_{X_x mt}(s) = X_x + \displaystyle\int_t^s D_q H(Y_{X_x mt}(r), Y_{X.mt}(r)\#m, r; Q_{X_x mt}(r)) dr, \\ Q_{X_x mt}(s) = D_x h(Y_{X_x mt}(T), Y_{X.mt}(T)\#m) \\ \qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dh}{d\nu}(Y_{X_y mt}(T), Y_{X.mt}(T)\#m)(Y_{X_x mt}(T)) dm(y) \\ \qquad + \displaystyle\int_s^T \Bigg[D_x H(Y_{X_x mt}(r), Y_{X.mt}(r)\#m, r; Q_{X_x mt}(r)) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^n} D_\xi \frac{dH}{d\nu}(Y_{X_y mt}(r), Y_{X.mt}(r)\#m, r; Q_{X_y mt}(r)) \\ \qquad\qquad\qquad (Y_{X_x mt}(r)) dm(y) \Bigg] dr. \end{cases}$$

We denote by

$$u_{X_x mt}(s) = \hat{v}\left(Y_{X_x mt}(s), Y_{X.mt}(s)\#m, s; Q_{X_x mt}(s)\right), \tag{96}$$

then, $u_{X.mt}(\cdot) \in L^2([t,T]; L^2_m(\mathbb{R}^n;\mathbb{R}^d))$ is the unique optimal control of problem $(\mathscr{P}^{t,m,X})$ when λ is large enough. As a consequence of Lemmas 5.11 and 5.12, we have the following regularity of $(Y_{X.mt}(\cdot), u_{X.mt}(\cdot), Q_{X.mt}(\cdot))$ with respect to the initial $(t,X) \in [0,T] \times L^2_m(\mathbb{R}^n;\mathbb{R}^n)$.

Lemma 7.7. *Under Assumptions (B2)–(B4), there exists a constant Λ depending only on (l,T), such that when $\lambda \geq \Lambda$, we have for any $X, X' \in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$ and $0 \leq t \leq t' \leq T$,*

$$\begin{aligned}
&\|Y_{X.mt}(\cdot)\|_{C([t,T];L^2_m)} + \|u_{X.mt}\|_{C([t,T];L^2_m)} + \|Q_{X.mt}(\cdot)\|_{C([t,T];L^2_m)} \\
\leq& C(l,T)\left(1 + \|X\|_{L^2_m}\right), \\
&\|Y_{X'.mt'}(\cdot) - Y_{X.mt}(\cdot)\|_{C([t,T];L^2_m)} + \|u_{X'.mt'}(\cdot) - u_{X.mt}(\cdot)\|_{C([t,T];L^2_m)} \\
&\quad + \|Q_{X'.mt'}(\cdot) - Q_{X.mt}(\cdot)\|_{C([t,T];L^2_m)} \\
\leq& C(l,T)\|X' - X\|_{L^2_m} + C(l,T)\left(1 + \|X\|_{L^2_m} + \|X'\|_{L^2_m}\right)|t' - t|.
\end{aligned}$$

7.6. Linear quadratic problems

As an example, we discuss the linear quadratic case: for any $(x,m,v) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^d$,

$$\begin{aligned}
g(x,m,v) &= Ax + \bar{A}\int_{\mathbb{R}^n} \xi dm(\xi) + Bv, \\
f(x,m,v) &= \frac{1}{2}\left[x^* M x + \left(\int_{\mathbb{R}^n} \xi dm(\xi)\right)^* \bar{M} \int_{\mathbb{R}^n} \xi dm(\xi) + v^* N v\right], \\
h(x,m) &= \frac{1}{2}\left[x^* M_T x + \left(\int_{\mathbb{R}^n} \xi dm(\xi)\right)^* \bar{M}_T \int_{\mathbb{R}^n} \xi dm(\xi)\right],
\end{aligned} \tag{97}$$

where $A, \bar{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$, and $M, \bar{M}, M_T, \bar{M}_T \in \mathbb{R}^{n\times n}$ are positive definite and $N \geq \lambda I_d$ for some $\lambda > 0$. It is easy to check that functionals defined in (97) satisfy Assumptions (B1)–(B4). From Theorem 7.6, we know that when λ is large enough, there is a unique solution $(X,Q) \in C([t,T]; L^2_m(\mathbb{R}^n;\mathbb{R}^n)) \times C([t,T]; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$ of the forward-backward equations:

$$\begin{cases} \dfrac{dX_x(s)}{ds} = AX_x(s) + \bar{A}\displaystyle\int_{\mathbb{R}^n} X_y(s)dm(y) - BN^{-1}B^*Q_x(s), \ s \in (t,T]; \\ \dfrac{dQ_x(s)}{ds} = -\Big[Q_x(s)A + \displaystyle\int_{\mathbb{R}^n} Q_y(s)\bar{A}dm(y) + X_x^*(s)M \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^n} X_y^*(s)dm(y)\bar{M}\Big], \ s \in [t,T); \\ X_x(t) = X_x, \quad Q_x(T) = X_x^*(T)M_T + \displaystyle\int_{\mathbb{R}^n} X_y^*(T)dm(y)\bar{M}_T, \quad x \in \mathbb{R}^n, \end{cases}$$

and

$$v_x(s) = -N^{-1}B^*Q_x(s), \quad s \in [t,T], \quad x \in \mathbb{R}^n,$$

is an optimal control for $(\mathscr{P}^{t,m,X})$ corresponding to (g,f,h) defined in (97).

8. Value function for mean field control problem

8.1. Regularity and Bellman equation for $\mathscr{V}$

For $(t,m) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^n)$, $X \in L_m^2(\mathbb{R}^n)$ and $v.(\cdot) \in L^2([t,T]; L_m^2(\mathbb{R}^n;\mathbb{R}^d))$, we denote by $X_{X.mt}^v(\cdot)$ the corresponding controlled state and $J_{Xmt}(v)$ the corresponding cost for the control problem $(\mathscr{P}^{t,m,X})$. We define the value function $\mathscr{V}$ for the control problem $(\mathscr{P}^{t,m,X})$ as

$$\mathscr{V}(X,m,t) := \inf_{v.(\cdot) \in L^2([t,T];L_m^2(\mathbb{R}^n;\mathbb{R}^d))} J_{Xmt}(v).$$

In view of Subsection 7.5, we have

$$\begin{aligned} \mathscr{V}(X,m,t) &= J_{Xmt}(u_{X.mt}(\cdot)) \\ &= \int_t^T \int_{\mathbb{R}^n} f(Y_{X_xmt}(s), Y_{X.mt}(s)\#m, u_{X_xmt}, s)dm(x)ds \\ &\quad + \int_{\mathbb{R}^n} h(Y_{X_xmt}(T), Y_{X.mt}(T)\#m)dm(x), \end{aligned}$$

where $(Y_{X.mt}(\cdot), u_{X.mt}(\cdot), Q_{X.mt}(\cdot))$ is the solution of forward-backward equations (95)–(96) (when λ is large enough). Moreover, we need the following additional assumption on coefficients (g,f) in t.

(B5) The functions (g_0, g_1, g_2) are L-Lipschitz continuous in t and g_3 is independent of t, and for $(x, m, v) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^d$ and $0 \le t \le t' \le T$,

$$|f(x,m,v,t') - f(x,m,v,t)| \le l\left(1 + |x|^2 + W_2^2(m,\delta_0) + |v|^2\right)|t'-t|,$$
$$\left|D_v f(x,m,v,t') - D_v f(x,m,v,t)\right| \le L\left(1 + |x| + W_2(m,\delta_0) + |v|\right)|t'-t|.$$

From Assumption (B5) and Lemma 7.2, we know that G defined in (73) and F defined in (74) satisfy (A5). As a consequence of Lemma 6.1 and Theorem 6.4, we have the following properties for $\mathscr{V}$.

