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A class of IMEX schemes and their error analysis for the Navier-Stokes Cahn-Hilliard system

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Dedicated to the Memory of Professor Roland Glowinski

We construct a class of implicit-explicit (IMEX) schemes for the Navier-Stokes Cahn-Hilliard (NSCH) system and carry out a rigorous error analysis for both semi-discrete and fully discrete (with a Fourier spectral approximation in space) schemes in the space periodic case. The schemes are based on the consistent splitting approach for the Navier-Stokes equations to decouple the computation of velocity and pressure, and the generalized scalar auxiliary variable (GSAV) approach to provide uniform bound for the numerical solutions. Our IMEX schemes are fully decoupled and linear, only requiring to solve a sequence of Poisson type equations at each time step. With help of the uniform bound for the numerical solutions, we derive global error estimates in the two dimensional case as well as local error estimates in the three dimensional case for temporal orders one to five. We also present some numerical examples to validate the schemes.

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1. Introduction

We consider in this paper the construction and error analysis of efficient high-order numerical schemes for the following Navier-Stokes Cahn-Hilliard (NSCH) system:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \nabla \phi = M \Delta \mu, \tag{1.1a}$$

$$\mu = -\lambda(\Delta \phi - f(\phi)), \tag{1.1b}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v} + \gamma \mu \nabla \phi, \tag{1.1c}$$

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$$\nabla \cdot \boldsymbol{v} = 0, \tag{1.1d}$$

with suitable initial conditions and boundary conditions. The above NSCH system is a phase-field model for a two-phase incompressible and immiscible fluid flow [2, 3, 21, 22, 25, 31]. The unknowns are the phase function ϕ, the velocity $\boldsymbol{v}$ and the pressure p, which is assumed to have zero mean for uniqueness, the chemical potential μ is a function of ϕ. Here, $f = \frac{\delta F}{\delta \phi}$ with $F(\phi) = \frac{1}{4\varepsilon^2}(1-\phi^2)^2$ where ε is the interfacial width, $M > 0$ is the mobility constant, $\lambda > 0$ is the mixing coefficient, $\nu > 0$ is the fluid viscosity and $\gamma > 0$ corresponds to the surface tension. The above system, with suitable boundary conditions, satisfies an energy dissipation law:

$$\frac{dE(\phi, \boldsymbol{v})}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\boldsymbol{v}\|^2 \le 0, \tag{1.2}$$

where the total energy $E(\phi, \boldsymbol{v})$ is

$$E(\phi, \boldsymbol{v}) = \int_\Omega \Big\{\frac{\lambda}{2}|\nabla\phi|^2 + \lambda F(\phi) + \frac{1}{2}|\boldsymbol{v}|^2\Big\} d\boldsymbol{x}. \tag{1.3}$$

Numerical issues in dealing with the NSCH system (1.1) include in particular the challenges inherited from the Navier-Stokes equations and Cahn-Hilliard equation. For the Navier-Stokes equations, the main difficulties are associated with the nonlinear term and the incompressibility constraint. An effective way for solving Navier-Stokes equation is to adopt an operator splitting (or fractional step) approach [26] for which two options are generally used. The first option is the projection type methods [10, 38] which decouple the computations of velocity and pressure: it is very efficient as one only needs to solve a sequence of Poisson type equations at each time step, the drawback is that most of these schemes, with the exception of consistent splitting schemes [20, 27], suffer from the splitting error which limits the accuracy of the projection type schemes [19]. (ii) The second option is advocated by R. Glowinski [6, 7, 18] and seeks to separate the difficulties due to the nonlinear term and the incompressibility constraint, i.e., in the first substep, it solves a linearized or nonlinear elliptic equation without the incompressibility constraint, while in the second substep, it solves a (generalized) Stokes system without the nonlinear term. This option is robust and does not suffer the splitting error as in projection type schemes, but it requires solving a non-definite Stokes system at each time step. For the Cahn-Hilliard equation, the main difficulty is to deal with the nonlinear stiffness due to the interfacial width parameter ε. A simple explicit treatment of the nonlinear term leads to severe time step constraints while an implicit treatment of the nonlinear term requires solving a nonlinear system, whose well-posedness also requires a time step constraint, at each time step. We refer to [4, 12], and the references therein, for a recent review on various numerical methods for Cahn-Hilliard equation and related nonlinear systems.

To design efficient numerical schemes for the NSCH system (1.1), in addition to deal with the above difficulties associated with the Navier-Stokes equations and Cahn-Hilliard equation, we also need to treat the nonlinear coupling terms in (1.1a) and (1.1c) in a way such that the resulting schemes are easy to implement while the energy dissipation law (1.3) is somewhat respected in a discrete sense.

The well-posedness of the NSCH system (1.1) has been well established, see [5, 16], also see [17] for (1.1) with logarithmic potential. The NSCH system (1.1) has been widely used in numerical simulation of two phase flows, there are also a few rigorous error analysis for numerical approximations of (1.1). For examples, in [11, 13, 14, 28], the authors established convergence and/or error estimates for some coupled nonlinear schemes based on finite-element or finite-difference for (1.1); more recently in [8, 9, 30, 41], the authors established error estimates for some fully decoupled linear schemes for (1.1). All results in these papers are restricted to first- or second-order schemes.

The main goals of this paper are two-folds: (i) to develop a class of fully decoupled IMEX schemes based on the consistent splitting approach for the Navier-Stokes equations [20, 27], which allows us to decouple the computation of velocity and pressure while being free of splitting errors, and a generalized scalar auxiliary variable (GSAV) approach [23, 24], which enables us to show that the numerical solutions are unconditionally stable for any order; (ii) to derive rigorously optimal error estimates for the schemes up to fifth-order in the space periodic case.

Our IMEX schemes are fully decoupled and linear, only requiring to solve a sequence of Poisson type equations at each time step. In particular, in the case of periodic boundary conditions with a Fourier approximation in space, the Poisson type equations lead to diagonal systems in the frequency space so these schemes are extremely efficient. Moreover, the GSAV approach provides a unconditionally uniform bound for the numerical solutions which plays an essential role in our error analysis. Our uniform error analysis for temporal orders of one to five, to the best of our knowledge, provides the first rigorous error estimates for the NSCH system (1.1) with temporal order greater than two. While the general process of the error analysis is similar to that in [23, 24] for the gradient flows and for Navier-Stokes equations separately, it is much more difficult as we have to deal with the additional difficulties caused by the nonlinear coupling between the Cahn-Hillard part and Navier-Stokes part in (1.1).

The rest of the paper is organized as follows. In the next section, we provide some preliminaries to be used in the sequel. In Section 3, we describe our semi-discrete and fully discrete with Fourier-Galerkin SAV schemes for the Cahn-Hilliard-Navier-Stokes system with periodic boundary conditions, prove its unconditionally stability. In section 4, we present detailed error analysis for the kth-order schemes ($k = 1, 2, 3, 4, 5$) in a unified form. In section 5, we provide numerical examples to demonstrate the convergence rates and validate the accuracy of our schemes.

2. Preliminaries

We first introduce some notations. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm in $L^2(\Omega)$, where $\Omega\subset\mathbb{R}^d (d=2,3)$ is a rectangular domain with periodic boundary conditions, and denote

$$\boldsymbol{H}_p^k(\Omega)=\{u^{(j)}\,(j=0,1,\cdots,k)\in L^2(\Omega): u^{(j)}\,\text{periodic}\,(j=0,1,\cdots,k{-}1)\},$$

with norm $\|\cdot\|_k$. For non-integer $s>0$, $H_p^s(\Omega)$ and the corresponding norm $\|\cdot\|_s$ are defined by space interpolation [1]. In particular, we set $H_p^0(\Omega)=L^2(\Omega)$.

Let V be a Banach space, we shall also use the standard notations $L^p(0,T;V)$ and $C([0,T];V)$. To simplify the notation, we often omit the spatial dependence for the exact solution u, i.e., $u(x,t)$ is often denoted by $u(t)$. We shall use bold faced letters to denote vectors and vector spaces, and use C to denote a generic positive constant independent of the discretization parameters.

We now define the following spaces which are particularly used in the study of Navier-Stokes equations:

$$\mathbf{H}=\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega):\nabla\cdot\boldsymbol{v}=0\},\quad \mathbf{V}=\{\boldsymbol{v}\in\boldsymbol{H}_p^1(\Omega):\nabla\cdot\boldsymbol{v}=0\}.$$

Let $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)$, we define $w:=\Delta^{-1}\boldsymbol{v}$ as the solution of

$$\Delta\boldsymbol{w}=\boldsymbol{v}\quad \boldsymbol{x}\in\Omega;\quad \boldsymbol{w}\text{ periodic with zero mean.}$$

Note that in the periodic case, we can define the operators ∇, $\nabla\cdot$ and Δ^{-1} in the Fourier space by expanding functions and their derivatives in Fourier series, and one can easily show that these operators commute with each other.

We define a linear operator $\mathbf{A}$ in $\boldsymbol{L}^2(\Omega)$ by

$$\mathbf{A}\boldsymbol{v} := \nabla\times\nabla\times\Delta^{-1}\boldsymbol{v}, \quad \forall \boldsymbol{v}\in \boldsymbol{L}^2(\Omega). \tag{2.1}$$

Since

$$\|\Delta\boldsymbol{w}\|^2 = \|\nabla\times\nabla\times\boldsymbol{w}\|^2 + \|\nabla\nabla\cdot\boldsymbol{w}\|^2, \quad \forall\boldsymbol{w}\in\boldsymbol{H}_p^2(\Omega),$$

we derive immediately from the above that

$$\|\mathbf{A}\boldsymbol{v}\|^2 = \|\Delta\Delta^{-1}\boldsymbol{v}\|^2 - \|\nabla\nabla\cdot\Delta^{-1}\boldsymbol{v}\|^2 \le \|\boldsymbol{v}\|^2,\ \forall\boldsymbol{v}\in\boldsymbol{L}^2(\Omega). \tag{2.2}$$

Next, we define the trilinear form $b(\cdot,\cdot,\cdot)$ and $b_{\mathbf{A}}(\cdot,\cdot,\cdot)$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}d\boldsymbol{x},\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \mathbf{A}((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{w}d\boldsymbol{x}.$$

In particular, we have

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),\ \forall\boldsymbol{u}\in\mathbf{H},\ \boldsymbol{v},\boldsymbol{w}\in\boldsymbol{H}_p^1(\Omega),$$

which implies

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0,\ \forall\boldsymbol{u}\in\mathbf{H},\ \boldsymbol{v}\in\boldsymbol{H}_p^1(\Omega). \tag{2.3}$$

Using (2.2), Hölder inequality and Sobolev inequality, we have [39]

(2.4a)
$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le c\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|^{1/2}\|\mathbf{v}\|_1^{1/2}\|\mathbf{w}\|_1, \quad d=2,$$
(2.4b)
$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le c\|\boldsymbol{u}\|_1\|\nabla\boldsymbol{v}\|_{1/2}\|\boldsymbol{w}\|, \quad d=3,$$
(2.4c)
$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1^{1/2}\|\boldsymbol{v}\|_2^{1/2}\|\boldsymbol{w}\|, \quad d=3.$$

We also use frequently the following inequalities [39]:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\ b_{\mathbf{A}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le \begin{cases} c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1; \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_0\|\boldsymbol{w}\|_1; \\ c\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_0; \\ c\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_0; \\ c\|\boldsymbol{u}\|_0\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|_1; \end{cases} \quad d\le 4. \tag{2.5}$$

3. The SAV schemes and stability results

In this section, we construct semi-discrete and fully discrete SAV schemes for the Navier-Stokes Cahn-Hilliard system (1.1) with periodic boundary conditions, and establish stability results for both semi-discrete and fully discrete schemes.

3.1. The SAV schemes

Following the ideas in [24] for the general dissipative systems and in [23] for dealing with Navier-Stokes equations with periodic boundary conditions, we construct below unconditionally energy stable schemes for (1.1) with periodic boundary conditions. These schemes can be easily extended to the case of non-periodic boundary conditions using a similar formulation as in [40].

For Navier-Stokes equations with periodic boundary conditions, we can explicitly eliminate the pressure from (1.1c). Indeed, taking the divergence on both sides of (1.1c), we find

$$-\Delta p = \nabla \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \gamma\mu\nabla\phi), \tag{3.1}$$

from which we derive

$$\begin{aligned}\nabla p &= \nabla\Delta^{-1}\Delta p = -\nabla\Delta^{-1}\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi)\\ &= -\nabla\nabla\cdot\Delta^{-1}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi)\\ &= -(\Delta + \nabla\times\nabla\times)\Delta^{-1}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi)\\ &= -\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \gamma\mu\nabla\phi - \nabla\times\nabla\times\Delta^{-1}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi)\\ &= -\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \gamma\mu\nabla\phi - \mathbf{A}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi),\end{aligned} \tag{3.2}$$

where $\mathbf{A}$ is defined in (2.1). Hence, (1.1c) is equivalent to (3.1) and

$$\frac{\partial \boldsymbol{v}}{\partial t} - \nu\Delta\boldsymbol{v} - \mathbf{A}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi) = 0. \tag{3.3}$$

In order to apply the SAV approach, we introduce a SAV,

$$r(t) = E(\phi(t), \boldsymbol{v}(t)) + 1,$$

and expand (1.1) as

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v}\cdot\nabla\phi = M\Delta\mu, \tag{3.4a}$$

$$\mu = -\lambda(\Delta\phi - f(\phi)), \tag{3.4b}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} - \nu\Delta\boldsymbol{v} - \mathbf{A}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \gamma\mu\nabla\phi) = 0, \tag{3.4c}$$

$$\frac{dr}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\boldsymbol{v}\|^2. \tag{3.4d}$$

We construct below semi-discrete and fully discrete schemes for the expanded system (3.4).

3.1.1. Semi-discrete SAV schemes. We consider first the time discretization of (3.4) based on the implicit-explicit BDF-k formulae in the following unified form:

Given r^n, $\phi^j, \boldsymbol{v}^j$ $(j = n, n-1, \cdots, n-k+1)$, we compute $\bar{\phi}^{n+1}$, $\bar{\boldsymbol{v}}^{n+1}$, r^{n+1}, p^{n+1}, ξ^{n+1}, ϕ^{n+1} and $\boldsymbol{v}^{n+1}$ consecutively by

(3.5a)
$$\frac{\alpha_k\bar{\phi}^{n+1} - A_k(\bar{\phi}^n)}{\delta t} + B_k(\boldsymbol{v}^n)\cdot\nabla B_k(\phi^n) = M\Delta\mu^{n+1},$$

(3.5b)
$$\mu^{n+1} = -\lambda(\Delta\bar{\phi}^{n+1} - f(B_k(\phi^n))),$$

(3.5c)
$$\frac{\alpha_k\bar{\boldsymbol{v}}^{n+1} - A_k(\bar{\boldsymbol{v}}^n)}{\delta t} - \nu\Delta\bar{\boldsymbol{v}}^{n+1} - \mathbf{A}(B_k(\boldsymbol{v}^n)\cdot\nabla B_k(\boldsymbol{v}^n) - \gamma B_k(\mu^n)\nabla B_k(\phi^n)) = 0,$$

(3.5d)
$$\frac{r^{n+1} - r^n}{\delta t} = -\frac{r^{n+1}}{E(\bar{\phi}^{n+1}, \bar{\boldsymbol{v}}^{n+1}) + 1}(M\|\nabla\mu^{n+1}\|^2 + \nu\|\nabla\bar{\boldsymbol{v}}^{n+1}\|^2),$$

(3.5e)
$$\xi^{n+1} = \frac{r^{n+1}}{E(\bar{\phi}^{n+1}, \bar{\boldsymbol{v}}^{n+1}) + 1}, \quad \eta^{n+1} = 1 - (1 - \xi^{n+1})^k,$$

(3.5f)
$$\phi^{n+1} = \eta^{n+1}\bar{\phi}^{n+1}, \quad \boldsymbol{v}^{n+1} = \eta^{n+1}\bar{\boldsymbol{v}}^{n+1}.$$

Whenever pressure is needed, it can be computed from

$$\Delta p^{n+1} = -\nabla\cdot(\boldsymbol{v}^{n+1}\cdot\nabla\boldsymbol{v}^{n+1} - \gamma\mu^{n+1}\nabla\phi^{n+1}). \tag{3.6}$$

In the above, α_k, the operators A_k and B_k $(k = 1, 2, 3, 4)$ are given by:

- first-order:

$$\alpha_1 = 1, \quad A_1(\phi^n) = \phi^n, \quad B_1(\phi^n) = \phi^n; \tag{3.7}$$

- second-order:

$$\alpha_2 = \frac{3}{2}, \quad A_2(\phi^n) = 2\phi^n - \frac{1}{2}\phi^{n-1}, \quad B_2(\phi^n) = 2\phi^n - \phi^{n-1}; \tag{3.8}$$

- third-order:

$$\begin{aligned} &\alpha_3 = \frac{11}{6}, \quad A_3(\phi^n) = 3\phi^n - \frac{3}{2}\phi^{n-1} + \frac{1}{3}\phi^{n-2}, \\ &B_3(\phi^n) = 3\phi^n - 3\phi^{n-1} + \phi^{n-2}. \end{aligned} \tag{3.9}$$

The formulae for $k = 4, 5, 6$ can also be readily derived by Taylor expansion.

Several remarks are in order:

- We observe from (3.5d) that r^{n+1} is a first-order approximation to $E(\phi(\cdot, t^{n+1}), \boldsymbol{v}(\cdot, t^{n+1})) + 1$ which implies that ξ^{n+1} is a first-order approximation to 1.
- (3.5a)-(3.5c) are kth-order approximations to (3.4a)-(3.4c) with kth-order BDF for the linear terms and kth-order Adams-Bashforth extrapolation for the nonlinear terms. Hence, $\bar{\phi}^{n+1}$ and $\bar{\boldsymbol{v}}^{n+1}$ are kth-order approximation to $\phi(\cdot, t^{n+1})$ and $\boldsymbol{v}(\cdot, t^{n+1})$ which, along with (3.5e) and (3.5f), implies that ϕ^{n+1} and $\boldsymbol{v}^{n+1}$ are kth-order approximations to $\phi(\cdot, t^{n+1})$ and $\boldsymbol{v}(\cdot, t^{n+1})$.
- The main computational cost is to solve the Poisson type equation (3.5a)- (3.5c).

