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Constructive exact controls for semi-linear wave equations

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Dedicated to the memory of Professor Roland Glowinski

The exact distributed controllability of the semi-linear wave equation $\partial_{tt}y - \Delta y + g(y) = f 1_\omega$ posed over multi-dimensional and bounded domains, assuming that $g \in \mathcal{C}^1(\mathbb{R})$ satisfies the growth condition $\limsup_{|r|\to\infty} g(r)/(|r| \ln^{1/2}|r|) = 0$ has been obtained by Fu, Yong and Zhang in 2007. The proof based on a non constructive Leray-Schauder fixed point theorem makes use of precise estimates of the observability constant for a linearized wave equation. Assuming that the derivative of g does not grow faster than $\beta \ln^{1/2}|r|$ at infinity for $\beta > 0$ small enough and is uniformly Hölder continuous on $\mathbb{R}$ with exponent $s \in (0,1]$, we design a constructive proof yielding an explicit sequence converging to a controlled solution for the semi-linear equation, at least with order $1+s$ after a finite number of iterations. Numerical experiments in the two-dimensional case illustrate the results. This work extends to a multi-dimensional case, enriches with additional results and completes with some numerical experiments the study in 2021 by Münch and Trélat devoted to the one-dimensional situation.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \in \{2,3\}$ with $\mathcal{C}^{1,1}$ boundary and $\omega \subset\subset \Omega$ be a non-empty open set. Let $T > 0$ and denote $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$. We consider the semi-linear wave

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equation

$$
\begin{cases} Ly + g(y) = f1_\omega, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \tag{1}
$$

where $L := \partial_{tt} - \Delta$ denotes the wave operator, $(u_0, u_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$ is the initial state of y and $f \in L^2(q_T)$ is a *control* function. Here and throughout the paper, $g : \mathbb{R} \to \mathbb{R}$ is a function of class $\mathcal{C}^1$ such that $|g(r)| \leq C(1 + |r|)\ln(2 + |r|)$ for every $r \in \mathbb{R}$ and some $C > 0$. Then, (1) has a unique global weak solution in $\mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ (see [6, 9]).

The exact controllability for (1) in time T is formulated as follows: for any $(u_0, u_1), (z_0, z_1) \in \boldsymbol{V}$, find a control function $f \in L^2(q_T)$ such that the weak solution of (1) satisfies $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$. Assuming a growth condition on the non-linearity g at infinity, this problem has been solved in [21].

Theorem 1.1. *[21, Theorem 2.2] For any $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$, let $\Gamma_0 = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) > 0\}$ and, for any $\epsilon > 0$, $\mathcal{O}_\epsilon(\Gamma_0) = \{y \in \mathbb{R}^d \mid |y - x| \leq \epsilon \text{ for } x \in \Gamma_0\}$. Assume*

($\mathbf{H_0}$) *$T > 2\max_{x \in \overline{\Omega}} |x - x_0|$ and $\omega \supseteq \mathcal{O}_\epsilon(\Gamma_0) \cap \Omega$ for some $\epsilon > 0$.*

If g satisfies

($\mathbf{H_1}$) $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r| \ln^{1/2}|r|} = 0$

then (1) is exactly controllable in time T.

This result improves [32] where a stronger condition of the support ω is made, namely that ω is a neighborhood of $\partial\Omega$ and that $T > \operatorname{diam}(\Omega \setminus \omega)$. In Theorem 1.1, Γ_0 is the usual star-shaped part of the whole boundary of Ω introduced in [33] and $\nu(x)$ denotes the outward normal derivative at any point $x \in \partial\Omega$.

A special case of Theorem 1.1 is when g is globally Lipschitz continuous, which gives the main result of [43], later generalized to an abstract setting in [26] using a global version of the inverse function theorem and improved in [41] for control domains ω satisfying the classical multiplier method of Lions [33]. Theorem 1.1 extends to the multi-dimensional case the result of [44] under the condition $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r|\ln^2|r|} = 0$, relaxed later on in [6], following [16], and in [34]. The exact controllability for subcritical non-linearities is obtained in [14] assuming the sign condition $rg(r) \geq 0$ for every

$r \in \mathbb{R}$. This latter assumption has been weakened in [25] to an asymptotic sign condition leading to a semi-global controllability result in the sense that the final data (z_0, z_1) is prescribed in a precise subset of $\boldsymbol{V}$.

The proof given in [21, 32] is based on a fixed-point argument introduced in [42, 44] that reduces the exact controllability problem to the obtention of suitable *a priori* estimates for the linearized wave equation with a potential (see Proposition A.1 in Appendix A). More precisely, it is shown that the operator $\Lambda : L^\infty(0,T;L^d(\Omega)) \to L^\infty(0,T;L^d(\Omega))$ where $y := \Lambda(z)$ is a controlled solution through the control function f of the linear boundary value problem

$$(2)\quad \begin{cases} Ly + \widehat{g}(z)y = -g(0) + f1_\omega, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \quad \widehat{g}(r) := \begin{cases} \dfrac{g(r) - g(0)}{r} & \text{if } r \neq 0, \\ g'(0) & \text{if } r = 0, \end{cases}$$

satisfying $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$ has a fixed point. The control f is chosen in [32] as the one of minimal $L^2(q_T)$-norm. The existence of a fixed point for the compact operator Λ is obtained by using the Leray-Schauder's degree theorem. In particular, under the growth assumption $(\mathbf{H_1})$, it is shown a stability property of the operator Λ, i.e. the existence of a constant $M = M(\|(u_0,u_1)\|_{\boldsymbol{V}}, \|(z_0,z_1)\|_{\boldsymbol{V}})$ such that $\Lambda(B_{L^\infty(0,T;L^d(\Omega))}(0,M)) \subset B_{L^\infty(0,T;L^d(\Omega))}(0,M)$.

This article is concerned with the determination of strongly convergent sequences $(y_k, f_k)_{k\in\mathbb{N}}$ toward a state-control pair for the nonlinear system (1). The controllability of nonlinear partial differential equations has attracted a large number of works in the last decades (see the monograph [13] and references therein). However, very few are concerned with the approximation of exact controls for nonlinear partial differential equations, and the construction of convergent control approximations for controllable nonlinear equations remains in general an open question. This is notably due to the fact that the available controllability results are based on non constructive fixed arguments. Thus, the Picard iterates $(y_k)_{k\in\mathbb{N}}$ associated with the operator Λ, defined for any $y_0 \in L^\infty(0,T;L^d(\Omega))$ by $y_{k+1} = \Lambda(y_k)$, $k \geq 0$, remains bounded in $L^\infty(0,T;L^d(\Omega))$ but has no reason to converge (we refer to [20] where divergence is observed numerically in a parabolic case).

Recently, two constructions of convergent sequences have been proposed in the one-dimensional case with $\Omega = (0,1)$: the first one in [39] is based on a least-squares approach: the extremal problem

$$\min_{(y,f)\in\mathcal{A}} E(y,f), \qquad E(y,f) := \|\partial_{tt}y - \partial_{xx}y + g(y) - f 1_\omega\|^2_{L^2(Q_T)}$$

is considered where $\mathcal{A}$ is a closed subset of $L^2(Q_T)\times L^2(q_T)$ containing the initial condition and controllability requirement at the initial and final time respectively. Assuming notably that $g\in C^1(\mathbb{R})$ satisfies for some $\beta>0$ small enough the asymptotic condition

$(\mathbf{H_1'})$ $\limsup_{|r|\to\infty} \frac{|g'(r)|}{\ln^{1/2}|r|} \le \beta$

a minimizing sequence for E is constructed and proved to converge strongly to a state-control pair for (1). We refer to [39, Theorem 2.3] for a precise statement. The least-squares approach turns out to be related to the operator $\Lambda_N : L^\infty(Q_T)\to L^\infty(Q_T)$, where for any $z\in L^\infty(Q_T)$, $y=\Lambda_N(z)$ is a controlled solution through the control f of

$$\text{(3)} \qquad \begin{cases} \partial_{tt}y - \partial_{xx}y + g'(z)\,y = f\,1_\omega + g'(z)z - g(z), & \text{in } Q_T,\\ y=0, & \text{on } \Sigma_T,\\ (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \end{cases}$$

satisfying $(y(\cdot,T),\partial_t y(\cdot,T)) = (z_0,z_1)$. Again, for each z, the control of minimal $L^2(q_T)$ norm is considered. A similar strategy has been successfully applied in [30] for a semi-linear 1D heat equation. The second one in [2] focuses on the boundary controllability and considers the operator $\Lambda_F : L^\infty(Q_T)\to L^\infty(Q_T)$, where for any $z\in L^\infty(Q_T)$, $y=\Lambda_F(z)$ is a controlled solution through the control f of

$$\text{(4)} \qquad \begin{cases} \partial_{tt}y - \partial_{xx}y = -g(z), & \text{in } Q_T,\\ y(0,\cdot)=0, \quad y(1,\cdot)=f, & \text{on } (0,T),\\ (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \end{cases}$$

satisfying $(y(\cdot,T),\partial_t y(\cdot,T)) = (z_0,z_1)$. For each z, the state-control pair (y,f) is chosen as the minimizer of an L^2 functional involving parametrized Carleman weights. Under the asymptotic condition $(\mathbf{H_1'})$, it is shown that the operator Λ_F is contracting for a small enough parameter β and large enough Carleman parameters (we refer to [2, Theorem 8] for a precise statement). This provides a convergent sequence $(y_k,f_k)_{k\in\mathbb{N}}$ to a state-control pair for the nonlinear equation. Remark that a similar operator has also been used recently for a semi-linear heat equation in [17].

The objective of the present paper is two-fold: first, we extend the least-squares approach introduced in [39] to a multi-dimensional case. With respect to the one-dimensional case studied in [39], the controlled solution is not anymore in $L^\infty(Q_T)$ but in $L^\infty(0,T;L^p(\Omega))$ for some p related to the dimension of Ω. This requires a finer analysis in order to estimate the observability constant. Second, we give some numerical illustrations of the method (not provided in [39]) both in the one and two dimensional cases. This requires the approximation of exact controls for linear wave equations, which is known to be a delicate issue, since the works of Glowinski (see the monograph [23] and the recent review [37]).

The paper is organized as follows. In Sections 2 and 3, we adapt [39] to the higher dimension without reproducing all the arguments. More precisely, we define the non-convex optimization problem (5) involving the least-squares functional E. We show that any critical point (y,f) for E such that $g'(y)\in L^\infty(0,T;L^d(\Omega))$ is also a zero of E. This is done by introducing a descent direction (Y^1,F^1) for E at any (y,f) for which $E'(y,f)\cdot(Y^1,F^1)$ is proportional to $\sqrt{E(y,f)}$. A minimizing sequence based on (Y^1,F^1) is then proved to converge to a controlled pair for the semi-linear wave equation (1) under assumptions on g similar to $(\mathbf{H_1'})$. We refer to Theorem 3.1 for a precise statement of our result. Section 4 provides several comments: in particular, we emphasize that the minimizing sequence still converge without any asymptotic property on the nonlinearity g if the initial condition and target are small enough (see Proposition 4.1). Section 5 then illustrates the result with some numerical experiments in one and two dimensions. Section 6 concludes. In Appendix A, we recall some *a priori* estimates for the linearized wave equation with potential in $L^\infty(0,T;L^d(\Omega))$ and source term in $L^2(Q_T)$.

As far as we know, the method introduced and analyzed in this work is the first one providing an explicit, algorithmic construction of exact controls for semi-linear wave equations with non-Lipschitz non-linearity and defined over multi-dimensional bounded domains.

We also mention some recent constructive approach but assuming smallness assumptions on the initial data to be controlled (see [8] devoted to the one-dimension case) or Lipschitz properties on the nonlinearity (see [40]).

This work extends the one-dimensional study addressed in [39], for which the solution is uniformly bounded with respect to both the time and space variable. In contrast, the multi-dimensional case for which the solution does not belong to $L^\infty(Q_T)$, requires finer analysis: we refer for instance to Lemma 3.3. With respect to [39], some proofs very closed to the one dimensional case are omitted; on the contrary, the proof of Proposition 4.1, left to the reader in [39], is given in the present work. With respect to [39], we also provide some numerical experiments (including for $d = 1$).

For parabolic equations with Lipschitz non-linearity, we mention [27]. Actually, this work devoted to controllability problems takes their roots in earlier works, namely [28, 29], concerned with the approximation of solution of Navier-Stokes type problem, through least-squares methods: they refine the analysis performed in [31, 36] inspired from the seminal contribution [3].

Throughout, we denote by $\|\cdot\|_\infty$ the usual norm in $L^\infty(\mathbb{R})$, by $(\cdot,\cdot)_X$ the scalar product of X (if X is a Hilbert space) and by $\langle\cdot,\cdot\rangle_{X,Y}$ the duality product between X and Y. The notation $\|\cdot\|_{2,q_T}$ stands for $\|\cdot\|_{L^2(q_T)}$ and $\|\cdot\|_p$ for $\|\cdot\|_{L^p(Q_T)}$, $p \in \mathbb{N}^\star$. We also denote by C a positive constant depending only on Ω and T that may vary from line to line.

In the rest of the paper, we assume that the open set ω and the time T satisfy $(\mathbf{H_0})$.

2. The least-squares functional and its properties

2.1. The least-squares problem

We define the Hilbert space

$$\mathcal{H} = \big\{(y,f) \in L^2(Q_T) \times L^2(q_T),\, y \in \mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)) \\ \mid Ly \in L^2(Q_T)\big\}$$

endowed with the inner product

$$((y,f),(\overline{y},\overline{f}))_{\mathcal{H}} = (y,\overline{y})_2 + \big((y(\cdot,0),\partial_t y(\cdot,0)),(\overline{y}(\cdot,0),\partial_t\overline{y}(\cdot,0))\big)_{\boldsymbol{V}} \\ + (Ly, L\overline{y})_2 + (f,\overline{f})_{2,q_T}$$

and the norm $\|(y,f)\|_{\mathcal{H}} := \sqrt{((y,f),(y,f))_{\mathcal{H}}}$.

Remark 2.1. *Endowed with the norm*

$$\|(y,\partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})} := \|y\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\partial_t y\|_{L^\infty(0,T;L^2(\Omega))},$$

the space $\mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ *is a Banach space and* $\mathcal{H} \hookrightarrow \big(\mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))\big) \times L^2(q_T)$ *continuously. Indeed, if* $(y,f) \in \mathcal{H}$*, we get from [33, Lemme 3.6, p. 39] that*

$$\|(y,\partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})} \le C\Big(\|Ly\|_{L^2(Q_T)} + \|(y(\cdot,0),\partial_t y(\cdot,0))\|_{\boldsymbol{V}}\Big)$$

from which we deduce that $\|(y,\partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})} + \|f\|_{L^2(q_T)} \le C\|(y,f)\|_{\mathcal{H}}$.

Let $(u_0,u_1),(z_0,z_1)\in \boldsymbol{V}$. We define the non-empty subspaces of $\mathcal{H}$

$$\mathcal{A}=\Big\{(y,f)\in\mathcal{H}\mid (y(\cdot,0),\partial_t y(\cdot,0))=(u_0,u_1),\ (y(\cdot,T),\partial_t y(\cdot,T))=(z_0,z_1)\Big\},$$
$$\mathcal{A}_0=\Big\{(y,f)\in\mathcal{H}\mid (y(\cdot,0),\partial_t y(\cdot,0))=(0,0),\ (y(\cdot,T),\partial_t y(\cdot,T))=(0,0)\Big\}.$$

Remark that $(0,0)\in\mathcal{A}_0$ while $\mathcal{A}$ contains the controlled pairs for the linear wave equation.