Lemma 8.1. *Under Assumptions (B2)–(B5) be satisfied, there exists a constant* Λ *depending only on* (l, T)*, such that when* $\lambda \ge \Lambda$*,* $\mathscr{V}$ *is differentiable in* t*, and for any* $m \in \mathcal{P}_2(\mathbb{R}^n)$*,* $\mathscr{V}$ *is Gâteaux differentiable in* $X \in L_m^2(\mathbb{R}^n;\mathbb{R}^n)$*, with the derivatives*

$$\begin{aligned}
&\frac{\partial \mathscr{V}}{\partial t}(X,m,t) = -\int_{\mathbb{R}^n} H(X_x, X\#m, t; Q_{X_x mt}(t))dm(x),\\
&D_X\mathscr{V}(X,m,t) \xlongequal{L_m^2(\mathbb{R}^n;\mathbb{R}^n)} Q_{X.mt}(t),
\end{aligned} \tag{98}$$

and satisfies the growth conditions

$$\begin{aligned}
&|\mathscr{V}(X,m,t)| + \left|\frac{\partial \mathscr{V}}{\partial t}(X,m,t)\right| + \|D_X\mathscr{V}(X,m,t)\|^2_{L_m^2}\\
\le& C(l,T)\left(1 + \|X\|^2_{L_m^2}\right),
\end{aligned} \tag{99}$$

and the continuity conditions

$$\begin{aligned}
&\left|\frac{\partial \mathscr{V}}{\partial t}(X',m,t') - \frac{\partial \mathscr{V}}{\partial t}(X,m,t)\right|\\
&\quad \le C(l,T)\left(1 + \|X\|^2_{L_m^2} + \|X'\|^2_{L_m^2}\right)|t'-t|\\
&\qquad + C(l,T)\left(1 + \|X\|_{L_m^2} + \|X'\|_{L_m^2}\right)\|X'-X\|_{L_m^2},\\
&\left\|D_X\mathscr{V}(X',m,t') - D_X\mathscr{V}(X,m,t)\right\|_{L_m^2}\\
&\quad \le C(l,T)\|X'-X\|_{L_m^2}\\
&\qquad + C(l,T)\left(1 + \|X\|_{L_m^2} + \|X'\|_{L_m^2}\right)|t'-t|.
\end{aligned} \tag{100}$$

Moreover, $\mathscr{V}$ *is the unique solution of the following equation (subject to*

(99)–(100)): *for* $(t,m)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^n)$ *and* $X\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$,

$$\begin{cases}\dfrac{\partial\mathscr{V}}{\partial t}(X,m,t)+\displaystyle\int_{\mathbb{R}^n}H\left(X_x,X\#m,t;D_X\mathscr{V}(X,m,t)|_x\right)dm(x)=0,\\ \mathscr{V}(X,m,T)=\displaystyle\int_{\mathbb{R}^n}h(X_x,X\#m)dm(x).\end{cases}\tag{101}$$

As a corollary, we have for $(s,x)\in[t,T]\times\mathbb{R}^n$,

$$u_{X_xmt}(s)=\hat{v}\left(Y_{X_xmt}(s),Y_{X.mt}(s)\#m,s;D_X\mathscr{V}(Y_{Xmt}(s),m,s)|_x\right).\tag{102}$$

8.2. Regularity and Bellman equation for $\mathcal{V}$

We now define the value function $\mathcal{V}:\mathcal{P}_2(\mathbb{R}^n)\times[0,T]\to\mathbb{R}$ for the mean field type control problem $(\mathscr{P}^{t,m})$ as

$$\mathcal{V}(m,t):=\mathscr{V}(I,m,t),$$

where I is the identity. For $m\in\mathcal{P}_2(\mathbb{R}^n)$ and $X\in L^2_m(\mathbb{R}^n)$, we first give a connection of $\mathcal{V}(X\#m,t)$ and $\mathscr{V}(X,m,t)$. The following lemma is proved in Appendix A.4.1.

Lemma 8.2. *Under assumptions in Lemma 8.1, for any* $(m,t)\in\mathcal{P}_2(\mathbb{R}^n)\times[0,T]$, *we have*

$$\mathcal{V}(X\#m,t)=\mathscr{V}(X,m,t),\quad X\in L^2_m(\mathbb{R}^n;\mathbb{R}^n),\tag{103}$$

and $u_{\cdot mt}(\cdot)$ *is the solution of mean field type control problem* $(\mathscr{P}^{t,X\#m})$.

Now we consider the Gâteaux differentiability of $\mathcal{V}$ in m. For $(m,t)\in\mathcal{P}_2(\mathbb{R}^n)\times[0,T]$ and $\mathcal{F}\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, the Gâteaux derivative of $\mathcal{V}$ at (t,m) along the direction $\mathcal{F}$ is defined as

$$D_m\mathcal{V}(m,t)(\mathcal{F}):=\lim_{\epsilon\to0}\left[\frac{\mathcal{V}((I+\epsilon\mathcal{F})\#m,t)-\mathcal{V}(m,t)}{\epsilon}\right].$$

We denote by $D_m\mathcal{V}(m,t)\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$ the derivative of $\mathcal{V}$ at (m,t) if

$$D_m\mathcal{V}(m,t)(\mathcal{F})=\int_{\mathbb{R}^n}D_m\mathcal{V}(m,t)|_x\cdot\mathcal{F}_x dm(x),$$

for all $\mathcal{F}\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$. This definition is also used in Bonnet-Frankowska [6] by another name as Dini differentiability. The following theorem is a strong version of Bonnet-Frankowska [6, Theorem 4.10], where we allow the running cost to exist.

Theorem 8.3. *Under the assumptions in Lemma 8.1, $\mathcal{V}$ is differentiable in t and Gâteaux differentiable in m, with the derivatives*

$$\frac{\partial \mathcal{V}}{\partial t}(m,t) = \frac{\partial \mathscr{V}}{\partial t}(I,m,t) = -\int_{\mathbb{R}^n} H(x,m,t;Q_{xmt}(t))dm(x), \tag{104}$$

$$D_m\mathcal{V}(t,m) = D_X\mathscr{V}(I,m,t) \overset{L^2_m(\mathbb{R}^n;\mathbb{R}^n)}{=\!=\!=\!=} Q_{Imt}(t), \tag{105}$$

and satisfies the growth condition

$$|\mathcal{V}(m,t)| + \left|\frac{\partial \mathcal{V}}{\partial t}(m,t)\right| + \|D_m\mathcal{V}(m,t)\|^2_{L^2_m} \leq C(l,T)(1+W_2^2(m,\delta_0)), \tag{106}$$

and the continuity conditions

$$\begin{aligned} &\left|\frac{\partial \mathcal{V}}{\partial t}(m,t') - \frac{\partial \mathcal{V}}{\partial t}(m,t)\right| \leq C(l,T)\left(1+W_2^2(m,\delta_0)\right)|t'-t|, \\ &\left\|D_m\mathcal{V}(m,t') - D_m\mathcal{V}(m,t)\right\|_{L^2_m} \leq C(l,T)(1+W_2(m,\delta_0))|t'-t|, \end{aligned} \tag{107}$$

and satisfies the following equation:

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(m,t) + \displaystyle\int_{\mathbb{R}^n} H\left(x,m,t;\, D_m\mathcal{V}(m,t)|_x\right)dm(x) = 0, \;\; t\in[0,T), \\ \mathcal{V}(m,T) = \displaystyle\int_{\mathbb{R}^n} h(x,m)dm(x), \quad m\in\mathcal{P}_2(\mathbb{R}^n). \end{cases} \tag{108}$$

Moreover, for $(t,m)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^n)$ and $X\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$D_m\mathcal{V}(X\#m,t)\circ X = D_X\mathscr{V}(X,m,t). \tag{109}$$

Proof. For $(t,m)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^n)$, from Lemma 8.2 we know that

$$\frac{\partial \mathcal{V}}{\partial t}(m,t) = \lim_{\epsilon\to 0}\left[\frac{\mathscr{V}(I,m,t+\epsilon) - \mathscr{V}(I,m,t)}{\epsilon}\right] = \frac{\partial \mathscr{V}}{\partial t}(I,m,t),$$

from which we obtain (104). For any $\mathcal{F}\in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, we know that $(I+\epsilon\mathcal{F})$ is an invertible map when ϵ is small enough. From Lemmas 8.1 and 8.2, we have

$$D_m\mathcal{V}(m,t)(\mathcal{F}) = \lim_{\epsilon\to 0}\left[\frac{\mathscr{V}(I+\epsilon\mathcal{F},m,t) - \mathscr{V}(I,m,t)}{\epsilon}\right]$$

$$= \int_{\mathbb{R}^n} D_X \mathscr{V}(I, m, t)|_x \cdot \mathcal{F}_x dm(x),$$

from which we obtain (105). Estimates (106)–(108) are direct consequences of (104)–(105) and Lemma 8.1. For $X \in L^2_m(\mathbb{R}^n; \mathbb{R}^n)$ and $\mathcal{F} \in L^2_{X\#m}(\mathbb{R}^n; \mathbb{R}^n)$, from Lemmas 8.1 and 8.2, we have

$$\begin{aligned} D_m\mathcal{V}(X\#m, t)(\mathcal{F}) &= \lim_{\epsilon \to 0} \left[\frac{\mathcal{V}((I + \epsilon\mathcal{F})\#(X\#m), t) - \mathcal{V}(X\#m, t)}{\epsilon} \right] \\ &= \lim_{\epsilon \to 0} \left[\frac{\mathscr{V}(X + \epsilon\mathcal{F} \circ X, m, t) - \mathscr{V}(X, m, t)}{\epsilon} \right] \\ &= \int_{\mathbb{R}^n} D_X \mathscr{V}(X, m, t)|_x \cdot (\mathcal{F} \circ X)|_x dm(x). \end{aligned} \tag{110}$$

On the other hand, from the definition of $D_m\mathcal{V}(X\#m, t)$, we have

$$\begin{aligned} D_m\mathcal{V}(X\#m, t)(\mathcal{F}) &= \int_{\mathbb{R}^n} D_m\mathcal{V}(X\#m, t)|_x \cdot \mathcal{F}_x d(X\#m)(x) \\ &= \int_{\mathbb{R}^n} (D_m\mathcal{V}(X\#m, t) \circ X)|_x \cdot (\mathcal{F} \circ X)|_x dm(x). \end{aligned} \tag{111}$$

From (110) and (111), we obtain (109). □

Now we consider the case when $\mathcal{V}$ is differentiable in m with the concept of linear functional derivative. The following theorem is proved in Appendix A.4.2.