3.1.2. Fully discrete schemes with the Fourier spectral method in space. We now consider $\Omega = [0, L_x) \times [0, L_y) \times [0, L_z)$ with periodic boundary conditions. We partition the domain $\Omega = (0, L_x) \times (0, L_y) \times (0, L_z)$ uniformly with size $h_x = L_x/N_x, h_y = L_y/N_y, h_z = L_z/N_z$ and Nx, Ny, Nz are positive even integers. Then the Fourier approximation space can be defined as

$$\begin{aligned} &S_N := \\ &\text{span}\{e^{i\xi_j x} e^{i\eta_k y} e^{i\tau_l z} : -\frac{N_x}{2} \le j \le \frac{N_x}{2} - 1, -\frac{N_y}{2} \le k \le \frac{N_y}{2} - 1, \\ &\qquad -\frac{N_z}{2} \le l \le \frac{N_z}{2} - 1\} \backslash \mathbb{R}, \end{aligned}$$

where $i = \sqrt{-1}$, $\xi_j = 2\pi j/L_x$, $\eta_k = 2\pi k/L_y$ and $\tau_l = 2\pi l/L_z$. Then, any function $u(x,y,z) \in L^2(\Omega)$ can be approximated by:

$$u(x,y,z) \approx u_N(x,y,z) = \sum_{j=-\frac{N_x}{2}}^{\frac{N_x}{2}-1} \sum_{k=-\frac{N_y}{2}}^{\frac{N_y}{2}-1} \sum_{l=-\frac{N_z}{2}}^{\frac{N_z}{2}-1} \hat{u}_{j,k,l} e^{i\xi_j x} e^{i\eta_k y} e^{i\tau_l z},$$

with the Fourier coefficients defined as

$$\hat{u}_{j,k,l} = \frac{1}{|\Omega|} \int_\Omega u e^{-i(\xi_j x + \eta_k y + \tau_l z)} d\boldsymbol{x}.$$

In the following, we fix $N_x = N_y = N_z = N$ for simplicity, and also set $\boldsymbol{S}_N = S_N^d$.

Define the L^2-orthogonal projection operator $\Pi_N : L^2(\Omega) \to S_N$ by

$$(\Pi_N u - u, \Psi) = 0, \quad \forall\, \Psi \in S_N, \quad u \in L^2(\Omega), \tag{3.10}$$

then we have the following approximation results (cf. [29]):

Lemma 1. *For any $0 \le k \le m$, there exists a constant C such that*

$$\|\Pi_N u - u\|_k \le C \|u\|_m N^{k-m}, \forall\, u \in \boldsymbol{H}_p^m(\Omega). \tag{3.11}$$

We are now ready to describe our fully discrete schemes. Given r^n, $\phi_N^j, \boldsymbol{v}_N^j$ $(j = n, n-1, \cdots, n-k+1)$, we compute $\bar{\phi}_N^{n+1}$, $\bar{\boldsymbol{v}}_N^{n+1}$, r^{n+1}, p_N^{n+1}, ξ^{n+1}, ϕ_N^{n+1} and $\boldsymbol{v}_N^{n+1}$ consecutively by

(3.12a)
$$\Big(\frac{\alpha_k \bar{\phi}_N^{n+1} - A_k(\bar{\phi}_N^n)}{\delta t}, \psi_N\Big) + \big(B_k(\boldsymbol{v}_N^n) \cdot \nabla B_k(\phi_N^n), \psi_N\big) = M\big(\Delta \mu_N^{n+1}, \psi_N\big), \quad \forall \psi_N \in S_N,$$

(3.12b)
$$\big(\mu_N^{n+1}, \psi_N\big) = -\lambda\big(\Delta \bar{\phi}_N^{n+1} - f(B_k(\phi_N^n)), \psi_N\big), \quad \forall \psi_N \in S_N,$$

(3.12c)
$$\Big(\frac{\alpha_k \bar{\boldsymbol{v}}_N^{n+1} - A_k(\bar{\boldsymbol{v}}_N^n)}{\delta t}, \boldsymbol{\psi}_N\Big) - \nu\big(\Delta \bar{\boldsymbol{v}}_N^{n+1}, \boldsymbol{\psi}_N\big) - \big(\mathbf{A}(B_k(\boldsymbol{v}_N^n) \cdot \nabla B_k(\boldsymbol{v}_N^n) - \gamma B_k(\mu_N^n) \nabla B_k(\phi_N^n)), \boldsymbol{\psi}_N\big) = 0, \quad \forall \boldsymbol{\psi}_N \in \boldsymbol{S}_N,$$

(3.12d)
$$\frac{r^{n+1} - r^n}{\delta t} = -\frac{r^{n+1}}{E(\bar{\phi}_N^{n+1}, \bar{\boldsymbol{v}}_N^{n+1}) + 1} (M \|\nabla \mu_N^{n+1}\|^2 + \nu \|\nabla \bar{\boldsymbol{v}}_N^{n+1}\|^2),$$

(3.12e)
$$\xi^{n+1} = \frac{r^{n+1}}{E(\bar{\phi}_N^{n+1}, \bar{\boldsymbol{v}}_N^{n+1}) + 1}, \quad \eta^{n+1} = 1 - (1 - \xi^{n+1})^k,$$
(3.12f)
$$\phi_N^{n+1} = \eta^{n+1} \bar{\phi}_N^{n+1}, \quad \boldsymbol{v}_N^{n+1} = \eta^{n+1} \bar{\boldsymbol{v}}_N^{n+1}.$$

Note that Fourier approximation of Poisson type equations leads to diagonal matrix in the frequency space, so the above scheme can be efficiently implemented as follows:

(i) Compute $\bar{\phi}_N^{n+1}$ and μ_N^{n+1} from (3.12a) and (3.12b), $\bar{\boldsymbol{v}}_N^{n+1}$ from (3.12c), which are Poisson-type equations;
(ii) With $\bar{\phi}_N^{n+1}$, μ_N^{n+1} and $\bar{\boldsymbol{v}}_N^{n+1}$ known, determine r^{n+1} explicitly from (3.12d);
(iii) Compute ξ^{n+1} and η_k^{n+1} from (3.12e);
(iv) Update ϕ_N^{n+1} and $\boldsymbol{v}_N^{n+1}$ from (3.12f), go to the next step.

Finally, whenever pressure is needed, it can be computed from

$$\Delta p_N^{n+1} = -\Pi_N \nabla \cdot (\boldsymbol{v}_N^{n+1} \cdot \nabla \boldsymbol{v}_N^{n+1} - \gamma \mu_N^{n+1} \nabla \phi^{n+1}). \tag{3.13}$$

3.2. Stability results

We have the following results concerning the stability of the above schemes.

Theorem 1. *Let $\{r^k, \xi^k, \bar{\phi}_N^k, \phi_N^k, \bar{\boldsymbol{v}}_N^k, \boldsymbol{v}_N^k\}$ be the solution of the fully discrete scheme (3.12). Then, given $r^n \geq 0$, we have $r^{n+1} \geq 0$, $\xi^{n+1} \geq 0$, and for any k, the scheme (3.12) is unconditionally energy stable in the sense that*

$$r^{n+1} - r^n = -\delta t \xi^{n+1} (M \|\nabla \mu_N^{n+1}\|^2 + \nu \|\nabla \bar{\boldsymbol{v}}_N^{n+1}\|^2) \leq 0, \quad \forall n. \tag{3.14}$$

Furthermore, there exists $M_k > 0$ such that

$$\|\nabla \phi_N^{n+1}\|^2 \leq \frac{M_k^2}{\lambda^2}, \; \|\boldsymbol{v}_N^{n+1}\|^2 \leq M_k^2, \quad \forall n. \tag{3.15}$$

It is clear that same results hold for the semi-discrete schemes (3.5).

Proof. Since the proofs for the fully discrete scheme (3.12) and for the semi-discrete scheme (3.5) are essentially the same, we shall only give the proof for the fully discrete scheme (3.12) below.

Given $r^n \geq 0$. Since $E(\bar{\phi}_N^{n+1}, \bar{\boldsymbol{v}}_N^{n+1}) + 1 > 0$, it follows from (3.12d) that

$$r^{n+1} = \frac{r^n}{1 + \delta t \frac{M\|\nabla \mu_N^{n+1}\|^2 + \nu \|\nabla \bar{\boldsymbol{v}}_N^{n+1}\|^2}{E(\bar{\phi}_N^{n+1}, \bar{\boldsymbol{v}}_N^{n+1}) + 1}} \geq 0. \tag{3.16}$$

Then we derive from (3.12e) that $\xi^{n+1} \geq 0$ and obtain (3.14).

Denote $M_r := r^0 = E[\phi(\cdot, 0), \boldsymbol{v}(\cdot, 0)]$, then (3.14) implies $r^n \leq M_r$, $\forall n$. It then follows from (3.12e) that

$$|\xi^{n+1}| = \frac{r^{n+1}}{E(\bar{\phi}_N^{n+1}, \bar{\boldsymbol{v}}_N^{n+1}) + 1} \leq \frac{2M_r}{\lambda \|\nabla \bar{\phi}_N^{n+1}\|^2 + \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 + 2}. \tag{3.17}$$

Since $\eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k$, we have $\eta_k^{n+1} = \xi^{n+1} P_k(\xi^{n+1})$ with P_k being a polynomial of degree $k - 1$. Then, we derive from (3.17) that there exists $M_k > 0$ such that

$$|\eta_k^{n+1}| = |\xi^{n+1} P_k(\xi^{n+1})| \leq \frac{M_k}{\lambda \|\nabla \bar{\phi}_N^{n+1}\|^2 + \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 + 2}, \tag{3.18}$$

which, along with $\phi^{n+1} = \eta_k^{n+1} \bar{\phi}^{n+1}$ and $\boldsymbol{v}_N^{n+1} = \eta_k^{n+1} \bar{\boldsymbol{v}}_N^{n+1}$, implies

$$\begin{aligned} \|\nabla \phi_N^{n+1}\|^2 &= (\eta_k^{n+1})^2 \|\nabla \bar{\phi}_N^{n+1}\|^2 \\ &\leq \Big(\frac{M_k}{\lambda \|\nabla \bar{\phi}_N^{n+1}\|^2 + \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 + 2}\Big)^2 \|\nabla \bar{\phi}_N^{n+1}\|^2 \leq \frac{M_k^2}{\lambda^2}, \end{aligned} \tag{3.19}$$

and

$$\|\boldsymbol{v}_N^{n+1}\|^2 = (\eta_k^{n+1})^2 \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 \leq \Big(\frac{M_k}{\lambda \|\nabla \bar{\phi}_N^{n+1}\|^2 + \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 + 2}\Big)^2 \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 \leq M_k^2. \tag{3.20}$$

The proof is complete. □

We note the above results would hold for corresponding schemes with non-periodic boundary conditions, see [40] for the case of Navier-Stokes equations with no-slip boundary conditions.

4. Error analysis

In this section, we carry out a unified error analysis for the fully discrete schemes (3.12) with $1 \leq k \leq 5$, and state, as corollaries, similar results for the semi-discrete schemes (3.5). We denote

$$\begin{aligned}
&t^n = n\,\delta t, \quad s^n = r^n - r(t^n),\\
&\bar{e}^n_{\phi,N}{=}\bar{\phi}^n_N - \Pi_N\phi(\cdot,t^n),\ e^n_{\phi,N}{=}\phi^n_N - \Pi_N\phi(\cdot,t^n),\ e^n_{\phi,\Pi}{=}\Pi_N\phi(\cdot,t^n) - \phi(\cdot,t^n),\\
&\bar{\boldsymbol{e}}^n_{\boldsymbol{v},N}{=}\bar{\boldsymbol{v}}^n_N - \Pi_N\boldsymbol{v}(\cdot,t^n),\ \boldsymbol{e}^n_{\boldsymbol{v},N}{=}\boldsymbol{v}^n_N - \Pi_N\boldsymbol{v}(\cdot,t^n),\ \boldsymbol{e}^n_{\boldsymbol{v},\Pi}{=}\Pi_N\boldsymbol{v}(\cdot,t^n) - \boldsymbol{v}(\cdot,t^n),\\
&e^n_{\mu,N} = \mu^n_N - \Pi_N\mu(\cdot,t^n),\ e^n_{\mu,\Pi} = \Pi_N\mu(\cdot,t^n) - \mu(\cdot,t^n),\ e^n_\mu = e^n_{\mu,N} + e^n_{\mu,\Pi},\\
&\bar{e}^n_\phi = \bar{\phi}^n_N - \phi(\cdot,t^n) = \bar{e}^n_{\phi,N} + e^n_{\phi,\Pi},\ e^n_\phi = \phi^n_N - \phi(\cdot,t^n) = e^n_{\phi,N} + e^n_{\phi,\Pi},\\
&\bar{\boldsymbol{e}}^n_{\boldsymbol{v}} = \bar{\boldsymbol{v}}^n_N - \boldsymbol{v}(\cdot,t^n) = \bar{\boldsymbol{e}}^n_{\boldsymbol{v},N} + \boldsymbol{e}^n_{\boldsymbol{v},\Pi},\ \boldsymbol{e}^n_{\boldsymbol{v}} = \boldsymbol{v}^n_N - \boldsymbol{v}(\cdot,t^n) = \boldsymbol{e}^n_{\boldsymbol{v},N} + \boldsymbol{e}^n_{\boldsymbol{v},\Pi}.
\end{aligned}$$

To simplify the notations, we dropped the dependence on N for $\bar{e}^n_\phi$, e^n_ϕ, $\bar{\boldsymbol{e}}^n_{\boldsymbol{v}}$ and $\boldsymbol{e}^n_{\boldsymbol{v}}$ in the above, and will do so for some other quantities in the sequel. We also assume the positive constants $M = \gamma = \nu = \lambda = 1$ in (3.12) for simplicity.

4.1. Several useful lemmas

We will frequently use the following two discrete versions of the Gronwall lemma.

Lemma 2. ***(Discrete Gronwall Lemma 1 [36])*** *Let y^k, h^k, g^k, f^k be four nonnegative sequences satisfying*

$$y^n + \delta t\sum_{k=0}^{n} h^k \le B + \delta t\sum_{k=0}^{n}(g^k y^k + f^k) \ \ \textit{with}\ \ \delta t\sum_{k=0}^{T/\delta t} g^k \le M,\ \forall\, 0 \le n \le T/\delta t.$$

We assume $\delta t\, g^k < 1$ for all k, and let $\sigma = \max_{0\le k\le T/\delta t}(1-\delta t g^k)^{-1}$. Then

$$y^n + \delta t\sum_{k=1}^{n} h^k \le \exp(\sigma M)(B + \delta t\sum_{k=0}^{n} f^k),\ \forall\, n \le T/\delta t.$$

Lemma 3. ***(Discrete Gronwall Lemma 2 [35])*** *Let ,a_n, b_n, c_n, and d_n be four nonnegative sequences satisfying*

$$a_m + \tau\sum_{n=1}^{m} b_n \le \tau\sum_{n=0}^{m-1} a_n d_n + \tau\sum_{n=0}^{m-1} c_n + C,\ m \ge 1,$$

where C and τ are two positive constants. Then

$$a_m + \tau\sum_{n=1}^{m} b_n \le \exp\big(\tau\sum_{n=0}^{m-1} d_n\big)\big(\tau\sum_{n=0}^{m-1} c_n + C\big),\ m \ge 1.$$

Based on Dahlquist's G-stability theory, Nevanlinna and Odeh [33] proved the following result which plays an essential role in our error analysis.

Lemma 4. *For $1 \leq k \leq 5$, there exist $0 \leq \tau_k < 1$, a positive definite symmetric matrix $G = (g_{ij}) \in \mathcal{R}^{k,k}$ and real numbers $\delta_0, ..., \delta_k$ such that*

$$\Big(\alpha_k u^{n+1} - A_k(u^n), u^{n+1} - \tau_k u^n\Big) = \sum_{i,j=1}^{k} g_{ij}(u^{n+1+i-k}, u^{n+1+j-k}) - \sum_{i,j=1}^{k} g_{ij}(u^{n+i-k}, u^{n+j-k}) + \|\sum_{i=0}^{k} \delta_i u^{n+1+i-k}\|^2,$$

where the smallest possible values of τ_k are

$$\tau_1 = \tau_2 = 0, \quad \tau_3 = 0.0836, \quad \tau_4 = 0.2878, \quad \tau_5 = 0.8160.$$

We also recall the following lemma [32] which will be used to prove local error estimates in the three-dimensional case.

Lemma 5. *Let $\Phi : (0, \infty) \to (0, \infty)$ be continuous and increasing, and let $M > 0$. Given T_* such that $0 < T_* < \int_M^\infty dz/\Phi(z)$, there exists $C_* > 0$ independent of $\delta t > 0$ with the following property. Suppose that quantities z_n, $w_n \geq 0$ satisfy*

$$z_n + \sum_{k=0}^{n-1} \delta t w_k \leq y_n := M + \sum_{k=0}^{n-1} \delta t \Phi(z_k), \, \forall n \leq n_*,$$

with $n_* \delta t \leq T_*$. Then $y_{n_*} \leq C_*$.

We also recall the following result (see Lemma 2.3 in [37]) which we shall use to deal with the nonlinear term.

