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A note on the family of synchronizations for a coupled system of wave equations*

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We show that a coupled system of wave equations can be exactly synchronized by p-groups with respect to different groupings under the same control matrix.

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1. Introduction

Let Ω be a bounded domain with a smooth boundary Γ in $\mathbb{R}^n$. Denote by χ_ω the characteristic function of a subdomain ω of Ω. Let A be a matrix of order N and D be a full column-rank matrix of order $N \times (N-p)$, both with constant elements. Consider the following system for the state variable $U = (u^{(1)}, \dots, u^{(N)})^T$ with the internal control $H = (h^{(1)}, \dots, h^{(N-p)})^T$:

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma. \end{cases} \tag{1}$$

Recall that for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$ and any given function $H \in (L^1_{loc}(0, +\infty; L^2(\Omega)))^{N-p}$, system (1) admits a unique weak solution U in the space

$$(C^0_{loc}([0, +\infty); H_0^1(\Omega)) \cap C^1_{loc}([0, +\infty); L^2(\Omega)))^N \tag{2}$$

with continuous dependence (Proposition 2.1 in [5]).

Let

$$0 = n_0 < n_1 < \cdots < n_p = N \tag{3}$$

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be a partition with $n_r - n_{r-1} \geqslant 2$ for $1 \leqslant r \leqslant p$. We arrange the components of the state variable U into p groups:

$$(u^{(1)}, \dots, u^{(n_1)}),\ (u^{(n_1+1)}, \dots, u^{(n_2)}), \dots, (u^{(n_{p-1}+1)}, \dots, u^{(n_p)}). \tag{4}$$

System (1) is exactly synchronizable by p-groups at time $T > 0$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, there exist a control function $H \in (L^1_{loc}(0, +\infty; L^2(\Omega)))^{N-p}$ and some functions u_r for $1 \leqslant r \leqslant p$:

$$u_r \in C^0_{loc}([0, +\infty); H_0^1(\Omega)) \cap C^1_{loc}([0, +\infty); L^2(\Omega)), \tag{5}$$

such that the corresponding solution U to system (1) satisfies

$$t \geqslant T: \quad u^{(k)} = u_r, \quad n_{r-1} + 1 \leqslant k \leqslant n_r, \quad 1 \leqslant r \leqslant p. \tag{6}$$

Accordingly, $(u_1, \dots, u_r)^T$ will be called the exactly synchronizable state by p-groups.

Let S_r be a full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \leqslant r \leqslant p. \tag{7}$$

Define the $(N - p) \times N$ matrix C_p by

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{8}$$

Let

$$Ker(C_p) = Span\{e_1, \dots, e_p\}, \tag{9}$$

where

$$e_r = (0, \dots, 0, \overset{(n_{r-1}+1)}{1}, \dots, \overset{(n_r)}{1}, 0, \dots, 0)^T, \quad 1 \leqslant r \leqslant p. \tag{10}$$

(6) can be rewritten as

$$t \geqslant T: \quad U = \sum_{r=1}^{p} u_r e_r \tag{11}$$

or equivalently,

$$t \geqslant T: \quad C_p U = 0. \tag{12}$$

The matrix A satisfies the condition of C_p-compatibility if there exists a unique matrix A_p of order $(N-p)$, such that

$$C_p A = A_p C_p. \tag{13}$$

The synchronization by p-groups of system (1) with respect to the specific grouping (4) was studied in [5]. In particular, we have

Theorem 1.1 (Theorem 3.1 in [5])**.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ satisfying the usual multiplier control condition and $\omega \subset \Omega$ be a neighbourhood of Γ. Assume that A satisfies the condition of C_p-compatibility* (13)*. Then system* (1) *is exactly synchronizable by p-groups at the time $T > 2d_0(\Omega)$, where $d_0(\Omega)$ denotes the diameter of Ω, if and only if*

$$\mathrm{rank}(C_p D) = N - p. \tag{14}$$

However, when $p \geqslant 2$, the situation of the grouping (4) could be very complicated. The goal of this work is to extend the previous study on the specific grouping (4) to the general one.