Theorem 8.4. *Under the assumptions in Lemma 8.1, suppose that $\mathcal{V}$ is differentiable in m with the concept of linear functional derivative such that the map $\xi \mapsto \frac{d\mathcal{V}}{d\nu}(m, t)(\xi)$ is differentiable with the derivative $D_\xi \frac{d\mathcal{V}}{d\nu}(m, t)(\xi)$ being continuous and*

$$D_\xi \frac{d\mathcal{V}}{d\nu}(m, t)(\xi) \le c(m, t)(1 + |\xi|), \quad (m, t, \xi) \in \mathcal{P}_2(\mathbb{R}^n) \times [0, T] \times \mathbb{R}^n.$$

Then, $\mathcal{V}$ satisfies the following Bellman equation:

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(m, t) + \displaystyle\int_{\mathbb{R}^n} H\left(x, m, t; D_\xi \frac{d\mathcal{V}}{d\nu}(m, t)(x)\right) dm(x) = 0, \ t \in [0, T); \\ \mathcal{V}(m, T) = \displaystyle\int_{\mathbb{R}^n} h(x, m) dm(x), \quad m \in \mathcal{P}_2(\mathbb{R}^n), \end{cases} \tag{112}$$

and for any $(t,m) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^n)$ *and* $X \in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$,

$$D_\xi \frac{d\mathcal{V}}{d\nu}(X\#m,t)(X_x) = D_X \mathscr{V}(X,m,t)|_x = Q_{X_x mt}(t), \; x \in \mathbb{R}^n. \tag{113}$$

As a corollary, the solution for Problem $(\mathscr{P}^{t,m,X})$ *is a feedback.*

Let $\hat{v}.(\cdot) \in L^2([0,T];L^2_m(\mathbb{R}^n))$ be the solution of mean field type control problem $(\mathscr{P}^{0,m})$, $\hat{X}.(\cdot)$ be the corresponding controlled state and $\hat{Q}.(\cdot)$ be the corresponding adjoint process. We denote by

$$m(t) := \hat{X}.(t)\#m, \quad t \in [0,T].$$

As a consequence of Theorems 8.3 and 8.4, We have the following relations.

Corollary 8.5. *Under the assumptions in Theorem 8.4, we have the following relations*

$$\begin{aligned} &\frac{\partial \mathcal{V}}{\partial t}(m(t),t) = -\int_{\mathbb{R}^n} H\left(\hat{X}_x(t), m(t), t; \hat{Q}_x(t)\right) dm(x), \\ &D_\xi \frac{d\mathcal{V}}{d\nu}(t,m(t))\left(\hat{X}_x(t)\right) = \hat{Q}_x(t), \quad x \in \mathbb{R}^n. \end{aligned} \tag{114}$$

Proof. (114) is a direct consequence of (113) and the fact that

$$v_{\hat{X}.(t),m,t}(t) = \hat{v}.(t), \quad \hat{Q}_{\hat{X}.(t),m,t}(t) = \hat{Q}(t). \qquad \square$$

Now we build a connection of $\hat{v}.(\cdot)$ and $u_{\cdot,m(t),t}$, where $u_{\cdot,m(t),t}$ is the solution of the mean field type control problem $(\mathscr{P}^{t,m(t)})$. The following corollary is proved in Appendix A.4.3.

Corollary 8.6. *Under the assumptions in Theorem 8.4, for any* $t \in [0,T]$, *we have*

$$u_{\hat{X}\cdot(t),m(t),t}(s) = \hat{v}.(s), \quad s \in [t,T]. \tag{115}$$

Appendix A

A.1. Proof of statements in Section 5

A.1.1. Proof of Lemma 5.3. We first show the uniformly boundedness of the norm $\|Y^\epsilon(s)\|_H$. From the equation in (17) and Assumption (A1), we have for $s \in [t, T]$,

$$Y^\epsilon(s) = \int_t^s \left[\int_0^1 D_X G\left(X^V(r) + \lambda\epsilon Y^\epsilon(r), V^\epsilon(r), r\right)(Y^\epsilon(r))\, d\lambda \right.$$
$$\left. + \int_0^1 D_V G\left(X^V(r), V(r) + \lambda\epsilon\tilde{V}(r), r\right)\left(\tilde{V}(r)\right) d\lambda \right] dr. \tag{116}$$

From Assumption (A1) and Cauchy's inequality, we have for $s \in [t, T]$,

$$\|Y^\epsilon(s)\|_H^2 \le C(L, T) \int_t^s \left(\|Y^\epsilon(r)\|_H^2 + \|\tilde{V}(r)\|_U^2\right) dr.$$

By applying Grönwall's inequality, we have

$$\sup_{t \le s \le T} \|Y^\epsilon(s)\|_H \le C(L, T) \left\|\tilde{V}\right\|_{L^2([t,T];U)}. \tag{117}$$

We denote by $\Delta^\epsilon := Y^\epsilon - \mathcal{D}_{\tilde{V}} X^V$. From (19), (116) and Assumption (A1), we have for $s \in [t, T]$,

$$\begin{aligned}
\Delta^\epsilon(s) = \int_t^s \Bigg[\int_0^1 \Big[& D_X G\left(X^V(r) + \lambda\epsilon Y^\epsilon(r), V^\epsilon(r), r\right)(Y^\epsilon(r)) \\
& - D_X G\left(X^V(r) + \lambda\epsilon Y^\epsilon(r), V^\epsilon(r), r\right)\left(\mathcal{D}_{\tilde{V}} X^V(r)\right) \Big] d\lambda \\
+ \int_0^1 \Big[& D_X G\left(X^V(r) + \lambda\epsilon Y^\epsilon(r), V^\epsilon(r), r\right)\left(\mathcal{D}_{\tilde{V}} X^V(r)\right) \\
& - D_X G\left(X^V(r), V(r), r\right)\left(\mathcal{D}_{\tilde{V}} X^V(r)\right) \Big] d\lambda \\
+ \int_0^1 \Big[& D_V G(X^V(r), V(r) + \lambda\epsilon\tilde{V}(r), r)(\tilde{V}(r)) \\
& - D_V G(X^V(r), V(r), r)(\tilde{V}(r)) \Big] d\lambda \Bigg] dr.
\end{aligned}$$

From Assumption (A1) and Cauchy's inequality, we have

$$\|\Delta^\epsilon(s)\|_H^2 \le C(L, T) \left(\int_t^s \|\Delta^\epsilon(r)\|_H^2 dr + I(\epsilon) \right),$$

where

$$I(\epsilon) := \int_t^T \int_0^1 \left\| D_X G\left(X^V(s) + \lambda\epsilon Y^\epsilon(s), V^\epsilon(s), s\right)\left(\mathcal{D}_{\tilde{V}} X^V(s)\right) \right.$$

$$- D_X G\left(X^V(s), V(s), s\right)\left(\mathcal{D}_{\tilde{V}} X^V(s)\right)\Big\|_H^2 d\lambda ds$$

$$+ \int_t^T \int_0^1 \left\| D_V G\left(X^V(s), V(s) + \lambda\epsilon\tilde{V}(s), s\right)\left(\tilde{V}(s)\right) - D_V G\left(X^V(s), V(s), s\right)\left(\tilde{V}(s)\right)\right\|_H^2 d\lambda ds.$$

From Gronwall's inequality, we have

$$\sup_{t\leq s\leq T} \|\Delta^\epsilon(s)\|_H^2 \leq C(L,T) I(\epsilon). \tag{118}$$

From Lemma 5.2, estimate (117) and the dominated convergence theorem, we have $\lim_{\epsilon\to 0} I(\epsilon) = 0$. From (118), we obtain (21).