Lemma 6. *Assume that $\|u\|_{H^1} \leq M_u$, and*

(4.1)
$$|g'(x)| < C(|x|^p + 1), \quad p > 0 \text{ arbitrary} \quad \text{if } n = 1, 2; \quad 0<p<4 \quad \text{if} \quad n=3.$$
$$|g''(x)| < C(|x|^{p'} + 1), \quad p' > 0 \text{ arbitrary} \quad \text{if } n = 1, 2; \quad 0<p'<3 \quad \text{if} \quad n=3.$$

Then for any $u \in H^4$*, there exist* $0 \le \sigma < 1$ *and a constant* $C(M_u)$ *such that the following inequality holds:*

$$\|\Delta g(u)\|^2 \le C(M_u)(1 + \|\Delta^2 u\|^{2\sigma}).$$

4.2. Error analysis for the phase function and the velocity in 2D

To carry out the error analysis, we need to assume that the solution has enough regularity. For the Navier-Stokes equations, it is shown in [39] that in the periodic case, $\boldsymbol{v}_0 \in \boldsymbol{H}_p^m$ implies that $\boldsymbol{v}(\cdot, t) \in \boldsymbol{H}_p^m$ for all $t \le T$, and furthermore, it is shown in [15] that $\boldsymbol{v}$ has Gevrey class regularity. Similar results hold for solutions of the Cahn-Hilliard equation [34]. It is therefore reasonable to expect that solutions of the NSCH system (1.1) should also have Gevrey class regularity, and we can assume that

$$\begin{aligned} &\boldsymbol{v} \in C([0,T]; \boldsymbol{H}_p^m),\ m \ge 4, \\ &\frac{\partial^j \boldsymbol{v}}{\partial t^j} \in L^2(0,T; \boldsymbol{H}_p^2)\ 1 \le j \le k,\ \frac{\partial^{k+1} \boldsymbol{v}}{\partial t^{k+1}} \in L^2(0,T; L_0^2), \end{aligned} \tag{4.2}$$

$$\phi \in C([0,T]; H^4),\ \frac{\partial^j \phi}{\partial t^j} \in L^2(0,T; H^2)\ 1 \le j \le k,\ \frac{\partial^{k+1} \phi}{\partial t^{k+1}} \in L^2(0,T; H^1). \tag{4.3}$$

Theorem 2. *Let* $d = 2$*,* $T > 0$*,* $\boldsymbol{v}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^m$*,* $\phi_0 \in \boldsymbol{H}_p^m$*,* $m \ge 4$ *and* ϕ*,* $\boldsymbol{v}$ *be the solution of* (1.1)*. We assume that* $\bar{\phi}_N^i$*,* ϕ_N^i*,* $\bar{\boldsymbol{v}}_N^i$ *and* $\boldsymbol{v}_N^i$ $(i = 1, \cdots, k-1)$ *are computed with a proper initialization procedure such that*

$$\begin{aligned} &\|\bar{\boldsymbol{v}}_N^i - \boldsymbol{v}(\cdot, t_i)\|,\ \|\boldsymbol{v}_N^i - \boldsymbol{v}(t_i)\| = O(\delta t^k + N^{-m}), \\ &\|\bar{\boldsymbol{v}}_N^i - \boldsymbol{v}(\cdot, t_i)\|_1, \|\boldsymbol{v}_N^i - \boldsymbol{v}(t_i)\|_1, \|\bar{\phi}_N^i - \phi(\cdot, t_i)\|_1, \|\phi_N^i - \phi(t_i)\|_1 = O(\delta t^k + N^{1-m}), \\ &\|\bar{\phi}_N^i - \phi(\cdot, t_i)\|_2,\ \|\phi_N^i - \phi(t_i)\|_2 = O(\delta t^k + N^{2-m}), \end{aligned} \tag{4.4}$$

for $i = 1, 2, 3, 4, 5$*. Let* $\bar{\phi}_N^{n+1}$*,* ϕ_N^{n+1}*,* $\bar{\boldsymbol{v}}_N^{n+1}$ *and* $\boldsymbol{v}_N^{n+1}$ *be computed with the* k*th-order scheme* (3.12) $(1 \le k \le 5)$*, and*

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k\ (k = 2, 3, 4, 5).$$

We assume (4.1)*,* (4.2) *and* (4.3)*. Then for* $n + 1 \le T/\delta t$ *with* $\delta t \le \frac{1}{1 + 2^{k+2} C_0^{k+1}}$ *and* $N \ge 2^{k+2} C_\Pi^{k+1} + 1$*, we have*

$$\|\bar{\phi}_N^n - \phi(\cdot, t^n)\|_2^2,\ \|\phi_N^n - \phi(\cdot, t^n)\|_2^2 \le C\delta t^{2k} + CN^{2(2-m)},$$

$$\|\bar{\boldsymbol{v}}_N^n - \boldsymbol{v}(\cdot, t^n)\|^2,\ \|\boldsymbol{v}_N^n - \boldsymbol{v}(\cdot, t^n)\|^2 \le C\delta t^{2k} + CN^{-2m},$$

and

$$\delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{v}}_N^{q+1} - \boldsymbol{v}(\cdot, t^{q+1})\|_1^2,\ \delta t \sum_{q=0}^{n} \|\boldsymbol{v}_N^{q+1} - \boldsymbol{v}(\cdot, t^{q+1})\|_1^2 \le C\delta t^{2k} + CN^{2(1-m)},$$

where the constants C_0, C_Π and C are dependent on T, Ω, the $k \times k$ matrix $G = (g_{ij})$ in Lemma 4 and the exact solution ϕ, $\boldsymbol{v}$, but are independent of δt and N.

Proof. To simplify the presentation, we assume $\bar{\phi}_N^i = \phi_N^i = \Pi_N \boldsymbol{\phi}(t_i)$, $\bar{\boldsymbol{v}}_N^i = \boldsymbol{v}_N^i = \Pi_N \boldsymbol{v}(t_i)$ and $r^i = E[\phi_N^i, \boldsymbol{v}_N^i] + 1$ for $i = 1, \cdots, k-1$ so that (4.4) is obviously satisfied.

The main task is to prove by induction,

$$|1 - \xi^q| \le C_0\, \delta t + C_\Pi N^{3-m},\ \forall q \le T/\delta t, \tag{4.5}$$

where the constant C_0 and C_Π will be defined in the induction process below. In the following, we shall use C to denote a constant which can change from one step to another and we may introduce $C_i > 0$, $i = 1, 2, ...$, to denote the upper bound for some specific terms while they are independent of $\delta t, N, C_0$ and C_Π.

Under the assumption, (4.5) certainly holds for $q = 0$. Now suppose we have

$$|1 - \xi^q| \le C_0\, \delta t + C_\Pi N^{3-m},\ \forall q \le n, \tag{4.6}$$

we shall prove below

$$|1 - \xi^{n+1}| \le C_0 \delta t + C_\Pi N^{3-m}. \tag{4.7}$$

We shall first consider $k = 2, 3, 4, 5$, and point out the necessary modifications for the case $k = 1$ later.

Step 1: Bounds for $\|\bar{\phi}^q\|_2$, $\|\phi^q\|_2$, $\|\bar{\boldsymbol{v}}_N^q\|$, $\|\bar{\boldsymbol{v}}_N^q\|_1$ and $\|\boldsymbol{v}_N^q\|_1$, $\forall q \le n$. We first recall the inequality

$$(a + b)^k \le 2^k (a^k + b^k), \quad \forall a, b > 0,\ k \ge 1. \tag{4.8}$$

Under the assumption (4.6), if we choose δt small enough and N large enough such that

$$\delta t \le \min\{\frac{1}{2^{k+2} C_0^k}, 1\}, \quad N \ge \max\{2^{k+2} C_\Pi^k, 1\}, \tag{4.9}$$

we have

(4.10)
$$1-(\frac{1}{2^{k+2}C_0^{k-1}}+\frac{N^{3-m}}{2^{k+2}C_\Pi^{k-1}})\le|\xi^q|\le 1+(\frac{1}{2^{k+2}C_0^{k-1}}+\frac{N^{3-m}}{2^{k+2}C_\Pi^{k-1}}),\ \forall q\le n,$$

and

$$(1-\xi^q)^k\le\frac{\delta t^{k-1}}{4}+\frac{N^{k(3-m)+1}}{4},\ \forall q\le n,$$

and

(4.11)
$$\frac{1}{2}\le 1-(\frac{\delta t^{k-1}}{4}+\frac{N^{k(3-m)+1}}{4})\le|\eta_k^q|\le 1+\frac{\delta t^{k-1}}{4}+\frac{N^{k(3-m)+1}}{4}<2,\ \forall q\le n.$$

Then it follows from the above and (3.15) that

$$\|\phi_N^q\|_1,\ \|\boldsymbol{v}_N^q\|\le M_k,\ \|\bar{\phi}_N^q\|_1,\ \|\bar{\boldsymbol{v}}_N^q\|\le 2M_k,\ \forall q\le n. \tag{4.12}$$

Moveover, (3.14), (4.10) and $m\ge 4$ imply that

$$\delta t\sum_{q=1}^n(\|\nabla\mu_N^q\|^2+\|\nabla\bar{\boldsymbol{v}}_N^q\|^2)\le 4r^0,\ \text{with}\quad C_0\ge 1,\ C_\Pi\ge 1, \tag{4.13}$$

and

$$\delta t\sum_{q=1}^n(\|\nabla\mu_N^q\|^2+\|\nabla\boldsymbol{v}_N^q\|^2)\le 16r^0,\ \text{with}\quad C_0\ge 1,\ C_\Pi\ge 1. \tag{4.14}$$

Consider (3.12) at step q and taking $\psi_N=\Delta^2\bar{\phi}_N^q-\tau_k\Delta^2\bar{\phi}_N^{q-1}$ in (3.12a), combining with (3.12b), it follows from Lemma 4 that there exist $0\le\tau_k<1$, a positive definite symmetric matrix $G=(g_{ij})\in\mathcal{R}^{k,k}$ and $\delta_0,...,\delta_k$ such that

(4.15)
$$\begin{aligned}&\frac{1}{\delta t}\Big(\sum_{i,j=1}^k g_{ij}(\Delta\bar{\phi}_N^{q+i-k},\Delta\bar{\phi}_N^{q+j-k})-\sum_{i,j=1}^k g_{ij}(\Delta\bar{\phi}_N^{q-1+i-k},\Delta\bar{\phi}_N^{q-1+j-k})\Big)\\&+\frac{1}{\delta t}\Big\|\sum_{i=0}^k\delta_i\Delta\bar{\phi}_N^{q+i-k}\Big\|^2+\frac{1}{2}\|\Delta^2\bar{\phi}_N^q\|^2\\\le&\big(\Delta f[B_k(\phi_N^{q-1})],\Delta^2\bar{\phi}_N^q-\tau_k\Delta^2\bar{\phi}_N^{q-1}\big)\end{aligned}$$

$$+ \big(B_k(\boldsymbol{v}_N^{q-1}) \cdot \nabla B_k(\phi_N^{q-1}), \Delta^2\bar{\phi}_N^q - \tau_k \Delta^2 \bar{\phi}_N^{q-1}\big)$$
$$+ \frac{\tau_k}{2}\|\Delta^2\bar{\phi}_N^{q-1}\|^2.$$

In the following, we bound the right hand side of (4.15) as follows:

Firstly, thanks to Lemma 6, we have

$$\begin{aligned}\|\Delta f[B_k(\phi_N^{q-1})]\|^2 &\le C(M_k)(\|\Delta^2 B_k(\phi_N^{q-1})\|^{2\sigma} + 1)\\ &\le \bar{\gamma}_k\|\Delta^2 B_k(\phi_N^{q-1})\|^2 + C(M_k, \bar{\gamma}_k)\\ &\le 40\bar{\gamma}_k \sum_{i=1}^{k} \|\Delta^2\bar{\phi}_N^{q-i}\|^2 + C(M_k,\bar{\gamma}_k)\\ &\le \gamma_k \sum_{i=1}^{k} \|\Delta^2\bar{\phi}_N^{q-i}\|^2 + C(M_k,\gamma_k),\end{aligned} \tag{4.16}$$

where $\bar{\gamma}_k$ can be any positive constant and the constant 40 comes from the coefficients in B_k and $\eta_k^q < 2$. To simplify the notation, we let $\gamma_k = 40\bar{\gamma}_k$. Then (4.16) implies

$$\begin{aligned}&\big|\eta_k^q\big(\Delta f[B_k(\phi_N^{q-1})], \Delta^2\bar{\phi}_N^q - \tau_k\Delta^2\bar{\phi}_N^{q-1}\big)\big|\\ \le& C(\varepsilon_k)\|\Delta f[B_k(\phi_N^{q-1})]\|^2 + \varepsilon_k\big(\|\Delta^2\bar{\phi}_N^q\|^2 + \|\Delta^2\bar{\phi}_N^{q-1}\|^2\big)\\ \le& C(M_k,\,\varepsilon_k,\,\gamma_k) + \big(C(\varepsilon_k)\gamma_k + \varepsilon_k\big)\sum_{i=0}^{k}\|\Delta^2\bar{\phi}_N^{q-i}\|^2.\end{aligned} \tag{4.17}$$

Next, it follows from the Cauchy-Schwarz inequality and the Sobolev inequality that:

$$(ab, c) \le \|a\|_{L^4}\|b\|_{L^4}\|c\| \le C\|a\|_1\|b\|_1\|c\|, \tag{4.18}$$

combining with (4.11) and (4.12), we can bound the second term in (4.15) as

$$\begin{aligned}&\big|\big(B_k(\boldsymbol{v}_N^{q-1}) \cdot \nabla B_k(\phi_N^{q-1}), \Delta^2\bar{\phi}_N^q - \tau_k\Delta^2\bar{\phi}_N^{q-1}\big)\big|\\ \le& C\|B_k(\boldsymbol{v}_N^{q-1})\|_{L^4}\|\nabla B_k(\phi_N^{q-1})\|_{L^4}\|\Delta^2\phi_N^q - \tau_k\Delta^2\phi_N^{q-1}\|\\ \le& C\|B_k(\boldsymbol{v}_N^{q-1})\|_1\|\nabla B_k(\phi_N^{q-1})\|_1\|\Delta^2\bar{\phi}_N^q - \tau_k\Delta^2\bar{\phi}_N^{q-1}\|\\ \le& C(\varepsilon_k)\|B_k(\bar{\boldsymbol{v}}_N^{q-1})\|_1^2\|\nabla B_k(\bar{\phi}_N^{q-1})\|_1^2 + \varepsilon_k\big(\|\Delta^2\bar{\phi}_N^q\|^2 + \|\Delta^2\bar{\phi}_N^{q-1}\|^2\big)\end{aligned} \tag{4.19}$$

$$\le C(M_k,\varepsilon_k)+\varepsilon_k\big(\|\Delta^2\bar\phi_N^q\|^2+\|\Delta^2\bar\phi_N^{q-1}\|^2\big)$$
$$+C(\varepsilon_k)\Big(\|\nabla B_k(\bar{\boldsymbol v}_N^{q-1})\|^2\|\Delta B_k(\bar\phi_N^{q-1})\|^2+\|\nabla B_k(\bar{\boldsymbol v}_N^{q-1})\|^2+\|\Delta B_k(\bar\phi_N^{q-1})\|^2\Big).$$

Now, combining (4.15), (4.17) and (4.19), we obtain

(4.20)
$$\sum_{i,j=1}^{k} g_{ij}(\Delta\bar\phi_N^{q+i-k},\Delta\bar\phi_N^{q+j-k})-\sum_{i,j=1}^{k} g_{ij}(\Delta\bar\phi_N^{q-1+i-k},\Delta\bar\phi_N^{q-1+j-k})$$
$$+\frac{\delta t}{2}\|\Delta^2\bar\phi_N^q\|^2$$
$$\le C(M_k,\,\varepsilon_k,\,\gamma_k)\delta t+\big(C(\varepsilon_k)\gamma_k+\varepsilon_k\big)\delta t\sum_{i=0}^{k}\|\Delta^2\bar\phi_N^{q-i}\|^2+\frac{\delta t\tau_k}{2}\|\Delta^2\bar\phi_N^{q-1}\|^2$$
$$+C(\varepsilon_k)\delta t\|\nabla B_k(\bar{\boldsymbol v}_N^{q-1})\|^2\|\Delta B_k(\bar\phi_N^{q-1})\|^2$$
$$+C(\varepsilon_k)\delta t\big(\|\nabla B_k(\bar{\boldsymbol v}_N^{q-1})\|^2+\|\Delta B_k(\bar\phi_N^{q-1})\|^2\big),$$

where ε_k above can be any positive constant. After taking the sum on (4.20), we are supposed to choose suitable δt, ε_k and γ_k such that

(4.21)
$$\frac{1}{2}\sum_{q=1}^{n}\big(\|\Delta^2\bar\phi_N^q\|^2-\tau_k\|\Delta^2\bar\phi_N^{q-1}\|^2\big)+C_I\ge\sum_{q=1}^{n}\Big(\big(C(\varepsilon_k)\gamma_k+\varepsilon_k\big)\sum_{i=0}^{k}\|\Delta^2\bar\phi_N^{q-i}\|^2\Big),$$

with C_I is a constant only depending on the initial data.