Let σ be a permutation of the set $\{1, \dots, N\}$. We arrange the components of the state variable U into p groups:

$$(u^{\sigma(1)}, \dots, u^{\sigma(n_1)}),\ (u^{\sigma(n_1+1)}, \dots, u^{\sigma(n_2)}), \dots, (u^{\sigma(n_{p-1}+1)}, \dots, u^{\sigma(n_p)}). \tag{15}$$

When the solution U to system (1) satisfies

$$t \geqslant T: \quad u^{\sigma(k)} = u_r, \quad n_{r-1}+1 \leqslant k \leqslant n_r, \quad 1 \leqslant r \leqslant p \tag{16}$$

for some functions u_r with $1 \leqslant r \leqslant p$, system (1) will be called exactly synchronizable by p-groups with respect to the grouping (15).

Accordingly, define the matrix $C_p^{(\sigma)}$ by

$$Ker(C_p^{(\sigma)}) = Span(e_1^{(\sigma)}, \dots, e_p^{(\sigma)}), \tag{17}$$

where

$$(e_r^{(\sigma)})_i = (e_r)_{\sigma(i)}, \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant r \leqslant p. \tag{18}$$

Then the exact synchronization by p-groups (16) can be rewritten as

$$t \geqslant T: \quad U = \sum_{r=1}^{p} u_r e_r^{(\sigma)}, \tag{19}$$

or equivalently,

$$t \geqslant T: \quad C_p^{(\sigma)} U = 0. \tag{20}$$

For $1 \leqslant r \leqslant p$, the size $(n_r - n_{r-1})$ of each group varies following the repartition (3), and the components in each group $(u^{\sigma(n_{r-1}+1)}, \ldots, u^{\sigma(n_r)})$ is determined by the permutation σ.

We will show in Theorem 3.1 that there exists a control matrix D of order $N \times (N-p)$, such that system (1) is exactly synchronizable by p-groups with respect to any given $C_p^{(\sigma)}$ and $C_p^{(\sigma')}$ groupings by (17) on some appropriate basis.

2. Family of generalized synchronizations

Denote by $\theta_i^{(j)}$ the Jordan chains associated with the eigenvalue a_i of A:

$$\theta_i^{(0)} = 0, \quad (A - a_i I)\theta_i^{(j)} = \theta_i^{(j-1)}, \quad 1 \leqslant i \leqslant d, \quad 1 \leqslant j \leqslant \widehat{\mu}_i. \tag{21}$$

Let $\mathcal{M}_p$ be the set of multi-indices $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ of length p:

$$0 \leqslant \mu_i \leqslant \widehat{\mu}_i, \quad |\mu| = \sum_{i=1}^{d} \mu_i = p. \tag{22}$$

For any given $\mu \in \mathcal{M}_p$, we define a matrix Θ_μ of order $(N-p) \times N$ by

$$Ker(\Theta_\mu) = Span(\theta_1^{(1)}, \ldots, \theta_1^{(\mu_1)}; \ldots; \theta_d^{(1)}, \ldots, \theta_d^{(\mu_d)}). \tag{23}$$

Instead of (12), if the solution U satisfies

$$t \geqslant T: \quad \Theta_\mu U = 0, \tag{24}$$

we say that system (1) is exactly Θ_μ-synchronizable, or generalized synchronizable with respect to Θ_μ. We refer to [4] for a systematic study and some interesting results on this topic.

Clearly, the matrix Θ_μ gives an equivalence class for the relation

$$\Theta_\mu \sim \Theta_{\mu'} \Longleftrightarrow Ker(\Theta_\mu) = Ker(\Theta_{\mu'}). \tag{25}$$

As μ runs through the set $\mathcal{M}_p$, (23) provides the family of all the matrices which satisfy the condition of Θ_μ-compatibility:

$$AKer(\Theta_\mu) \subseteq Ker(\Theta_\mu). \tag{26}$$

By [2, Proposition 2.15], there exists a matrix A_μ of order $(N-p)$, such that $\Theta_\mu A = A_\mu \Theta_\mu$. Applying Θ_μ to system (1) and setting $W_\mu = \Theta_\mu U$, we get the reduced system

$$\begin{cases} W_\mu'' - \Delta W_\mu + A_\mu W_\mu = \Theta_\mu D \chi_\omega H & \text{in } (0,+\infty)\times\Omega, \\ W_\mu = 0 & \text{on } (0,+\infty)\times\Gamma. \end{cases} \tag{27}$$