We consider the following non convex extremal problem:

$$\inf_{(y,f)\in\mathcal{A}} E(y,f),\qquad E(y,f):=\frac{1}{2}\big\|Ly+g(y)-f1_\omega\big\|_2^2. \tag{5}$$

The functional E is well-defined in $\mathcal{A}$. Precisely,

Lemma 2.1. *There exists a positive constant $C>0$ such that $E(y,f)\le C(1+\|(y,f)\|_{\mathcal{H}}^3)$ for any $(y,f)\in\mathcal{A}$.*

Proof. A priori estimate for the linear wave equation reads as

$$\|(y,\partial_t y)\|^2_{L^\infty(0,T;\boldsymbol{V})}\le C\big(\|Ly\|_2^2+\|(u_0,u_1)\|^2_{\boldsymbol{V}}\big)$$

for any y such that $(y,f)\in\mathcal{A}$. Using that $|g(r)|\le C(1+|r|)\log(2+|r|)$ for every $r\in\mathbb{R}$ and some $C>0$, we infer that

$$\begin{aligned}\|g(y)\|_2^2&\le C^2\int_{Q_T}\Big((1+|y|)\log(2+|y|)\Big)^2\\&\le C^2\int_{Q_T}(1+|y|)^3\le C^2(|Q_T|^3+\|y\|^3_{L^3(Q_T)})\\&\le C^2\big(|Q_T|^3+\|y\|^3_{L^\infty(0,T;H_0^1(\Omega))}\big)\end{aligned}$$

from which we get $E(y,f)\le C\big(\|Ly\|_2^2+\|f\|^2_{2,q_T}+|Q_T|^3+\|y\|^3_{L^\infty(0,T;H_0^1(\Omega))}\big)$ and the result. $\square$

Within the hypotheses of Theorem 1.1, the infimum of the functional of E is zero and is reached by at least one pair $(y,f)\in\mathcal{A}$, solution of (1) and satisfying $(y(\cdot,T),\partial_t y(\cdot,T))=(z_0,z_1)$. Conversely, any pair $(y,f)\in\mathcal{A}$ for which $E(y,f)$ vanishes is solution of (1). In spite of the lack of convexity of the functional E, we are going to construct a minimizing sequence which always converges to a zero of E. In this respect, we formally look, for any $(y, f) \in \mathcal{A}$, for a pair $(Y^1, F^1) \in \mathcal{A}_0$ solution of

$$\begin{cases} LY^1 + g'(y) \cdot Y^1 = F^1 1_\omega + \big(Ly + g(y) - f\, 1_\omega\big), & \text{in } Q_T, \\ Y^1 = 0, & \text{on } \Sigma_T, \\ (Y^1(\cdot, 0), \partial_t Y^1(\cdot, 0)) = (0, 0), & \text{in } \Omega. \end{cases} \tag{6}$$

Since (Y^1, F^1) belongs to $\mathcal{A}_0$, F^1 is a null control for Y^1. Among the controls of this linear equation, we select the control of minimal $L^2(q_T)$ norm. In the sequel, we shall call the corresponding solution $(Y^1, F^1) \in \mathcal{A}_0$ the solution of *minimal control norm*. We have the following property.

Proposition 2.1. *For any $(y, f) \in \mathcal{A}$, there exists a pair $(Y^1, F^1) \in \mathcal{A}_0$ solution of (6). Moreover, the pair (Y^1, F^1) of minimal control norm satisfies the following estimates:*

$$\|(Y^1, \partial_t Y^1)\|_{L^\infty(0,T;\boldsymbol{V})} + \|F^1\|_{2,q_T} \le C e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \sqrt{E(y, f)}, \tag{7}$$

and

$$\|(Y^1, F^1)\|_{\mathcal{H}} \le C\big(1 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big) e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \sqrt{E(y, f)} \tag{8}$$

for some positive constant $C > 0$.

Proof. The first estimate follows Proposition A.2 and the equality $\|Ly + g(y) - f\, 1_\omega\|_2 = \sqrt{2E(y, f)}$. The second one follows from

$$\begin{aligned} \|(Y^1, F^1)\|_{\mathcal{H}} &\le \|LY^1\|_2 + \|Y^1\|_2 + \|F^1\|_{2,q_T} + \|Y^1(\cdot, 0), \partial_t Y^1(\cdot, 0)\|_{\boldsymbol{V}} \\ &\le \|Y^1\|_2 + \|g'(y)Y^1\|_2 + 2\|F^1\|_{2,q_T} + \sqrt{2}\sqrt{E(y, f)} \\ &\le C\big(1 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big) e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \sqrt{E(y, f)} \end{aligned}$$

using that

$$\begin{aligned} \|g'(y)Y^1\|_2^2 &\le \int_0^T \|g'(y)\|^2_{L^3(\Omega)} \|Y^1\|^2_{L^6(\Omega)} \\ &\le C\|g'(y)\|^2_{L^\infty(0,T;L^3(\Omega))} \|Y^1\|^2_{L^\infty(0,T;H^1_0(\Omega))}. \end{aligned} \qquad \square$$

2.2. Main properties of the functional E

Given any $s \in (0, 1]$, we introduce for any $g \in \mathcal{C}^1(\mathbb{R})$ the following hypothesis:

($\overline{\mathbf{H}}_{\mathbf{s}}$) $[g']_s := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|g'(a)-g'(b)|}{|a-b|^s} < +\infty$

meaning that g' is uniformly Hölder continuous with exponent s. In particular, g satisfies ($\overline{\mathbf{H}}_{\mathbf{1}}$) if and only if g' is Lipschitz continuous (in this case, g' is almost everywhere differentiable and $g'' \in L^\infty(\mathbb{R})$, and we have $[g']_s \leq \|g''\|_\infty$).

The interest of the pair $(Y^1, F^1) \in \mathcal{A}_0$ lies in the following result.

Proposition 2.2. *Assume that g satisfies ($\overline{\mathbf{H}}_{\mathbf{s}}$) for some $s \in (0,1]$. Let $(y,f) \in \mathcal{A}$ and let $(Y^1, F^1) \in \mathcal{A}_0$ be a solution of (6). Then the derivative of E at the point $(y,f) \in \mathcal{A}$ along the direction (Y^1, F^1) satisfies*

$$E'(y,f)\cdot(Y^1,F^1) = 2E(y,f). \tag{9}$$

Proof. We check that for all $(Y,F) \in \mathcal{A}_0$ the functional E is differentiable at the point $(y,f) \in \mathcal{A}$ along the direction $(Y,F) \in \mathcal{A}_0$. For any $\lambda \in \mathbb{R}$, simple computations lead to the equality

$$E(y+\lambda Y, f+\lambda F) = E(y,f) + \lambda E'(y,f)\cdot(Y,F) + h((y,f),\lambda(Y,F))$$

with

$$E'(y,f)\cdot(Y,F) := \big(Ly + g(y) - f\,1_\omega, LY + g'(y)Y - F\,1_\omega\big)_2 \tag{10}$$

and

$$\begin{aligned} h((y,f),\lambda(Y,F)) :=& \frac{\lambda^2}{2}\big(LY + g'(y)Y - F\,1_\omega, LY + g'(y)Y - F\,1_\omega\big)_2 \\ &+ \lambda\big(LY + g'(y)Y - F\,1_\omega, l(y,\lambda Y)\big)_2 \\ &+ \big(Ly + g(y) - f\,1_\omega, l(y,\lambda Y)\big) + \frac{1}{2}(l(y,\lambda Y), l(y,\lambda Y)) \end{aligned}$$

where $l(y,\lambda Y) := g(y+\lambda Y) - g(y) - \lambda g'(y)Y$. The application $(Y,F) \to E'(y,f)\cdot(Y,F)$ is linear and continuous from $\mathcal{A}_0$ to $\mathbb{R}$ as it satisfies

$$\begin{aligned} &|E'(y,f)\cdot(Y,F)| \\ &\quad\leq \|Ly + g(y) - f\,1_\omega\|_2 \|LY + g'(y)Y - F\,1_\omega\|_2 \\ &\quad\leq \sqrt{2E(y,f)}\Big(\|LY\|_2 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\|Y^1\|_{L^\infty(0,T;H_0^1(\Omega))} + \|F\|_{2,q_T}\Big) \\ &\quad\leq \sqrt{2E(y,f)}\max\big(1, \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)\|(Y,F)\|_{\mathcal{H}}. \end{aligned} \tag{11}$$

Similarly, for all $\lambda \in \mathbb{R}^\star$,

$$\begin{aligned}\left|\frac{1}{\lambda}h((y,f),\lambda(Y,F))\right| \leq & \frac{|\lambda|}{2}\|LY+g'(y)Y-F\,1_\omega\|_2^2 \\ & + \Big(|\lambda|\|LY+g'(y)Y-F\,1_\omega\|_2 \\ & + \sqrt{2E(y,f)}+\frac{1}{2}\|l(y,\lambda Y)\|_2\Big)\frac{1}{|\lambda|}\|l(y,\lambda Y)\|_2.\end{aligned}$$

For any $(x,y)\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$, we then write $g(x+\lambda y)-g(x)=\int_0^\lambda yg'(x+\xi y)d\xi$ leading to

$$\begin{aligned}|g(x+\lambda y)-g(x)-\lambda g'(x)y| &\leq \left|\int_0^\lambda |y||g'(x+\xi y)-g'(x)|d\xi\right| \\ &\leq \left|\int_0^\lambda |y|^{1+s}|\xi|^s\frac{|g'(x+\xi y)-g'(x)|}{|\xi y|^s}d\xi\right| \\ &\leq [g']_s|y|^{1+s}\frac{|\lambda|^{1+s}}{1+s}.\end{aligned}$$

It follows that $|l(y,\lambda Y)|=|g(y+\lambda Y)-g(y)-\lambda g'(y)Y|\leq [g']_s\frac{|\lambda|^{1+s}}{1+s}|Y|^{1+s}$ and

$$\frac{1}{|\lambda|}\big\|l(y,\lambda Y)\big\|_2 \leq [g']_s\frac{|\lambda|^s}{1+s}\big\||Y|^{1+s}\big\|_2. \tag{12}$$

But $\big\||Y|^{1+s}\big\|_2^2=\|Y\|_{2(s+1)}^{2(s+1)}\leq C\|Y\|^{2(s+1)}_{L^\infty(0,T;L^4(\Omega))}$; consequently, $|\frac{1}{\lambda}|\|l(y,\lambda Y)\|_2 \to 0$ as $\lambda\to 0$ and $|h((y,f),\lambda(Y,F))|=o(\lambda)$. Eventually, the equality (9) follows from the definition of the pair (Y^1,F^1) given in (6). $\square$

Remark that from the equality (10), the derivative $E'(y,f)$ is independent of (Y,F). We can then define the norm

$$\|E'(y,f)\|_{\mathcal{A}_0'} := \sup_{(Y,F)\in\mathcal{A}_0\setminus\{0\}}\frac{E'(y,f)\cdot(Y,F)}{\|(Y,F)\|_{\mathcal{H}}}$$

associated with $\mathcal{A}_0'$, the topological dual of $\mathcal{A}_0$.

Combining the equality (9) and the inequality (7), we deduce the following estimate of $E(y,f)$ in term of the norm of $E'(y,f)$.

Proposition 2.3. *For any $(y, f) \in \mathcal{A}$, the following inequalities hold true:*

$$\begin{aligned}
&\frac{1}{\sqrt{2}\max\left(1, \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\right)}\|E'(y,f)\|_{\mathcal{A}_0'} \\
(13)\qquad &\leq \sqrt{E(y,f)} \\
&\leq \frac{1}{\sqrt{2}}C\left(1 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\right)e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\|E'(y,f)\|_{\mathcal{A}_0'}
\end{aligned}$$

where C is the positive constant from Proposition 2.1.

Proof. (9) rewrites $E(y, f) = \frac{1}{2}E'(y, f) \cdot (Y^1, F^1)$ where $(Y^1, F^1) \in \mathcal{A}_0$ is solution of (6) and therefore, with (8)

$$\begin{aligned}
E(y,f) &\leq \frac{1}{2}\|E'(y,f)\|_{\mathcal{A}_0'}\|(Y^1,F^1)\|_{\mathcal{A}_0} \\
&\leq \frac{1}{2}C\big(1+\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\|E'(y,f)\|_{\mathcal{A}_0'}\sqrt{E(y,f)}.
\end{aligned}$$

On the other hand, the left inequality follows from (11). □

Consequently, any *critical* point $(y, f) \in \mathcal{A}$ of E (i.e., $E'(y, f)$ vanishes) such that $\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}$ is finite is a zero for E, a pair solution of the controllability problem. In other words, any sequence $(y_k, f_k)_{k>0}$ satisfying $\|E'(y_k, f_k)\|_{\mathcal{A}_0'} \to 0$ as $k \to \infty$ and for which $\|g'(y_k)\|_{L^\infty(0,T;L^3(\Omega))}$ is uniformly bounded is such that $E(y_k, f_k) \to 0$ as $k \to \infty$. We insist that this property does not imply the convexity of the functional E (and *a fortiori* the strict convexity of E, which actually does not hold here in view of the multiple zeros for E) but show that a minimizing sequence for E can not be stuck in a local minimum.

On the other hand, the left inequality indicates the functional E is flat around its zero set. As a consequence, gradient-based minimizing sequences may achieve a low speed of convergence (we refer to [38] and also [31] devoted to the Navier-Stokes equation where this phenomenon is observed).

We end this section with the following fundamental estimate.

Lemma 2.2. *Assume that g satisfies $(\overline{\mathbf{H}}_{\mathbf{s}})$ for some $s \in (0, 1]$. For any $(y, f) \in \mathcal{A}$, let $(Y^1, F^1) \in \mathcal{A}_0$ be defined by (6). For any $\lambda \in \mathbb{R}$ the following estimate holds*

$$(14)\qquad E\big((y,f) - \lambda(Y^1,F^1)\big) \leq E(y,f)\left(|1-\lambda| + |\lambda|^{1+s}\,c(y)E(y,f)^{s/2}\right)^2$$

with $c(y) := \frac{C}{(1+s)\sqrt{2}}[g']_s d(y)^{1+s}$ and $d(y) := Ce^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}$.