Note that $0\le\tau_k<1$, we can choose δt, ε_k and γ_k small enough such that

$$\varepsilon_k<\frac{1-\tau_k}{4(k+1)},\quad \gamma_k<\frac{1-\tau_k}{4(k+1)C(\varepsilon_k)},\tag{4.22}$$

we have

$$C(\varepsilon_k)\gamma_k+\varepsilon_k\le\frac{1-\tau_k}{4(k+1)}+\frac{1-\tau_k}{4(k+1)}\le\frac{1-\tau_k}{2(k+1)}.$$

Then, taking the sum on (4.20) for q from k to n, we obtain

$$\sum_{i,j=1}^{k} g_{ij}(\Delta\bar\phi_N^{n+i-k},\Delta\bar\phi_N^{n+j-k})$$
$$\le C(M_k,\,\tau_k)T+C(\bar\phi_N^0,\ldots,\bar\phi_N^{k-1})+C\delta t\sum_{q=0}^{n-1}\|\nabla B_k(\bar{\boldsymbol v}_N^q)\|^2\|\Delta B_k(\bar\phi_N^q)\|^2$$

$$+C\delta t\sum_{q=0}^{n-1}\left(\|\nabla B_k(\bar{\boldsymbol{v}}_N^q)\|^2+\|\Delta B_k(\bar{\phi}_N^q)\|^2\right),$$

where $C(M_k,\,\tau_k)$ is a constant only depends on M_k, τ_k, $C(\bar{\phi}_N^0,...,\bar{\phi}_N^{k-1})$ only depends on $\bar{\phi}_N^0,...,\bar{\phi}_N^{k-1}$. Since $G=(g_{ij})$ is a positive definite symmetric matrix, we have

$$\begin{aligned}
\lambda_G\|\Delta\phi_N^n\|^2\leq&\sum_{i,j=1}^{k}g_{ij}(\Delta\phi_N^{n+i-k},\Delta\phi_N^{n+j-k})\\
\leq& C(M_k,\,\tau_k)T+C(\phi_N^0,...,\phi_N^{k-1})\\
&+C\delta t\sum_{q=0}^{n-1}\|\nabla B_k(\boldsymbol{v}_N^q)\|^2\|\Delta B_k(\phi_N^q)\|^2\\
&+C\delta t\sum_{q=0}^{n-1}\left(\|\nabla B_k(\boldsymbol{v}_N^q)\|^2+\|\Delta B_k(\phi_N^q)\|^2\right)\\
\leq& C(M_k,\,\tau_k)T+C(\phi_N^0,...,\phi_N^{k-1})\\
&+C\delta t\sum_{q=0}^{n-1}\left(\|\nabla B_k(\bar{\boldsymbol{v}}_N^q)\|^2+1\right)\|\Delta\phi_N^q\|^2\\
&+C\delta t\sum_{q=0}^{n-1}\|\nabla B_k(\bar{\boldsymbol{v}}_N^q)\|^2,
\end{aligned}\tag{4.23}$$

where $\lambda_G>0$ is the minimum eigenvalue of $G=(g_{ij})$. It follows from (4.13) that

$$\delta t\sum_{q=0}^{n-1}\|\nabla B_k(\bar{\boldsymbol{v}}_N^q)\|^2<C_{H^2},$$

where the constant C_{H^2} is independent of $C_0, C_\Pi, \delta t$ and N. As a result, we can apply discrete Gronwall Lemma 3 to (4.23) and obtain:

$$\|\Delta\bar{\phi}_N^n\|^2\leq C.$$

Together with (4.12), the above implies

$$\|\bar{\phi}_N^q\|_2\leq C_1,\ \forall q\leq n.\tag{4.24}$$

for a constant C_1 independent of δt and N. Noting that

$$\|\bar{\phi}_N^q\|_2 = |\eta_k^q|\|\phi_N^q\|_2,$$

then (4.11) implies

$$\|\phi_N^q\|_2 \le 2C_1, \ \forall q \le n. \tag{4.25}$$

Step 2: Estimates for $\|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{n+1}\|$ and $\|\bar{e}_\phi^{n+1}\|_2$. By the assumptions on the exact solution ϕ, $\boldsymbol{v}$ and (4.25), we can choose C large enough such that

$$\|\boldsymbol{v}(t)\|_2^2,\ \|\phi^{(j)}(t)\|_2^2 \le C, \forall t \le T,\ j = 0, 1, 2;\ \|\bar{\phi}_N^q\|_2^2 \le C,\ \forall q \le n. \tag{4.26}$$

Since $H^2 \subset L^\infty$, without loss of generality, we can adjust C such that for $i = 0, 1, 2, 3$ and $j = 0, 1, 2$,

$$|\boldsymbol{v}(t)|,\ |\phi^{(j)}(t)|,\ |f^{(i)}[\phi(t)]| \le C,\ \forall t \le T;\ |f^{(i)}(\bar{\phi}_N^q)| \le C,\ \forall q \le n. \tag{4.27}$$

From (3.12a) and (3.12b), we can write down the error equation for $\bar{\phi}_N^{q+1}$ as

$$\begin{aligned} &\big(\alpha_k \bar{e}_\phi^{q+1} - A_k(\bar{e}_\phi^q), \psi_N\big) + \delta t\big(\Delta^2 \bar{e}_\phi^{q+1}, \psi_N\big) \\ =&\big(R_{\phi,k}^q, \psi_N\big) + \delta t\big(P_{\phi,k}^q, \psi_N\big) + \delta t\big(\Delta Q_{\phi,k}^q, \psi_N\big), \quad \forall \psi_N \in S_N, \end{aligned} \tag{4.28}$$

where $P_{\phi,k}^q$, $Q_{\phi,k}^q$ and $R_{\phi,k}^q$ are given by

$$P_{\phi,k}^q = -B_k(\boldsymbol{v}_N^q) \cdot \nabla B_k(\phi_N^q) + \boldsymbol{v}(t^{q+1}) \cdot \nabla \phi(t^{q+1}), \tag{4.29}$$

$$Q_{\phi,k}^q = f(B_k(\phi_N^q)) - f[\phi(t^{q+1})], \tag{4.30}$$

and

$$\begin{aligned} R_{\phi,k}^q &= -\alpha_k \phi(t^{q+1}) + A_k(\phi(t^q)) + \delta t \phi_t(t^{q+1}) \\ &= \sum_{i=1}^k a_i \int_{t^{q+1-i}}^{t^{q+1}} (t^{q+1-i} - s)^k \frac{\partial^{k+1}\phi}{\partial t^{k+1}}(s) ds, \end{aligned} \tag{4.31}$$

with a_i being some fixed and bounded constants determined by the truncation errors. Let $\psi_N = \Delta^2 \bar{e}_{\phi,N}^{q+1} - \tau_k \Delta^2 \bar{e}_{\phi,N}^q$ in (4.28), it follows from Lemma 4 and (3.10) that

$$
\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\Delta\bar{e}_{\phi,N}^{q+1+i-k},\Delta\bar{e}_{\phi,N}^{q+1+j-k})-\sum_{i,j=1}^{k} g_{ij}(\Delta\bar{e}_{\phi,N}^{q+i-k},\Delta\bar{e}_{\phi,N}^{q+j-k})\\
&+\Big\|\sum_{i=0}^{k}\delta_i\Delta\bar{e}_{\phi,N}^{q+1+i-k}\Big\|^2+\delta t\|\Delta^2\bar{e}_{\phi,N}^{q+1}\|^2\\
=&\delta t(\Delta^2\bar{e}_{\phi,N}^{q+1},\tau_k\Delta^2\bar{e}_{\phi,N}^{q})+(R_{\phi,k}^q,\Delta^2\bar{e}_{\phi,N}^{q+1}-\tau_k\Delta^2\bar{e}_{\phi,N}^{q})\\
&+\delta t(P_{\phi,k}^q,\Delta^2\bar{e}_{\phi,N}^{q+1}-\tau_k\Delta^2\bar{e}_{\phi,N}^{q})+\delta t(\Delta Q_{\phi,k}^q,\Delta^2\bar{e}_{\phi,N}^{q+1}-\tau_k\Delta^2\bar{e}_{\phi,N}^{q}).
\end{aligned}\tag{4.32}
$$

In the following, we bound the righthand side of (4.32). Firstly, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|\delta t(\Delta^2\bar{e}_{\phi,N}^{q+1},\tau_k\Delta^2\bar{e}_{\phi,N}^{q})|&\le\delta t\|\Delta^2\bar{e}_{\phi,N}^{q+1}\|\|\tau_k\Delta^2\bar{e}_{\phi,N}^{q}\|\\
&\le\frac{\delta t}{2}\|\Delta^2\bar{e}_{\phi,N}^{q+1}\|^2+\frac{\delta t\tau_k^2}{2}\|\Delta^2\bar{e}_{\phi,N}^{q}\|^2.
\end{aligned}\tag{4.33}
$$

It follows from (4.31) and the assumption on the exact solution ϕ that

$$
\|R_{\phi,k}^q\|^2\le C\delta t^{2k+1}\int_{t^{q+1-k}}^{t^{q+1}}\Big\|\frac{\partial^{k+1}\phi}{\partial t^{k+1}}(s)\Big\|^2ds\le C\delta t^{2k+2}.\tag{4.34}
$$

Therefore,

$$
\begin{aligned}
&\Big|(R_{\phi,k}^q,\Delta^2\bar{e}_{\phi,N}^{q+1}-\tau_k\Delta^2\bar{e}_{\phi,N}^{q})\Big|\\
&\le\frac{C(\varepsilon)}{\delta t}\|R_{\phi,k}^q\|^2+\delta t\varepsilon\|\Delta^2\bar{e}_{\phi,N}^{q+1}-\tau_k\Delta^2\bar{e}_{\phi,N}^{q}\|^2\\
&\le\frac{C(\varepsilon)}{\delta t}\|R_{\phi,k}^q\|^2+2\delta t\varepsilon\|\Delta^2\bar{e}_{\phi,N}^{q+1}\|^2+2\delta t\varepsilon\|\Delta^2\bar{e}_{\phi,N}^{q}\|^2\\
&\le2\delta t\varepsilon\|\Delta^2\bar{e}_{\phi,N}^{q+1}\|^2+2\delta t\varepsilon\|\Delta^2\bar{e}_{\phi,N}^{q}\|^2+C(\varepsilon)\delta t^{2k+1}.
\end{aligned}\tag{4.35}
$$

For the term with $P_{\phi,k}^q$, we first split $P_{\phi,k}^q$ as

$$
\begin{aligned}
P_{\phi,k}^q=&B_k(\boldsymbol{v}_N^q)\cdot\big(-\nabla B_k(\phi_N^q)+\nabla B_k(\phi(t^q))\big)-\big(B_k(\boldsymbol{v}_N^q)\\
&-B_k(\boldsymbol{v}(t^q))\big)\cdot\nabla B_k(\phi(t^q))\\
&-\big(B_k(\boldsymbol{v}(t^q))-\boldsymbol{v}(t^{q+1})\big)\cdot\nabla B_k(\phi(t^q))-\boldsymbol{v}(t^{q+1})\cdot\big(\nabla B_k(\phi(t^q))\\
&-\nabla\phi(t^{q+1})\big).
\end{aligned}\tag{4.36}
$$

Since we have

$$\|ab\|^2 \le \|a\|_{L^4}^2\|b\|_{L^4}^2 \le C\|a\|_1^2\|b\|_1^2, \tag{4.37}$$

then it follows from (4.12) and (4.27) that

$$\begin{aligned}
\|P_{\phi,k}^q\|^2 \le& C\|B_k(\boldsymbol{v}_N^q)\|_1^2\|\nabla B_k(e_\phi^q)\|_1^2 + \|\nabla B_k(\phi(t^q))\|_{L^\infty}^2\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2 \\
&+ \|\nabla B_k(\phi(t^q))\|_{L^\infty}^2\Big\|\sum_{i=1}^k b_i\int_{t^{q+1-i}}^{t^{q+1}}(t^{q+1-i}-s)^{k-1}\frac{\partial^k\boldsymbol{v}}{\partial t^k}(s)ds\Big\|^2 \\
&+ \|\boldsymbol{v}(t^{q+1})\|_{L^\infty}^2\Big\|\sum_{i=1}^k b_i\int_{t^{q+1-i}}^{t^{q+1}}(t^{q+1-i}-s)^{k-1}\nabla\frac{\partial^k\phi}{\partial t^k}(s)ds\Big\|^2 \\
\le& C\|B_k(\boldsymbol{v}_N^q)\|_1^2\|\nabla B_k(e_\phi^q)\|_1^2 + C\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2 \\
&+ C\delta t^{2k-1}\int_{t^{q+1-k}}^{t^{q+1}}\Big\|\frac{\partial^k\phi}{\partial t^k}(s)\Big\|_1^2ds \\
&+ C\delta t^{2k-1}\int_{t^{q+1-k}}^{t^{q+1}}\Big\|\frac{\partial^k\boldsymbol{v}}{\partial t^k}(s)\Big\|^2ds \\
\le& C\|B_k(\boldsymbol{v}_N^q)\|_1^2\|B_k(e_\phi^q)\|_2^2 + C\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2 + C\delta t^{2k},
\end{aligned} \tag{4.38}$$

where b_i above are some fixed and bounded constants determined by the truncation error. For example, in the case $k=3$, we have

$$\begin{aligned}
B_3(\boldsymbol{v}(t^q)) - \boldsymbol{v}(t^{q+1}) = &-\frac{3}{2}\int_{t^q}^{t^{q+1}}(t^q-s)^2\frac{\partial^3\boldsymbol{v}}{\partial t^3}(s)ds \\
&+\frac{3}{2}\int_{t^{q-1}}^{t^{q+1}}(t^{q-1}-s)^2\frac{\partial^3\boldsymbol{v}}{\partial t^3}(s)ds \\
&-\frac{1}{2}\int_{t^{q-2}}^{t^{q+1}}(t^{q-2}-s)^2\frac{\partial^3\boldsymbol{v}}{\partial t^3}(s)ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\Big|\delta t\big(P_{\phi,k}^q, \Delta^2\bar{e}_\phi^{q+1} - \tau_k\Delta^2\bar{e}_\phi^q\big)\Big| \\
\le& C(\varepsilon)\delta t\|P_{\phi,k}^q\|^2 + \delta t\varepsilon\|\Delta^2\bar{e}_\phi^{q+1} - \tau_k\Delta^2\bar{e}_\phi^q\|^2 \\
\le& C(\varepsilon)\delta t\|P_{\phi,k}^q\|^2 + 2\delta t\varepsilon\|\Delta^2\bar{e}_\phi^{q+1}\|^2 + 2\delta t\varepsilon\|\Delta^2\bar{e}_\phi^q\|^2 \\
\le& C(\varepsilon)\delta t\big(\|B_k(\boldsymbol{v}_N^q)\|_1^2\|B_k(e_\phi^q)\|_2^2 + \|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\big) + C(\varepsilon)\delta t^{2k+1}
\end{aligned} \tag{4.39}$$

$$+ 2\delta t\varepsilon\|\Delta^2\bar{e}_\phi^{q+1}\|^2 + 2\delta t\varepsilon\|\Delta^2\bar{e}_\phi^{q}\|^2.$$

Similarly, for the term with $\Delta Q^q_{\phi,k}$, we first split $\Delta Q^q_{\phi,k}$ as

$$\begin{aligned}\Delta Q^q_{\phi,k} = &\big(\Delta f(B_k(\phi_N^q)) - \Delta f[B_k(\phi(t^q))]\big) + \big(\Delta f[B_k(\phi(t^q))] \\ &- \Delta f[B_k(\phi(t^{q+1}))]\big) \\ =:& \Delta Q^q_{\phi,k1} + \Delta Q^q_{\phi,k2},\end{aligned} \tag{4.40}$$

and note that

$$\Delta f(\phi) = f''(\phi)|\nabla\phi|^2 + f'(\phi)\Delta\phi,$$

then, by using (4.12), (4.26) and (4.27), we have

$$\begin{aligned}|\Delta Q^q_{\phi,k1}| \le &\big|f''(B_k(\phi_N^q))(|\nabla B_k(\phi_N^q)|^2 - |\nabla B_k(\phi(t^q))|^2)\big| \\ &+ \big||\nabla B_k(\phi(t^q))|^2(f''(B_k(\phi_N^q)) - f''[B_k(\phi(t^q))])\big| \\ &+ |f'(B_k(\phi_N^q))(\Delta B_k(\phi_N^q) - \Delta B_k(\phi(t^q)))| \\ &+ |\Delta B_k(\phi(t^q))(f'(B_k(\phi_N^q)) - f'[B_k(\phi(t^q))])| \\ \le& C\big(|\nabla B_k(e_\phi^q)| + |B_k(e_\phi^q)| + |\Delta B_k(e_\phi^q)|\big),\end{aligned} \tag{4.41}$$

and hence,

$$\|\Delta Q^q_{\phi,k1}\|^2 \le C\|B_k(e^q_\phi)\|_2^2. \tag{4.42}$$

For $Q^q_{k,2}$, we have

$$\begin{aligned}|\Delta Q^q_{\phi,k2}| \le& C\Big|\sum_{i=1}^k b_i \int_{t^{q+1-i}}^{t^{q+1}} (t^{q+1-i} - s)^{k-1}\frac{\partial^k\phi}{\partial t^k}(s)ds\Big| \\ &+ C\Big|\sum_{i=1}^k b_i \int_{t^{q+1-i}}^{t^{q+1}} (t^{q+1-i} - s)^{k-1}\nabla\frac{\partial^k\phi}{\partial t^k}(s)ds\Big| \\ &+ C\Big|\sum_{i=1}^k b_i \int_{t^{q+1-i}}^{t^{q+1}} (t^{q+1-i} - s)^{k-1}\Delta\frac{\partial^k\phi}{\partial t^k}(s)ds\Big|,\end{aligned}$$

and hence,

$$\|\Delta Q^q_{\phi,k2}\|^2 \le C\delta t^{2k-1}\int_{t^{q+1-k}}^{t^{q+1}} \Big\|\frac{\partial^k\phi}{\partial t^k}(s)\Big\|_2^2 ds \le C\delta t^{2k}. \tag{4.43}$$

Therefore,

$$
\begin{aligned}
&\Big|\delta t\big(\Delta Q^q_{\phi,k},\Delta^2\bar e^{q+1}_{\phi,N}-\tau_k\Delta^2\bar e^q_{\phi,N}\big)\Big|\\
\le& C(\varepsilon)\delta t\|\Delta Q^q_{\phi,k}\|^2+\delta t\varepsilon\|\Delta^2\bar e^{q+1}_{\phi,N}-\tau_k\Delta^2\bar e^q_{\phi,N}\|^2\\
\le& C(\varepsilon)\delta t\|\Delta Q^q_{\phi,k}\|^2+2\delta t\varepsilon\|\Delta^2\bar e^{q+1}_{\phi,N}\|^2+2\delta t\varepsilon\|\Delta^2\bar e^q_{\phi,N}\|^2\\
\le& C(\varepsilon)\delta t\big(\|\Delta Q^q_{\phi,k1}\|^2+\|\Delta Q^q_{\phi,k2}\|^2\big)+2\delta t\varepsilon\|\Delta^2\bar e^{q+1}_{\phi,N}\|^2+2\delta t\varepsilon\|\Delta^2\bar e^q_{\phi,N}\|^2\\
\le& C(\varepsilon)\delta t\|B_k(e^q_\phi)\|_2^2+C(\varepsilon)\delta t^{2k+1}+2\delta t\varepsilon\|\Delta^2\bar e^{q+1}_{\phi,N}\|^2+2\delta t\varepsilon\|\Delta^2\bar e^q_{\phi,N}\|^2.
\end{aligned}
\tag{4.44}
$$

Now, we choose ε small enough such that

$$
\frac{1-\tau_k^2}{4}\ge 12\varepsilon,
\tag{4.45}
$$

which is possible as $\tau_k<1$. Combining (4.32), (4.33), (4.35), (4.39), (4.44) and dropping some unnecessary terms, we can obtain:

$$
\begin{aligned}
&\sum_{i,j=1}^k g_{ij}(\Delta\bar e^{q+1+i-k}_{\phi,N},\Delta\bar e^{q+1+j-k}_{\phi,N})-\sum_{i,j=1}^k g_{ij}(\Delta\bar e^{q+i-k}_{\phi,N},\Delta\bar e^{q+j-k}_{\phi,N})\\
&+\frac{1-\tau_k^2}{4}\delta t\|\Delta^2\bar e^{q+1}_{\phi,N}\|^2\\
\le& C\delta t\big((\|B_k(\boldsymbol{v}^q_N)\|_1^2+1)\|B_k(e^q_\phi)\|_2^2+\|B_k(e^q_{\boldsymbol{v}})\|^2\big)+C\delta t^{2k+1}.
\end{aligned}
\tag{4.46}
$$