Noting (24), the exact Θ_μ-synchronization of system (1) is equivalent to the exact controllability of the reduced system (27), which contains $(N-p)$ equations. By [5, Theorem 2.7], we have

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ satisfying the usual multiplier control condition and $\omega \subset \Omega$ be a neighbourhood of Γ. Assume that A satisfies the condition of Θ_μ-compatibility (26). Then the original system (1) is exactly Θ_μ-synchronizable if and only if*

$$\text{rank}(D) = \text{rank}(\Theta_\mu D) = N - p. \tag{28}$$

Our objective is to find a matrix D such that the rank condition (28) will be satisfied by all the multi-indices $\mu \in \mathcal{M}_p$ for the same system. Note that the cardinal of $\mathcal{M}_p$ could be much bigger than rank(D), it is not trivial (even it is surprising) that only one control matrix D can satisfy the rank condition (28) for all $\mu \in \mathcal{M}_p$.

The following result is a simplified version of the generalized Vandermonde matrix, the proof of which is adapted from a collection of Ecole Polytechnique.

Proposition 2.2. *Let* $0 < a_1 < \cdots < a_n$ *and* $0 < \gamma_1 < \cdots < \gamma_n$. *The determinant of the following generalized Vandermonde matrix is strictly positive:*

$$\det \begin{pmatrix} a_1^{\gamma_1} & a_2^{\gamma_1} & \cdots & a_n^{\gamma_1} \\ a_1^{\gamma_2} & a_2^{\gamma_2} & \cdots & a_n^{\gamma_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{\gamma_n} & a_2^{\gamma_n} & \cdots & a_n^{\gamma_n} \end{pmatrix} > 0. \tag{29}$$

Proof. We first show that for any given real coefficients $c_1, \ldots, c_n$ not all zero, the function

$$f_n(x) = c_1 x^{\gamma_1} + c_2 x^{\gamma_2} + \cdots + c_n x^{\gamma_n}, \quad x > 0$$

has at most $(n-1)$ strictly positive zeros.

The conclusion is trivial for $n = 1$. Assume that it is true for the value $(n-1)$. Assume by contradiction that f_n has n strictly positive zeros, so has the function

$$g_n(x) = \frac{f_n(x)}{x^{\gamma_1}} = c_1 + c_2 x^{\gamma_2 - \gamma_1} + \cdots + c_n x^{\gamma_n - \gamma_1}, \quad x > 0.$$

By Rolle's Theorem, the derivative

$$g_n'(x) = c_2(\gamma_2 - \gamma_1) x^{\gamma_2 - \gamma_1 - 1} + \cdots + c_n(\gamma_n - \gamma_1) x^{\gamma_n - \gamma_1 - 1}$$

has $(n-1)$ strictly positive zeros, so has the function

$$x^{\gamma_1 + 1} g_n'(x) = c_2(\gamma_2 - \gamma_1) x^{\gamma_2} + \cdots + c_n(\gamma_n - \gamma_1) x^{\gamma_n},$$

which contradicts the induction's hypothesis for the value $(n-1)$.

Now we return to prove (29), which is obviously true for $n = 1$. Assume that it is true for the value $(n-1)$. Let

$$f(x) = \det \begin{pmatrix} a_1^{\gamma_1} & a_2^{\gamma_1} & \cdots & a_{n-1}^{\gamma_1} & x^{\gamma_1} \\ a_1^{\gamma_2} & a_2^{\gamma_2} & \cdots & a_{n-1}^{\gamma_2} & x^{\gamma_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^{\gamma_{n-1}} & a_2^{\gamma_{n-1}} & \cdots & a_{n-1}^{\gamma_{n-1}} & x^{\gamma_{n-1}} \\ a_1^{\gamma_n} & a_2^{\gamma_n} & \cdots & a_{n-1}^{\gamma_n} & x^{\gamma_n} \end{pmatrix}.$$