Proof. Estimate (12) applied with $Y = Y^1$ reads

$$\big\|l(y,\lambda Y^1)\big\|_2 \le [g']_s \frac{|\lambda|^{1+s}}{1+s} \big\||Y^1|^{1+s}\big\|_2.$$

But $\||Y^1|^{1+s}\|_2^2 = \|Y^1\|_{2(s+1)}^{2(s+1)} \le C\|Y^1\|_{L^\infty(0,T;H_0^1(\Omega))}^{2(s+1)}$ with (7) lead to

$$\big\||Y^1|^{1+s}\big\|_2 \le C\Big(Ce^{C\|g'(y)\|_{L^\infty(0,T;L^d(\Omega))}}\Big)^{1+s} E(y,f)^{\frac{1+s}{2}}. \tag{15}$$

Eventually, we write

$$\begin{aligned}
&2E\big((y,f)-\lambda(Y^1,F^1)\big)\\
&= \Big\|\big(Ly+g(y)-f\,1_\omega\big)-\lambda\big(LY^1+g'(y)Y^1-F\,1_\omega\big)+l(y,-\lambda Y^1)\Big\|_2^2\\
&= \Big\|(1-\lambda)\big(Ly+g(y)-f\,1_\omega\big)+l(y,-\lambda Y^1)\Big\|_2^2\\
&\le \Big(\big\|(1-\lambda)\big(Ly+g(y)-f\,1_\omega\big)\big\|_2+\big\|l(y,-\lambda Y^1)\big\|_2\Big)^2\\
&\le 2\bigg(|1-\lambda|\sqrt{E(y,f)}+[g']_s\frac{|\lambda|^{1+s}}{1+s}\big\||Y^1|^{1+s}\big\|_2\bigg)^2\\
&\le 2\bigg(|1-\lambda|\sqrt{E(y,f)}+[g']_s\frac{|\lambda|^{1+s}}{1+s}C\Big(Ce^{C\|g'(y)\|_{L^\infty(0,T;L^d(\Omega))}}\Big)^{1+s}E(y,f)^{\frac{1+s}{2}}\bigg)^2
\end{aligned} \tag{16}$$

and we get the result. □

3. Convergence of a minimizing sequence for E

Equality (9) shows that $-(Y^1,F^1)$ given by the solution of (6) is a descent direction for E. Therefore, we define, for any fixed $m \ge 1$, the following minimizing sequence $(y_k,f_k)_{k>0} \in \mathcal{A}$

$$\begin{cases}(y_0,f_0)\in\mathcal{A},\\ (y_{k+1},f_{k+1})=(y_k,f_k)-\lambda_k(Y_k^1,F_k^1), \quad k\in\mathbb{N},\\ \lambda_k=\operatorname{argmin}_{\lambda\in[0,m]}E\big((y_k,f_k)-\lambda(Y_k^1,F_k^1)\big),\end{cases} \tag{17}$$

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\text{(18)} \qquad \begin{cases} LY_k^1 + g'(y_k) \cdot Y_k^1 = F_k^1 1_\omega + Ly_k + g(y_k) - f_k 1_\omega, & \text{in } Q_T, \\ Y_k^1 = 0, & \text{on } \Sigma_T, \\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0), & \text{in } \Omega. \end{cases}$$

The real $m \geq 1$ is arbitrarily fixed and is introduced in order to keep the sequence $(\lambda_k)_{k\in\mathbb{N}}$ bounded.

Given any $s \in (0,1]$, we set

$$\text{(19)} \qquad \beta^\star(s) := \sqrt{\frac{s}{2C(2s+1)}}$$

where $C > 0$, only depending on Ω and T, is the constant appearing in Proposition A.2. In this section, we prove our main result.

Theorem 3.1. *Assume that g' satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for some $s \in (0,1]$ and*

- $(\mathbf{H_2})$ *There exists $\alpha \geq 0$ and $\beta \in [0, \beta^\star(s))$ such that $|g'(r)| \leq \alpha + \beta \ln^{1/2}(1+|r|)$ for every $r \in \mathbb{R}$.*

Then, for any $(y_0, f_0) \in \mathcal{A}$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (17) strongly converges to a pair $(y, f) \in \mathcal{A}$ satisfying (1) and the condition

$$(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1), \textit{for all } (u_0,u_1), (z_0,z_1) \in \boldsymbol{V}.$$

Moreover, the convergence is at least linear and is at least of order $1+s$ after a finite number of iterations

This result remains true for $s = 0$ (remark that $[g']_0 < \infty$ is equivalent to $g' \in L^\infty(\mathbb{R})$) under the additional smallness assumption on $\|g'\|_\infty$:

- $(\mathbf{H_3})$ $\sqrt{2}C\|g'\|_\infty e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} < 1$

with C the constant appearing in Proposition A.2. We refer to [39] devoted to the case $d = 1$ for the details.

The proof of Theorem 3.1 consists in showing that the decreasing sequence $(E(y_k, f_k))_{k\in\mathbb{N}}$ converges to zero. In view of (13), this property is related to the uniform property of the observability constant

$$e^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$$

with respect to k. In order to fix some notations and the main ideas of the proof of Theorem 3.1, we first prove in Section 3.1 the convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ under the stronger condition that $g' \in L^\infty(\mathbb{R})$, sufficient to ensure the boundedness of the sequence $\big(e^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}\big)_{k\in\mathbb{N}}$. Then, in Section 3.2, we prove Theorem 3.1 by showing that under the assumption **($\mathbf{H_2}$)**, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ is actually bounded in $\mathcal{A}$. This implies the same property for the real sequence $e^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$, and then the announced convergence.

3.1. Proof of the convergence under the additional assumption $g' \in L^\infty(\mathbb{R})$

We establish the following preliminary result which coincides with Theorem 3.1 in the simpler case $\beta = 0$.

Proposition 3.1. *Assume that g' satisfies* ($\overline{\mathbf{H}}_\mathbf{s}$) *for some $s \in (0,1]$ and that $g' \in L^\infty(\mathbb{R})$. For any $(y_0, f_0) \in \mathcal{A}$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (17) strongly converges to a pair $(y, f) \in \mathcal{A}$ satisfying* (1) *and the condition $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$. Moreover, the convergence is at least linear and is at least of order $1+s$ after a finite number of iterations.*

Proceeding as in [29, 39], Proposition 3.1 follows from the following lemma.

Lemma 3.1. *Under the hypotheses of Proposition 3.1, for any $(y_0, f_0) \in \mathcal{A}$, there exists a $k_0 \in \mathbb{N}$ such that the sequence $(E(y_k, f_k))_{k\geq k_0}$ tends to 0 as $k \to \infty$ with at least a rate $s+1$.*

Proof. Since $g' \in L^\infty(\mathbb{R})$, the nonnegative constant $c(y_k)$ in (14) is uniformly bounded w.r.t. k: we introduce the real $c > 0$ as follows

$$c(y_k) \leq c := \frac{C}{(1+s)\sqrt{2}}[g']_s\Big(Ce^{C\|g'\|^2_\infty |\Omega|^{2/d}}\Big)^{1+s}, \quad \forall k \in \mathbb{N}. \tag{20}$$

$|\Omega|$ denotes the measure of the domain Ω. For any $(y_k, f_k) \in \mathcal{A}$, let us then denote the real function p_k by

$$p_k(\lambda) := |1-\lambda| + \lambda^{1+s} c\, E(y_k, f_k)^{s/2}, \quad \forall \lambda \in [0, m].$$

Lemma 2.2 with $(y, f) = (y_k, f_k)$ then allows to write that

$$\sqrt{E(y_{k+1}, f_{k+1})} = \min_{\lambda\in[0,m]} \sqrt{E((y_k, f_k) - \lambda(Y_k^1, F_k^1))} \leq p_k(\widetilde{\lambda_k})\sqrt{E(y_k, f_k)} \tag{21}$$

with $p_k(\widetilde{\lambda_k}) := \min_{\lambda\in[0,m]} p_k(\lambda)$. Assume first that $s>0$. The optimal $\widetilde{\lambda_k}$ is given by

$$\widetilde{\lambda_k} := \begin{cases} \frac{1}{(1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)}}, & \text{if } (1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)} \geq 1, \\ 1, & \text{if } (1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)} < 1 \end{cases}$$

leading to

(22)
$$p_k(\widetilde{\lambda_k}) = \begin{cases} 1 - \frac{s}{(1+s)^{\frac{1}{s}+1}} \frac{1}{c^{1/s}\sqrt{E(y_k,f_k)}}, & \text{if } (1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)} \geq 1, \\ c\, E(y_k,f_k)^{s/2}, & \text{if } (1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)} < 1. \end{cases}$$

Accordingly, we may distinguish two cases:

• If $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)} < 1$, then $c^{1/s}\sqrt{E(y_0,f_0)} < 1$, and thus $c^{1/s}\sqrt{E(y_k,f_k)} < 1$ for all $k\in\mathbb{N}$ since the sequence $(E(y_k,f_k))_{k\in\mathbb{N}}$ is decreasing. Hence (21) implies that

$$c^{1/s}\sqrt{E(y_{k+1},f_{k+1})} \leq \left(c^{1/s}\sqrt{E(y_k,f_k)}\right)^{1+s} \quad \forall k\in\mathbb{N}.$$

It follows that $c^{1/s}\sqrt{E(y_k,f_k)} \to 0$ as $k\to\infty$ with a rate equal to $1+s$.

• If $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)} \geq 1$ then we check that the set $I := \{k \in \mathbb{N},\ (1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)} \geq 1\}$ is a finite subset of $\mathbb{N}$; indeed, for all $k\in I$, (21) implies that

(23)
$$\begin{aligned} c^{1/s}\sqrt{E(y_{k+1},f_{k+1})} &\leq \left(1 - \frac{s}{(1+s)^{\frac{1}{s}+1}} \frac{1}{c^{1/s}\sqrt{E(y_k,f_k)}}\right) c^{1/s}\sqrt{E(y_k,f_k)} \\ &\leq c^{1/s}\sqrt{E(y_k,f_k)} - \frac{s}{(1+s)^{\frac{1}{s}+1}} \end{aligned}$$

and the strict decrease of the sequence $(c^{1/s}\sqrt{E(y_k,f_k)})_{k\in I}$. Thus there exists $k_0\in\mathbb{N}$ such that for all $k\geq k_0$, $(1+s)^{1/s}c^{1/s}\sqrt{E(y_k,f_k)} < 1$, that is I is a finite subset of $\mathbb{N}$. Arguing as in the first case, it follows that $\sqrt{E(y_k,f_k)} \to 0$ as $k\to\infty$.

It follows in particular from (22) that the sequence $(p_k(\widetilde{\lambda_k}))_{k\in\mathbb{N}}$ decreases as well. $\square$

Proof of Proposition 3.1. In view of (8), we write

$$\begin{aligned}\big(1+\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\\ &\leq (1+\|g'\|_\infty|\Omega|^{1/3})e^{C\|g'\|^2_\infty|\Omega|^{2/d}}\\ &\leq e^{2C\|g'\|^2_\infty|\Omega|^{2/d}}\end{aligned}$$

using that $(1+u)e^{u^2}\leq e^{2u^2}$ for all $u\in\mathbb{R}^+$. It follows that

$$\sum_{n=0}^{k}|\lambda_n|\|(Y_n^1,F_n^1)\|_{\mathcal{H}}\leq m\,Ce^{C\|g'\|^2_\infty|\Omega|^{2/d}}\sum_{n=0}^{k}\sqrt{E(y_n,f_n)}. \tag{24}$$

Using that $p_n(\widetilde{\lambda}_n)\leq p_0(\widetilde{\lambda}_0)$ for all $n\geq 0$, we can write for $n>0$,

$$\begin{aligned}\sqrt{E(y_n,f_n)}&\leq p_{n-1}(\widetilde{\lambda}_{n-1})\sqrt{E(y_{n-1},f_{n-1})}\\ &\leq p_0(\widetilde{\lambda}_0)\sqrt{E(y_{n-1},f_{n-1})}\\ &\leq (p_0(\widetilde{\lambda}_0))^n\sqrt{E(y_0,f_0)}.\end{aligned} \tag{25}$$

Then, using that $p_0(\widetilde{\lambda}_0)=\min_{\lambda\in[0,m]}p_0(\lambda)<1$ (since $p_0(0)=1$ and $p_0'(0)<0$), we finally obtain the uniform estimate

$$\sum_{n=0}^{k}|\lambda_n|\|(Y_n^1,F_n^1)\|_{\mathcal{H}}\leq m\,Ce^{C\|g'\|^2_\infty|\Omega|^{2/d}}\frac{\sqrt{E(y_0,f_0)}}{1-p_0(\widetilde{\lambda}_0)}$$

for which we deduce (since $\mathcal{H}$ is a complete space) that the series $\sum_{n\geq 0}\lambda_n(Y_n^1,F_n^1)$ converges in $\mathcal{A}_0$. Writing from (17) that $(y_{k+1},f_{k+1})=(y_0,f_0)-\sum_{n=0}^{k}\lambda_n(Y_n^1,F_n^1)$, we conclude that (y_k,f_k) strongly converges in $\mathcal{A}$ to $(y,f):=(y_0,f_0)+\sum_{n\geq 0}\lambda_n(Y_n^1,F_n^1)$.

Let us now pass to the limit in (18). We write that $\|g(y_k)-g(y)\|_{L^2(Q_T)}\leq\|g'\|_\infty\|y_k-y\|_{L^2(Q_T)}$ and thus $g(y_k)\to g(y)$ in $L^2(Q_T)$. Moreover, $(g'(y_k))_{k\in\mathbb{N}}$ is a bounded sequence of $L^2(Q_T)$ since $g'\in L^\infty$. Then, using that (Y_k^1,F_k^1) goes to zero as $k\to\infty$ in $\mathcal{A}_0$, we pass to the limit in (18) and get that $(y,f)\in\mathcal{A}$ solves (1). Moreover, since the limit (y,f) belongs to $\mathcal{A}$, we have that $(y(\cdot,T),y_t(\cdot,T))=(z_0,z_1)$ in Ω. Eventually, for all $k>0$

(26)

$$\|(y,f)-(y_k,f_k)\|_{\mathcal{H}}=\Big\|\sum_{p=k+1}^{\infty}\lambda_p(Y_p^1,F_p^1)\Big\|_{\mathcal{H}}\leq m\sum_{p=k+1}^{\infty}\|(Y_p^1,F_p^1)\|_{\mathcal{H}}$$

$$\leq m\,C\sum_{p=k+1}^{\infty}\sqrt{E(y_p,f_p)}\leq m\,C\sum_{p=k+1}^{\infty}p_0(\widetilde{\lambda}_0)^{p-k}\sqrt{E(y_k,f_k)}$$
$$\leq m\,C\frac{p_0(\widetilde{\lambda}_0)}{1-p_0(\widetilde{\lambda}_0)}\sqrt{E(y_k,f_k)}$$

and conclude from Lemma 3.1 to the convergence of order at least $1+s$ after a finite number of iterates. □

Remark 3.1. *In particular, along the sequence $(y_k,f_k)_{k\in\mathbb{N}}$ defined by (17), the inequality (26) is a coercivity type property for the functional E; we emphasize, in view of the non uniqueness of the zeros of E, that an estimate (similar to (26)) of the form $\|(\overline{y},\overline{f})-(y,f)\|_{\mathcal{H}}\leq C\sqrt{E(\overline{y},\overline{f})}$ does not hold for all $(\overline{y},\overline{f})\in\mathcal{A}$. We also emphasize that the sequence $(y_k,f_k)_{k\in\mathbb{N}}$ and its limits (y,f) are uniquely determined from the initialization $(y_0,f_0)\in\mathcal{A}$ and from the selection criterion chosen for the state-control pair (Y_k^1,F_k^1).*

Remark 3.2. *Estimate (24) implies the uniform estimate on the sequence $(\|(y_k,f_k)\|_{\mathcal{H}})_{k\in\mathbb{N}}$:*

$$\|(y_k,f_k)\|_{\mathcal{H}}\leq\|(y_0,f_0)\|_{\mathcal{H}}+m\,Ce^{C\|g'\|_\infty^2|\Omega|^{2/d}}\sum_{n=0}^{k-1}\sqrt{E(y_n,f_n)}$$
$$\leq\|(y_0,f_0)\|_{\mathcal{H}}+m\,Ce^{C\|g'\|_\infty^2|\Omega|^{2/d}}\frac{\sqrt{E(y_0,f_0)}}{1-p_0(\widetilde{\lambda}_0)}.$$

In particular, for the less favorable case for which $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)}\geq 1$, we get $\frac{\sqrt{E(y_0,f_0)}}{1-p_0(\widetilde{\lambda}_0)}=\frac{(1+s)^{\frac{1}{s}+1}}{s}c^{1/s}E(y_0,f_0)$, (see (22)) leading to

$$\|(y_k,f_k)\|_{\mathcal{H}}\leq\|(y_0,f_0)\|_{\mathcal{H}}+m\,Ce^{C\|g'\|_\infty^2|\Omega|^{2/d}}\frac{(1+s)^{\frac{1}{s}+1}}{s}c^{1/s}E(y_0,f_0),$$

and then, in view of (20), to the explicit estimate in term of the data

$$\|(y_k,f_k)\|_{\mathcal{H}}\leq\|(y_0,f_0)\|_{\mathcal{H}}+m\,\frac{(1+s)}{s}\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s}\left(Ce^{C\|g'\|_\infty^2|\Omega|^{2/d}}\right)^{\frac{2s+1}{s}}E(y_0,f_0).$$

Remark 3.3. *Recalling that the constant c is defined in (20), if*

$(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)}\geq 1$, *inequality* (23) *implies that*

$$c^{1/s}\sqrt{E(y_k,f_k)}\leq c^{1/s}\sqrt{E(y_0,f_0)}-k\frac{s}{(1+s)^{\frac{1}{s}+1}},\quad \forall k\in I.$$

Hence, the number of iteration k_0 *to achieve a rate* $1+s$ *is estimated as follows:*

$$k_0=\left\lfloor (1+s)\left(c^{1/s}(1+s)^{1/s}\sqrt{E(y_0,f_0)}\right)-\frac{1}{s}\right\rfloor+1$$

where $\lfloor x\rfloor$ *denotes the integer part of* x. *As expected, this number increases with* $\sqrt{E(y_0,f_0)}$ *and* $\|g'\|_\infty$. *If* $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)}<1$, *then* $k_0=0$. *In particular, as* $s\to 0^+$, $k_0\to\infty$ *if* $c>1$, *i.e. if* $(\mathbf{H_3})$ *does not hold.*

The following convergence also holds, independently of the dimension of Ω. We refer to [39, Section 3, step 2] for the proof.