Taking the sum of (4.46) on q from $k-1$ to n, noting that $G=(g_{ij})$ is a symmetric positive definite matrix with minimum eigenvalue λ_G, we obtain:

$$
\begin{aligned}
&\lambda_G\|\Delta\bar e^{n+1}_{\phi,N}\|^2+\frac{\delta t(1-\tau_k^2)}{4}\sum_{q=0}^{n+1}\|\Delta^2\bar e^q_{\phi,N}\|^2\\
\le&\sum_{i,j=1}^k g_{ij}(\Delta\bar e^{n+1+i-k}_{\phi,N},\Delta\bar e^{n+1+j-k}_{\phi,N})+\frac{1-\tau_k^2}{4}\delta t\sum_{k=0}^{n+1}\|\Delta^2\bar e^q_{\phi,N}\|^2\\
\le& C\delta t\Big(\sum_{q=0}^n(\|B_k(\boldsymbol{v}^q_N)\|_1^2+1)\|B_k(e^q_\phi)\|_2^2+\sum_{q=0}^n\|B_k(\boldsymbol{e}^q_{\boldsymbol{v}})\|^2\Big)+C\delta t^{2k},
\end{aligned}
\tag{4.47}
$$

On the other hand, if we take $\psi_N=-\Delta\bar e^{q+1}_{\phi,N}+\tau_k\Delta\bar e^q_{\phi,N}$ in (4.28), we can obtain the following by the same procedure as above:

$$
\begin{aligned}
&\lambda_G\|\nabla\bar{e}_{\phi,N}^{n+1}\|^2+\frac{\delta t(1-\tau_k^2)}{4}\sum_{q=0}^{n+1}\|\nabla\Delta\bar{e}_{\phi,N}^q\|^2\\
\le & C\delta t\big(\sum_{q=0}^{n}(\|B_k(\boldsymbol{v}_N^q)\|_1^2+1)\|B_k(e_\phi^q)\|_2^2+\sum_{q=0}^{n}\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\big)+C\delta t^{2k},
\end{aligned}
\tag{4.48}
$$

and if we take $\psi_N=\bar{e}_{\phi,N}^{q+1}-\tau_k\bar{e}_{\phi,N}^q$ in (4.28), we can obtain:

$$
\lambda_G\|\bar{e}_{\phi,N}^{n+1}\|^2\le C\delta t\big(\sum_{q=0}^{n}(\|B_k(\boldsymbol{v}_N^q)\|_1^2+1)\|B_k(e_\phi^q)\|_2^2+\sum_{q=0}^{n}\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\big)+C\delta t^{2k}.
\tag{4.49}
$$

From (3.12c), we can write down the error equation for $\bar{\boldsymbol{v}}_N^{q+1}$ as

$$
\begin{aligned}
&\big(\alpha_k\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}-A_k(\bar{\boldsymbol{e}}_{\boldsymbol{v}}^q),\boldsymbol{\psi}_N\big)+\delta t\big(\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1},\nabla\boldsymbol{\psi}_N\big)\\
=&\big(R_{\boldsymbol{v},k}^q,\boldsymbol{\psi}_N\big)+\delta t\big(Q_{\boldsymbol{v},k}^q,\boldsymbol{\psi}_N\big)+\delta t\big(P_{\boldsymbol{v},k}^q,\boldsymbol{\psi}_N\big),\quad\forall\psi_N\in\boldsymbol{S}_N,
\end{aligned}
\tag{4.50}
$$

where $P_{\phi,k}^q$, $Q_{\phi,k}^q$ and $R_{\phi,k}^q$ are given by

$$
P_{\boldsymbol{v},k}^q=\mathbf{A}\big(B_k(\boldsymbol{v}_N^q)\cdot\nabla B_k(\boldsymbol{v}_N^q)\big)-\mathbf{A}\big(\boldsymbol{v}(t^{q+1})\cdot\nabla\boldsymbol{v}(t^{q+1})\big),
\tag{4.51}
$$

$$
Q_{\boldsymbol{v},k}^q=-\mathbf{A}\big(B_k(\mu_N^q)\nabla B_k(\phi_N^q)\big)+\mathbf{A}\big(\mu(t^{q+1})\nabla\phi(t^{q+1})\big),
\tag{4.52}
$$

and

$$
\begin{aligned}
R_{\boldsymbol{v},k}^q&=-\alpha_k\boldsymbol{v}(t^{q+1})+A_k(\boldsymbol{v}(t^q))+\delta t\boldsymbol{v}_t(t^{q+1})\\
&=\sum_{i=1}^{k}a_i\int_{t^{q+1-i}}^{t^{q+1}}(t^{q+1-i}-s)^k\frac{\partial^{k+1}\boldsymbol{v}}{\partial t^{k+1}}(s)ds,
\end{aligned}
\tag{4.53}
$$

with a_i being some fixed and bounded constants determined by the truncation errors as before. Let $\boldsymbol{\psi}_N=\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q$ in (4.50), it follows from Lemma 4 and (3.10) that

(4.54)

$$
\sum_{i,j=1}^{k}g_{ij}(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+i-k},\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+j-k})-\sum_{i,j=1}^{k}g_{ij}(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+i-k},\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+j-k})+\Big\|\sum_{i=0}^{k}\delta_i\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+i-k}\Big\|^2
$$

$$
\begin{aligned}
&+\delta t\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|^2\\
=&\delta t(\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1},\tau_k\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})+(R_{\boldsymbol{v},k}^q,\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})+\delta t(P_{\boldsymbol{v},k}^q,\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})\\
&+\delta t(Q_{\boldsymbol{v},k}^q,\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}).
\end{aligned}
$$

In the following, we bound the righthand side of (4.54). Firstly, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|\delta t(\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1},\tau_k\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})|&\le\delta t\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|\|\tau_k\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\|\\
&\le\frac{\delta t}{2}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|^2+\frac{\delta t\tau_k^2}{2}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\|^2.
\end{aligned}\tag{4.55}
$$

For the term with $R_{\boldsymbol{v},k}^q$, we can obtain the following estimate by the same way as in (4.34) and (4.35):

$$
\left|(R_{\boldsymbol{v},k}^q,\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})\right|\le 2\delta t\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|^2+2\delta t\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\|^2+C(\varepsilon)\delta t^{2k+1}.\tag{4.56}
$$

For the term with $Q_{\boldsymbol{v},k}^q$, we split $Q_{\boldsymbol{v},k}^q$ as

$$
\begin{aligned}
Q_{\boldsymbol{v},k}^q=&-\mathbf{A}\big(B_k(\mu_N^q)\nabla B_k(e_\phi^q)\big)-\mathbf{A}\big(B_k(e_\mu^q)\nabla B_k(\phi(t^q))\big)\\
&-\mathbf{A}\big((B_k(\mu(t^q))-\mu(t^{q+1}))\nabla B_k(\phi(t^q))\big)\\
&-\mathbf{A}\big(\mu(t^{q+1}))(\nabla B_k(\phi(t^q))-\nabla\phi(t^{q+1}))\big).
\end{aligned}\tag{4.57}
$$

Then, by the similar ways as in (4.38), (4.39), and making use of (2.2), (4.26), we have

$$
\begin{aligned}
&\left|\delta t(Q_{\boldsymbol{v},k}^q,\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})\right|\\
\le&C(\varepsilon)\delta t\|Q_{\boldsymbol{v},k}^q\|^2+\delta t\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\|^2\\
\le&2\delta t\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|^2+2\delta t\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\|^2+C(\varepsilon)\delta t\|B_k(\mu_N^q)\|_1^2\|B_k(e_\phi^q)\|_2^2\\
&+C(\varepsilon)\delta t\|B_k(e_\mu^q)\|^2+C(\varepsilon)\delta t^{2k+1}.
\end{aligned}\tag{4.58}
$$

For the term with $P_{\boldsymbol{v},k}^q$, we split it as

$$
\begin{aligned}
&(P_{\boldsymbol{v},k}^q,\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q})\\
=&\Big(\mathbf{A}\big([B_k(\boldsymbol{v}_N^q)-\boldsymbol{v}(t^{q+1})]\cdot\nabla\boldsymbol{v}(t^{q+1})\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\Big)\\
&+\Big(\mathbf{A}\big(B_k(\boldsymbol{v}_N^q)\cdot\nabla[B_k(\boldsymbol{v}(t^q))-\boldsymbol{v}(t^{q+1})]\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\Big)\\
&+\Big(\mathbf{A}\big(B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\cdot\nabla B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q}\Big)
\end{aligned}\tag{4.59}
$$

$$+\Big(\mathbf{A}\big(B_k(\boldsymbol{v}(t^q))\cdot\nabla B_k(\boldsymbol{e}_v^q)\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\Big).$$

We bound the terms on the right hand side of (4.59) with the help of (2.4a), (2.5) and (4.26):

(4.60)

$$\begin{aligned}
&\Big|\Big(\mathbf{A}\big([B_k(\boldsymbol{v}_N^q)-\boldsymbol{v}(t^{q+1})]\cdot\nabla\boldsymbol{v}(t^{q+1})\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\Big)\Big|\\
\le& C\|B_k(\boldsymbol{v}_N^q)-\boldsymbol{v}(t^{q+1})\|\|\boldsymbol{v}(t^{q+1})\|_2\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1\\
\le& C(\varepsilon)\|B_k(\boldsymbol{v}_N^q)-\boldsymbol{v}(t^{q+1})\|^2\|\boldsymbol{v}(t^{q+1})\|_2^2+\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2\\
\le& C(\varepsilon)\|B_k(\boldsymbol{v}(t^q))-\boldsymbol{v}(t^{q+1})\|^2\|\boldsymbol{v}(t^{q+1})\|_2^2+C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\|\boldsymbol{v}(t^{q+1})\|_2^2\\
&+\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2\\
\le& C(\varepsilon)\Big\|\sum_{i=1}^k b_i\int_{t^{q+1-i}}^{t^{q+1}}(t^{q+1-i}-s)^{k-1}\frac{\partial^k\boldsymbol{v}}{\partial t^k}(s)ds\Big\|^2+C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\\
&+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2\\
\le& C(\varepsilon)\delta t^{2k}+C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2,
\end{aligned}$$

where b_i are some fixed and bounded constants determined by the truncation error as before. For the other terms in the righthand side of (4.59), we have

$$\begin{aligned}
&\Big|\Big(\mathbf{A}\big(B_k(\boldsymbol{v}_N^q)\cdot\nabla[B_k(\boldsymbol{v}(t^q))-\boldsymbol{v}(t^{q+1})]\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\Big)\Big|\\
\le& C\|B_k(\boldsymbol{v}_N^q)\|\|B_k(\boldsymbol{v}(t^q))-\boldsymbol{v}(t^{q+1})\|_2\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1\\
\le& C(\varepsilon)\|B_k(\boldsymbol{v}_N^q)\|^2\|B_k(\boldsymbol{v}(t^q))-\boldsymbol{v}(t^{q+1})\|_2^2+\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2\\
\le& C(\varepsilon)\delta t^{2k-1}\int_{t^{q+1-k}}^{t^{q+1}}\Big\|\frac{\partial^k\boldsymbol{v}}{\partial t^k}(s)\Big\|_2^2ds+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2\\
\le& C(\varepsilon)\delta t^{2k}+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2.
\end{aligned}\tag{4.61}$$

Since $d=2$, we can use (2.4a) to obtain

(4.62)

$$\begin{aligned}
&\Big|\Big(\mathbf{A}\big(B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\cdot\nabla B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\Big)\Big|\\
\le& C\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^{1/2}\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^{1/2}\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^{1/2}\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^{1/2}\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1\\
\le& C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2+\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2\\
\le& C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2;
\end{aligned}$$

Note that the above inequality is the only place restricted to the two-dimensional case.

Thanks to (2.5), we have

$$
\begin{aligned}
&\Big|\Big(\mathbf{A}\big(B_k(\boldsymbol{v}(t^q))\cdot\nabla B_k(\boldsymbol{e}_v^q)\big),\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\Big)\Big|\\
\le&C\|B_k(\boldsymbol{v}(t^q))\|_2\|B_k(\boldsymbol{e}_v^q)\|\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1\\
\le&C(\varepsilon)\|B_k(\boldsymbol{v}(t^q))\|_2^2\|B_k(\boldsymbol{e}_v^q)\|^2+\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}-\tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2\\
\le&C(\varepsilon)\|B_k(\boldsymbol{e}_v^q)\|^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2+2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2.
\end{aligned}\tag{4.63}
$$

We combine (4.54), (4.55), (4.56), (4.58), (4.60)-(4.63) and choose ε small enough, we can obtain:

$$
\begin{aligned}
&\sum_{i,j=1}^k g_{ij}(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+i-k},\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+j-k})-\sum_{i,j=1}^k g_{ij}(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+i-k},\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+j-k})+\frac{1-\tau_k^2}{4}\delta t\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|^2\\
\le&C\delta t\big((\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2+1)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2+\|B_k(\mu_N^q)\|_1^2\|B_k(e_\phi^q)\|_2^2+\|B_k(e_\mu^q)\|^2\big)\\
&+C\delta t^{2k+1}.
\end{aligned}\tag{4.64}
$$

Taking the sum of (4.64) on q from $k-1$ to n, noting that $G=(g_{ij})$ is a symmetric positive definite matrix with minimum eigenvalue λ_G, we obtain:

$$
\begin{aligned}
&\lambda_G\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{n+1}\|^2+\frac{\delta t(1-\tau_k^2)}{4}\sum_{q=0}^{n+1}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|^2\\
\le&\sum_{i,j=1}^k g_{ij}(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+i-k},\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1+j-k})+\frac{\delta t(1-\tau_k^2)}{4}\sum_{q=0}^{n+1}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|^2\\
\le&C\delta t\big(\sum_{q=0}^n(\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2+1)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\\
&+\sum_{q=0}^n\|B_k(\mu_N^q)\|_1^2\|B_k(e_\phi^q)\|_2^2+\sum_{q=0}^n\|B_k(e_\mu^q)\|^2\big)\\
&+C\delta t^{2k}.
\end{aligned}\tag{4.65}
$$

Since $\mu=-\Delta\phi+f(\phi)$, we have

$$
\|B_k(e_\mu^q)\|^2\le C(\|\Delta B_k(e_\phi^q)\|^2+\|B_k(e_\phi^q)\|^2),\quad\forall q\le n.\tag{4.66}
$$

On the other hand, we derive from (4.8) and (4.6) that

$$|\eta_k^q - 1| \leq 2^k C_0^k \,\delta t^k + 2^k C_\Pi^k N^{k(3-m)}, \quad \forall q \leq n.$$

Note that $\phi_N^q = \eta_k^q \bar{\phi}_N^q$ and $\boldsymbol{v}_N^q = \eta_k^q \bar{\boldsymbol{v}}_N^q$, we can estimate $\|B_k(e_\phi^q)\|_2^2$ and $\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2$ as

$$\begin{aligned} &\|B_k(e_\phi^q)\|_2^2 \\ =&\|B_k(\phi_N^q - \bar{\phi}_N^q) + B_k(\bar{e}_{\phi,N}^q) + B_k(e_{\phi,\Pi}^q)\|_2^2 \\ \leq& CC_0^{2k}\delta t^{2k} + CC_\Pi^{2k} N^{2k(3-m)} + C\|B_k(\bar{e}_{\phi,N}^q)\|_2^2 + C\|\phi(t^q)\|_m^2 N^{4-2m}. \end{aligned} \tag{4.67}$$

and

$$\begin{aligned} &\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2 \\ =&\|B_k(\boldsymbol{v}_N^q - \bar{\boldsymbol{v}}_N^q) + B_k(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q) + B_k(\boldsymbol{e}_{\boldsymbol{v},\Pi}^q)\|^2 \\ \leq& CC_0^{2k}\delta t^{2k} + CC_\Pi^{2k} N^{2k(3-m)} + C\|B_k(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q)\|^2 + C\|\boldsymbol{v}(t^q)\|_m^2 N^{-2m}. \end{aligned} \tag{4.68}$$

It follows from (4.12) and (4.14) that there exists a constant C_2 independent of C_0, C_Π, δt and N such that

$$\delta t \sum_{q=0}^{n} \|B_k(\mu_N^q)\|_1^2,\ \delta t \sum_{q=0}^{n} \|B_k(\boldsymbol{v}_N^q)\|_1^2,\ \delta t \sum_{q=0}^{n} \|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2 \leq C_2. \tag{4.69}$$

Now, we put (4.47)-(4.49), (4.65)-(4.69) together:

(4.70)

$$\begin{aligned} &\lambda_G(\|\bar{e}_{\phi,N}^{n+1}\|_2^2 + \|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{n+1}\|^2) + \frac{\delta t(1-\tau_k^2)}{4}(\sum_{q=0}^{n+1}\|\nabla\Delta\bar{e}_{\phi,N}^q\|^2 + \sum_{q=0}^{n+1}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|^2) \\ \leq& C\delta t\big(\sum_{q=0}^{n}(\|B_k(\boldsymbol{v}_N^q)\|_1^2 + \|B_k(\mu_N^q)\|_1^2 + 1)\|B_k(e_\phi^q)\|_2^2 \\ &+ \sum_{q=0}^{n}(\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2 + 1)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\big) + C\delta t^{2k} \\ \leq& C\delta t\big(\sum_{q=0}^{n}(\|B_k(\boldsymbol{v}_N^q)\|_1^2 + \|B_k(\mu_N^q)\|_1^2 + 1)\|B_k(\bar{e}_{\phi,N}^q)\|_2^2 \end{aligned}$$

$$+\sum_{q=0}^{n}(\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_1^2+1)\|B_k(\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q)\|^2)+CC_0^{2k}\delta t^{2k}+CC_\Pi^{2k}N^{2k(3-m)}$$
$$+CN^{4-2m}.$$

With the upper bound in (4.69), we can apply discrete Gronwall Lemma 3 on (4.70), we obtain

$$\begin{aligned}
&\|\bar{e}_{\phi,N}^{n+1}\|_2^2+\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{n+1}\|^2+\delta t\Big(\sum_{q=0}^{n+1}\|\nabla\Delta\bar{e}_{\phi,N}^q\|^2+\sum_{q=0}^{n+1}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|^2\Big)\\
&\le C\exp(3C_2+T)(C_0^{2k}\delta t^{2k}+C_\Pi^{2k}N^{2k(3-m)}+N^{4-2m})\\
&=C(C_0^{2k}\delta t^{2k}+C_\Pi^{2k}N^{2k(3-m)}+N^{4-2m}),
\end{aligned}\tag{4.71}$$

where we still use $C:=C\exp(3C_2+T)$ to denote a general constant independent of C_0, C_Π, δt and N.