Developing the determinant according to the last column, we get

$$f(x) = c_1 x^{\gamma_1} + c_2 x^{\gamma_2} + \cdots + c_n x^{\gamma_n},$$

where

$$c_n = \det \begin{pmatrix} a_1^{\gamma_1} & a_2^{\gamma_1} & \cdots & a_{n-1}^{\gamma_1} \\ a_1^{\gamma_2} & a_2^{\gamma_2} & \cdots & a_{n-1}^{\gamma_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{\gamma_{n-1}} & a_2^{\gamma_{n-1}} & \cdots & a_{n-1}^{\gamma_{n-1}} \end{pmatrix} > 0$$

by the induction hypothesis for the value $(n-1)$. Since f vanishes at $x = a_1, \ldots, a_{n-1}$, by what has been shown in the first step, f has at most $(n-1)$ strictly positive zeros, then $a_1, \ldots, a_{n-1}$ exhaust the zeros on the interval $(0, +\infty)$. By continuity, f keeps a constant sign for $x > a_{n-1}$, in particular, we get $f(a_n) > 0$, which gives (29) for the value n. The proof is then complete. □

Proposition 2.3. *Let*

$$G = \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_{N-p}^1 \\ a_1^2 & a_2^2 & \cdots & a_{N-p}^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^i & a_2^i & \cdots & a_{N-p}^i \\ \vdots & \vdots & \vdots & \vdots \\ a_1^N & a_2^N & \cdots & a_{N-p}^N \end{pmatrix}, \tag{30}$$

where

$$0 < a_1 < a_2 < \cdots < a_{N-p}. \tag{31}$$

Let

$$Q = (\theta_1^{(1)}, \ldots, \theta_1^{(\widehat{\mu}_1)}, \ldots, \theta_d^{(1)}, \ldots, \theta_d^{(\widehat{\mu}_d)}). \tag{32}$$

Defining

$$D = QG, \tag{33}$$

condition (28) *holds for all* $\mu \in \mathcal{M}_p$.

Proof. By Lemma 2.2, we have $\operatorname{rank}(D) = N - p$.

Let

$$Q^{-T} = (\eta_1^{(1)}, \ldots, \eta_1^{(\widehat{\mu}_1)}, \ldots, \eta_d^{(1)}, \ldots, \eta_d^{(\widehat{\mu}_d)}). \tag{34}$$

Then

$$(\theta_i^{(k)}, \eta_j^{(l)}) = \delta_{ij}\delta_{kl}.$$

It follows that

(35)

$$Im(\Theta_\mu^T) = Ker(\Theta_\mu)^\perp = Span(\eta_1^{(\mu_1+1)}, \dots, \eta_1^{(\widehat{\mu}_1)}; \dots; \eta_d^{(\mu_d+1)}, \dots, \eta_d^{(\widehat{\mu}_d)}).$$

Let

$$m_i = \sum_{j=1}^{i-1} \widehat{\mu}_i, \quad 1 \leqslant i \leqslant d.$$

Defining

$$\mathcal{E}_i^{(j)} = (0, \dots, 0, \overset{(m_i+j-1)}{1}, 0, \dots, 0)^T, \quad 1 \leqslant j \leqslant \widehat{\mu}_i, \quad 1 \leqslant i \leqslant d \tag{36}$$

and

$$E_{\bar{\mu}} = Span(\mathcal{E}_1^{(\mu_1+1)}, \dots, \mathcal{E}_1^{(\widehat{\mu}_1)}; \dots; \mathcal{E}_d^{(\mu_d+1)}, \dots, \mathcal{E}_d^{(\widehat{\mu}_d)}),$$

we have

$$\Theta_\mu^T = Q^{-T} E_{\bar{\mu}}.$$

It follows that

$$\Theta_\mu D = E_{\bar{\mu}}^T Q^{-1} Q G = E_{\bar{\mu}}^T G.$$

The matrix $E_{\bar{\mu}}^T G$ is in fact the extraction of $(N-p)$ rows from the matrix G, namely, we have