A.1.2. Proof of Lemma 5.10. From Assumption (A4), we have

$$\left(D_V F\left(0, \hat{V}(0,s;0), s\right) - D_V F(0,0,s), \hat{V}(0,s;0)\right)_U \geq 2\lambda \|\hat{V}(0,0,s)\|_U^2.$$

From the definition of $\hat{V}$, we have $D_V F\left(0, \hat{V}(0,s;0), s\right) = 0$. Therefore, from Assumption (A2), we have

$$2\lambda \left\|\hat{V}(0,s;0)\right\|_U \leq \|D_V F(0,0,s)\|_U \leq L, \quad s\in[0,T].$$

From Assumption (A4), we have for any $X^1, X^2, Q^1, Q^2 \in H$ and $s\in[0,T]$,

$$\begin{aligned}&\Big(D_V F\left(X^2, \hat{V}(X^2,s;Q^2), s\right) - D_V F\left(X^2, \hat{V}(X^1,s;Q^1), s\right), \\ &\quad \hat{V}\left(X^2,s;Q^2\right) - \hat{V}\left(X^1,s;Q^1\right)\Big)_U \\ \geq& 2\lambda \left\|\hat{V}\left(X^2,s;Q^2\right) - \hat{V}\left(X^1,s;Q^1\right)\right\|_U^2.\end{aligned}$$

From the definition of $\hat{V}$ and (49), we have for any $s\in[0,T]$,

$$\begin{aligned}&\left(D_V F\left(X^i, \hat{V}(X^i,s;Q^i), s\right), \hat{V}(X^2,s;Q^2) - \hat{V}(X^1,s;Q^1)\right)_U \\ =& -\left(\mathcal{G}_2(s)\left(\hat{V}(X^2,s;Q^2) - \hat{V}(X^1,s;Q^1)\right), Q^i\right)_H, \qquad i=1,2.\end{aligned}$$

Therefore, from Assumptions (A3)–(A4), we have

$$2\lambda \left\|\hat{V}(X^2,s;Q^2) - \hat{V}(X^1,s;Q^1)\right\|_U^2$$

$$\le \left(\frac{\partial F}{\partial V}\left(X^2, \hat{V}(X^2, s; Q^2 s), s\right) - \frac{\partial F}{\partial V}\left(X^1, \hat{V}(X^1, s; Q^1), s\right), \right.$$
$$\left. \hat{V}(X^2, s; Q^2) - \hat{V}(X^1, s; Q^1) \right)_U$$
$$+ \left(\frac{\partial F}{\partial V}\left(X^1, \hat{V}(X^1, s; Q^1), s\right) - \frac{\partial F}{\partial V}\left(X^2, \hat{V}(X^1, s; Q^1), s\right), \right.$$
$$\left. \hat{V}(X^2, s; Q^2) - \hat{V}(X^1, s; Q^1) \right)_U$$
$$\le - \left(\mathcal{G}_2(s) \left(\hat{V}(X^2, s; Q^2) - \hat{V}(X^1, s; Q^1) \right),\ Q^2 - Q^1 \right)_H$$
$$+ \left(\frac{\partial F}{\partial V}\left(X^1, \hat{V}(X^1, s; Q^1), s\right) - \frac{\partial F}{\partial V}\left(X^2, \hat{V}(X^1, s; Q^1), s\right), \right.$$
$$\left. \hat{V}(X^2, s; Q^2) - \hat{V}(X^1, s; Q^1) \right)_U$$
$$\le C(L) \left(\left\| Q^2 - Q^1 \right\|_H + \left\| X^2 - X^1 \right\|_H \right) \left\| \hat{V}(X^2, s; Q^2) - \hat{V}(X^1, s; Q^1) \right\|_U .$$

From the average inequality, we obtain (53).

A.1.3. Proof of Lemma 5.11. We first prove (56). From the definition of U_{Xt} and Assumption (A3), we know that for $s \in (t, T)$,

$$(Q_{Xt}(s), \mathcal{G}_2(s) U_{Xt}(s))_H + (D_V F(Y_{Xt}(s), U_{Xt}(s), s), U_{Xt}(s))_U = 0.$$

Then, from Assumption (A4), we have for any $s \in [t, T]$,

$$\begin{aligned}
&\frac{d}{ds} (Q_{Xt}(s), Y_{Xt}(s))_H \\
&= - (D_V F(Y_{Xt}(s), U_{Xt}(s), s), U_{Xt}(s))_U \\
&\quad - (D_X F(Y_{Xt}(s), U_{Xt}(s), s), Y_{Xt}(s))_H \\
&\le - 2\lambda \|U_{Xt}(s)\|_U^2 - (D_V F(Y_{Xt}(s), 0, s), U_{Xt}(s))_U \\
&\quad - (D_X F(Y_{Xt}(s), U_{Xt}(s), s), Y_{Xt}(s))_H .
\end{aligned}$$

Therefore, from Assumption (A2) and the average inequality, we have

$$\begin{aligned}
&2\lambda \|U_{Xt}\|^2_{L^2[t,T];U} \\
&\le (Q_{Xt}(t), X)_H - (D_X F_T(Y_{Xt}(T)), Y_{Xt}(T))_H \\
&\quad - \int_t^T \Big[(D_V F(Y_{Xt}(s), 0, s), U_{Xt}(s))_U
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + (D_X F(Y_{Xt}(s), U_{Xt}(s), s), Y_{Xt}(s))_H \Big] ds \\
\leq & C(L,T)\Big(1 + \|X\|_H^2 + \|Y_{Xt}(\cdot)\|_{C([t,T];H)}^2 \\
&\qquad\qquad + \|Q_{Xt}(\cdot)\|_{C([t,T];H)}^2 + \|U_{Xt}(\cdot)\|_{L^2([t,T];U)}^2\Big).
\end{aligned}
$$

Substituting Lemmas 5.1 and 5.4 into the last inequality, we obtain

$$
2\lambda \|U_{Xt}\|_{L^2[t,T];U}^2 \leq C(L,T)\left(1 + \|X\|_H^2 + \|U_{Xt}\|_{L^2([t,T];U)}^2\right).
$$

So there exists a constant Λ depending only on (L,T), such that when $\lambda \geq \Lambda$, we have

$$
\|U_{Xt}\|_{L^2([t,T];U)} \leq C(L,T)(1 + \|X\|_H).
$$

Substituting the last inequality back to Lemmas 5.1 and 5.4 we have

$$
\|Y_{Xt}\|_{C([t,T];H)} + \|Q_{Xt}\|_{C([t,T];H)} \leq C(L,T)(1 + \|X\|_H).
$$

By applying Lemma 5.10, we obtain (56). We next prove (57). We set $\Delta Y(s) := Y_{X't}(s) - Y_{Xt}(s)$, $\Delta U(s) := U_{X't}(s) - U_{Xt}(s)$, $\Delta Q(s) := Q_{X't}(s) - Q_{Xt}(s)$ and $\Delta X := X' - X$. Then $(\Delta Y(s), \Delta U(s), \Delta Q(s))$ satisfy the following equations: for $s \in [t,T]$,

$$
\begin{cases}
\Delta Y(s) = \Delta X + \displaystyle\int_t^s [\mathcal{G}_1(r)\Delta X(r) + \mathcal{G}_2(r)\Delta V(r)]\, dr, \\
\Delta Q(s) = [D_X F_T(Y_{X't}(T)) - D_X F_T(Y_{Xt}(T))] \\
\qquad\qquad + \displaystyle\int_s^T \big[D_X \mathcal{H}(Y_{X't}(r), U_{X't}(r), Q_{X't}(r), r) \\
\qquad\qquad\qquad - D_X \mathcal{H}(Y_{Xt}(r), U_{Xt}(r), Q_{Xt}(r), r)\big]\, dr.
\end{cases}
$$

Similar as Lemma 5.1, we have the following estimates for ΔY,

$$
\sup_{t \leq s \leq T} \|\Delta Y(s)\|_H \leq C(L,T)\left(\|\Delta X\|_H + \|\Delta U\|_{L^2([t,T];U)}\right). \tag{119}
$$

Similar as Lemma 5.4, from estimate (119), we have the following estimate for ΔQ,

$$
\sup_{t \leq s \leq T} \|\Delta Q(s)\|_H \leq C(L,T)\left(\|\Delta X\|_H + \|\Delta U\|_{L^2([t,T];U)}\right). \tag{120}
$$

From Assumptions (A2)–(A3) and (54), we have for $s \in (t,T)$,

$$\begin{aligned}
&\frac{d}{ds}\left(\Delta Q(s), \Delta Y(s)\right)_H \\
=& -\left(D_V F(Y_{X't}(s), U_{X't}(s), s) - D_V F(Y_{Xt}(s), U_{Xt}(s), s), \Delta U(s)\right)_U \\
& - \left(D_X F(Y_{X't}(s), U_{X't}(s), s) - D_X F(Y_{Xt}(s), U_{Xt}(s), s), \Delta Y(s)\right)_H .
\end{aligned}$$

Therefore, from Assumption (A4), we have

$$\begin{aligned}
&\frac{d}{ds}\left(\Delta Q(s), \Delta Y(s)\right)_H \\
=& -\left(D_V F(Y_{X't}(s), U_{X't}(s), s) - D_V F(Y_{X't}(s), U_{Xt}(s), s),\ \Delta U(s)\right)_U \\
& - \left(D_V F(Y_{X't}(s), U_{Xt}(s), s) - D_V F(Y_{Xt}(s), U_{Xt}(s), s),\ \Delta U(s)\right)_U \\
& - \left(D_X F(Y_{X't}(s), U_{X't}(s), s) - D_X F(Y_{Xt}(s), U_{Xt}(s), s),\ \Delta Y(s)\right)_H \\
\le& - 2\lambda \|\Delta U(s)\|_U^2 + 2L \|\Delta Y(s)\|_H \|\Delta U(s)\|_U + \|\Delta Y(s)\|_H^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{d}{ds}\left(\Delta Q(s), \Delta Y(s)\right)_H \\
=& \left(D_X F_T(Y_{X't}(T)) - D_X F_T(Y_{Xt}(T)), \Delta Y(T)\right)_H - \left(\Delta Q(t), \Delta X\right)_H \\
\ge& - L\|\Delta Y(T)\|_H^2 - \left(\Delta Q(t), \Delta X\right)_H .
\end{aligned}$$