Since $\bar{e}_\phi^q=\bar{e}_{\phi,N}^q+e_{\phi,\Pi}^q$ and $\bar{\boldsymbol{e}}_{\boldsymbol{v}}^q=\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q+\boldsymbol{e}_{\boldsymbol{v},\Pi}^q$, $\forall q$, it follows from the triangle inequality that

$$\|\bar{e}_\phi^{n+1}\|_2^2\le C(C_0^{2k}\delta t^{2k}+C_\Pi^{2k}N^{2k(3-m)}+N^{4-2m})+CN^{2(2-m)},\tag{4.72}$$

$$\delta t\sum_{q=0}^{n+1}\|\nabla\Delta\bar{e}_\phi^q\|^2\le C(C_0^{2k}\delta t^{2k}+C_\Pi^{2k}N^{2k(3-m)}+N^{4-2m})+CN^{2(3-m)},\tag{4.73}$$

$$\|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{n+1}\|^2\le C(C_0^{2k}\delta t^{2k}+C_\Pi^{2k}N^{2k(3-m)}+N^{4-2m})+CN^{-2m},\tag{4.74}$$

and

$$\delta t\sum_{q=0}^{n+1}\|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^q\|_1^2\le C(C_0^{2k}\delta t^{2k}+C_\Pi^{2k}N^{2k(3-m)}+N^{4-2m})+CN^{2(1-m)}.\tag{4.75}$$

Under the condition (4.9) and $m\ge 4$, the above also imply there exists a constant C_3 independent of $\delta t, N, C_0, C_\Pi$ such that

$$\|\bar{\phi}_N^{n+1}\|_2^2,\ \delta t\sum_{q=0}^{n+1}\|\Delta^2\bar{\phi}_N^q\|^2,\ \|\bar{\boldsymbol{v}}_N^{n+1}\|^2,\ \delta t\sum_{q=0}^{n+1}\|\bar{\boldsymbol{v}}_N^q\|_1^2\le C_3.\tag{4.76}$$

Note that $H^2 \subset L^\infty$, without loss of generality, we assume C_3 above also satisfies

$$|f(\bar{\phi}_N^{n+1})|, \, |f'(\bar{\phi}_N^{n+1})| \le C_3. \tag{4.77}$$

Step 3: Estimate for $|1-\xi^{n+1}|$. It follows from (3.12d) that the equation for $\{s^j\}$ can be written as

$$\begin{aligned} s^{q+1} - s^q =& \delta t\big(\|\nabla\mu(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1}) + 1}\|\nabla\mu_N^{q+1}\|^2\big) \\ &+ \delta t\big(\|\nabla\boldsymbol{v}(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1}) + 1}\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|^2\big) + T_q, \ \forall q \le n, \end{aligned} \tag{4.78}$$

where T_q is the truncation error

$$T_q = r(t^q) - r(t^{q+1}) + \delta t r_t(t^{q+1}) = \int_{t^q}^{t^{q+1}} (s - t^q) r_{tt}(s) ds. \tag{4.79}$$

Taking the sum of (4.78) for q from 0 to n, and noting that $s^0 = 0$, we have

$$\begin{aligned} s^{n+1} =& \delta t \sum_{q=0}^{n} \big(\|\nabla\mu(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1}) + 1}\|\nabla\mu_N^{q+1}\|^2\big) \\ &+ \delta t \sum_{q=0}^{n} \big(\|\nabla\boldsymbol{v}(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1}) + 1}\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|^2\big) + \sum_{q=0}^{n} T_q. \end{aligned} \tag{4.80}$$

We bound the righthand side of (4.80) as follows. By direct calculation, we have

$$r_{tt} = \int_\Omega \big(|\nabla\phi_t|^2 + \nabla\phi\cdot\nabla\phi_{tt} + f'(\phi)\phi_t^2 + f(\phi)\phi_{tt} + \boldsymbol{v}_t^2 + \boldsymbol{v}\boldsymbol{v}_{tt}\big)dx, \tag{4.81}$$

then from (4.79), we have

$$|T_q| \le C\delta t \int_{t^q}^{t^{q+1}} |r_{tt}| ds \le C\delta t^2, \, \forall q \le n.$$

By triangular inequality,

$$\begin{aligned}&\Big|\|\nabla\boldsymbol{v}(t^{q+1})\|^2-\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|^2\Big|\\ \le&\|\nabla\boldsymbol{v}(t^{q+1})\|^2\Big|1-\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\Big|\\ &+\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\big|\|\nabla\boldsymbol{v}(t^{q+1})\|^2-\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|^2\big|=:K_1^q+K_2^q.\end{aligned}\tag{4.82}$$

It follows from (4.26) and Theorem 1 that

$$\begin{aligned}K_1^q\le&C\Big|1-\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\Big|\\ =&C\Big|\frac{r(t^{q+1})}{E(\phi(t^{q+1}),\boldsymbol{v}(t^{q+1}))+1}-\frac{r^{q+1}}{E(\phi(t^{q+1}),\boldsymbol{v}(t^{q+1}))+1}\Big|\\ &+C\Big|\frac{r^{q+1}}{E(\phi(t^{q+1}),\boldsymbol{v}(t^{q+1}))+1}-\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\Big|\\ \le&C\big(|E(\phi(t^{q+1}),\boldsymbol{v}(t^{q+1}))-E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})|+|s^{q+1}|\big),\ \forall q\le n,\end{aligned}\tag{4.83}$$

and

$$\begin{aligned}K_2^q&\le C\big|\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|^2-\|\nabla\boldsymbol{v}(t^{q+1})\|^2\big|\\ &\le C\|\nabla\bar{\boldsymbol{v}}_N^{q+1}-\nabla\boldsymbol{v}(t^{q+1})\|(\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|+\|\nabla\boldsymbol{v}(t^{q+1})\|)\\ &\le C\|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|+C\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|,\ \forall q\le n.\end{aligned}\tag{4.84}$$

We can bound the term with $\nabla\mu$ by the similar ways:

$$\begin{aligned}&\big|\|\nabla\mu(t^{q+1})\|^2-\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\|\nabla\mu_N^{q+1}\|^2\big|\\ \le&\|\nabla\mu(t^{q+1})\|^2\Big|1-\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\Big|\\ &+\frac{r^{q+1}}{E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})+1}\big|\|\nabla\mu(t^{q+1})\|^2-\|\nabla\mu_N^{q+1}\|^2\big|=:K_3^q+K_4^q.\end{aligned}\tag{4.85}$$

Same as (4.83), we have

$$K_3^q\le C\big(|E(\phi(t^{q+1}),\boldsymbol{v}(t^{q+1}))-E(\bar\phi_N^{q+1},\bar{\boldsymbol{v}}_N^{q+1})|+|s^{q+1}|\big),\ \forall q\le n.\tag{4.86}$$

For K_4^q, we can derive the following same as (4.84)

$$K_4^q \le C\|\nabla \mu_N^{q+1}\|\|\nabla e_\mu^{q+1}\| + C\|\nabla e_\mu^{q+1}\|, \ \forall q \le n. \tag{4.87}$$

On the other hand, we can bound $\|\nabla \bar{e}_\mu^{q+1}\|$ by the definition of μ and (4.77) as

$$\|\nabla \bar{e}_\mu^{q+1}\| \le CC_3(\|\nabla\Delta \bar{e}_\phi^{q+1}\| + \|\nabla \bar{e}_\phi^{q+1}\|), \ \forall q \le n. \tag{4.88}$$

For the term with $E(\phi, \boldsymbol{v})$, we have

$$\begin{aligned}
&|E(\phi(t^{q+1}), \boldsymbol{v}(t^{q+1})) - E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1})| \\
\le&\frac{1}{2}(\|\nabla\phi(t^{q+1})\| + \|\nabla\bar{\phi}_N^{q+1}\|)\|\nabla\phi(t^{q+1}) - \nabla\bar{\phi}_N^{q+1}\| \\
&+ \int \big(F[\phi(t^{q+1})] - F(\bar{\phi}_N^{q+1})\big)dx + \frac{1}{2}(\|\boldsymbol{v}(t^{q+1})\| + \|\bar{\boldsymbol{v}}_N^{q+1}\|)\|\boldsymbol{v}(t^{q+1}) - \bar{\boldsymbol{v}}_N^{q+1}\| \\
\le& CC_3(\|\bar{e}_\phi^{q+1}\|_1 + \|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|).
\end{aligned} \tag{4.89}$$

It follows from (4.13), (4.72), (4.73), (4.88) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\delta t\sum_{q=0}^n \|\nabla\mu_N^{q+1}\|\|\nabla e_\mu^{q+1}\| \le \big(\delta t\sum_{q=0}^n \|\nabla\mu_N^{q+1}\|^2 \delta t\sum_{q=0}^n \|\nabla e_\mu^{q+1}\|^2\big)^{1/2} \\
\le& C\sqrt{C_0^{2k}\delta t^{2k} + C_\Pi^{2k} N^{2k(3-m)} + N^{4-2m} + N^{2(3-m)}},
\end{aligned} \tag{4.90}$$

and it follows from (4.13), (4.75) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\delta t\sum_{q=0}^n \|\nabla\bar{\boldsymbol{v}}_N^{q+1}\|\|\nabla \bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\| \le \big(\delta t\sum_{q=0}^n \|\nabla\bar{\boldsymbol{v}}^{q+1}\|^2 \delta t\sum_{q=0}^n \|\nabla \bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|^2\big)^{1/2} \\
\le& C\sqrt{C_0^{2k}\delta t^{2k} + C_\Pi^{2k} N^{2k(3-m)} + N^{4-2m} + N^{2(1-m)}},
\end{aligned} \tag{4.91}$$

Now, we are ready to estimate s^{n+1}. Combining the estimates obtained above, (4.80) leads to

(4.92)
$$
\begin{aligned}
|s^{n+1}| =&\delta t \sum_{q=0}^{n} \Big| \|\nabla \mu(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1})+1} \|\nabla \mu_N^{q+1}\|^2 \Big| \\
&+ \delta t \sum_{q=0}^{n} \big| \|\nabla \boldsymbol{v}(t^{q+1})\|^2 - \frac{r^{q+1}}{E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1})+1} \|\nabla \bar{\boldsymbol{v}}_N^{q+1}\|^2 \big| + \sum_{q=0}^{n} |T_q| \\
\le& C\delta t \sum_{q=0}^{n} |s^{q+1}| + CC_3 \delta t \sum_{q=0}^{n} (\|\bar{e}_\phi^{q+1}\|_1 + \|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|_1 + \|\nabla \Delta \bar{e}_\phi^{q+1}\|) \\
&+ C\delta t \sum_{q=0}^{n} (\|\nabla \mu_N^{q+1}\| \|\nabla e_\mu^{q+1}\| + \|\nabla \bar{\boldsymbol{v}}_N^{q+1}\| \|\nabla \bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|) + C\delta t \\
\le& C\delta t \sum_{q=0}^{n} |s^{q+1}| + C\sqrt{C_0^{2k} \delta t^{2k} + C_\Pi^{2k} N^{2k(3-m)} + N^{2(3-m)}} + C\delta t.
\end{aligned}
$$

Finally, applying Lemma 2 on (4.92) with $\delta t < \frac{1}{2C}$, we obtain the following estimate for s^{n+1}:

(4.93)
$$
\begin{aligned}
|s^{n+1}| &\le C \exp((1-\delta t C)^{-1} T) \big(\sqrt{C_0^{2k} \delta t^{2k} + C_\Pi^{2k} N^{2k(3-m)} + N^{2(3-m)}} + \delta t \big) \\
&\le C_4 \big(\sqrt{C_0^{2k} \delta t^{2k} + C_\Pi^{2k} N^{2k(3-m)} + N^{2(3-m)}} + \delta t \big) \\
&\le C_4 C_0^k \delta t^k + C_4 C_\Pi^k N^{k(3-m)} + C_4 N^{3-m} + C_4 \delta t,
\end{aligned}
$$

where C_4 is independent of $\delta t, N, C_0, C_\Pi$ and can be defined as

$$
C_4 := C\max\{\exp(2T), 2\},
$$

then $\delta t < \frac{1}{2C}$ can be guaranteed by

$$
\delta t < \frac{1}{C_4}. \tag{4.94}
$$

Thanks to (4.72), (4.74), (4.83), (4.89) and (4.93), we have

$$
\begin{aligned}
|1-\xi^{n+1}| \le& C\big(|E(\phi(t^{q+1}), \boldsymbol{v}(t^{q+1})) - E(\bar{\phi}_N^{q+1}, \bar{\boldsymbol{v}}_N^{q+1})| + |s^{q+1}| \big) \\
\le& C(C_3(\|\bar{e}_\phi^{q+1}\|_1 + \|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|) + |s^{n+1}|)
\end{aligned} \tag{4.95}
$$

$$
\begin{aligned}
&\le CC_3\sqrt{C_0^{2k}\delta t^{2k} + C_\Pi^{2k}N^{2k(3-m)} + N^{4-2m}} \\
&\quad + C_4C_0^k\delta t^k + C_4C_\Pi^k N^{k(3-m)} + C_4N^{3-m} + C_4\delta t \\
&\le (CC_3 + C_4)C_0^k\delta t^k + (CC_3 + C_4)C_\Pi^k N^{k(3-m)} \\
&\quad + (CC_3 + C_4)N^{3-m} + C_4\delta t \\
&\le C_5\delta t(C_0^k\delta t^{k-1} + 1) + C_5N^{3-m}\big(C_\Pi^k N^{(3-m)(k-1)} + 1\big),
\end{aligned}
$$

where the constant $C_5 := CC_3 + C_4$ is independent of C_0, C_Π, δt and N.

For the cases $k = 2, 3, 4, 5$, we can choose $C_0 = 2C_5$ and $\delta t \le \frac{1}{C_0^k}$ to obtain

$$
C_5(C_0^k\delta t^{k-1} + 1) \le C_5[C_0^k\delta t + 1] \le 2C_5 = C_0, \tag{4.96}
$$

and since $m \ge 4$, we can choose $C_\Pi = 2C_5$ and $N \ge C_\Pi^k$ to obtain

$$
C_5\big(C_\Pi^k N^{(3-m)(k-1)} + 1\big) \le C_5\big(C_\Pi^k N^{3-m} + 1\big) \le 2C_5 = C_\Pi. \tag{4.97}
$$

For the case $k = 1$, since $\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2$, we can estimate $|1 - \xi^{n+1}|$ by exactly the same way as above and (4.95) becomes

$$
|1 - \xi^{n+1}| \le C_5\delta t(C_0^2\delta t + 1) + C_5N^{3-m}\big(C_\Pi^2 N^{3-m} + 1\big). \tag{4.98}
$$

Then we can choose $C_0 = 2C_5$, $\delta t \le \frac{1}{C_0^2}$ and $C_\Pi = 2C_5$ and $N \ge C_\Pi^2$ to obtain

$$
|1 - \xi^{n+1}| \le C_0\delta t + C_\Pi N^{3-m}.
$$

To summarize, combining the above with (4.95), we derive from (4.95) that

$$
|1 - \xi^{n+1}| \le C_0\delta t + C_\Pi N^{3-m}, \tag{4.99}
$$

under the conditions

$$
\delta t \le \frac{1}{2^{k+2}C_0^{k+1} + 1}, \quad N \ge 2^{k+2}C_\Pi^{k+1} + 1, \quad 1 \le k \le 5. \tag{4.100}
$$

Note that the above implies (4.9), and with $C_5 > C_4$, it also implies (4.94). The induction process for (4.5) is complete.

We derive from (3.12f) and (4.76) that

$$
\|\phi_N^{n+1} - \bar{\phi}_N^{n+1}\|_2^2 \le |\eta_k^{n+1} - 1|^2\|\bar{\phi}_N^{n+1}\|_2^2 \le |\eta_k^{n+1} - 1|^2 C, \tag{4.101}
$$

$$\|\boldsymbol{v}_N^{n+1} - \bar{\boldsymbol{v}}_N^{n+1}\|^2 \le |\eta_k^{n+1} - 1|^2 \|\bar{\boldsymbol{v}}_N^{n+1}\|^2 \le |\eta_k^{n+1} - 1|^2 C, \tag{4.102}$$

and

$$\begin{aligned}\delta t \sum_{q=0}^{n} \|\boldsymbol{v}_N^{q+1} - \bar{\boldsymbol{v}}_N^{q+1}\|_1^2 &\le \delta t \sum_{q=0}^{n} |\eta_k^{q+1} - 1|^2 \|\bar{\boldsymbol{v}}_N^{q+1}\|_1^2 \\ &\le \max_q |\eta_k^{q+1} - 1|^2 \delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{v}}_N^{q+1}\|_1^2 \\ &\le \max_q |\eta_k^{q+1} - 1|^2 C.\end{aligned} \tag{4.103}$$

On the other hand, we derive from (4.5) that

$$|\eta_1^{q+1} - 1| \le 2^2 C_0^2 \delta t^2 + 2^2 C_\Pi^2 N^{2(3-m)}, \qquad \forall q \le n \quad k = 1, \tag{4.104a}$$

$$|\eta_k^{q+1} - 1| \le 2^k C_0^k \delta t^k + 2^k C_\Pi^k N^{k(3-m)}, \qquad \forall q \le n \quad k = 2, 3, 4, 5. \tag{4.104b}$$

Therefore, we derive from (4.72), (4.73), (4.75), (4.101)-(4.104) and the triangle inequality that

$$\|e_\phi^{n+1}\|_2^2 \le \|\bar{e}_\phi^{n+1}\|_2^2 + \|\phi_N^{n+1} - \bar{\phi}_N^{n+1}\|_2^2,$$

$$\|\boldsymbol{e}_{\boldsymbol{v}}^{n+1}\|^2 \le \|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{n+1}\|^2 + \|\boldsymbol{v}_N^{n+1} - \bar{\boldsymbol{v}}_N^{n+1}\|^2,$$

and

$$\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2 \le \|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^{q+1}\|_1^2 + \|\boldsymbol{v}_N^{q+1} - \bar{\boldsymbol{v}}_N^{q+1}\|_1^2, \quad \forall q \le n,$$

under the condition (4.100) on δt and N. The proof is now complete since we already proved (4.72), (4.73) and (4.75). □

Using exactly the same procedure above without the spatial discretization, we can prove the following result for the semi-discrete schemes (3.5).