$$E_{\bar{\mu}}^T G = \begin{pmatrix} a_1^{j_1} & a_2^{j_1} & \cdots & a_{N-p}^{j_1} \\ a_1^{j_2} & a_2^{j_2} & \cdots & a_{N-p}^{j_2} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{j_{N-p}} & a_2^{j_{N-p}} & \cdots & a_{N-p}^{j_{N-p}} \end{pmatrix}, \tag{37}$$

where the indices $j_i (i = 1, \dots, N-p)$ are defined by (36) so that

$$1 \leqslant j_1 < j_2 < \cdots < j_{N-p} \leqslant N. \tag{38}$$

Still by Lemma 2.2, under conditions (31) and (38), the determinant of (37) is strictly positive. The proof is achieved. □

Finally, combining Propositions 2.1 and 2.3, we get

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ satisfying the usual multiplier control condition and $\omega \subset \Omega$ be a neighbourhood of Γ. Assume that the control matrix D is given by (33). Then system (1) is exactly Θ_μ-synchronizable for all $\mu \in \mathcal{M}_p$.*

3. Family of synchronizations by groups

We will establish the relationship between the exact Θ_μ-synchronization and the exact synchronization by p-groups with respect to a $C_p^{(\sigma)}$-grouping. We first show the following

Proposition 3.1. *Let the matrices Θ_μ and $C_p^{(\sigma)}$ be given by (23) and (17) respectively. If system (1) is exactly Θ_μ-synchronizable, then it is exactly synchronizable by p-groups with respect to the $C_p^{(\sigma)}$-grouping on some appropriate basis.*

Proof. Since $\operatorname{rank}(\Theta_\mu) = \operatorname{rank}(C_p^{(\sigma)}) = N - p$, there exists an invertible matrix P such that

$$\Theta_\mu = C_p^{(\sigma)} P. \tag{39}$$

Applying P to system (1) and setting $\widetilde{U} = PU$, we get

$$\begin{cases} \widetilde{U}'' - \Delta\widetilde{U} + PAP^{-1}\widetilde{U} = PD\chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ \widetilde{U} = 0 & \text{on } (0, +\infty) \times \Gamma. \end{cases} \tag{40}$$

Accordingly, the final condition (24) becomes

$$t \geqslant T: \quad C_p^{(\sigma)}\widetilde{U} = 0. \tag{41}$$

Since system (1) is exactly Θ_μ-synchronizable under the control matrix D, therefore, system (40) is exactly synchronizable by p-groups with respect to the $C_p^{(\sigma)}$-grouping under the control matrix PD. □

We next examine the possibility to convert every exact Θ_μ-synchronization to some exact synchronizations by p-groups with respect to different $C_p^{(\sigma)}$-groupings. This requires certain additional conditions on the structure of the coupling matrix. We first give some preliminaries.

Proposition 3.2. *The matrix A admits two Jordan chains if and only if there exist $\mu, \mu' \in \mathcal{M}_p$, such that $Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'}) \neq \{0\}$.*

Proof. Assume that A admits d Jordan chains with $d \geqslant 2$.

If $\widehat{\mu}_1 > p$, we define

$$Ker(\Theta_\mu) = Span\{\theta_1^{(1)}, \dots, \theta_1^{(p)}\},$$
$$Ker(\Theta_{\mu'}) = Span\{\theta_1^{(1)}, \dots, \theta_1^{(p-1)}, \theta_2^{(1)}\}.$$

Clearly, $\theta_1^{(1)} \in Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'})$.