Therefore, from the average inequality, we have

(121)
$$\begin{aligned}
&2\lambda \|\Delta U\|^2_{L^2([t,T];U)} \\
\le& L\|\Delta Y(T)\|_H^2 + \left(\Delta Q(t), \Delta X\right)_H \\
&+ \int_t^T \left[2L\|\Delta Y(s)\|_H \|\Delta U(s)\|_U + \|\Delta Y(s)\|_H^2\right] ds \\
\le& C(L,T)\Big(\|\Delta X\|_H^2 + \|\Delta Y\|^2_{C([t,T];H)} + \|\Delta Q\|^2_{C([t,T];H)} + \|\Delta U\|^2_{L^2([t,T];U)}\Big).
\end{aligned}$$

Substituting (119) and (120) into (121), we have

$$2\lambda \|\Delta U\|^2_{L^2([t,T];U)} \le C(L,T)\left(\|\Delta X\|_H^2 + \|\Delta U\|^2_{L^2([t,T];U)}\right).$$

So there exists a constant Λ depending only on (L,T), such that when $\lambda \ge \Lambda$, we have

$$\|\Delta U\|_{L^2([t,T];U)} \le C(L,T)\|\Delta X\|_H. \tag{122}$$

Substituting the last inequality back to (119) and (120), we deduce that

$$\|Y_{X't} - Y_{Xt}\|_{C([t,T];H)} + \|Q_{X't} - Q_{Xt}\|_{C([t,T];H)} \le C(L,T)\|\Delta X\|_H.$$

By applying Lemma 5.10, we obtain (57).

A.2. Proof of statements in Section 6

A.2.1. Proof of Lemma 6.1. We first prove (63). From (60) and (56), we have

$$\begin{aligned} |\mathscr{V}(X,t)| &= \left| \int_t^T F(Y_{Xt}(s), U_{Xt}(s), s) ds + F_T(Y_{Xt}(T)) \right| \\ &\le C(L,T) \left(1 + \|Y_{Xt}\|^2_{C([t,T];H)} + \|U_{Xt}\|^2_{L^2([t,T];U)}\right) \\ &\le C(L,T) \left(1 + \|X\|_H^2\right). \end{aligned}$$

We next prove (61). From (60) and the definition of $\mathscr{V}$, we have

(123)
$$J_{X't}(U_{X't}) - J_{Xt}(U_{X't}) \le \mathscr{V}(X',t) - \mathscr{V}(X,t) \le J_{X't}(U_{Xt}) - J_{Xt}(U_{Xt}).$$

From Assumption (A4), we have

(124)
$$\begin{aligned} & J_{X't}(U_{Xt}) - J_{Xt}(U_{Xt}) \\ =& \int_t^T \left[F\left(X_{X't}^{U_{Xt}}(s), U_{Xt}(s), s\right) - F(Y_{Xt}(s), U_{Xt}(s), s)\right] ds \\ & + \left[F_T\left(X_{X't}^{U_{Xt}}(T)\right) - F_T(Y_{Xt}(T))\right] \\ =& \int_t^T \int_0^1 \Big(D_X F\Big((1-\epsilon)Y_{Xt}(s) + \epsilon X_{X't}^{U_{Xt}}(s), U_{Xt}(s), s\Big), X_{X't}^{U_{Xt}}(s) - Y_{Xt}(s)\Big)_H d\epsilon ds \\ & + \int_0^1 \left(D_X F_T\left((1-\epsilon)Y_{Xt}(T) + \epsilon X_{X't}^{U_{Xt}}(T)\right),\ X_{X't}^{U_{Xt}}(T) - Y_{Xt}(T)\right)_H d\epsilon \\ \le& \int_t^T \left(D_X F(Y_{Xt}(s), U_{Xt}(s), s),\ X_{X't}^{U_{Xt}}(s) - Y_{Xt}(s)\right)_H ds \\ & + \left(D_X F_T(Y_{Xt}(T)), X_{X't}^{U_{Xt}}(T) - Y_{Xt}(T)\right)_H + C(L,T)\left\|X_{X't}^{U_{Xt}} - Y_{Xt}\right\|^2_{C([t,T];H)}. \end{aligned}$$

From Assumption (A3) and Grönwall's inequality, we have

$$\left\| X_{X't}^{U_{Xt}} - Y_{Xt} \right\|_{C([t,T];H)} \leq C(L,T)\|X' - X\|_H. \tag{125}$$

From the definition of Q_{Xt}, we have for $s \in (t,T)$,

$$\begin{aligned} &\frac{d}{ds}\left(Q_{Xt}(s),\ X_{X't}^{U_{Xt}}(s) - Y_{Xt}(s)\right)_H \\ =& -\left(D_X F(Y_{Xt}(s), U_{Xt}(s), s),\ X_{X't}^{U_{Xt}}(s) - Y_{Xt}(s)\right)_H, \end{aligned}$$

then

$$\begin{aligned} &\int_t^T \left(D_X F(Y_{Xt}(s), U_{Xt}(s), s), X_{X't}^{U_{Xt}}(s) - Y_{Xt}(s)\right)_H ds \\ &+ \left(D_X F_T(Y_{Xt}(T)), X_{X't}^{U_{Xt}}(T) - Y_{Xt}(T)\right)_H \\ =& \left(Q_{Xt}(t), X' - X\right)_H. \end{aligned} \tag{126}$$

Substituting (125) and (126) back to (124), we have

$$J_{X't}(U_{Xt}) - J_{Xt}(U_{Xt}) \leq \left(Q_{Xt}(t), X' - X\right)_H + C(L,T)\|X' - X\|_H^2. \tag{127}$$

In a similar way, we can also have

$$\begin{aligned} &J_{X't}(U_{X't}) - J_{Xt}(U_{X't}) \\ \geq& \left(Q_{X't}(t), X' - X\right)_H - C(L,T)\|X' - X\|_H^2. \end{aligned} \tag{128}$$

From Lemma 5.11, we have

$$\|Q_{X't}(t) - Q_{Xt}(t)\|_H \leq C(L,T)\|X' - X\|_H.$$

Substituting the last inequality back to (128), we have

$$\begin{aligned} &J_{X't}(U_{X't}) - J_{Xt}(U_{X't}) \\ \geq& \left(Q_{Xt}(t), X' - X\right)_H - C(L,T)\|X' - X\|_H^2. \end{aligned} \tag{129}$$

From (123), (127) and (129), we have

$$\left|\mathscr{V}(X',t) - \mathscr{V}(X,t) - \left(Q_{Xt}(t), X' - X\right)_H\right| \leq C(L,T)\|X' - X\|_H^2, \tag{130}$$

from which we obtain (61). We next prove (62). It is well-known that for any $\epsilon \in [0, T-t]$,

$$\mathscr{V}(X,t) = \int_t^{t+\epsilon} F(Y_{Xt}(s), U_{Xt}(s), s)ds + \mathscr{V}(Y_{Xt}(t+\epsilon), t+\epsilon).$$

So we have

$$\begin{aligned} \frac{1}{\epsilon}\left[\mathscr{V}(X,t+\epsilon) - \mathscr{V}(X,t)\right] =& \frac{1}{\epsilon}\left[\mathscr{V}(X,t+\epsilon) - \mathscr{V}(Y_{Xt}(t+\epsilon), t+\epsilon)\right] \\ & - \frac{1}{\epsilon}\int_t^{t+\epsilon} F(Y_{Xt}(s), U_{Xt}(s), s)ds. \end{aligned} \tag{131}$$

From (130), we have

$$\begin{aligned} &\Big| \mathscr{V}(X,t+\epsilon) - \mathscr{V}(Y_{Xt}(t+\epsilon), t+\epsilon) \\ &\quad - \left(Q_{Y_{X_t(t+\epsilon)},t+\epsilon}(t+\epsilon), X - Y_{Xt}(t+\epsilon)\right)_H \Big| \\ \leq& C(L,T)\|X - Y_{Xt}(t+\epsilon)\|_H^2. \end{aligned}$$

From the uniqueness of solution of the forward-backward equations (47)–(48), we know that

$$Q_{Y_{Xt}(t+\epsilon),t+\epsilon}(t+\epsilon) = Q_{Xt}(t+\epsilon),$$

therefore,

$$\begin{aligned} &\Bigg| \frac{1}{\epsilon}[\mathscr{V}(X,t+\epsilon) - \mathscr{V}(Y_{Xt}(t+\epsilon), t+\epsilon)] \\ &\quad + \left(Q_{Xt}(t+\epsilon), \frac{1}{\epsilon}\int_t^{t+\epsilon} G(Y_{Xt}(s), U_{Xt}(s), s)ds\right)_H \Bigg| \\ \leq& C(L,T)\epsilon \left\| \frac{1}{\epsilon}\int_t^{t+\epsilon} G(Y_{Xt}(s), U_{Xt}(s), s)ds \right\|_H^2. \end{aligned} \tag{132}$$