Lemma 3.2. *Assume that* g' *satisfies* $(\overline{\mathbf{H}}_\mathbf{s})$ *for some* $s\in(0,1]$ *and that* $g'\in L^\infty(\mathbb{R})$. *The sequence* $(\lambda_k)_{k>k_0}$ *defined in (17) converges to* 1 *as* $k\to\infty$ *at least with order* $1+s$.

3.2. Proof of Theorem 3.1

In this section, we relax the condition $g'\in L^\infty(\mathbb{R})$ and prove Theorem 3.1 under the assumption $(\mathbf{H_2})$. This assumption implies notably that $|g(r)|\leq C(1+|r|)\ln(2+|r|)$ for every $r\in\mathbb{R}$, mentioned in the introduction to state the well-posedness of (1). The case $\beta=0$ corresponds to the case developped in the previous section, i.e. $g'\in L^\infty(\mathbb{R})$.

Within this more general framework, the difficulty is to have a uniform control with respect to k of the observability constant $Ce^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$ appearing in the estimates for (Y_k^1,F_k^1), see Proposition 2.1. In other terms, we have to show that the sequence $(y_k,f_k)_{k\in\mathbb{N}}$ uniquely defined in (17) is uniformly bounded in $\mathcal{A}$, for any $(y_0,f_0)\in\mathcal{A}$.

The following intermediate result is crucial as it gives an estimate of the observability constant in term of an $L^\infty(0,T,L^p(\Omega))$ norm of the state.

Lemma 3.3. *Let* $C>0$, *only depending on* Ω *and* T *be the constant appearing in Proposition A.2. Assume that* g *satisfies* $(\mathbf{H_2})$ *and* $2C\beta^2\leq 1$. *Then for any* $(y,f)\in\mathcal{A}$,

$$e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\leq 2\,C\max(1,e^{2C\alpha^2}|\Omega|^2)\left(1+\frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\beta^2}$$

for any $p^\star \in \mathbb{N}^\star$ *with* $p^\star < \infty$ *if* $d = 2$ *and* $p^\star \leq 6$ *if* $d = 3$.

Proof. We use the following inequality (direct consequence of the inequality (3.8) in [32]):

$$(27)\qquad e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \leq C\Big(1 + \sup_{t\in(0,T)} \int_\Omega e^{C|g'(y)|^2}\Big), \quad \forall (y,f) \in \mathcal{A}.$$

Writing that $|g'(y)|^2 \leq 2\big(\alpha^2 + \beta^2 \ln(1+|y|)\big)$, we get that $\int_\Omega e^{C|g'(y)|^2} \leq e^{2C\alpha^2}\int_\Omega (1+|y|)^{2C\beta^2}$. Assuming $2C\beta^2 \leq p^\star$, Hölder inequality leads to

$$\begin{aligned}\int_\Omega e^{C|g'(y)|^2} &\leq e^{2C\alpha^2}\Big(\int_\Omega (1+|y|)^{p^\star}\Big)^{\frac{2C\beta^2}{p^\star}} |\Omega|^{1-\frac{2C\beta^2}{p^\star}}\\ &\leq e^{2C\alpha^2}|\Omega|\Big(1 + \frac{\|y\|_{L^{p^\star}(\Omega)}}{|\Omega|^{1/p^\star}}\Big)^{2C\beta^2}.\end{aligned}$$

It follows from (27) that for every $(y,f) \in \mathcal{A}$,

$$\begin{aligned}e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} &\leq C\bigg(1 + e^{2C\alpha^2}|\Omega|\Big(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\Big)^{2C\beta^2}\bigg)\\ &\leq C\max(1, e^{2C\alpha^2}|\Omega|)\bigg(1 + \Big(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\Big)^{2C\beta^2}\bigg)\\ &\leq 2^{2C\beta^2}\, C\max(1, e^{2C\alpha^2}|\Omega|)\Big(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\Big)^{2C\beta^2}\end{aligned}$$

and the result. □

Lemma 3.4. *Assume that* g *satisfies* **(H₂)** *and* $2C\beta^2 \leq 1$. *For any* $(y,f) \in \mathcal{A}$, *the unique solution* $(Y^1, F^1) \in \mathcal{A}_0$ *of* (6) *satisfies*

$$\|(Y^1, \partial_t Y^1)\|_{L^\infty(0,T;\boldsymbol{V})} + \|F^1\|_{2,q_T} \leq d(y)\sqrt{E(y,f)}$$

with $d(y) := C_3(\alpha)\big(1 + \frac{\|y\|_{L^\infty(0,T;L^1(\Omega))}}{|\Omega|}\big)^{2C\beta^2}$ *and* $C_3(\alpha) := 2\, C\max(1, e^{2C\alpha^2}|\Omega|)$.

Proof. Lemma 3.3 with $p^\star = 1$ and estimate (7) lead to the result. □

Proof of Theorem 3.1. If the initialization $(y_0, f_0) \in \mathcal{A}$ is such that $E(y_0, f_0) = 0$, then the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ constant equal to (y_0, f_0) is convergent. We assume in the sequel that $E(y_0, f_0) > 0$.

We are going to prove that, for any $\beta < \beta^\star(s)$, there exists a constant $M > 0$ such that the sequence $(y_k)_{k\in\mathbb{N}}$ defined by (17) enjoys the uniform property

$$\|y_k\|_{L^\infty(0,T;L^1(\Omega))} \le M, \quad \forall k \in \mathbb{N}. \tag{28}$$

The convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ will then follow by proceeding as in Section 3.1. Remark preliminary that the assumption $\beta < \beta^\star(s)$ implies $2C\beta^2 < \frac{s}{2s+1} \le 1$ since $s \in (0, 1]$.

Proof of the uniform property (28) *for some M large enough.* As for $n = 0$, from any initialization (y_0, f_0) chosen in $\mathcal{A}$, it suffices to take M larger than $M_1 := \|y_0\|_{L^\infty(0,T;L^1(\Omega))}$. We then proceed by induction and assume that, for some $n \in \mathbb{N}$, $\|y_k\|_{L^\infty(0,T;L^1(\Omega))} \le M$ for all $k \le n$. This implies in particular that,

$$d(y_k) \le d_M(\beta) := C_3(\alpha)\left(1 + \frac{M}{|\Omega|}\right)^{2C\beta^2}, \quad \forall k \le n$$

and then

$$c(y_k) \le c_M(\beta) := \frac{C}{(1+s)\sqrt{2}}[g']_s\, d_M^{1+s}(\beta), \quad \forall k \le n. \tag{29}$$

Then, we write that

$$\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))} \le \|y_0\|_{L^\infty(0,T;L^1(\Omega))} + \sum_{k=0}^{n} \lambda_k \|Y_k^1\|_{L^\infty(0,T;L^1(\Omega))}.$$

But, Lemma 3.4 implies that $\|Y_k^1\|_{L^\infty(0,T;L^1(\Omega))} \le d_M(\beta)\sqrt{E(y_k, f_k)}$ for all $k \le n$ leading to

$$\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))} \le \|y_0\|_{L^\infty(0,T;L^1(\Omega))} + m\, d_M(\beta) \sum_{k=0}^{n} \sqrt{E(y_k, f_k)}. \tag{30}$$

Moreover, inequality (25) implies that

$$\sum_{k=0}^{n} \sqrt{E(y_k, f_k)} \le \frac{1}{1 - p_0(\widetilde{\lambda_0})} \sqrt{E(y_0, f_0)}$$

where $p_0(\widetilde{\lambda_0})$ is given by (22) with $c = c_M(\beta)$.

Now, we take M large enough so that $(1+s)^{1/s}c_M^{1/s}(\beta)\sqrt{E(y_0,f_0)}\geq 1$ i.e.

$$\left(\frac{C}{\sqrt{2}}[g']_s\right)^{1/s}C_3(\alpha)^{2/s}\left(1+\frac{M}{|\Omega|}\right)^{\frac{4C\beta^2}{s}}\sqrt{E(y_0,f_0)}\geq 1. \tag{31}$$

Such M exists since $\sqrt{E(y_0,f_0)}>0$ is independent of M and since the left hand side is of order $\mathcal{O}(M^{\frac{4C\beta^2}{s}})$ with $\frac{4C\beta^2}{s}>0$. We denote by M_2 the smallest value of M such that (31) hold true.

Then, from (22), we get that $p_0(\widetilde{\lambda_0})=1-\frac{s}{(1+s)^{\frac{1}{s}+1}}\frac{1}{c_M^{1/s}(\beta)\sqrt{E(y_0,f_0)}}$ and therefore

$$\frac{1}{1-p_0(\widetilde{\lambda_0})}=\frac{(1+s)^{\frac{1}{s}+1}}{s}c_M^{1/s}(\beta)\sqrt{E(y_0,f_0)}$$

so that $\sum_{k=0}^n\sqrt{E(y_k,f_k)}\leq\frac{(1+s)^{\frac{1}{s}+1}}{s}c_M^{1/s}(\beta)E(y_0,f_0)$. It follows from (30) that

$$\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))}\leq\|y_0\|_{L^\infty(0,T;L^1(\Omega))}+m\,d_M(\beta)\frac{(1+s)^{\frac{1}{s}+1}}{s}c_M^{1/s}(\beta)E(y_0,f_0).$$

The definition of $c_M(\beta)$ (see (29)) then gives

$$\begin{aligned}\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))}&\leq\|y_0\|_{L^\infty(0,T;L^1(\Omega))}\\&\quad+\frac{m(1+s)}{s}\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s}\Big(C_3(\alpha)\Big)^{1+\frac{2}{s}}E(y_0,f_0)\left(1+\frac{M}{|\Omega|}\right)^{\frac{(2C\beta^2)(2s+1)}{s}}.\end{aligned}$$

Now, we take $M>0$ large enough so that the right hand side is bounded by M, i.e.

$$\begin{aligned}&\|y_0\|_{L^\infty(0,T;L^1(\Omega))}\\&+\frac{m(1+s)}{s}\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s}\Big(C_3(\alpha)\Big)^{1+\frac{2}{s}}E(y_0,f_0)\left(1+\frac{M}{|\Omega|}\right)^{\frac{(2C\beta^2)(2s+1)}{s}}\leq M.\end{aligned} \tag{32}$$

Such M exists under the assumption $\beta<\beta^\star(s)$ equivalent to $\frac{(2C\beta^2)(2s+1)}{s}<1$. We denote by M_3 the smallest value of M such that (32) holds true. Eventually, taking $M:=\max(M_1,M_2,M_3)$, we get that $\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))}\leq M$ as well. We have then proved by induction the uniform property (28) for some M large enough.

Proof of the convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$. In view of Lemma 3.3 with $p^\star = 1$, the uniform property (28) implies that the observability constant $Ce^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$ appearing in the estimates for (Y_k^1, F_k^1) (see Proposition 2.1) is uniformly bounded with respect to the parameter k. As a consequence, the constant $c(y_k)$ appearing in the instrumental estimate (14) is bounded by $c_M(\beta)$ given by (29). Consequently, the developments of Section 3.1 apply with $c = c_M(\beta)$. Theorem 3.1 then follows from the proof of Proposition 3.1 except for the limit in (18) with respect to k (since g' is not anymore in $L^\infty(Q_T)$). Since $g \in \mathcal{C}^1(\mathbb{R})$, a.e in Q_T there exists $0 \le \theta(x,t) \le 1$ such that

$$\begin{aligned}
&|g(y_k(x,t)) - g(y(x,t))|\\
&\qquad = |g'(y(x,t) + \theta(x,t)y_k(x,t))||y_k(x,t) - y(x,t)|\\
&\qquad \le (\alpha + \beta \ln^{1/2}\big(1 + |y(x,t) + \theta(x,t)y_k(x,t)|\big))|y_k(x,t) - y(x,t)|\\
&\qquad \le (\alpha + \beta(|y(x,t)|^{1/2} + |y_k(x,t)|^{1/2}))|y_k(x,t) - y(x,t)|
\end{aligned}$$

and thus

$$\|g(y_k) - g(y)\|_2 \le \big(\alpha |Q_T|^{1/4} + \beta(\|y\|_2^{1/2} + \|y_k\|_2^{1/2})\big)\|y_k - y\|_4.$$

Since $y_k \to y$ in $L^4(Q_T)$, it follows that $g(y_k) \to g(\overline{y})$ in $L^2(Q_T)$. Moreover, since $(y_k)_{k\in\mathbb{N}}$ is a bounded sequence of $L^4(Q_T)$, the estimate

$$\|g'(y_k)\|_2 \le C(\alpha + \beta\|y_k\|_2^{1/2})\|y_k\|_4$$

implies that $(g'(y_k))_{k\in\mathbb{N}}$ is a bounded sequence of $L^2(Q_T)$. Then, using that (Y_k^1, F_k^1) goes to zero as $k \to \infty$ in $\mathcal{A}_0$, we pass to the limit in (18) and get that $(y, f) \in \mathcal{A}$ solves (1). $\square$

Remark 3.4. *Remark that* $M := \max(M_2, M_3)$ *since* $M_3 \ge M_1$. *The constant* M_2 *can be made explicit since the constraint (31) implies that*

$$\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s} C_3(\alpha)^{2/s}\left(1 + \frac{M}{|\Omega|}\right)^{\frac{4C\beta^2}{s}}\sqrt{E(y_0, f_0)} \ge 1,$$

equivalent to

$$\left(1 + \frac{M}{|\Omega|}\right)^{2C\beta^2} \ge C_3(\alpha)^{-1}\sqrt{E(y_0, f_0)}^{-s/2}\left(\frac{C}{\sqrt{2}}[g']_s\right)^{-1/2}.$$

In particular, M_2 is large for small values of $\sqrt{E(y_0, f_0)}$, for any $s > 0$. On the other hand, the constant M_3 is no explicit, hence whether $M_2 > M_3$ or $M_3 > M_2$ depend on the values of $\sqrt{E(y_0, f_0)}$ and $\|y_0\|_{L^\infty(0,T;L^1(\Omega))}$. Remark that $\sqrt{E(y_0, f_0)}$ can be large and $\|y_0\|_{L^\infty(0,T;L^1(\Omega))}$ small, and vice versa.