If $\widehat{\mu}_1 \leqslant p$, noting $\sum_{i=1}^d \widehat{\mu}_i = N > p$, there exists an integer d_0 with $1 \leqslant d_0 < d$, such that $\widehat{\mu}_1 + \dots + \widehat{\mu}_{d_0} = p_0 \leqslant p$, but $\widehat{\mu}_1 + \dots + \widehat{\mu}_{d_0} + \widehat{\mu}_{d_0+1} > p$. We define

$$Ker(\Theta_\mu) = Span\{\theta_1^{(1)}, \dots, \theta_1^{(\widehat{\mu}_1)}, \dots, \theta_{d_0}^{(1)}, \dots, \theta_{d_0}^{(\widehat{\mu}_{d_0})}, \theta_{d_0+1}^{(1)}, \dots, \theta_{d_0+1}^{(p-p_0)}\},$$
$$Ker(\Theta_\mu) = Span\{\theta_1^{(1)}, \dots, \theta_1^{(\widehat{\mu}_1)}, \dots, \theta_{d_0}^{(1)}, \dots, \theta_{d_0}^{(\widehat{\mu}_{d_0}-1)}, \theta_{d_0+1}^{(1)}, \dots, \theta_{d_0+1}^{(p-p_0+1)}\}.$$

Then, we get again $\theta_1^{(1)} \in Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'})$. Another sense is trivial. □

Proposition 3.3. *Let $C_p^{(\sigma)}$ and $C_p^{(\sigma')}$ be given by (17), respectively, Θ_μ and $\Theta_{\mu'}$ be given by (23). There exists an invertible matrix P such that*

$$\Theta_\mu = C_p^{(\sigma)} P, \qquad \Theta_{\mu'} = C_p^{(\sigma')} P \tag{42}$$

if and only if

$$dim(Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'})) = dim(Ker(C_p^{(\sigma)}) \cap Ker(C_p^{(\sigma')})) := q > 0. \tag{43}$$

Proof. Assume that condition (42) holds. Then we have

$$P(Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'})) = Ker(C_p^{(\sigma)}) \cap Ker(C_p^{(\sigma')}).$$

Noting

$$\sum_{r=1}^{p} e_r^{(\sigma)} = e \quad \text{with} \quad e = (1, 1, \dots, 1)^T \in Ker(C_p^{(\sigma)}) \cap Ker(C_p^{(\sigma')}),$$

we get thus (43).

Conversely, assume that (43) holds, then we have

$$Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'}) = Span(\theta_1, \ldots, \theta_q).$$

We complete $(\theta_1, \ldots, \theta_q)$ to get a basis of $Ker(\Theta_\mu)$:

$$Ker(\Theta_\mu) = Span(\theta_1, \ldots, \theta_q, \theta_{q+1}^{(\mu)}, \ldots, \theta_p^{(\mu)}),$$

respectively a basis of $Ker(\Theta_{\mu'})$:

$$Ker(\Theta_{\mu'}) = Span(\theta_1, \ldots, \theta_q, \theta_{q+1}^{(\mu')}, \ldots, \theta_p^{(\mu')}).$$

We easily check the linear independence of the family

$$(\theta_1, \ldots, \theta_q, \theta_{q+1}^{(\mu)}, \ldots, \theta_p^{(\mu)}, \theta_{q+1}^{(\mu')}, \ldots, \theta_p^{(\mu')}). \tag{44}$$

In fact, let $a_l (l = 1, \ldots q)$ and $b_l, c_l (l = 1 + q, \ldots p)$ be some coefficients such that

$$\sum_{l=1}^{q} a_l \theta_l + \sum_{l=q+1}^{p} b_l \theta_l^{(\mu)} + \sum_{l=q+1}^{p} c_l \theta_l^{(\mu')} = 0.$$

Then

$$\sum_{l=1}^{q} a_l \theta_l + \sum_{l=q+1}^{p} b_l \theta_l^{(\mu)} \in Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'}),$$

therefore

$$\sum_{l=q+1}^{p} b_l \theta_l^{(\mu)} \in Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'}).$$

Since $(\theta_{q+1}^{(\mu)}, \ldots, \theta_p^{(\mu)}) \notin Ker(\Theta_{\mu'})$, we get $\sum_{l=q+1}^{p} b_l \theta_l^{(\mu)} = 0$. Similarly, we have $\sum_{l=q+1}^{p} c_l \theta_l^{(\mu')} = 0$. We get thus the linear independence of the family (44).