From Lemma 6.1 and Assumption (A5), we have

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_t^{t+\epsilon} G(Y_{Xt}(s), U_{Xt}(s), s)ds \overset{H}{=} G(X, U_{Xt}(t), t), \\ &\lim_{\epsilon\to 0} Q_{Xt}(t+\epsilon) \overset{H}{=} Q_{Xt}(t). \end{aligned} \tag{133}$$

Substituting (133) back to (132), we have

$$\begin{aligned}(134)\qquad &\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\mathscr{V}(X,t+\epsilon)-\mathscr{V}(Y_{Xt}(t+\epsilon),t+\epsilon)\right]\\ &=-\left(Q_{Xt}(t),G(X,U_{Xt}(t),t)\right)_H.\end{aligned}$$

From Lemma 6.1 and Assumption (A5), we also have

$$(135)\qquad \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_t^{t+\epsilon}F(Y_{Xt}(s),U_{Xt}(s),s)ds=F(X,U_{Xt}(t),t).$$

Substituting (134) and (135) back to (131), we obtain (62). As a consequence of (61), (62) and Lemma 5.11, we have (64), and for $t\in[0,T]$, $X,X'\in H$,

$$\begin{aligned}(136)\qquad &\left\|D_X\mathscr{V}(X',t)-D_X\mathscr{V}(X,t)\right\|_H\le C(L,T)\|X'-X\|_H,\\ &\left|\frac{\partial\mathscr{V}}{\partial t}(X',t)-\frac{\partial\mathscr{V}}{\partial t}(X,t)\right|\le C(L,T)(1+\|X\|_H+\|X'\|_H)\|X'-X\|_H.\end{aligned}$$

Now it remains to prove the continuity of derivatives in t. From Lemma 6.1, we have for any $0\le t\le t'\le T$ and $X\in H$,

$$\begin{aligned}(137)\qquad &\|Q_{Xt'}(t')-Q_{Xt}(t)\|_H\\ \le&\|Q_{Xt'}(t')-Q_{Xt}(t')\|_H+\|Q_{Xt}(t')-Q_{Xt}(t)\|_H\\ \le&C(L,T)(1+\|X\|_H)|t'-t|\\ &+\left\|\int_t^{t'}D_X\mathcal{L}(Y_{Xt}(s),U_{Xt}(s),s;Q_{Xt}(s))ds\right\|_H.\end{aligned}$$

From Assumptions (A2) and (A3) and Lemma 5.11, we have

$$\begin{aligned}&\left\|\int_t^{t'}D_X\mathcal{L}(Y_{Xt}(s),U_{Xt}(s),s;Q_{Xt}(s))ds\right\|_H\\ \le&L\int_t^{t'}\left[1+\|Y_{Xt}(s)\|_H+\|U_{Xt}(s)\|_U+\|Q_{Xt}(s)\|_H\right]ds\\ \le&L\left(1+\|Y_{Xt}\|_{C([t,T];H)}+\|U_{Xt}\|_{C([t,T];U)}+\|Q_{Xt}\|_{C([t,T];H)}\right)|t'-t|\\ \le&C(L,T)(1+\|X\|_H)|t'-t|.\end{aligned}$$

Substituting the last inequality back to (137) and applying (61), we have

$$\|D_X\mathscr{V}(X,t') - D_X\mathscr{V}(X,t)\|_H = \|Q_{Xt'}(t') - Q_{Xt}(t)\|_H \\ \leq C(L,T)(1+\|X\|_H)|t'-t|. \tag{138}$$

Now we prove the continuity of $\frac{\partial \mathscr{V}}{\partial t}$ in t. From Assumptions (A2)–(A5) and Lemma 5.11, we have for $0 \leq t \leq t' \leq T$ and $X \in H$,

$$\begin{aligned}
&|\mathcal{L}(X, U_{Xt'}(t'), t'; Q_{Xt'}(t')) - \mathcal{L}(X, U_{Xt}(t), t; Q_{Xt}(t))| \\
\leq& L\left(1+\|X\|_H^2+\|U_{Xt}(t)\|_U^2+\|U_{Xt'}(t')\|_U^2+\|Q_{Xt}(t)\|_H^2 + \|Q_{Xt'}(t')\|_H^2\right)|t'-t| \\
&+ L\left(1+\|X\|_H + \|U_{Xt}(t)\|_U + \|U_{Xt'}(t')\|_U + \|Q_{Xt}(t)\|_H + \|Q_{Xt'}(t')\|_H\right) \\
&\quad \left(\|U_{Xt'}(t') - U_{Xt}(t)\|_U + \|Q_{Xt'}(t') - Q_{Xt}(t)\|_H\right) \\
\leq& C(L,T)\left(1+\|X\|_H^2\right)|t'-t| \\
&+ C(L,T)(1+\|X\|_H)\left(\|U_{Xt'}(t') - U_{Xt}(t)\|_U + \|Q_{Xt'}(t') - Q_{Xt}(t)\|_H\right).
\end{aligned} \tag{139}$$

From Assumption (A4), we have

$$\begin{aligned}
&\left(D_V F(X, U_{Xt'}(t'), t') - D_V F(X, U_{Xt}(t), t'),\ U_{Xt'}(t') - U_{Xt}(t)\right)_U \\
\geq& 2\lambda \left\|U_{Xt'}(t') - U_{Xt}(t)\right\|_U^2.
\end{aligned} \tag{140}$$

From Assumption (A5), we have

$$\begin{aligned}
&\left(D_V F(X, U_{Xt}(t), t') - D_V F(X, U_{Xt}(t), t),\ U_{Xt'}(t') - U_{Xt}(t)\right)_U \\
\geq& - L\|U_{Xt'}(t') - U_{Xt}(t)\|_U\left(1+\|X\|_H + \|U_{Xt}(t)\|_U\right)|t'-t|.
\end{aligned} \tag{141}$$

Recall that

$$\begin{aligned}
&\left(D_V F(X, U_{Xt'}(t'), t') - D_V F(X, U_{Xt}(t), t),\ U_{Xt'}(t') - U_{Xt}(t)\right)_U \\
=& -\left(\mathcal{G}_2(U_{Xt'}(t') - U_{Xt}(t)),\ Q_{Xt'}(t') - Q_{Xt}(t)\right)_H.
\end{aligned} \tag{142}$$

From (140)–(142), Assumption (A3), Lemma 5.11 and estimate (138), we have

$$\begin{aligned}
&2\lambda \left\|U_{Xt'}(t') - U_{Xt}(t)\right\|_U \\
\leq& \left\|Q_{Xt'}(t') - Q_{Xt}(t)\right\|_H + L\left(1+\|X\|_H + \|U_{Xt}(t)\|_U\right)|t'-t| \\
\leq& C(L,T)(1+\|X\|_H)|t'-t|.
\end{aligned} \tag{143}$$

Substituting (138) and (143) back to (139) and applying (62), we have

$$
\begin{aligned}
&\left|\frac{\partial \mathscr{V}}{\partial t}(X,t') - \frac{\partial \mathscr{V}}{\partial t}(X,t)\right| \\
&= \left|\mathcal{L}\left(X, U_{Xt'}(t'), t'; Q_{Xt'}(t')\right) - \mathcal{L}(X, U_{Xt}(t), t; Q_{Xt}(t))\right| \\
&\leq C(L,T)\left(1 + \|X\|_H^2\right)|t'-t|.
\end{aligned} \tag{144}
$$

From (136), (138) and (144), we obtain (65).