4. Comments

Asymptotic condition. The asymptotic condition $(\mathbf{H_2})$ on g' is slightly stronger than the asymptotic condition $(\mathbf{H_1})$ made in [21]: this is due to our linearization of (1) which involves $r \to g'(r)$ while the linearization (2) in [21] involves $r \to (g(r) - g(0))/r$. There exist cases covered by Theorem 1.1 in which exact controllability for (1) is true but that are not covered by Theorem 3.1. Note however that the example $g(r) = a + br + cr \ln^{1/2}(1 + |r|)$, for any $a, b \in \mathbb{R}$ and for any $c > 0$ small enough (which is somehow the limit case in Theorem 1.1) satisfies $(\mathbf{H_2})$ as well as $(\overline{\mathbf{H}}_\mathbf{s})$ for any $s \in (0, 1]$.

Link with Newton method. Defining $F : \mathcal{A} \to L^2(Q_T)$ by $F(y, f) := (Ly + g(y) - f\,1_\omega)$, we have $E(y, f) = \frac{1}{2}\|F(y, f)\|_2^2$ and we observe that, for $\lambda_k = 1$, the algorithm (17) coincides with the Newton algorithm associated to the mapping F. This explains the super-linear convergence property in Theorem 3.1, in particular the quadratic convergence when $s = 1$. The optimization of the parameter λ_k gives to a global convergence property of the algorithm and leads to the so-called damped Newton method applied to F (we refer to [15, chapter 8]). Section 5.3 provides some numerical illustrations of this property.

Initialization with the controlled pair of the linear equation. The number of iterates to achieve convergence (notably to enter in a super-linear regime) depends on the size of the value $E(y_0, f_0)$. A natural example of an initialization $(y_0, f_0) \in \mathcal{A}$ is the unique solution of minimal control norm of (1) with $g = 0$ (i.e., in the linear case). Under the assumption $(\mathbf{H_2})$, this leads to the estimate

$$E(y_0, f_0) = \frac{1}{2}\|g(y_0)\|_2^2 \leq |g(0)|^2|Q_T| + 2\int_{Q_T} |y_0|^2\big(\alpha^2 + \beta^2 \ln(1 + |y_0|)\big).$$

Local controllability when removing the growth condition $(\mathbf{H_2})$. If the real $E(y_0, f_0)$ is small enough, we may remove the growth condition $(\mathbf{H_2})$ on g'.

Proposition 4.1. *Assume g' satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for some $s \in (0, 1]$. Let $(y_k, f_k)_{k>0}$ be the sequence of $\mathcal{A}$ defined in (17). There exists a constant*

$C([g']_s)$ such that if $E(y_0,f_0)\le C([g']_s)$, then $(y_k,f_k)_{k\in\mathbb{N}}\to(\overline{y},\overline{f})$ in $\mathcal{A}$ where $\overline{f}$ is a null control for $\overline{y}$ solution of (1). Moreover, the convergence is at least linear and is at least of order $1+s$ after a finite number of iterations.

Proof. In this proof, the notation $\|\cdot\|_{\infty,d}$ stands for $\|\cdot\|_{L^\infty(0,T;L^d(\Omega))}$. We note $D:=\frac{C}{(1+s)\sqrt{2}}[g']_s$ and $e_k:=c(y_k)E(y_k,f_k)^{s/2}$ with $c(y):=Dd(y)^{1+s}$ and $d(y):=Ce^{C\|g'(y)\|^2_{\infty,d}}$. (21) then reads

$$\sqrt{E(y_{k+1},f_{k+1})}\le\min_{\lambda\in[0,m]}\big(|1-\lambda|+\lambda^{1+s}e_k\big)\sqrt{E(y_k,f_k)}. \tag{33}$$

We write $|g'(y_k)-g'(y_k-\lambda_kY_k^1)|\le[g']_s|\lambda_kY_k^1|^s$ so that

$$\begin{aligned}&\|g'(y_{k+1})\|^2_{\infty,d}\\&\quad\le\|g'(y_k)\|^2_{\infty,d}+\big([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\big)^2+2\|g'(y_k)\|_{\infty,d}[g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\end{aligned}$$

and

$$\begin{aligned}&e^{C\|g'(y_{k+1})\|^2_{\infty,d}}\\&\quad\le e^{C\|g'(y_k)\|^2_{\infty,d}}e^{C\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)^2}e^{2C\|g'(y_k)\|_{\infty,d}\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)}\end{aligned}$$

leading to

$$\frac{c(y_{k+1})}{c(y_k)}\le\left(e^{C\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)^2}e^{2C\|g'(y_k)\|_{\infty,d}\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)}\right)^{1+s}.$$

We infer that $\|(Y_k^1)^s\|_{\infty,d}=\|Y_k^1\|^s_{\infty,sd}$. Moreover, estimate (7) leads to

$$\begin{aligned}\|Y_k^1\|^s_{\infty,sd}&\le d^s(y_k)E(y_k,f_k)^{s/2}=\frac{c(y_k)^{\frac{s}{1+s}}}{D^{\frac{s}{1+s}}}E(y_k,f_k)^{s/2}\\&\le D^{-\frac{s}{1+s}}c(y_k)E(y_k,f_k)^{s/2}\end{aligned}$$

using that $c(y_k)\ge1$ (by increasing the constant C if necessary). Consequently,

$$e^{C\left([g']_s\lambda^s\|(Y_k^1)^s\|_{\infty,d}\right)^2}\le e^{C\left([g']_s\lambda^sD^{-\frac{s}{1+s}}e_k\right)^2}:=e^{C_1e_k^2}.$$

Similarly,

$$\|g'(y_k)\|_{\infty,d}\|(Y_k^1)^s\|_{\infty,d}\le\|g'(y_k)\|_{\infty,d}d^s(y_k)E(y_k,f_k)^{s/2}$$

$$
\begin{aligned}
&\leq \|g'(y_k)\|_{\infty,d}\Big(Ce^{C\|g'(y)\|^2_{\infty,d}}\Big)^s E(y_k,f_k)^{s/2}\\
&\leq \Big(Ce^{C\|g'(y)\|^2_{\infty,d}}\Big)^{s+1} E(y_k,f_k)^{s/2}\\
&\leq \frac{c(y_k)}{D}E(y_k,f_k)^{s/2}=\frac{e_k}{D}
\end{aligned}
$$

using that $a\leq Ce^{Ca^2}$ for all $a\geq 0$ and $C>0$ large enough. It follows that

$$
e^{2C\|g'(y_k)\|_{\infty,d}\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)}\leq e^{2C[g']_s\lambda_k^s\frac{e_k}{D}}:=e^{C_2e_k}
$$

and then $\frac{c(y_{k+1})}{c(y_k)}\leq (e^{C_1e_k^2+C_2e_k})^{1+s}$. By multiplying (33) by $c(y_{k+1})$, we obtain the inequality

$$
e_{k+1}\leq \min_{\lambda\in[0,m]}\left(|1-\lambda|+e_k\lambda^{1+s}\right)\quad (e^{C_1e_k^2+C_2e_k})^{1+s}\,e_k.
$$

If $2e_k<1$, the minimum is reached for $\lambda=1$ leading $\frac{e_{k+1}}{e_k}\leq e_k(e^{C_1e_k^2+C_2e_k})^{1+s}$. Consequently, if the initial guess (y_0,f_0) belongs to the set $\{(y_0,f_0)\in\mathcal{A}, e_0<1/2, e_0(e^{C_1e_0^2+C_2e_0})^{1+s}<1\}$, the sequence $(e_k)_{k>0}$ goes to zero as $k\to\infty$. Since $c(y_k)\geq 1$ for all $k\in\mathbb{N}$, this implies that the sequence $(E(y_k,f_k))_{k>0}$ goes to zero as well. Moreover, from (7), we get

$$
D\|(Y_k^1,F_k^1)\|_{\mathcal{H}}\leq e_k\sqrt{E(y_k,f_k)}
$$

and repeating the arguments of the proof of Proposition 3.1, we conclude that the sequence $(y_k,f_k)_{k>0}$ converges to a controlled pair for (1). □

These computations do not assume **($\mathbf{H_2}$)** for g. However, the smallness assumption on e_0 requires a smallness assumption on $E(y_0,f_0)$ (since $c(y_0)>1$). This is equivalent to assume the controllability of (1). Alternatively, in the case $g(0)=0$, the smallness assumption on $E(y_0,f_0)$ is achieved as soon as $\|(u_0,u_1)\|_{\boldsymbol{V}}$ is small enough. Therefore, the convergence result stated in Proposition 4.1 is equivalent to the local controllability property for (1). Proposition 4.1 can also be seen as a consequence of the usual convergence of the Newton method: when $E(y_0,f_0)$ is small enough, i.e., when the initialization is close enough to the solution, then $\lambda_k=1$ for every $k\in\mathbb{N}$ and we recover the standard Newton method.

Weakening of the condition **($\overline{\mathbf{H}}_{\mathbf{s}}$)**. Given any $s\in(0,1]$, we introduce for any $g\in\mathcal{C}^1(\mathbb{R})$ the following hypothesis:

($\overline{\mathbf{H}}'_{\mathbf{s}}$) There exist $\overline{\alpha}, \overline{\beta}, \gamma \in \mathbb{R}^+$ such that $|g'(a) - g'(b)| \leq |a-b|^s\big(\overline{\alpha} + \overline{\beta}(|a|^\gamma + |b|^\gamma)\big), \quad \forall a, b \in \mathbb{R}$

which coincides with ($\overline{\mathbf{H}}_{\mathbf{s}}$) if $\gamma = 0$ for $\overline{\alpha} + \overline{\beta} = [g']_s$. If $\gamma \in (0,1)$ is small enough and related to the constant β appearing in the growth condition ($\mathbf{H_2}$), Theorem 3.1 still holds if ($\overline{\mathbf{H}}_{\mathbf{s}}$) is replaced by the weaker hypothesis ($\overline{\mathbf{H}}'_{\mathbf{s}}$). Precisely, if g satisfies ($\mathbf{H_2}$) and ($\overline{\mathbf{H}}'_{\mathbf{s}}$) for some $s \in (0,1]$, then the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (17) fulfills the estimate

$$E(y_{k+1}, f_{k+1}) \leq E(y_k, f_k) \min_{\lambda \in [0,m]} \Big(|1-\lambda| + \lambda^{1+s} c(y_k) E(y_k, f_k)^{s/2}\Big)^2$$

with $c(y) := \frac{1}{(1+s)\sqrt{2}}\Big(\overline{\alpha} + 2\overline{\beta}\|y_k\|^\gamma_{\infty,6\gamma} + \overline{\beta} m^\gamma d(y)^\gamma E(y_0, f_0)^{\gamma/2}\Big) d(y)^{1+s}$ and $d(y) := Ce^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}$. Using Lemma 3.3 with $p^\star = 6\gamma \leq 6$ and proceeding as in the proof of Theorem 3.1, one may prove by induction that the sequence $(\|y_k\|_{L^\infty(0,T;L^6(\Omega))})_{k\in\mathbb{N}}$ is uniformly bounded under the condition $\frac{\gamma + 2C\beta^2(1+2s)}{s} < 1$ and then deduce the convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$.

5. Numerical illustrations

In this section, we illustrate our results of convergence. We provide some practical details about the algorithm (17), then discuss some experiments in one and two space dimension performed with the software FreeFem++ [24].

5.1. Algorithm

We introduce a cut-off χ of the form $\chi(x,t) = \chi_1(x)\chi_2(t)$, where $\chi_1 \in \mathcal{C}_0^\infty(\omega)$ and $\chi_2 \in \mathcal{C}_0^\infty(0,T)$ take values in $[0,1]$. In the sequel, we consider controls of minimal $L^2_\chi(q_T)$-norm, with $L^2_\chi(q_T) := \{f \mid \int_{q_T} f^2\chi^{-1} < +\infty\}$. Besides, for $k \in \mathbb{N}$, we denote $e_k := \partial_{tt} y_k - \Delta y_k + g(y_k) - f_k 1_\omega$. Then, algorithm (17) can be expanded as follows.

1. *Initialization* – We compute the state-control pair $(y_0, f_0) \in \mathcal{A}$ solution of

$$\begin{cases} Ly_0 = f_0 1_\omega, & \text{in } Q_T, \\ y_0 = 0, & \text{on } \Sigma_T, \\ (y_0(\cdot,0), \partial_t y_0(\cdot,0)) = (u_0, u_1), & \text{in } \Omega, \\ (y_0(\cdot,T), \partial_t y_0(\cdot,T)) = (z_0, z_1), & \text{in } \Omega, \end{cases} \tag{34}$$

where f_0 is the control of minimal $L^2_\chi(q_T)$-norm. We then evaluate $e_0 = g(y_0)$.

Assume now that $(y_k, f_k) \in \mathcal{A}$ and $e_n \in L^2(Q_T)$ are computed for some $k \geq 0$.

2. *Evaluation of the least-squares functional* – We compute the error functional $E(y_k, f_k) = \frac{1}{2}\|e_k\|_2^2$. If $\sqrt{2E(y_k, f_k)} \leq 10^{-5}$, then the algorithm stops.
3. *Descent direction* – We compute the state-control pair $(Y_k^1, F_k^1) \in \mathcal{A}_0$ solution of

$$
\begin{cases}
LY_k^1 + g'(y_k)Y_k^1 = F_k^1 1_\omega + e_k, & \text{in } Q_T,\\
Y_k^1 = 0, & \text{on } \Sigma_T,\\
(Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0), & \text{in } \Omega,\\
(Y_k^1(\cdot,T), \partial_t Y_k^1(\cdot,T)) = (0,0), & \text{in } \Omega,
\end{cases}
\tag{35}
$$

where F_k^1 is the control of minimal $L^2_\chi(q_T)$-norm.
4. *Optimal descent step* – We compute the optimal descent step λ_k as the minimizer in $[0,1]$ of $\lambda \mapsto E\big((y_k, f_k) - \lambda(Y_k^1, F_k^1)\big)$, evaluated using the expression

$$E\big((y_k, f_k) - \lambda(Y_k^1, F_k^1)\big) = \frac{1}{2}\big\|(1-\lambda)e_k + l_k(\lambda)\big\|_2^2,$$

where $l_k(\lambda) := g(y_k - \lambda Y_k^1) - g(y_k) + \lambda g'(y_k)Y_k^1$. This is done with 20 iterations of the trichotomy method on the interval $[0,1]$.
5. *Update* – We set $(y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k(Y_k^1, F_k^1)$. We then evaluate $e_{k+1} = (1-\lambda_k)e_k + l_k(\lambda_k)$ and return to step 2.