Similarly, we can write

$$Ker(C_p^{(\sigma)}) = Span(e_1, \ldots, e_q, e_{q+1}^{(\sigma)}, \ldots, e_p^{(\sigma)}), \tag{45}$$

$$Ker(C_p^{(\sigma')}) = Span(e_1, \ldots, e_q, e_{q+1}^{(\sigma')}, \ldots, e_p^{(\sigma')}), \tag{46}$$

where e_r for $1 \leqslant r \leqslant q$ are defined by (18).

Moreover, we easily check that the families

$$(\theta_1, \dots, \theta_q, \theta_{q+1}^{(\mu)}, \dots, \theta_p^{(\mu)}, \theta_{q+1}^{(\mu')}, \dots, \theta_p^{(\mu')}),$$
$$(e_1, \dots, e_q; e_{q+1}^{(\sigma)}, \dots, e_p^{(\sigma)}; e_{q+1}^{(\sigma')}, \dots, e_p^{(\sigma')})$$

are linearly independent. Therefore, there exists an invertible matrix P such that

$$\begin{aligned} P(\theta_1, \dots, \theta_q, \theta_{q+1}^{(\mu)}, \dots, \theta_p^{(\mu)}, \theta_{q+1}^{(\mu')}, \dots, \theta_p^{(\mu')}) \\ = (e_1, \dots, e_q; e_{q+1}^{(\sigma)}, \dots, e_p^{(\sigma)}; e_{q+1}^{(\sigma')}, \dots, e_p^{(\sigma')}), \end{aligned} \tag{47}$$

namely,

$$PKer(\Theta_\mu) = Ker(C_p^{(\sigma)}), \quad PKer(\Theta_{\mu'}) = Ker(C_p^{(\sigma')}).$$

Then, we get

$$Ker(\Theta_\mu) = Ker(C_p^{(\sigma)}P), \quad Ker(\Theta_{\mu'}) = Ker(C_p^{(\sigma')}P).$$

Noting (25), we get (42). □

Now we give the main theorem in this work.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ satisfying the usual multiplier control condition and $\omega \subset \Omega$ be a neighbourhood of Γ. Assume that A admits two Jordan chains. Then there exists a control matrix D of order $N \times (N-p)$, such that system (1) is exactly synchronizable by p-groups with respect to any given $C_p^{(\sigma)}$ and $C_p^{(\sigma')}$ groupings by (17) on some appropriate basis.*

Proof. Since A admits two Jordan chains, by Proposition 3.2, we have

$$dim(Ker(\Theta_\mu) \cap Ker(\Theta_{\mu'})) = q > 0. \tag{48}$$

We can thus arbitrarily chose the matrices $C_p^{(\sigma)}$ and $C_p^{(\sigma')}$ such that (43) holds. Then, by Proposition 3.3, there exists an invertible matrix P such that (42) holds. By Theorem 2.1, system (1) is exactly Θ_μ and Θ'_μ-synchronizable under the same control matrix D. Noting (42) and Proposition 3.1, system (1) is exactly synchronizable by p-groups with respect to the $C_p^{(\sigma)}$ and $C_p^{(\sigma')}$-groupings under the same control matrix D on the commun basis P. □

Remark 3.1. *Because of the restrictive condition* (43), *only a part of the exact* $C_p^{(\sigma)}$*-synchronizations can be realized under the same control matrix* D*. The number of* $C_p^{(\sigma)}$*-synchronizations depends on the structure of the coupling matrix* A*. For a better understanding, in Theorem 3.1, we only explicate the case of two Jordan chains. We will not deepen the discussion on the topic in this short note. Instead, we give one example of three Jordan chains for illustrating the preceding abstract results.*

Example. In this example, we have $N = 6$, $p = 2$. Let

$$A = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & 0 & 1 & & \\ & & 0 & 0 & & \\ & & & & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix}$$

with three Jordan chains of length 2:

$$\theta_1^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \theta_1^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \qquad \theta_2^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \theta_2^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\theta_3^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \theta_3^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since A is already Jordanized, we have $Q = I$ in (32) and $D = G$ in (33). We first exhaust the matrices of generalized synchronization Θ_μ given by (23).