A.3. Proof of statements in Section 7

A.3.1. Proof of Lemma 7.1. For any $X, \tilde{X} \in L^2_m(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$
\begin{aligned}
&\frac{1}{\epsilon}\left[F\left(X+\epsilon\tilde{X}\right) - F(X)\right] \\
=&\frac{1}{\epsilon}\left[f\left((X+\epsilon\tilde{X})\#m\right) - F(X\#m)\right] \\
=&\int_0^1\int_{\mathbb{R}^n}\frac{df}{d\nu}\left((1-\lambda)X\#m + \lambda(X+\epsilon\tilde{X})\#m\right)(x) \\
&\qquad d\left(\frac{(X+\epsilon\tilde{X})\#m - X\#m}{\epsilon}\right)(x)d\lambda \\
=&\int_0^1\int_{\mathbb{R}^n}\frac{1}{\epsilon}\Bigg[\frac{df}{d\nu}\left((1-\lambda)X\#m + \lambda(X+\epsilon\tilde{X})\#m\right)\left(X_x + \epsilon\tilde{X}_x\right) \\
&\qquad - \frac{df}{d\nu}\left((1-\lambda)X\#m + \lambda(X+\epsilon\tilde{X})\#m\right)(X_x)\Bigg]dm(x)d\lambda \\
=&\int_0^1\int_0^1\int_{\mathbb{R}^n}D_\xi\frac{df}{d\nu}\left((1-\lambda)X\#m + \lambda(X+\epsilon\tilde{X})\#m\right)\left(X_x + \delta\epsilon\tilde{X}_x\right) \\
&\qquad \cdot\tilde{X}_x dm(x)d\delta d\lambda.
\end{aligned}
$$

It is easy to check that for any $\lambda \in [0,1]$, the family of distributions

$$
\left\{m_\epsilon^\lambda := (1-\lambda)X\#m + \lambda(X+\epsilon\tilde{X})\#m,\ \epsilon \in [0,1]\right\}
$$

converges to m in the weak sense and $W_2(m_\epsilon^\lambda, \delta_0) \to W_2(X\#m, \delta_0)$ as $\epsilon \to 0$. Therefore, we have $W_2(m_\epsilon^\lambda, X\#m) \to 0$ as $\epsilon \to 0$. From the continuity of $\frac{\partial}{\partial\xi}\frac{\partial f}{\partial\nu}$ and the dominated convergence theorem, we have

$$
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[F(X+\epsilon\tilde{X}) - F(X)\right] = \int_{\mathbb{R}^n} D_\xi\frac{df}{d\nu}(X\#m)(X_x)\cdot\tilde{X}_x dm(x),
$$

from which we obtain (78).

A.3.2. Proof of Lemma 7.2. From (73) and Assumption (B1), we have for $(X, V, s) \in L^2_m(\mathbb{R}^n; \mathbb{R}^n) \times L^2_m(\mathbb{R}^n; \mathbb{R}^d) \times [0, T]$,

$$G(X, V, s)|_x \leq l[1 + |X_x| + |V_x| + W_2(X\#m, \delta_0)]$$

therefore, we have

$$\|G(X, V, s)\|_H \leq C(l) \left(1 + \|X\|_{L^2_m} + \|V\|_{L^2_m}\right). \tag{145}$$

From (74) and Assumption (B2), we have

$$\begin{aligned}|F(X, V, s)| &\leq l \int_{\mathbb{R}} \left(1 + |X_x|^2 + |V_x|^2 + W_2^2(X\#m, \delta_0)\right) dm(x) \\ &\leq C(l) \left(1 + \|X\|^2_{L^2_m} + \|V\|^2_{L^2_m}\right).\end{aligned}$$

For $X, \tilde{X} \in L^2_m(\mathbb{R}^n; \mathbb{R}^n)$ and $\epsilon \in [0, 1]$, similar as Lemma 7.1, we have

$$\begin{aligned}&\frac{1}{\epsilon}\left[G(X + \epsilon\tilde{X}, V, s)\Big|_x - G(X, V, s)|_x\right] \\ =&\frac{1}{\epsilon}\left[g\left(X_x + \epsilon\tilde{X}_x, (X + \epsilon\tilde{X})\#m, V_x, s\right) - g(X_x, X\#m, V_x, s)\right] \\ =&\int_0^1 (D_x g)^* \left(X_x + \lambda\epsilon\tilde{X}_x, (X + \epsilon\tilde{X})\#m, V_x, s\right)\tilde{X}_x d\lambda \\ &+ \int_0^1 \int_0^1 \int_{\mathbb{R}^n} \left(D_\xi \frac{dg}{d\nu}\right)^* \left(X_x, (1 - \lambda)X\#m + \lambda(X + \epsilon\tilde{X})\#m, V_x, s\right) \\ &\qquad \left(X_y + \delta\epsilon\tilde{X}_y\right)\tilde{X}_y dm(y) d\delta d\lambda.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^n} \left|\frac{1}{\epsilon}\left[G(X + \epsilon\tilde{X}, V, s)\Big|_x - G(X, V, s)|_x\right] - (D_x g)^*(X_x, X\#m, V_x, s)\tilde{X}_x \right. \\ &\qquad \left. - \int_{\mathbb{R}^n} \left(D_\xi \frac{dg}{d\nu}\right)^* (X_x, X\#m, V_x, s)(X_y)\tilde{X}_y dm(y)\right|^2 dm(x) \\ \leq& 2\int_{\mathbb{R}^n} \int_0^1 \left|D_x g\left(X_x + \lambda\epsilon\tilde{X}_x, (X + \epsilon\tilde{X})\#m, V_x, s\right)\right. \\ &\qquad \left. - D_x g(X_x, X\#m, V_x, s)\right|^2 \left|\tilde{X}_x\right|^2 d\lambda dm(x)\end{aligned}$$

$$+ 2\int_{\mathbb{R}^n}\int_0^1\int_0^1\int_{\mathbb{R}^n}\left|D_\xi\frac{dg}{d\nu}\left(X_x,(1-\lambda)X\#m+\lambda(X+\epsilon\tilde{X})\#m,V_x,s\right)\right.$$
$$\left.\times\left(X_y+\delta\epsilon\tilde{X}_y\right)-D_\xi\frac{dg}{d\nu}(X_x,X\#m,V_x,s)(X_y)\right|^2\left|\tilde{X}_y\right|^2 dm(y)d\delta d\lambda dm(x).$$

For any $\lambda\in[0,1]$, the family of distributions

$$\left\{m_\epsilon^\lambda := (1-\lambda)X\#m+\lambda(X+\epsilon\tilde{X})\#m,\ \epsilon\in[0,1]\right\}$$

converges to $X\#m$ in the weak sense and $W_2(m_\epsilon^\lambda,\delta_0)\to W_2(X\#m,\delta_0)$ as $\epsilon\to 0$. Therefore, we have $W_2(m_\epsilon^\lambda, X\#m)\to 0$ as $\epsilon\to 0$. From Assumption (B1) and the dominated convergence theorem, we have

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[G\left(X+\epsilon\tilde{X},V,s\right)-G(X,V,s)\right]\\=&(D_xg)^*(X_\cdot,X\#m,V_\cdot,s)\tilde{X}_\cdot\\&+\int_{\mathbb{R}^n}\left(D_\xi\frac{dg}{d\nu}\right)^*(X_\cdot,X\#m,V_\cdot,s)(X_y)\tilde{X}_y dm(y),\end{aligned}$$

from which we obtain (79). Similarly,

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[F(X+\epsilon\tilde{X},V,s)-F(X,V,s)\right]\\=&\int_{\mathbb{R}^n}\left[(D_xf)^*(X_x,X\#m,V_x,s)\tilde{X}_x\right.\\&\left.+\int_{\mathbb{R}^n}\left(D_\xi\frac{df}{d\nu}\right)^*(X_x,X\#m,V_x,s)(X_y)\tilde{X}_y dm(y)\right]dm(x)\\=&\int_{\mathbb{R}^n}\left[D_xf(X_y,X\#m,V_y,s)\right.\\&\left.+\int_{\mathbb{R}^n}D_\xi\frac{df}{d\nu}(X_x,X\#m,V_x,s)(X_y)dm(x)\right]^*\tilde{X}_y dm(y),\end{aligned}$$

from which we obtain (81). Arguments for (80), (82) and (83) are similar. Other arguments in (A1) and (A2) are direct consequences of (79)–(83). We next prove (84). For any $Q\in L_m^2(\mathbb{R}^n;\mathbb{R}^n)$, from Fubini's theorem, we have

$$\begin{aligned}&D_X\mathcal{L}(X,V,s;Q)(\tilde{X})\\=&\int_{\mathbb{R}^n}\left[Q_x^*\,D_XG(X,V,s)(\tilde{X})\Big|_x+\left(D_XF(X,V,s)|_x\right)^*\tilde{X}_x\right]dm(x)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \Bigg[Q_x^* (D_x g)^* (X_x, X\#m, V_x, s) \tilde{X}_x \\
&\qquad + \int_{\mathbb{R}^n} Q_x^* \left(D_\xi \frac{dg}{d\nu} \right)^* (X_x, X\#m, V_x, s)(X_y) \tilde{X}_y dm(y) \Bigg] dm(x) \\
&\quad + \int_{\mathbb{R}^n} \Bigg[D_x f(X_x, X\#m, V_x, s) \\
&\qquad + \int_{\mathbb{R}^n} D_\xi \frac{df}{d\nu}(X_y, X\#m, V_y, s)(X_x) dm(y) \Bigg]^* \tilde{X}_x dm(x) \\
&= \int_{\mathbb{R}^n} \Bigg[D_x g(X_x, X\#m, V_x, s) Q_x + D_x f(X_x, X\#m, V_x, s) \\
&\qquad + \int_{\mathbb{R}^n} D_\xi \frac{dg}{d\nu}(X_y, X\#m, V_y, s)(X_x) Q_y dm(y) \\
&\qquad + \int_{\mathbb{R}^n} D_\xi \frac{df}{d\nu}(X_y, X\#m, V_y, s)(X_x) dm(y) \Bigg]^* \tilde{X}_x dm(x),
\end{aligned}
$$

from which we obtain (84). Similarly, we have (85).