Corollary 1. *Let $d = 2$, $T > 0$, $\boldsymbol{v}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^2$ and ϕ, $\boldsymbol{v}$ be the solution of (1.1). We assume that $\bar{\phi}_N^i$, ϕ_N^i, $\bar{\boldsymbol{v}}_N^i$ and $\boldsymbol{v}_N^i$ $(i = 1, \cdots, k-1)$ are computed with a proper initialization procedure such that*

(4.105)

$$\begin{aligned}&\|\bar{\boldsymbol{v}}^i - \boldsymbol{v}(\cdot, t_i)\|,\ \|\boldsymbol{v}^i - \boldsymbol{v}(t_i)\| = O(\delta t^k),\\ &\|\bar{\boldsymbol{v}}^i - \boldsymbol{v}(\cdot, t_i)\|_1,\ \|\boldsymbol{v}^i - \boldsymbol{v}(t_i)\|_1,\ \|\bar{\phi}^i - \phi(\cdot, t_i)\|_1,\ \|\phi^i - \phi(t_i)\|_1 = O(\delta t^k),\\ &\|\bar{\phi}^i - \phi(\cdot, t_i)\|_2,\ \|\phi^i - \phi(t_i)\|_2 = O(\delta t^k),\end{aligned}$$

for $i = 1, 2, 3, 4, 5$. Let $\bar{\phi}^{n+1}$, ϕ^{n+1}, $\bar{\boldsymbol{v}}^{n+1}$ and $\boldsymbol{v}^{n+1}$ be computed with the kth-order scheme (3.5) ($1 \le k \le 5$), and

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k \ (k = 2, 3, 4, 5).$$

We assume (4.1), (4.2) and (4.3). Then for $n + 1 \le T/\delta t$ with $\delta t \le \frac{1}{1+2^{k+2}C_0^{k+1}}$, we have

$$\|\bar{\phi}^n - \phi(\cdot, t^n)\|_2^2, \ \|\phi^n - \phi(\cdot, t^n)\|_2^2 \le C\delta t^{2k}, \tag{4.106}$$

$$\|\bar{\boldsymbol{v}}^n - \boldsymbol{v}(\cdot, t^n)\|^2, \ \|\boldsymbol{v}^n - \boldsymbol{v}(\cdot, t^n)\|^2 \le C\delta t^{2k}, \tag{4.107}$$

and

$$\delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{v}}^{q+1} - \boldsymbol{v}(\cdot, t^{q+1})\|_1^2, \ \delta t \sum_{q=0}^{n} \|\boldsymbol{v}^{q+1} - \boldsymbol{v}(\cdot, t^{q+1})\|_1^2 \le C\delta t^{2k}, \tag{4.108}$$

where the constants C_0 and C are dependent on T, Ω, the $k \times k$ matrix $G = (g_{ij})$ in Lemma 4 and the exact solution ϕ, $\boldsymbol{v}$, but are independent of δt.

4.3. Error analysis for the phase function and the velocity in 3D

In the three-dimensional case, (4.62) no longer holds true. Instead, we shall derive local estimates with a stronger norm for the velocity in analogy to the local existence of strong solution for the 3-D Cahn-Hilliard Navier-Stokes equations [5].

Theorem 3. *Let $d = 3$, $T > 0$, $\boldsymbol{v}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^m$, $\phi_0 \in \boldsymbol{H}_p^m$, $m \ge 4$. We assume that (1.1) admits a unique strong solution $\boldsymbol{v}$ in $C([0, T]; \boldsymbol{H}_p^1) \cap L^2(0, T; \boldsymbol{H}_p^2)$. We assume (4.4), (4.2) and (4.3) as in* **Theorem 2**, *and Let $\bar{\phi}_N^{n+1}$, ϕ_N^{n+1}, $\bar{\boldsymbol{v}}_N^{n+1}$ and $\boldsymbol{v}_N^{n+1}$ be computed with the kth-order scheme (3.12) ($1 \le k \le 5$), and*

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k \ (k = 2, 3, 4, 5).$$

Then, there exits $T_* > 0$ such that for $0 < T < T_*$, $n + 1 \le T/\delta t$ and $\delta t \le \frac{1}{1+2^{k+2}C_0^{k+1}}$, $N \ge 2^{k+2}C_{\Pi}^{k+1} + 1$, we have

$$\|\bar{\phi}_N^n - \phi(\cdot, t^n)\|_2^2, \ \|\phi_N^n - \phi(\cdot, t^n)\|_2^2 \le C\delta t^{2k} + CN^{2(2-m)}, \tag{4.109}$$

(4.110) $$\|\bar{\boldsymbol{v}}_N^n - \boldsymbol{v}(\cdot,t^n)\|^2,\ \|\boldsymbol{v}_N^n - \boldsymbol{v}(\cdot,t^n)\|^2 \le C\delta t^{2k} + CN^{-2m},$$

and

(4.111) $$\delta t\sum_{q=0}^{n}\|\bar{\boldsymbol{v}}_N^{q+1} - \boldsymbol{v}(\cdot,t^{q+1})\|_1^2,\ \delta t\sum_{q=0}^{n}\|\boldsymbol{v}_N^{q+1} - \boldsymbol{v}(\cdot,t^{q+1})\|_1^2 \le C\delta t^{2k} + CN^{2(1-m)},$$

where the constants C_0, C_Π and C are dependent on T, Ω, the $k\times k$ matrix $G=(g_{ij})$ in Lemma 4 and the exact solution ϕ, $\boldsymbol{v}$, but are independent of δt and N.

Proof. The proof follows essentially the same procedure as the proof for **Theorem 2**. However, since (2.4a) is not valid when $d=3$, we have to derive an alternative for (4.62). To simplify the presentation, we shall only point out below how to derive an alternative for (4.62) in **step 2** of **Theorem 2**.

In **Step 1**, we still assume (4.6) holds and choose δt and N satisfies (4.9). Let $v_N = -\Delta\bar{\boldsymbol{v}}_N^{n+1} + \tau_k\Delta\bar{\boldsymbol{v}}_N^n$ in (3.12c), it follows from Lemma 4 that

(4.112)
$$\begin{aligned}
&\sum_{i,j=1}^{k} g_{ij}(\nabla\bar{\boldsymbol{v}}_N^{q+1+i-k},\nabla\bar{\boldsymbol{v}}_N^{q+1+j-k}) - \sum_{i,j=1}^{k} g_{ij}(\nabla\bar{\boldsymbol{v}}_N^{q+i-k},\nabla\bar{\boldsymbol{v}}_N^{q+j-k})\\
&+\Big\|\sum_{i=0}^{k}\delta_i\nabla\bar{\boldsymbol{v}}_N^{q+1+i-k}\Big\|^2 + \delta t\|\Delta\bar{\boldsymbol{v}}_N^{q+1}\|^2\\
=&\delta t(\Delta\bar{\boldsymbol{v}}_N^{q+1},\tau_k\Delta\bar{\boldsymbol{v}}_N^{q}) + \delta t\Big(\mathbf{A}\big((B_k(\boldsymbol{v}_N^q)\cdot\nabla)B_k(\boldsymbol{v}_N^q)\big), -\Delta\bar{\boldsymbol{v}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{v}}_N^{q}\Big)\\
&-\delta t\Big(\mathbf{A}\big((B_k(\mu_N^q)\cdot\nabla B_k(\phi_N^q)\big), -\Delta\bar{\boldsymbol{v}}_N^{q+1} + \tau_k\Delta\bar{\boldsymbol{v}}_N^{q})\Big).
\end{aligned}$$

We now bound the right hand side of (4.112). Note that (4.9) implies

(4.113) $$\frac12 < 1-\Big(\frac{\delta t^{k-1}}{4}+\frac{N^{k(3-m)+1}}{4}\Big) \le |\eta_k^q| \le 1+\frac{\delta t^{k-1}}{4}+\frac{N^{k(3-m)+1}}{4} < 2,\ \forall q\le n.$$

First, we have

(4.114) $$|\delta t(\Delta\bar{\boldsymbol{v}}_N^{q+1},\tau_k\Delta\bar{\boldsymbol{v}}_N^{q})| \le \frac{\delta t}{2}\|\Delta\bar{\boldsymbol{v}}_N^{q+1}\|^2 + \frac{\delta t\tau_k}{2}\|\Delta\bar{\boldsymbol{v}}_N^{q}\|^2.$$

Next, it follows from (2.4b) that

$$
\begin{aligned}
&|(\mathbf{A}\big((B_k(\boldsymbol{v}_N^q)\cdot\nabla)B_k(\boldsymbol{v}_N^q)\big),-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q)|\\
\le& C\|B_k(\boldsymbol{v}_N^q)\|_1\|B_k(\nabla\boldsymbol{v}_N^q)\|_{1/2}\|-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q\|\\
\le& C\|B_k(\boldsymbol{v}_N^q)\|_1\|B_k(\boldsymbol{v}_N^q)\|_1^{1/2}\|B_k(\boldsymbol{v}_N^q)\|_2^{1/2}\|-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q\|\\
\le& C(\varepsilon)\|B_k(\boldsymbol{v}_N^q)\|_1^3\|B_k(\boldsymbol{v}_N^q)\|_2+\varepsilon\|-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q\|^2\\
\le& C(\varepsilon)\|B_k(\boldsymbol{v}_N^q)\|_1^6+\varepsilon\|B_k(\boldsymbol{v}_N^q)\|_2^2+2\varepsilon\|\Delta\bar{\boldsymbol{v}}_N^{q+1}\|^2+2\varepsilon\|\Delta\bar{\boldsymbol{v}}_N^q\|^2,
\end{aligned}
\tag{4.115}
$$

and with the help of (4.13), (4.18) and (4.24), we have

$$
\begin{aligned}
&|\mathbf{A}\big((B_k(\mu_N^q)\cdot\nabla B_k(\phi_N^q)\big),-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q)|\\
\le& C\|B_k(\mu_N^q)\|_{L^4}\|\nabla B_k(\phi_N^q)\|_{L^4}\|-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q\|\\
\le& C\|B_k(\mu_N^q)\|_1\|\nabla B_k(\phi_N^q)\|_1\|-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q\|\\
\le& C(\varepsilon)\|B_k(\mu_N^q)\|_1^2+\varepsilon\|-\Delta\bar{\boldsymbol{v}}_N^{q+1}+\tau_k\Delta\bar{\boldsymbol{v}}_N^q\|^2\\
\le& C(\varepsilon)\|B_k(\mu_N^q)\|_1^2+2\varepsilon\|\Delta\bar{\boldsymbol{v}}_N^{q+1}\|^2+2\varepsilon\|\Delta\bar{\boldsymbol{v}}_N^q\|^2.
\end{aligned}
\tag{4.116}
$$

Now, combining (4.112)-(4.116) and noting that $\boldsymbol{v}_N^q=\eta_k^q\bar{\boldsymbol{v}}_N^q$, we find after dropping some unnecessary terms that

$$
\begin{aligned}
&\sum_{i,j=1}^k g_{ij}(\nabla\bar{\boldsymbol{v}}_N^{q+1+i-k},\nabla\bar{\boldsymbol{v}}_N^{q+1+j-k})-\sum_{i,j=1}^k g_{ij}(\nabla\bar{\boldsymbol{v}}_N^{q+i-k},\nabla\bar{\boldsymbol{v}}_N^{q+j-k})\\
&+\delta t(\frac{1}{2}-4\varepsilon)\|\Delta\bar{\boldsymbol{v}}_N^{q+1}\|^2\\
\le&\delta t(\frac{\tau_k}{2}+4\varepsilon)\|\Delta\bar{\boldsymbol{v}}_N^q\|^2+\varepsilon\delta t\|B_k(\boldsymbol{v}_N^q)\|_2^2+C(\varepsilon)\delta t\|B_k(\boldsymbol{v}_N^q)\|_1^6\\
&+C(\varepsilon)\delta t\|B_k(\mu_N^q)\|_1^2\\
\le&\delta t(\frac{\tau_k}{2}+4\varepsilon)\|\Delta\bar{\boldsymbol{v}}_N^q\|^2+2^2\varepsilon\delta t\|B_k(\bar{\boldsymbol{v}}_N^q)\|_2^2+2^6C(\varepsilon)\delta t\|B_k(\bar{\boldsymbol{v}}_N^q)\|_1^6\\
&+C(\varepsilon)\delta t\|B_k(\mu_N^q)\|_1^2.
\end{aligned}
\tag{4.117}
$$

It follows from (4.13) and (4.24) that

$$
\delta t\sum_{q=0}^{n-1}\|B_k(\mu_N^q)\|_1^2\le C.
$$

Now, taking the sum of (4.117) for q from $k-1$ to $n-1$, noting that $G=(g_{ij})$ is a symmetric positive definite matrix with the minimum eigenvalue

λ_G and $\tau_k < 1$, we can choose ε small enough such that:

$$\begin{aligned}
&\lambda_G\|\bar{\boldsymbol{v}}_N^n\|_1^2 + \frac{\delta t(1-\tau_k)}{4}\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{v}}_N^q\|^2 \\
\leq &\sum_{i,j=1}^{k} g_{ij}(\nabla\bar{\boldsymbol{v}}^{n+i-k}, \nabla\bar{\boldsymbol{v}}^{n+j-k}) + \frac{\delta t(1-\tau_k)}{4}\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{v}}_N^q\|^2 \\
\leq &C\delta t\sum_{q=0}^{n-1}\|\bar{\boldsymbol{v}}_N^q\|_1^6 + M_0,
\end{aligned}$$

where $M_0 > 0$ is a constant independent of δt, N, C_0 and C_Π. If we define Φ as $\Phi(x) = x^3$ and let

$$0 < T_* < \int_{M_0}^{\infty} dz/\Phi(z), \tag{4.118}$$

then Lemma 5 implies that there exists $C_* > 0$ independent of δt, N, C_0 and C_Π such that

$$\|\bar{\boldsymbol{v}}_N^n\|_1^2 + \delta t\sum_{q=0}^{n}\|\Delta\bar{\boldsymbol{v}}_N^q\|^2 \leq C_*. \tag{4.119}$$

Noting that $\boldsymbol{v}_N^q = \eta_k^q\bar{\boldsymbol{v}}_N^q$ and $\frac{1}{2} < |\eta_k^q| < 2$, we also have

$$\|\boldsymbol{v}_N^n\|_1^2 + \delta t\sum_{q=0}^{n}\|\Delta\boldsymbol{v}_N^q\|^2 \leq 2C_*. \tag{4.120}$$

In the three-dimensional case, we can bound (4.62) by using (2.5) as

$$\begin{aligned}
&\left|\Big(\mathbf{A}\big(B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\cdot\nabla B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\big), \bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1} - \tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\Big)\right| \\
\leq &C\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_2\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1} - \tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1 \\
\leq &C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_2^2 + \varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1} - \tau_k\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2 \\
\leq &C(\varepsilon)\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|^2\|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_2^2 + 2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^{q+1}\|_1^2 + 2\varepsilon\|\bar{\boldsymbol{e}}_{\boldsymbol{v},N}^q\|_1^2.
\end{aligned} \tag{4.121}$$

It follows from (4.120) that there exists a constant C independent of δt, N,

C_0 and C_Π such that

$$\delta t \sum_{q=0}^{n} \|B_k(\boldsymbol{e}_{\boldsymbol{v}}^q)\|_2^2 \le C. \tag{4.122}$$

Now, with (4.122) holding true, we can then prove (4.109), (4.110) and (4.111) by following the same procedures in **Step 2** and **Step 3** in the proof of **Theorem 2**. □

Similarly, we can prove the following result for the semi-discrete scheme (3.5).

Corollary 2. *Let $d = 3$, $T > 0$, $\boldsymbol{v}_0 \in \boldsymbol{V} \cap \boldsymbol{H}_p^m$, $\phi \in \boldsymbol{H}_p^m$, $m \ge 4$. We assume that (1.1) admits a unique strong solution $\boldsymbol{v}$ in $C([0,T];\boldsymbol{H}_p^1) \cap L^2(0,T;\boldsymbol{H}_p^2)$. We assume (4.4), (4.2) and (4.3) as in* **Theorem 2**, *and Let $\bar{\phi}^{n+1}$, ϕ^{n+1}, $\bar{\boldsymbol{v}}^{n+1}$ and $\boldsymbol{v}^{n+1}$ be computed with the kth-order scheme (3.5) $(1 \le k \le 5)$, and*

$$\eta_1^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad \eta_k^{n+1} = 1 - (1 - \xi^{n+1})^k \ (k = 2, 3, 4, 5).$$

Then, there exits $T_* > 0$ such that for $0 < T < T_*$, $n + 1 \le T/\delta t$ and $\delta t \le \frac{1}{1+2^{k+2}C_0^{k+1}}$, we have

$$\|\bar{\phi}^n - \phi(\cdot, t^n)\|_2^2,\ \|\phi^n - \phi(\cdot, t^n)\|_2^2 \le C\delta t^{2k}, \tag{4.123}$$

$$\|\bar{\boldsymbol{v}}^n - \boldsymbol{v}(\cdot, t^n)\|^2,\ \|\boldsymbol{v}^n - \boldsymbol{v}(\cdot, t^n)\|^2 \le C\delta t^{2k}, \tag{4.124}$$

and

$$\delta t \sum_{q=0}^{n} \|\bar{\boldsymbol{v}}^{q+1} - \boldsymbol{v}(\cdot, t^{q+1})\|_1^2,\ \delta t \sum_{q=0}^{n} \|\boldsymbol{v}^{q+1} - \boldsymbol{v}(\cdot, t^{q+1})\|_1^2 \le C\delta t^{2k},$$

where the constants C_0 and C are dependent on T, Ω, the $k \times k$ matrix $G = (g_{ij})$ in Lemma 4 and the exact solution ϕ, $\boldsymbol{v}$, but are independent of δt.

4.4. Error analysis for the pressure

With the established error estimates for the velocity $\boldsymbol{v}$ and ϕ, the error estimate for the pressure p can be derived directly from (3.6) or (3.13).