In the sequel, we denote by $k^\star = \min\big\{k \geq 0 \mid \sqrt{2E(y_k, f_k)} \leq 10^{-5}\big\}$ and define the corresponding approximation of the solution in $\mathcal{A}$ by

$$(y^\star, f^\star) = (y_0, f_0) - \sum_{k=0}^{k^\star} \lambda_k (Y_k^1, F_k^1).$$

Then, in order to measure *a posteriori* the quality of this approximation, we shall compute the relative term

$$\mathcal{E}_T := \frac{\|(y, \partial_t y)(\cdot, T; f^\star)\|_{\boldsymbol{V}}}{\|(y, \partial_t y)(\cdot, T; 0)\|_{\boldsymbol{V}}},$$

where $y(\cdot, \tau; f^\star)$ (resp. $y(\cdot, \tau; 0)$) is the solution at time τ of (1) with control equal to $f = f^\star$ (resp. $f = 0$).

The introduction of the cut-off χ together with regularity assumptions on the initial datum (u_0, u_1) make the state-control pairs (y_0, f_0) and (Y_k^1, F_k^1) regular as well (we refer to [18] extended in [2] for weighted functionals). This allows to give a meaning to $e_k = \partial_{tt} y_k - \Delta y_k + g(y_k) - f_k 1_\omega$ as an $L^2(Q_T)$ function. Moreover, this involves stability properties with respect to the discretization parameters for the standard finite-dimensional approximations of systems (34) to (37) below.

5.2. Experiments in 2D

We consider a two-dimensional case for which $\Omega = (0,1)^2$. The controllability time is equal to $T = 3$ and the control domain ω is depicted on Figure 1. The triplet (Ω, ω, T) satisfies $(\mathbf{H_0})$. Moreover, for any real constant c_g, we consider the non-linear function g defined by

$$g(r) = -c_g\, r \ln^{1/2}(2 + |r|), \quad \forall r \in \mathbb{R}.$$

We check that g satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for $s = 1$ and $(\mathbf{H_2})$ for $|c_g|$ small enough. Remark that the unfavorable situation in which the norm of the corresponding uncontrolled solution of (1) grows corresponds to positive values of c_g. As for the initial and final data, we consider $(u_0, u_1) = (100 \sin(\pi x_1) \sin(\pi x_2), 0)$ and $(z_0, z_1) = (0, 0)$ respectively.

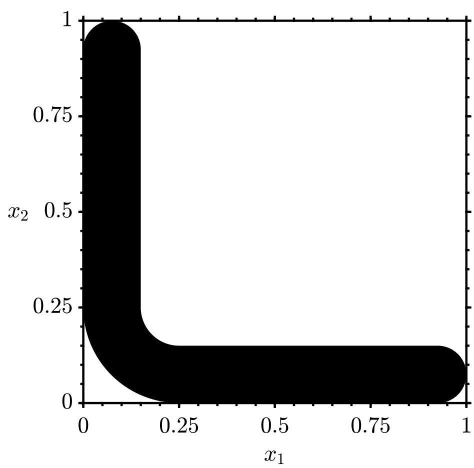

Figure 1: Control domain $\omega \subset \Omega = (0,1)^2$ (black part).

In order to determine the state-control pairs (y_0, f_0) and (Y_n^1, F_n^1) of (34) and (35) respectively, we employ the *discretize-then-control* method introduced by Glowinski-Li-Lions in the seminal work [22], based on the unconstrained minimization of the conjugate functional.

Concerning problem (34), for any $(w_0, w_1) \in \boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$, we consider the adjoint system

$$
\begin{cases}
\partial_{tt}\varphi - \Delta\varphi = 0, & \text{in } Q_T, \\
\varphi = 0, & \text{on } \Sigma_T, \\
(\varphi(\cdot, T), \partial_t\varphi(\cdot, T)) = (w_0, w_1), & \text{in } \Omega,
\end{cases} \tag{36}
$$

and the functional

$$
\begin{aligned}
J_0^\star(w_0, w_1) := \frac{1}{2}\int_{q_T} |\varphi|^2\chi - \langle u_0, \partial_t\varphi(\cdot, 0)\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + (u_1, \varphi(\cdot, 0))_{L^2(\Omega)} \\
+ \langle z_0, w_1\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - (z_1, w_0)_{L^2(\Omega)}.
\end{aligned}
$$

Then, the control f_0 of minimal $L^2_\chi(q_T)$-norm is given by $f_0 = \varphi_0\chi$, where φ_0 is the solution of (36) associated with the minimizer $(\widehat{w}_0, \widehat{w}_1)$ of $J_0^\star$ over $\boldsymbol{H}$. The resolution of this minimization problem is done using the Fletcher-Reeves conjugate gradient algorithm, initialized with $(w_0, w_1) = (0, 0)$. The stopping criterion is $\|g_p\|_{\boldsymbol{H}} \leq 10^{-5}\|g_0\|_{\boldsymbol{H}}$, where g_p denotes the gradient of $J_0^\star$ at iteration p.

Concerning problem (35), for any $(w_0, w_1) \in \boldsymbol{H}$, we consider the adjoint system

$$
\begin{cases}
L\varphi + g'(y_n)\varphi = 0, & \text{in } Q_T, \\
\varphi = 0, & \text{on } \Sigma_T, \\
(\varphi(\cdot, T), \partial_t\varphi(\cdot, T)) = (w_0, w_1), & \text{in } \Omega,
\end{cases} \tag{37}
$$

and, for all $k > 0$ the functional

$$
J_k^\star(w_0, w_1) := \frac{1}{2}\int_{q_T} |\varphi|^2\chi + \int_{Q_T} e_k\varphi.
$$

Then, the control F_k^1 of minimal $L^2_\chi(q_T)$-norm is given by $F_k^1 = \varphi_k\chi$, where φ_k is the solution of (37) associated with the minimizer $(\widehat{w}_0, \widehat{w}_1)$ of $J_k^\star$ over $\boldsymbol{H}$. The resolution of this minimization problem is done using the Fletcher-Reeves conjugate gradient algorithm, initialized with the minimizer of the functional $J_{k-1}^\star$. The stopping criterion is $\|g_p\|_{\boldsymbol{H}} \leq 10^{-5}\|g_0\|_{\boldsymbol{H}}$, where g_p denotes the gradient of $J_k^\star$ at iteration p.

To compute the solution of the state systems (34)–(35) and the adjoint systems (36)–(37), we use a time-marching method combining an explicit

Table 1: $c_g = 1$ – Norms of (y_k, f_k) w.r.t. k defined by the algorithm (17)

k	$\sqrt{2E(y_k,f_k)}$	λ_k	$\frac{\Vert y_k-y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert f_k-f_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert f_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert f_k\Vert_{L^2_\chi(q_T)}$
0	7.32×10^{1}	1.	–	–	37.653	1339.39
1	9.62×10^{-1}	1.	1.72×10^{-1}	3.29×10^{-1}	37.113	1265.62
2	1.03×10^{-5}	1.	3.83×10^{-4}	1.06×10^{-3}	37.115	1265.77
3	6.42×10^{-15}	–	4.44×10^{-9}	9.34×10^{-9}	37.115	1265.77

centered finite-difference scheme in time and a finite-element approximation in space. We consider a uniform discretization $(t_i)_{i=0,\dots,N}$ of the time interval $[0,T]$ and denote by $\delta t = T/N$ the time discretization parameter. Besides, we consider a family $\mathcal{T} = \{\mathcal{T}_h, h>0\}$ of regular triangulations of Ω such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. The family is indexed by $h = \max_{K\in\mathcal{T}_h}|K|$. For every time t_i, the variables $\varphi_0(\cdot,t_i)$, $\varphi_n(\cdot,t_i)$, $y_0(\cdot,t_i)$ and $Y^1_n(\cdot,t_i)$ are approximated in the space $P_h = \left\{p_h \in \mathcal{C}(\overline{Q_T}) \mid p_{h|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\right\}$ where $\mathbb{P}_1(K)$ denotes the space of polynomials of degree one. We refer to [5] for convergence results in this setting. We also refer to [1, 7, 19, 35]. In the sequel, we mainly use a regular triangulation $\mathcal{T}_h$ with fineness $h = 1/64$ and a time step equal to $\delta t = h/3$ in order to satisfy the CFL condition arising from the explicit scheme with respect to the time variable.

We now present some simulations for several values of the constant c_g.

• Case $c_g = 1$ – Table 1 collects some norms from the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ associated with the value $c_g = 1$. The convergence of the algorithm is observed after $k^\star = 3$ iterations. The optimal steps λ_k are equal to one so that the algorithm (17) coincides with the Newton algorithm (see the second item of Section 4). Figure 2-left depicts with respect to the time variable the $L^2(\Omega)$-norm of the controlled solution $y^\star = y_{k=k^\star}$ (red solid line) to be compared with the $L^2(\Omega)$-norm of the controlled solution $y_{k=0}$ of the linear equation (blue dash-dotted line) used to initialize the algorithm (equivalently, this controlled solution corresponds to $c_g = 0$). The effect of the non-linearity is reduced as the dynamics of the two controlled solutions are similar. The figure also depicts the $L^2(\Omega)$-norm of the uncontrolled solution (blue dashed line) and displays a periodic behavior. Similarly, Figure 2-right depicts the $L^2_\chi(q_T)$-norm of the null control $f^\star = f_{k=k^\star}$ (red solid line) and $f_{k=0}$ (blue dash-dotted line). By construction, these controls vanish at the initial and final times. The corresponding value of the relative error $\mathcal{E}_T = 2.53\times 10^{-4}$ indicates a notable reduction of the solution at time T thought the action of the control.

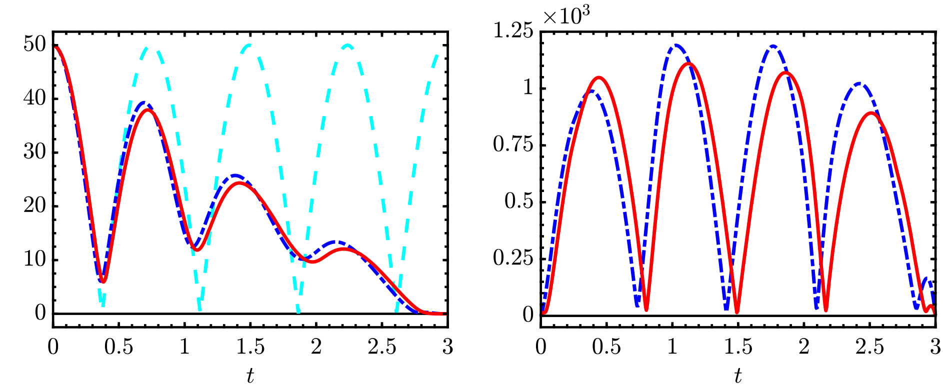

Figure 2: $c_g = 1$ – **Left**: – (red) $\|y^\star(\cdot,t)\|_{L^2(\Omega)}$; (blue) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (cyan) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$ vs t; **Right**: (red) $\|f^\star(\cdot,t)\|_{L^2_\chi(\omega)}$; (blue) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ vs t.

Table 2: $c_g = 5$ – Norms of (y_k, f_k) w.r.t. k defined by the algorithm (17)

k	$\sqrt{2E(y_k,f_k)}$	λ_k	$\frac{\|y_k-y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k-f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$
0	3.66×10^2	1.	–	–	37.653	1339.39
1	4.87×10^1	0.996	9.50×10^{-1}	1.05×10^0	31.353	1223.44
2	8.65×10^{-1}	1.	7.67×10^{-2}	1.50×10^{-1}	32.101	1348.12
3	5.82×10^{-5}	1.	3.91×10^{-4}	7.37×10^{-4}	32.104	1348.09
4	7.17×10^{-14}	–	1.45×10^{-8}	3.24×10^{-8}	32.104	1348.09

• Case $c_g = 5$ – Table 2 and Figures 3 collect the results obtained for the value $c_g = 5$. The relative error takes the value $\mathcal{E}_T = 2.36 \times 10^{-4}$. The convergence is quadratic and is obtained after $k^\star = 4$ iterations.

• Case $c_g = 10$ – Table 3 and Figures 4 collect the results obtained for the value $c_g = 10$. We compute the relative error $\mathcal{E}_T = 2.64 \times 10^{-5}$. The convergence is observed after $k^\star = 4$ iterations. As before, the optimal steps are very close to one. The main difference with the previous situations for which $c_g = 1$ and $c_g = 5$ is the behavior of the uncontrolled solution which grows exponentially with respect to the time variable, as shown in Figure 4-left. As expected, this larger value of c_g induces a larger gap between the non-linear control and the linear one. We observe notably that the non-linear control $f^\star$ acts stronger from the beginning, precisely in order to balance the initial exponential growth of the solution outside the subset ω. We also observe that the control reduces the oscillations of the corresponding controlled solution (in comparison with the solution of the linear equation). For larger values of c_g, we suspect a different dynamic yielding

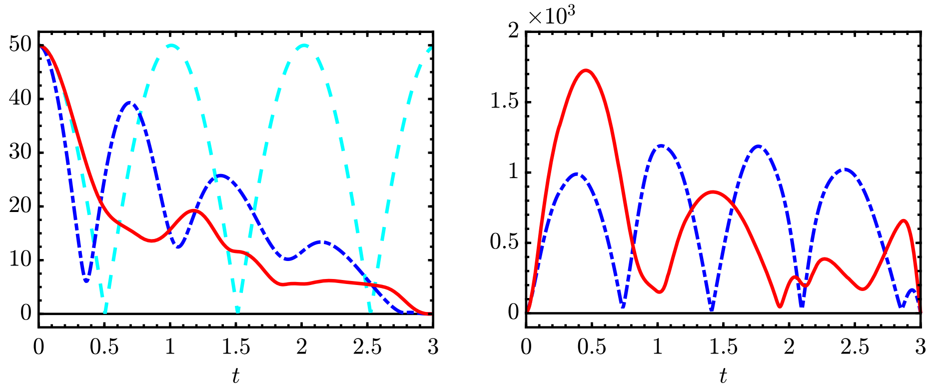

Figure 3: $c_g = 5$ – **Left**: – (—) $\|y^\star(\cdot,t)\|_{L^2(\Omega)}$; (- -) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (- -) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$ vs t; **Right**: (—) $\|f^\star(\cdot,t)\|_{L^2_\chi(\omega)}$; (- -) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ vs t.

Table 3: $c_g = 10$ – Norms of (y_k, f_k) w.r.t. k defined by the algorithm (17)

k	$\sqrt{2E(y_k,f_k)}$	λ_k	$\frac{\|y_k-y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k-f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$
0	7.32×10^2	1	–	–	37.653	1339.39
1	1.59×10^2	0.998	1.76×10^0	9.40×10^{-1}	57.718	1164.11
2	2.43×10^0	1	8.82×10^{-2}	1.30×10^{-1}	59.934	1138.75
3	4.09×10^{-3}	1	1.62×10^{-3}	5.66×10^{-3}	59.892	1141.01
4	1.80×10^{-9}	–	1.10×10^{-6}	2.96×10^{-6}	59.891	1141.01

to the first values of the optimal step λ_k being far from one (as observed in [29] for the resolution of the Navier-Stokes system with large values of the Reynolds number). However, for larger values of c_g (for instance $c_g = 20$), the exponential growth behavior of the free solution used to initialize the algorithm leads to numerical instabilities and overflows in the computation of the controlled pair (Y_k^1, F_k^1) solution of (35), where the potential $g'(y_k)$ appears. This leads to the divergence of the conjugate gradient algorithm including for very fine discretizations and the non-convergence of the least-squares algorithm. In the next section, we shall employ a different method of approximation allowing to consider larger values of c_g, as it fails within the strategy *control-then-discretize*.