A.4. Proof of statements in Section 8

A.4.1. Proof of Lemma 8.2. For any $v_\cdot(\cdot) \in L^2([t,T]; L^2_{X\#m}(\mathbb{R}^n;\mathbb{R}^d))$, we know that $v'_\cdot(s) := v_\cdot(s) \circ X_\cdot$, $s \in [t,T]$ belongs to $L^2([t,T]; L^2_m(\mathbb{R}^n))$, and it is an admissible control for $(\mathscr{P}^{t,m,X})$. We denote by $X^v_{\cdot,X\#m,t}(\cdot)$ the corresponding controlled state for mean field type control problem $(\mathscr{P}^{t,X\#m})$ with control $v_\cdot(\cdot)$, and denote by $X^{v'}_{X_\cdot mt}(\cdot)$ the corresponding controlled state for $(\mathscr{P}^{t,m,X})$ with control $v'_\cdot(\cdot)$. Note that for $s \in [t,T]$,

$$
X^v_{\cdot,X\#m,t}(s)\#(X\#m) = \left(X^v_{\cdot,X\#m,t}(s) \circ X_\cdot\right)\#m = X^v_{X_\cdot,X\#m,t}(s)\#m, \tag{146}
$$

so we know that $X^v_{X_\cdot,X\#m,t}(s) = X^v_{\cdot,X\#m,t}(s) \circ X_\cdot$ satisfies the following equation: for $(s,x) \in [t,T] \times \mathbb{R}^n$,

$$
X^v_{X_x,X\#m,t}(s) = X_x + \int_t^s g\left(X^v_{X_x,X\#m,t}(r), X^v_{X_\cdot,X\#m,t}(r)\#m, v'_x(r), r\right) dr.
$$

From the uniqueness of the equation of $X^{v'}_{Xmt}$, we have that

$$
X^v_{X_\cdot,X\#m,t}(s) \xlongequal{L^2_m(\mathbb{R}^n;\mathbb{R}^n)} X^{v'}_{X_\cdot mt}(s), \quad s \in [t,T]. \tag{147}
$$

From (146) and (147), we have

(148)
$$
\begin{aligned}
&J_{X\#m,t}(v.(\cdot))\\
=&\int_t^T\int_{\mathbb{R}^n} f\left(X^v_{x,X\#m,t}(s), X^v_{\cdot,X\#m,t}(s)\#(X\#m), v_x(s), s\right) d(X\#m)(x)ds\\
&+\int_{\mathbb{R}^n} h\left(X^v_{x,X\#m,t}(T), X^v_{\cdot,X\#m,t}(T)\#(X\#m)\right) d(X\#m)(x)\\
=&\int_t^T\int_{\mathbb{R}^n} f\left(X^v_{X_x,X\#m,t}(s), X^v_{X.,X\#m,t}(s)\#m, v'_x(s), s\right) dm(x)ds\\
&+\int_{\mathbb{R}^n} h(X^v_{X_x,X\#m,t}(T), X^v_{X.,X\#m,t}(T)\#m)dm(x)\\
=&\int_t^T\int_{\mathbb{R}^n} f(X^{v'}_{X_xmt}(s), X^{v'}_{X.mt}(s)\#m, v'_x(s), s)dm(x)ds\\
&+\int_{\mathbb{R}^n} h(X^{v'}_{X_xmt}(T), X^{v'}_{X.mt}(T)\#m)dm(x)\\
=&J_{Xmt}(v'_{\cdot}(\cdot))\\
\geq&J_{Xmt}(u_{X.mt}(\cdot)) = \mathscr{V}(X,m,t).
\end{aligned}
$$

And by setting $v.(\cdot) = u_{\cdot mt}(\cdot)$ in the last inequality, we get (103).

A.4.2. Proof of Theorem 8.4. From Lemmas 8.2 and 7.1, we have

$$D_X\mathscr{V}(I,m,t)|_x = D_\xi\frac{d\mathcal{V}}{d\nu}(m,t)(x), \quad x\in\mathbb{R}^n,$$

then, from Theorem 8.3, we know that

$$D_m\mathcal{V}(m,t)|_x = D_\xi\frac{d\mathcal{V}}{d\nu}(m,t)(x), \quad x\in\mathbb{R}^n, \tag{149}$$

and $\mathcal{V}$ satisfies the Bellman equation (112). From (109) and (149), we have

$$
\begin{aligned}
D_X\mathscr{V}(X,m,t)|_x &= \left(D_m\mathcal{V}(X\#m,t)\circ X\right)\big|_x\\
&= D_\xi\frac{d\mathcal{V}}{d\nu}(X\#m,t)(X_x), \quad x\in\mathbb{R}^n,
\end{aligned} \tag{150}
$$

from which we obtain (113). From (98) and (150), we have

$$Q_{X_xmt}(s) = Q_{Y_{X_xmt}(s),m,s}(s) = \left(D_X\mathscr{V}(Y_{X_xmt}(s),m,s)\right)\big|_x$$

$$= D_\xi \frac{d\mathcal{V}}{d\nu}(Y_{X.mt}(s)\#m, s)(Y_{X_xmt}(s)),\ x \in \mathbb{R}^n.$$

So from (102), we know that the solution of $(\mathscr{P}^{t,m,X})$ satisfies for $(s,x) \in [t,T] \times \mathbb{R}^n$,

$$u_{X_xmt}(s) = \hat{v}\Big(Y_{X_xmt}(s), Y_{Xmt}(s)\#m, s; D_\xi \frac{d\mathcal{V}}{d\nu}(Y_{Xmt}(s)\#m, s)(Y_{X_xmt}(s))\Big).$$

We define the map $\phi : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times [t,T]$ as

$$\phi(x,m,s) := \hat{v}\left(x, m, s; D_\xi \frac{d\mathcal{V}}{d\nu}(m,s)(x)\right),$$

then we know that

$$u_{X_xmt}(s) = \phi(Y_{X_xmt}(s), Y_{X.mt}(s)\#m, s), \quad (s,x) \in [t,T] \times \mathbb{R}^n, \tag{151}$$

is a feedback.

A.4.3. Proof of Corollary 8.6. From (151), we have

$$\hat{v}_x(s) = \phi\left(\hat{X}_x(s), m(s), s\right), \quad (s,x) \in [0,T] \times \mathbb{R}^n. \tag{152}$$

We define $\hat{g} : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times [0,T] \to \mathbb{R}^n$ as

$$\hat{g}(x,m,s) := g(x,m,\phi(x,m,s),s), \quad (x,m,s) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times [0,T],$$

and define $\hat{X}^t_\cdot(\cdot) \in L^2([t,T]; L^2_{m(t)}(\mathbb{R}^n;\mathbb{R}^n))$ as

$$\hat{X}^t_x(s) = x + \int_t^s \hat{g}\left(\hat{X}^t_x(r), \hat{X}^t_\cdot(r)\#m(t), r\right) dr, \quad (s,x) \in [t,T] \times \mathbb{R}^n,$$

From the fact that

$$\hat{X}^t_\cdot(s)\#m(t) = \hat{X}^t_\cdot(s)\#(\hat{X}_\cdot(t)\#m) = (\hat{X}^t(s) \circ \hat{X}(t))\#m, \quad s \in [t,T],$$

we have

$$\hat{X}_\cdot(s) = \hat{X}^t_\cdot(s) \circ \hat{X}_\cdot(t), \quad s \in [t,T]. \tag{153}$$

From (152) and (153), we know that

$$\hat{v}^t_\cdot(s) := \hat{v}_\cdot(s) \circ (\hat{X}_\cdot(t))^{-1}, \quad s \in [t,T], \tag{154}$$

is well-defined and belongs to $L^2([t,T];L^2_{m(t)}(\mathbb{R}^n;\mathbb{R}^d))$. Then, for any $v_\cdot(\cdot) \in L^2([t,T];L^2_{m(t)}(\mathbb{R}^n;\mathbb{R}^d))$, from (148), (153) and (154), we have

$$\begin{aligned} J_{I,m(t),t}(v_\cdot(\cdot)) &= J_{\hat{X}(t),m,t}\left((v_\cdot \circ \hat{X}_\cdot(t))(\cdot)\right) \\ &\geq J_{\hat{X}(t),m,t}\left(\hat{v}_\cdot(\cdot)|_{[t,T]}\right) \\ &= J_{\hat{X}(t),m,t}\left((\hat{v}^t_\cdot \circ \hat{X}_\cdot(t))(\cdot)\right) = J_{I,m(t),t}\left(\hat{v}^t_\cdot(\cdot)\right). \end{aligned}$$

Therefore, $\hat{v}^t_\cdot(\cdot)$ is a solution of the mean field type control problem $(\mathscr{P}^{t,m(t)})$. From the uniqueness of solution of $(\mathscr{P}^{t,m(t)})$, we have $\hat{v}^t_\cdot(\cdot) = u_{\cdot,m(t),t}(\cdot)$, from which we obtain (115).

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