$$Ker(\Theta_{(2,0,0)}) = Span(\theta_1^{(1)}, \theta_1^{(2)}) \quad \text{with } \mu_1 = 2, \mu_2 = \mu_3 = 0, \mu = (2,0,0),$$
$$Ker(\Theta_{(1,1,0)}) = Span(\theta_1^{(1)}, \theta_2^{(1)}) \quad \text{with } \mu_1 = 1, \mu_2 = 1, \mu_3 = 0, \mu = (1,1,0)$$

and

$$Ker(\Theta_{(0,2,0)}) = Span(\theta_2^{(1)}, \theta_2^{(2)}), \qquad Ker(\Theta_{(0,1,1)}) = Span(\theta_2^{(1)}, \theta_3^{(1)}),$$
$$Ker(\Theta_{(0,0,2)}) = Span(\theta_3^{(1)}, \theta_3^{(2)}), \qquad Ker(\Theta_{(1,0,1)}) = Span(\theta_1^{(1)}, \theta_3^{(1)}),$$

or equivalently by (35),

$$\Theta_{(2,0,0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Theta_{(1,1,0)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\Theta_{(0,2,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Theta_{(0,1,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\Theta_{(0,0,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \Theta_{(1,0,1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 2.1, system (1) is exactly Θ_μ-synchronizable for all $\mu \in \mathcal{M}_p$ under the same control matrix D:

$$D = \begin{pmatrix} 1 & a & b & c \\ 1 & a^2 & b^2 & c^2 \\ 1 & a^3 & b^3 & c^3 \\ 1 & a^4 & b^4 & c^4 \\ 1 & a^5 & b^5 & c^5 \\ 1 & a^6 & b^6 & c^6 \end{pmatrix}, \quad 1 < a < b < c.$$

Since

$$Ker(\Theta_{(2,0,0)}) \cap Ker(\Theta_{(1,1,0)}) \cap Ker(\Theta_{(1,0,1)}) = Span(\theta_1^{(1)}),$$

the three matrices $\Theta_{(2,0,0)}, \Theta_{(1,1,0)}$ and $\Theta_{(1,0,1)}$ satisfy condition (43). We can arbitrarily chose three matrices defined by (45)-(46) as follows:

$$Ker(C_2^{(\sigma)}) = Span(e_1, e_2^{(\sigma)}),$$
$$Ker(C_2^{(\sigma')}) = Span(e_1, e_2^{(\sigma')}),$$

$$Ker(C_2^{(\sigma'')}) = Span(e_1, e_2^{(\sigma'')}),$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix},$$
$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix},$$
$$\sigma'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \end{pmatrix},$$

or equivalently with

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad e_2^{(\sigma)} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2^{(\sigma')} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2^{(\sigma'')} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then we define the matrix P by (47) such that

$$P\theta_1^{(1)} = e_1, \quad P\theta_1^{(2)} = e_2^{(\sigma)}, \quad P\theta_2^{(1)} = e_2^{(\sigma')}, \quad P\theta_3^{(1)} = e_2^{(\sigma'')}.$$

By Proposition 3.3, we have

$$\theta_{(2,0,0)} = C_2^{(\sigma)}P, \qquad \theta_{(1,1,0)} = C_2^{(\sigma')}P, \qquad \theta_{(1,0,1)} = C_2^{(\sigma'')}P.$$

By Theorem 3.1, system (1) is exactly synchronized by 2-groups under the control matrix D with respect to the following three groupings:

$$\begin{aligned} &\sigma\text{-partition:} \quad u_1 = u_3 = u_5, \quad u_2 = u_4 = u_6, \\ &\sigma'\text{-partition:} \quad u_1 = u_2 = u_3, \quad u_4 = u_5 = u_6, \\ &\sigma''\text{-partition:} \quad u_1 = u_3 = u_6, \quad u_2 = u_4 = u_5. \end{aligned}$$

The above choice of basis is only an example, there are many other amusing groupings with large N.

Remark 3.2. *For clarity, we have only considered the internal controllability of Dirichlet problem (1). Obviously, the approach can be applied to the internal controllability of Neumann problem.*

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