• Case $c_g = -20$ – For negative values of c_g leading to $rg(r) \geq 0$ for every r, the situation is more favorable from a computational viewpoint. Table 4 and Figures 5 are concerned with the value $c_g = -20$. The convergence is observed after $k^\star = 4$ iterations and leads to $\mathcal{E}_T = 3.64 \times 10^{-4}$. We observe that the uncontrolled solution oscillates faster as c_g decreases. This leads to

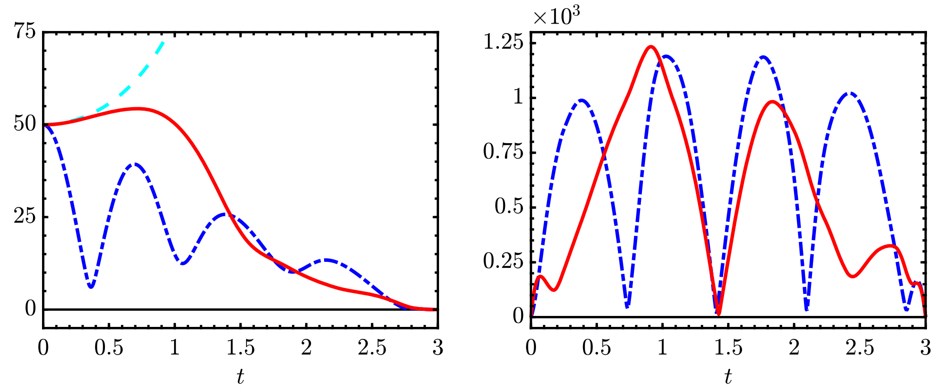

Figure 4: $c_g = 10$ – **Left**: – (red) $\|y^\star(\cdot,t)\|_{L^2(\Omega)}$; (blue dash-dot) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (cyan dashed) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$ vs t; **Right**: (red) $\|f^\star(\cdot,t)\|_{L^2_\chi(\omega)}$; (blue dash-dot) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ vs t.

Table 4: $c_g = -20$ – Norms of (y_k, f_k) w.r.t. k defined by the algorithm (17)

k	$\sqrt{2E(y_k,f_k)}$	λ_k	$\frac{\|y_k-y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k-f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$
0	1.46×10^{3}	1	–	–	37.653	1339.39
1	2.70×10^{2}	0.985	1.38×10^{0}	1.69×10^{0}	42.479	2601.16
2	1.55×10^{1}	1	1.57×10^{-1}	1.43×10^{-1}	44.309	2696.17
3	1.94×10^{-2}	1	3.68×10^{-3}	5.13×10^{-3}	44.34	2700.73
4	9.66×10^{-9}	–	2.88×10^{-6}	4.80×10^{-6}	44.34	2700.73

an oscillatory dynamic of the optimal control pair $(y^\star, f^\star)$. We also observe that the norm of the control $f^\star$ is significantly greater than the norm of f_0, the initial control associated with the linear case.

Table 5 associated with the value $c_g = 5$ provides a numerical evidence of the convergence of the approximation $(y_h^\star, f_h^\star)$ with respect to the value of h. Actually, in view of the inequality

$$\|f - f_k^h\| \leq \|f - f_k\| + \|f_k - f_k^h\|, \quad \forall k \in \mathbb{N}, \forall h > 0,$$

the convergence result stated in Theorem 3.1 for the sequence $(f_k)_{k\in\mathbb{N}}$ and the convergence, for any k, of the approximation $(f_k^h)_{h>0}$ of the linear control f_k implies that f_k^h is a finite dimensional approximation of f, a control for (1). We observe that the level of the discretization has no influence on the speed of convergence of the least-squares algorithm: we observe that $k^\star = 4$.

To end this section, we compare our least-squares approach with two fixed-point methods. We first consider the method associated with the oper-

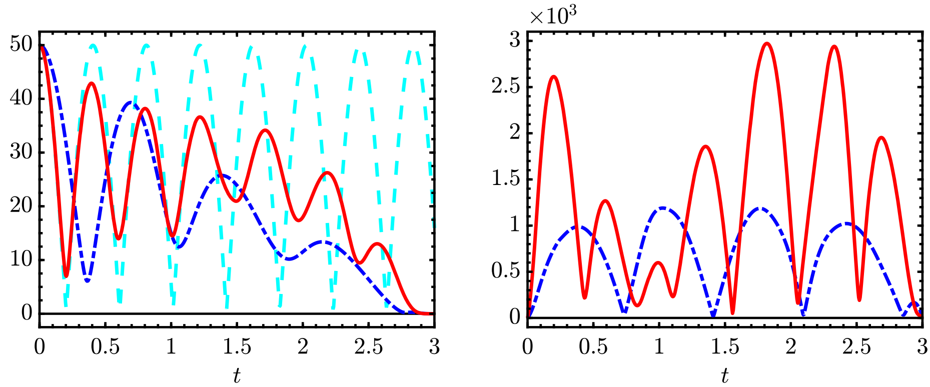

Figure 5: $c_g = -20$ – **Left**: – (red solid) $\|y^\star(\cdot,t)\|_{L^2(\Omega)}$; (blue dash-dot) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (cyan dashed) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$ vs t; **Right**: (red solid) $\|f^\star(\cdot,t)\|_{L^2_\chi(\omega)}$; (blue dash-dot) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ vs t.

Table 5: $c_g = 5$ – Norm of $(y_h^\star, f_h^\star)$ w.r.t. h

h	$k^\star$	$\|y_h^\star\|_{L^2(Q_T)}$	$\|f_h^\star\|_{L^2_\chi(q_T)}$	$\mathcal{E}_{T,h}$
1/10	4	27.278	753.111	8.88×10^{-2}
1/20	4	31.431	1397.7	1.00×10^{-2}
1/40	4	32.026	1353.28	8.91×10^{-4}
1/80	4	32.123	1350.24	1.37×10^{-4}
1/100	4	32.134	1350.11	8.27×10^{-5}
1/120	4	32.139	1350.06	5.71×10^{-5}

ator $\Lambda : L^\infty(0,T;L^d(\Omega)) \to L^\infty(0,T;L^d(\Omega))$ mentioned in the introduction where $y = \Lambda(z)$ solves (2). This leads to the algorithm:

$$y_0 \in L^2(Q_T), \quad y_{k+1} = \Lambda(y_k), \quad k \geq 0. \tag{38}$$

With the same data and initialization, Table 6 collects some norms with respect to k for $c_g = 5$. The $L^2_\chi(q_T)$-norm of the control is smaller than the one from the least-squares algorithm (967.97 vs 1348.09) but leads to a larger $L^2(Q_T)$-norm of the controlled solution (36.901 vs 32.104). The convergence is linear and reached after $k^\star = 10$ iterations leading to $\mathcal{E}_T = 2.15 \times 10^{-4}$. Figure 6 displays the time evolution of the norms of $y_{k^\star}$ and $f_{k^\star}$ for the final iteration. We observe that the approximation obtained differs from those of Figures 3. For these data, the sequence converges for $|c_g| < 15$ approximately. For larger values, we observe the non-convergence of the method suggesting that the operator Λ is not contracting in general.

Table 6: $c_g = 5$ – Norms for the sequence defined by the fixed-point algorithm (38)

k	$\sqrt{2E(y_k,f_k)}$	$\frac{\Vert y_k-y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert f_k-f_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert f_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert f_k\Vert_{L^2_\chi(q_T)}$
0	3.66×10^{2}	–	–	37.653	1339.39
1	4.73×10^{1}	1.04×10^{0}	1.31×10^{0}	37.828	1031.1
2	2.65×10^{0}	5.83×10^{-2}	1.71×10^{-1}	36.867	972.97
3	1.70×10^{-1}	3.74×10^{-3}	1.39×10^{-2}	36.901	967.508
4	1.05×10^{-2}	2.54×10^{-4}	1.02×10^{-3}	36.9	968.005
5	2.42×10^{-3}	5.27×10^{-5}	1.25×10^{-4}	36.901	967.973
6	5.20×10^{-4}	1.20×10^{-5}	2.63×10^{-5}	36.901	967.972
7	1.62×10^{-4}	3.57×10^{-6}	7.13×10^{-6}	36.901	967.97
8	4.39×10^{-5}	9.84×10^{-7}	1.96×10^{-6}	36.901	967.97
9	1.28×10^{-5}	2.83×10^{-7}	5.56×10^{-7}	36.901	967.97
10	3.59×10^{-6}	7.99×10^{-8}	1.57×10^{-7}	36.901	967.97

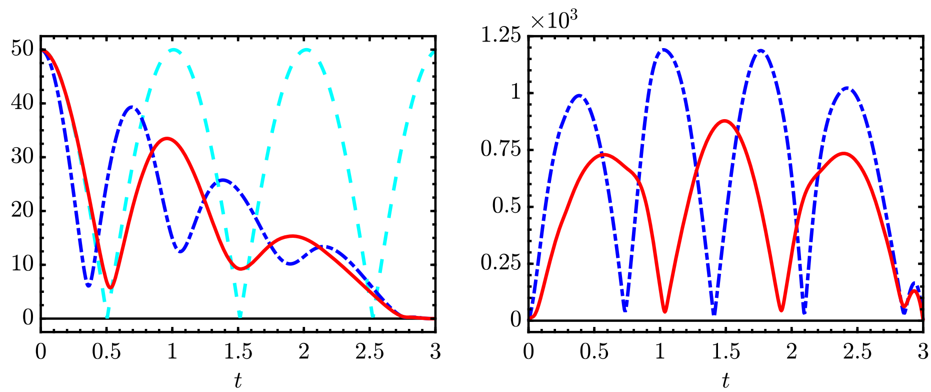

Figure 6: Fixed-point algorithm (38); $c_g = 5$ – **Left**: $\Vert y_{k^\star}(\cdot,t)\Vert_{L^2(\Omega)}$ (—) and $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$ (- -) vs t; **Right**: $\Vert f_{k^\star}(\cdot,t)\Vert_{L^2_\chi(\omega)}$ (—) and $\Vert f_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$ (- -) vs t.

The second fixed-point method is associated with the operator Λ_F : $L^2(Q_T)\to L^2(Q_T)$ and defined by $y=\Lambda_F(z)$, where y is a controlled solution of

$$\begin{cases} Ly = f1_\omega - g(z), & \text{in } Q_T,\\ y=0, & \text{on } \Sigma_T,\\ (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1), & \text{in } \Omega, \end{cases} \tag{39}$$

satisfying $(y(\cdot,T),\partial_t y(\cdot,T)) = (z_0,z_1)$. The function f is selected as the control of minimal $L^2_\chi(q_T)$-norm. Any fixed point of Λ_F is a controlled solution for (1). Theorem 1.1 implies the existence of at least one fixed point for Λ_F. The controllability of the system (39) allows to define the sequence $(y_k)_{k\in\mathbb{N}}$ as follows:

$$y_0 \in L^2(Q_T), \quad y_{k+1} = \Lambda_F(y_k), \quad k \geq 0. \tag{40}$$

With the same data and initialization, Table 7 collects some norms with respect to k for $c_g = 5$. The function f_k is the control of minimal $L^2_\chi(q_T)$-norm for y_k solution of (39). The L^2-norm of the controlled pair is greater than the one obtained from the least-squares algorithm. The convergence is significantly slower and reached after $k^\star = 50$ iterations leading to $\mathcal{E}_T = 2.02 \times 10^{-4}$. The convergence is again linear. Figures 7 depicts the time evolution of the norms of $y_{k^\star}$ and $f_{k^\star}$ for the final iteration. We check that the approximation obtained differs from those of Figures 3. For these data, the sequence converges for $|c_g| < 7$ approximately. For larger values, we observe the non-convergence of the method suggesting that the operator Λ_F is not contracting in general. We refer to [2] for $d = 1$ (extended in [12] and boundary controllability, the operator Λ_F is proved to be contracting when associated to a different control cost.

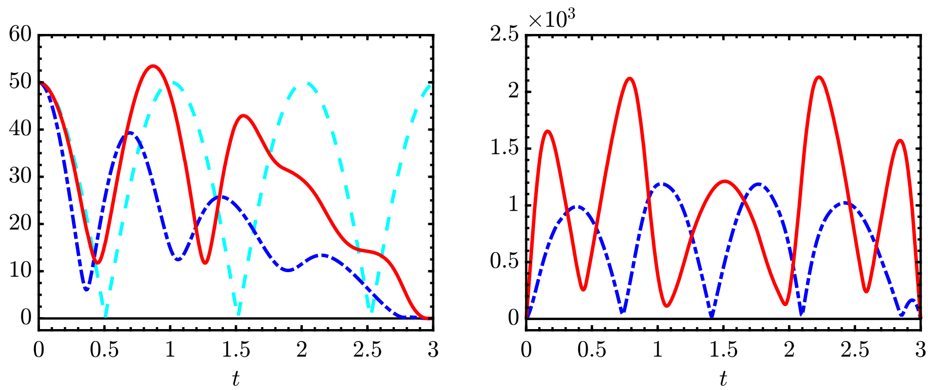

Figure 7: Fixed-point algorithm (40); $c_g = 5$ – **Left**: $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ (—) and $\|y_0(\cdot,t)\|_{L^2(\Omega)}$ (- -) vs t; **Right**: $\|f_{k^\star}(\cdot,t)\|_{L^2_\chi(\omega)}$ (—) and $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ (- -) vs t.

5.3. Experiments in 1D

In order to bypass the numerical instabilities observed for large values of c_g in Subsection 5.2, we employ, in the one-dimensional setting, a different

Table 7: $c_g = 5$ – Norms for the sequence defined by the fixed-point algorithm (40)

k	$\sqrt{2E(y_k, f_k)}$	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert f_k - f_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert f_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert f_k\Vert_{L^2_\chi(q_T)}$
0	3.66×10^2	–	–	37.653	1339.39
1	3.29×10^2	8.82×10^{-1}	1.64×10^0	50.427	2333.24
2	1.97×10^2	3.95×10^{-1}	7.55×10^{-1}	51.157	2339.62
3	7.41×10^1	1.41×10^{-1}	2.58×10^{-1}	51.492	2032.67
4	2.94×10^1	5.55×10^{-2}	8.26×10^{-2}	52.085	1988.9
5	1.58×10^1	2.91×10^{-2}	2.91×10^{-2}	52.667	1994.82
6	1.05×10^1	1.91×10^{-2}	2.05×10^{-2}	53.102	2003.63
7	7.31×10^0	1.31×10^{-2}	1.27×10^{-2}	53.43	2009.52
8	5.19×10^0	9.21×10^{-3}	9.58×10^{-3}	53.671	2014.45
9	3.71×10^0	6.53×10^{-3}	6.63×10^{-3}	53.848	2018.1
10	2.66×10^0	4.66×10^{-3}	4.84×10^{-3}	53.978	2020.95
...	...	...	...	...	...
46	2.87×10^{-5}	4.87×10^{-8}	5.37×10^{-8}	54.334	2029.72
47	2.10×10^{-5}	3.56×10^{-8}	3.93×10^{-8}	54.334	2029.72
48	1.54×10^{-5}	2.60×10^{-8}	2.87×10^{-8}	54.334	2029.72
49	1.13×10^{-5}	1.91×10^{-8}	2.10×10^{-8}	54.334	2029.72
50	8.23×10^{-6}	1.39×10^{-8}	1.54×10^{-8}	54.334	2029.72

method, not based on the minimization of $J^\star$ but on the direct approximation of the optimality condition associated with the controllability. We refer to [10, 11] where this method – falling in the framework *control-then-discretize* – has been introduced and to [4] for a numerical analysis.