We denote

$$e_{pN}^n := p_N^n - \Pi_N p(\cdot,t^n),\ e_{p\Pi}^n := \Pi_N p(\cdot,t^n) - p(\cdot,t^n),\ \text{ and } e_p^n = e_{pN}^n + e_{p\Pi}^n.$$

Theorem 4. *Under the same assumptions as in* **Theorem 2** *and* **Theorem 3**, *we have*

(4.125)
$$\delta t\sum_{q=0}^{n-1}\|p_N^{q+1} - p(\cdot,t^{q+1})\|^2 \le \begin{cases} C\delta t^{2k} + CN^{2(2-m)},\ \forall n\le T/\delta t, & d=2,\\ C\delta t^{2k} + CN^{2(2-m)},\ \forall n\le T_*/\delta t, & d=3,\end{cases}$$

where p_N^{n+1} is computed from (3.13), T_* is defined in (4.118) and C is a constant independent of δt and N.

Proof. From (3.13), we can write down the error equation for p_N^{q+1} with $q\le n-1$ as

(4.126)
$$\begin{aligned}\big(\nabla e_p^{q+1},\nabla \boldsymbol{\psi}_N\big) =&\big(\boldsymbol{v}_N^{q+1}\cdot\nabla\boldsymbol{v}_N^{q+1} - \boldsymbol{v}(t^{q+1})\cdot\nabla\boldsymbol{v}(t^{q+1}),\nabla\boldsymbol{\psi}_N\big)\\ &- \big(\mu_N^{q+1}\nabla\phi_N^{q+1} - \mu(t^{q+1})\nabla\phi(t^{q+1}),\nabla\boldsymbol{\psi}_N\big),\ \forall\boldsymbol{\psi}_N\in\boldsymbol{S}_N.\end{aligned}$$

To prove (4.125), we set $\boldsymbol{\psi}_N = \Delta^{-1}e_{pN}^{q+1}$ in (4.126) to obtain

(4.127)
$$\begin{aligned}\|e_{pN}^{q+1}\|^2 =&\Big(\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\cdot\nabla\boldsymbol{e}_{\boldsymbol{v}}^{q+1},\Delta^{-\frac12}e_{pN}^{q+1}\Big) + \Big(\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\cdot\nabla\boldsymbol{v}(t^{q+1}),\Delta^{-\frac12}e_{pN}^{q+1}\Big)\\ &+\Big(\boldsymbol{v}(t^{q+1})\cdot\nabla\boldsymbol{e}_{\boldsymbol{v}}^{q+1},\Delta^{-\frac12}e_{pN}^{q+1}\Big) + \Big(e_{\mu}^{q+1}\cdot\nabla\phi_N^{q+1},\Delta^{-\frac12}e_{pN}^{q+1}\Big)\\ &+\Big(\mu(t^{q+1})\cdot\nabla e_{\phi}^{q+1},\Delta^{-\frac12}e_{pN}^{q+1}\Big).\end{aligned}$$

We can bound the righthand side of (4.127) by using the stability result in Theorem 1 and error estimate for the phase function and velocity. We need to deal with the first term on the right hand side of (4.127) by different ways in 2D case and 3D case separately.

In the case d=2, we can make use of (2.4a) and obtain

(4.128)
$$\begin{aligned}&\Big|\Big(\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\cdot\nabla\boldsymbol{e}_{\boldsymbol{v}}^{q+1},\Delta^{-\frac12}e_{pN}^{q+1}\Big)\Big|\\ \le& C\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^{1/2}\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^{1/2}\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^{1/2}\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^{1/2}\|\Delta^{-\frac12}e_{pN}^{q+1}\|_1\\ \le& C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^2\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2 + \varepsilon\|\Delta^{-\frac12}e_{pN}^{q+1}\|_1^2\end{aligned}$$

$$\le C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^2\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2+\varepsilon\|e_{pN}^{q+1}\|^2$$
$$\le C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2+\varepsilon\|e_{pN}^{q+1}\|^2.$$

In the case d=3, it follows from (4.120) and the regularity of the exact solution that there exists a constant C independent of δt and N,

$$\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2\le C,\quad \forall q+1\le n\le T_*/\delta t,$$

then we can make use of (2.5) and obtain

$$\begin{aligned}
&\left|\left(\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\cdot\nabla\boldsymbol{e}_{\boldsymbol{v}}^{q+1},\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\right)\right|\\
\le& C\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_1\\
\le& C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2+\varepsilon\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_1^2\\
\le& C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2+\varepsilon\|e_{pN}^{q+1}\|^2\\
\le& C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|_1^2+\varepsilon\|e_{pN}^{q+1}\|^2,\quad \forall q+1\le n\le T_*/\delta t,\quad \text{for d=3.}
\end{aligned}\tag{4.129}$$

For the other terms on the right hand side of (4.127), we have

$$\begin{aligned}
\left|\left(\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\cdot\nabla\boldsymbol{v}(t^{q+1}),\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\right)\right|&\le C\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|\|\boldsymbol{v}(t^{q+1})\|_2\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_1\\
&\le C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^2\|\boldsymbol{v}(t^{q+1})\|_2^2+\varepsilon\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_1^2\\
&\le C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^2+\varepsilon\|e_{pN}^{q+1}\|^2,
\end{aligned}\tag{4.130}$$

and similarly,

$$\left|\left(\boldsymbol{v}(t^{q+1})\cdot\nabla\boldsymbol{e}_{\boldsymbol{v}}^{q+1},\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\right)\right|\le C(\varepsilon)\|\boldsymbol{e}_{\boldsymbol{v}}^{q+1}\|^2+\varepsilon\|e_{pN}^{q+1}\|^2.\tag{4.131}$$

On the other hand, by using (4.37), we have

$$\begin{aligned}
\left|\left(e_{\mu}^{q+1}\cdot\nabla\phi_N^{q+1},\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\right)\right|&\le C\|e_{\mu}^{q+1}\|\|\nabla\phi_N^{q+1}\|_{L^4}\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_{L^4}\\
&\le C\|e_{\mu}^{q+1}\|\|\nabla\phi_N^{q+1}\|_1\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_1\\
&\le C(\varepsilon)\|e_{\mu}^{q+1}\|^2\|\nabla\phi_N^{q+1}\|_1^2+\varepsilon\|\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\|_1^2\\
&\le C(\varepsilon)\|e_{\mu}^{q+1}\|^2+\varepsilon\|e_{pN}^{q+1}\|^2,
\end{aligned}\tag{4.132}$$

and similarly,

$$\left|\left(\mu(t^{q+1})\cdot\nabla e_{\phi}^{q+1},\Delta^{-\frac{1}{2}}e_{pN}^{q+1}\right)\right|\leq C(\varepsilon)\|\nabla e_{\phi}^{q+1}\|_1^2+\varepsilon\|e_{pN}^{q+1}\|^2. \tag{4.133}$$

Combining (4.127)-(4.133) with $\varepsilon=\frac{1}{6}$ and using the estimate in Theorem 2, we obtain

$$\begin{aligned}\delta t\sum_{q=0}^{n-1}\|e_{pN}^{q+1}\|^2 &\leq C\delta t\sum_{q=0}^{n-1}\|\bar{e}_{\boldsymbol{v}}^{q+1}\|_1^2+C\delta t\sum_{q=0}^{n-1}\|e_{\mu}^{q+1}\|^2+C\delta t\sum_{q=0}^{n-1}\|\nabla e_{\phi}^{q+1}\|_1^2\\ &\leq\begin{cases}C\delta t^{2k}+CN^{2(2-m)}, \ \forall n\leq T/\delta t, & d=2,\\ C\delta t^{2k}+CN^{2(2-m)}, \ \forall n\leq T_*/\delta t, & d=3.\end{cases}\end{aligned} \tag{4.134}$$

Finally, we can obtain (4.125) from (4.134) and

$$\|e_{p\Pi}^{q}\|^2\leq CN^{-2m},\forall q\leq n. \qquad \square$$

Similarly, we can derive the following results for the semi-discrete scheme (3.5).

Corollary 3. *Under the same assumptions as in* **Corollary 1 and Corollary 2**, *we have*

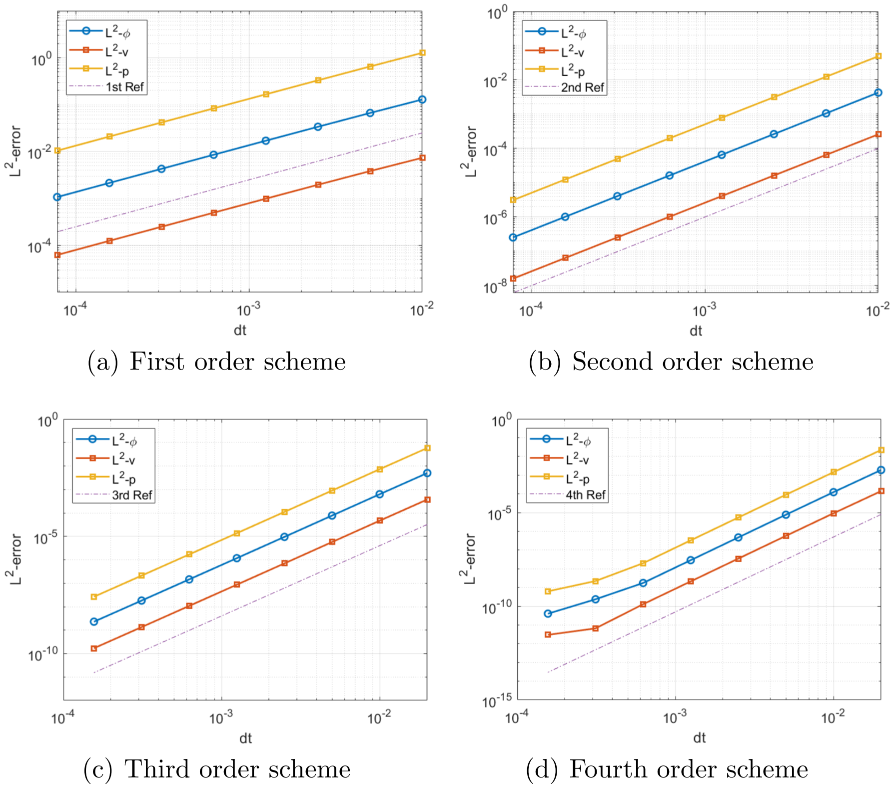
$$\delta t\sum_{q=0}^{n-1}\|p^{q+1}-p(\cdot,t^{q+1})\|^2\leq\begin{cases}C\delta t^{2k}, \ \forall n\leq T/\delta t, & d=2,\\ C\delta t^{2k}, \ \forall n\leq T_*/\delta t, & d=3,\end{cases}$$

where p^{n+1} is computed from (3.6), T_* is defined in (4.118) and C is a constant independent of δt.

5. Numerical examples

Example 1: Convergence test. Consider the Cahn-Hilliard Navier-Stokes system (1.1) in $\Omega=(-1,1)\times(-1,1)$ with periodic boundary condition and the initial conditions are given as

$$\begin{aligned}\phi(x,y,0)&=\sin(\pi x)\cos(\pi y),\\ \boldsymbol{v}_1(x,y,0)&=\sin(\pi x)\sin(\pi y),\\ \boldsymbol{v}_2(x,y,0)&=\cos(\pi x)\cos(\pi y).\end{aligned}$$



(a) First order scheme (b) Second order scheme

(c) Third order scheme (d) Fourth order scheme

Figure 1: Convergence test for the Navier-Stokes Cahn-Hilliard systems using SAV/BDF k ($k = 1, 2, 3, 4$).

We set $M = 10^{-4}$, $\nu = 1$, $\lambda = 0.02$, $\varepsilon = 0.02$, $\gamma = 1$ in (1.1), and use the Fourier spectral method with 128×128 modes for space discretization so that the spatial discretization error is negligible with respect to the time discretization error. The reference solution is generated by the fourth-order scheme with $\delta t = 0.0002$. In Figures 1, we plot the convergence rate of the L^2 error for the phase function ϕ, velocity $\boldsymbol{v}$ and the pressure p at $T = 1$ by using first- to fourth-order schemes. We observe the expected convergence rates for all the cases.

Example 2: Surface tension effects. In this example, we start with a square fluid bubble in the domain $(-1, 1) \times (-1, 1)$ and fix $M = 10^{-4}, \nu = 0.1, \varepsilon = \lambda = 0.02, \delta t = 0.002$. We adopt the second order version of the scheme (3.12) and use the Fourier spectral method with 128×128 modes for space discretization. The initial condition for the phase function is given

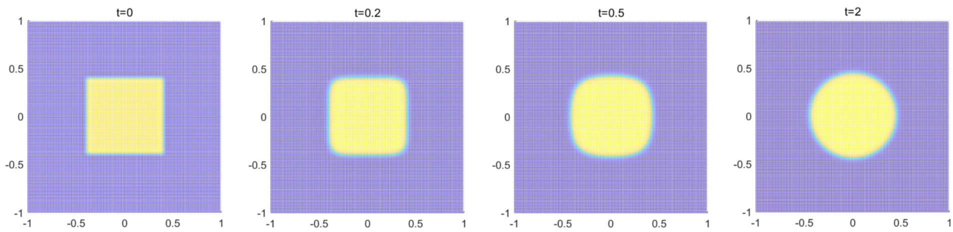


Figure 2: Example 2: phase evolution at t=0, 0.2, 0.5, 2 with $\gamma = 0.1$.

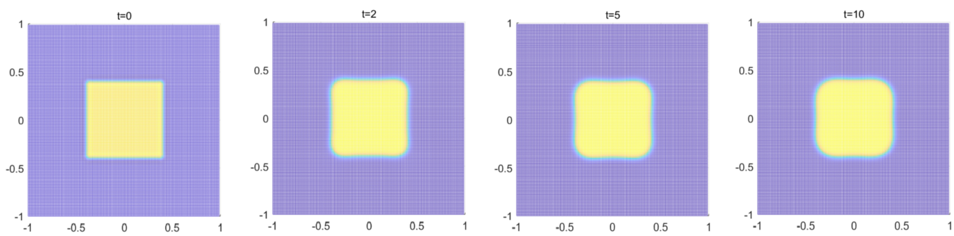


Figure 3: Example 2: phase evolution at t=0, 2, 5, 10 with $\gamma = 0$.

as

$$\phi(x,y,0) = \begin{cases} -1, & |x| > 0.4 \text{ or } |y| > 0.4, \\ 1, & \text{otherwise}, \end{cases}$$

and the initial velocity and pressure are set to be zero. In Figure 2, we choose $\gamma = 0.1$ and the square bubble quickly deforms into a circular bubble due to the surface tension. On the other hand, we choose $\gamma = 0$ (i.e. no fluid in the system) in Figure 3, the bubble will deform into a circular bubble.

Example 3: Boussinesq approximation. In this example, we use the Boussinesq approximation to model the case where the two fluids have different densities [31]. We rewrite (1.1c) as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v} + \gamma \mu \nabla \phi - \boldsymbol{g}(\rho_1 + \rho_2 - 2\rho_0) - \phi \boldsymbol{g}(\rho_1 - \rho_2), \tag{5.1}$$

where the constant ρ_0 is treated as the "background" density, ρ_1, ρ_2 are the corresponding density for two fluids and $\boldsymbol{g}$ is the gravitational acceleration. In the following, we set $\rho_1 - \rho_2 = -1$ and the term $\boldsymbol{g}(\rho_1 + \rho_2 - 2\rho_0)$ is a constant vector which can be absorbed into the pressure. We start with a circular bubble near the bottom of the domain $(-1,1) \times (-2,2)$ and fix $M =$

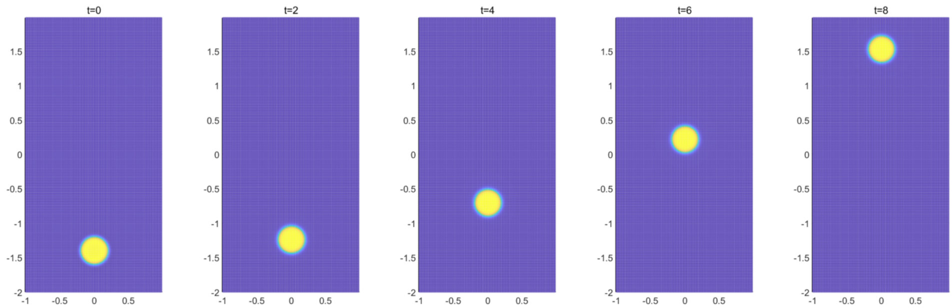


Figure 4: Example 3: Case A, $\boldsymbol{g} = (0, 0.1)^T$, $\nu = 0.1$ with $\delta t = 0.005$.

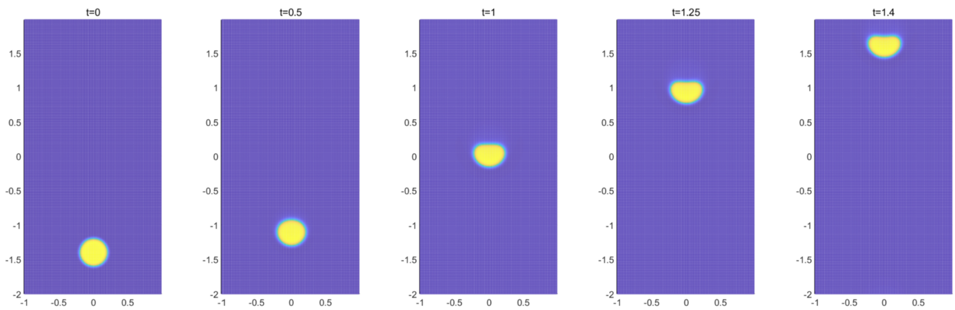


Figure 5: Example 3: Case B, $\boldsymbol{g} = (0, 4)^T$, $\nu = 0.1$ with $\delta t = 0.001$.

$10^{-4}, \nu = 0.1, \varepsilon = \lambda = 0.02, \gamma = 0.1$. We adopt the third order version of the scheme (3.12) and use the Fourier spectral method with 128×256 modes for space discretization. In Case A, we choose the gravitational constant vector $\boldsymbol{g} = (0, 0.1)^T$ and $\delta t = 0.005$, we can see the bubble rises due to the gravity differential but without noticeable shape deformation in Figure 4. In Case B, we choose $\boldsymbol{g} = (0, 4)^T$ and $\delta t = 0.001$, we can see the bubble rises with noticeable shape deformation in Figure 5.

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