We consider $\Omega = (0,1)$. The controllability time is equal to $T = 2.5$ and the control domain is the interval $\omega = (0.2, 0.4)$. For any $c_g \in \mathbb{R}$ and $\alpha > 0$, we consider the non-linear function g defined by

$$g(r) = -c_g\, r \ln^\alpha(2 + |r|), \quad \forall r \in \mathbb{R}.$$

For $\alpha = 1/2$, we check that g satisfies ($\overline{\mathbf{H}}_\mathbf{s}$) for $s = 1$ and ($\mathbf{H_2}$) for $|c_g|$ small enough. Moreover, in this one-dimensional setting, it is known (see [44]) that the semi-linear wave equation (1) is exactly controllable up to $\alpha = 2$. As for the initial and final data, we consider $(u_0, u_1) = (100\sin(\pi x_1)\sin(\pi x_2), 0)$ and $(z_0, z_1) = (0, 0)$ respectively.

In order to determine the state-control pairs (y_0, f_0) and (Y_k^1, F_k^1) of (34) and (35) respectively, we employ the space-time mixed formulation method used in [11].

Concerning problem (34), we set $M := L^2(0,T;H_0^1(\Omega))$,

$$\Phi_0 := \{\varphi \in L^2(Q_T);\ L\varphi \in L^2(0,T;H^{-1}(\Omega)) \text{ in } Q_T,\ \varphi = 0 \text{ on } \Sigma_T\},$$

and for $(\varphi,\mu) \in \Phi_0 \times M$, we consider the Lagrangian

$$\begin{aligned}\mathcal{L}_0(\varphi,\mu) := \frac{1}{2}\int_{q_T} |\varphi|^2\chi - \int_0^T \langle L\varphi,\mu\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ - \langle u_0, \partial_t\varphi(\cdot,0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + (u_1,\varphi(\cdot,0))_{L^2(\Omega)} \\ + \langle z_0, \partial_t\varphi(\cdot,T)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - (z_1,\varphi(\cdot,T))_{L^2(\Omega)}.\end{aligned}$$

Then, the control f_0 of minimal $L^2_\chi(q_T)$-norm is given by $f_0 = \varphi_0\chi$, where $(\varphi_0,\mu_0) \in \Phi_0 \times M$ is the unique saddle point of $\mathcal{L}_0$. Note also that it appears that μ_0 is the controlled solution y_0 of (34) associated with the control f_0.

Concerning problem (35), we set

$$\Phi_k := \{\varphi \in L^2(Q_T);\ L\varphi + g'(y_k)\varphi \in L^2(0,T;H^{-1}(\Omega)) \text{ in } Q_T,\ \varphi = 0 \text{ on } \Sigma_T\},$$

and for $(\varphi,\mu) \in \Phi_k \times M$, we consider the Lagrangian

$$\mathcal{L}_k(\varphi,\mu) := \frac{1}{2}\int_{q_T} |\varphi|^2\chi - \int_0^T \langle L\varphi + g'(y_k)\varphi,\mu\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_{Q_T} e_k\varphi.$$

Then, the control F_k^1 of minimal $L^2_\chi(q_T)$-norm is given by $F_k^1 = \varphi_k\chi$, where $(\varphi_k,\mu_k) \in \Phi_k \times M$ is the unique saddle point of $\mathcal{L}_k$. Note also that it appears that μ_k is the controlled solution Y_k^1 of (35) associated with the control F_k^1.

To approximate the saddle point of $\mathcal{L}_0$ and $\mathcal{L}_k$, we solve a finite-element discretization of the mixed formulation associated with these Lagrangians. We consider a family $\mathcal{T} = \{\mathcal{T}_h, h > 0\}$ of regular triangulations of Q_T such that $\overline{Q_T} = \bigcup_{K\in\mathcal{T}_h} K$. The family is indexed by $h = \max_{K\in\mathcal{T}_h}|K|$. The functions φ_0 and φ_k are approximated in the space $\Phi_h = \{\varphi_h \in \mathcal{C}^1(\overline{Q_T}) \mid \varphi_{h|K} \in \mathbb{P}(K), \forall K \in \mathcal{T}_h\}$ where $\mathbb{P}(K)$ denotes the reduced Hsieh-Clough-Tocher $\mathcal{C}^1$-element. The functions μ_0 and μ_k are approximated in the space $M_h = \{\mu_h \in \mathcal{C}(\overline{Q_T}) \mid \mu_{h|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\}$ where $\mathbb{P}_1(K)$ denotes the space of polynomials of degree 1. In the sequel, we mainly use a regular triangulation $\mathcal{T}_h$ with fineness $h = 1/64$.

We now present some simulations for large values of c_g in the case $\alpha = 1/2$ and $\alpha = 2$.

• Case $\alpha = 1/2$, $c_g = 50$ – We set $\alpha = 1/2$ and we compute the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ associated with the value $c_g = 50$. The convergence of the algorithm is observed after $k^\star = 21$ iterations. The corresponding value of the relative error is $\mathcal{E}_T = 6.24 \times 10^{-17}$. Figure 8-left depicts the evolution of the error $\sqrt{2E(y_k, f_k)}$ (red dots, left axis), as well as the evolution of the optimal steps λ_k (blue stars, right axis). Here, we can clearly see the two rates of convergence described in Theorem 3.1. At first, the values of λ_k are close to 0, while the error decreases linearly. Afterwards, around iteration $k = 16$, λ_k reaches the value 1 while the error decreases quadratically. Note that, due to numerical instabilities, the numerical method used in Subsection 5.2 does not converge for c_g greater than 7.

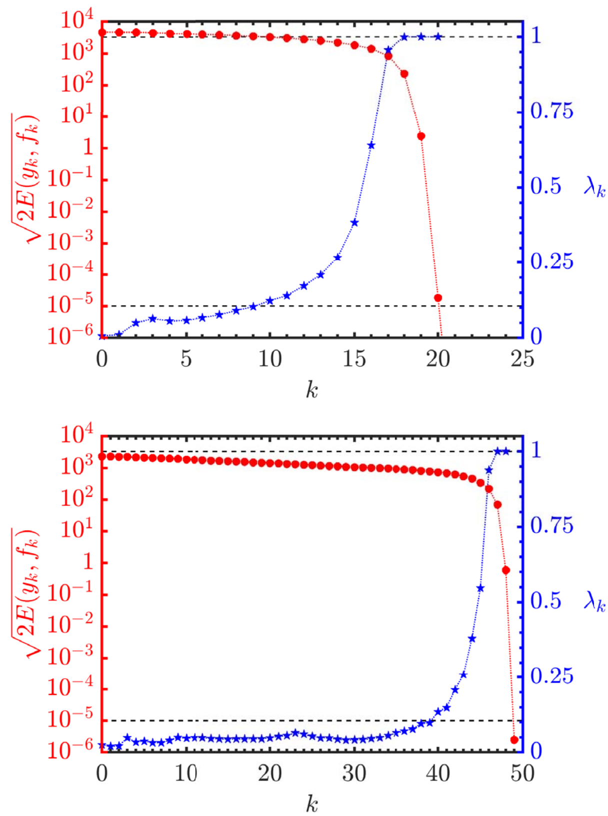

Figure 8: Evolution of $\sqrt{2E(y_k, f_k)}$ ($\bullet$, left axis) and λ_k ($\star$, right axis) w.r.t k; Top: $\alpha = 1/2$, $c_g = 50$; Bottom: $\alpha = 2$, $c_g = 3$.

• Case $\alpha = 2$, $c_g = 3$ – We set $\alpha = 2$ and we compute the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ associated with the value $c_g = 3$. The convergence of the algorithm is observed after $k^\star = 49$ iterations. The corresponding value of the relative error is $\mathcal{E}_T = 1.18 \times 10^{-17}$. Figure 8-right depicts the evolution of the error $\sqrt{2E(y_k, f_k)}$ (red dots, left axis), as well as the evolution of the optimal step λ_k (blue stars, right axis). In this case, the switch between the linear convergence (corresponding to the damped Newton regime) and the quadratic convergence (corresponding to the classical Newton regime) occurs around iteration $k = 45$.

6. Conclusion

Exact controllability of (1) has been established in [21], under a growth condition on g, by means of a Leray-Schauder fixed point argument that is not constructive. In this paper, under a slightly stronger growth condition and under the additional assumption that g' is uniformly Hölder continuous with exponent $s \in (0, 1]$, we have designed an explicit algorithm and proved its convergence of a controlled solution of (1). Moreover, the convergence is super-linear of order greater than or equal to $1 + s$ after a finite number of iterations. Our approach gives a new and constructive proof of the exact controllability of (1). The method is general and may be applied to any other equations or systems – not necessarily of hyperbolic nature – for which a precise observability estimate for the linearized problem is available: we refer to [27, 30, 17] addressing the case of the heat equation. We also mention the recent extension [2] of this constructive method to address controllability problem (initially investigated in [43]).

Numerical experiments reported are in agreement with the theoretical convergence: in particular, the convergence is, after a finite number of iterations, super-linear to be compared with the linear rate observed with algorithms derived from simpler linearizations. The experiments also confirm that the numerical method developed in [10, 11] and based on the direct resolution of the optimality system (35)–(37) turns out to be very robust with respect to the size of the potential and allows to consider large amplitudes of the nonlinearity, in contrast with the standard minimization of the corresponding conjugate functional (introduced in [22]). We also emphasized that the convergence of the least-squares algorithm still holds without growth assumptions on the nonlinear function if the initial condition and final target are small enough.

The case for which the nonlinearities enjoy a sign type condition as mentioned in the introduction and leading to uniform controllability results with respect to the initial data will be discussed in a future work.

Appendix A. Appendix: controllability results for the wave equation

We recall some *a priori* estimates for the linear wave equation with potential in $L^\infty(0,T;L^d(\Omega))$, $d\in\mathbb{N}^\star$ and right hand side in $L^2(Q_T)$.

Proposition A.1 ([32, Theorem 2.1])**.** *Assume that ω and T satisfy* $(\mathbf{H_0})$ *(see Theorem 1.1). For any $d\in\mathbb{N}^\star$, $A\in L^\infty(0,T;L^d(\Omega))$ and $(\phi_0,\phi_1)\in \boldsymbol{H}:=L^2(\Omega)\times H^{-1}(\Omega)$, the weak solution ϕ of*

$$\begin{cases} L\phi + A\phi = 0, & \textit{in } Q_T,\\ \phi = 0, & \textit{on } \Sigma_T,\\ (\phi(\cdot,0),\partial_t\phi(\cdot,0)) = (\phi_0,\phi_1), & \textit{in } \Omega, \end{cases} \tag{41}$$

satisfies the observability inequality $\|\phi_0,\phi_1\|_{\boldsymbol{H}} \le Ce^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}}\|\phi\|_{2,q_T}$ for some $C>0$ only depending on Ω and T.

Classical arguments then lead to following controllability result.

Proposition A.2. *Let d in $\mathbb{N}^\star$, $A\in L^\infty(0,T;L^d(\Omega))$, $B\in L^2(Q_T)$ and $(z_0,z_1)\in\boldsymbol{V}:=H^1_0(\Omega)\times L^2(\Omega)$. Assume that ω and T satisfy* $(\mathbf{H_0})$*. There exists a control function $u\in L^2(q_T)$ such that the weak solution of*

$$\begin{cases} Lz + Az = u1_\omega + B, & \textit{in } Q_T,\\ z = 0, & \textit{on } \Sigma_T,\\ (z(\cdot,0),\partial_t z(\cdot,0)) = (z_0,z_1), & \textit{in } \Omega, \end{cases} \tag{42}$$

satisfies $(z(\cdot,T),z_t(\cdot,T))=(0,0)$ in Ω. Moreover, the unique pair (u,z) of minimal control norm satisfies

$$\|u\|_{2,q_T} + \|(z,\partial_t z)\|_{L^\infty(0,T;\boldsymbol{V})} \le C\Big(\|B\|_2 + \|z_0,z_1\|_{\boldsymbol{V}}\Big)e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}} \tag{43}$$

for some constant $C>0$ only depending on Ω and T.

Let $p^\star\in\mathbb{N}^\star$ such that $p^\star<\infty$ if $d=2$ and $p^\star<6$ if $d=3$. We next discuss some properties of the operator $\Lambda: L^\infty(0,T;L^{p^\star}(\Omega))\to L^\infty(0,T;L^{p^\star}(\Omega))$ defined by $\Lambda(\xi)=y_\xi$, a null controlled solution of the linear boundary value problem (2) with the control f_ξ of minimal $L^2(q_T)$ norm. Proposition A.2 with $B=-g(0)$ gives

$$\|(y_\xi,\partial_t y_\xi)\|_{L^\infty(0,T;\boldsymbol{V})} \le C\Big(\|u_0,u_1\|_{\boldsymbol{V}} + \|g(0)\|_2\Big)e^{C\|\widehat{g}(\xi)\|^2_{L^\infty(0,T;L^d(\Omega))}} \tag{44}$$

where the function $\widehat{g}$ is defined in (2). We assume that $g \in \mathcal{C}^1(\mathbb{R})$ satisfies the following asymptotic condition (slightly weaker than $(\mathbf{H_1})$): there exists a $\overline{\beta}$ small enough such that $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r|\ln^{1/2}|r|} \leq \overline{\beta}$, i.e.

$\widehat{(\mathbf{H_1})}$ There exist $\overline{\alpha} \geq 0$ and $\overline{\beta} \geq 0$ small enough such that $|g(r)| \leq \overline{\alpha} + \overline{\beta}(1+|r|)\ln^{1/2}(1+|r|)$ for every $r \in \mathbb{R}$.

This implies that $\widehat{g}$ satisfies $|\widehat{g}(r)| \leq \overline{\alpha} + \overline{\beta}\ln^{1/2}(1+|r|)$ for every $r \in \mathbb{R}$ and some constant $\overline{\alpha} > 0$. This also implies that $\widehat{g}(\xi) \in L^\infty(0,T;L^d(\Omega))$ for any $\xi \in L^\infty(0,T;L^{p^\star}(\Omega))$. Assuming $2C\overline{\beta}^2 \leq 1$ and proceeding as in the proof of Lemma 3.3, we get, for all $\xi \in L^\infty(0,T;L^{p^\star}(\Omega))$,

$$e^{C\|\widehat{g}(\xi)\|^2_{L^\infty(0,T;L^d(\Omega))}} \leq C_1\left(1 + \frac{\|\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\overline{\beta}^2},$$

for some $C_1 = C_1(\alpha)$. Using (44), we then infer for all $\xi \in L^\infty(0,T;L^{p^\star}(\Omega))$ that

$$\|y_\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))} \leq C\Big(\|u_0,u_1\|_{\boldsymbol{V}} + \|g(0)\|_2\Big)C_1\left(1 + \frac{\|\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\overline{\beta}^2}.$$

Taking $\overline{\beta}$ small enough so that $2C\overline{\beta}^2 < 1$, we conclude that there exists $M > 0$ such that $\|\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))} \leq M$ implies $\|\Lambda(\xi)\|_{L^\infty(0,T;L^{p^\star}(\Omega))} \leq M$. This is the argument (introduced in [44] for the one-dimensional case and) used in [32] to prove the controllability of (1).

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