The variation of the maximal function of a radial function

Hannes Luiro

Abstract. It is shown for the non-centered Hardy-Littlewood maximal operator $M$ that $\|DMf\|_1 \leq C_n \|Df\|_1$ for all radial functions in $W^{1,1}(\mathbb{R}^n)$.

1. Introduction

The non-centered Hardy-Littlewood maximal operator $M$ is defined by setting for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ that

\begin{equation}
Mf(x) = \sup_{B(z,r) \ni x} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(y)| \, dy =: \sup_{B(z,r) \ni x} \int_{B(z,r)} |f(y)| \, dy
\end{equation}

for every $x \in \mathbb{R}^n$. The centered version of $M$, denoted by $M_c$, is defined by taking the supremum over all balls centered at $x$. The classical theorem of Hardy, Littlewood and Wiener asserts that $M$ (and $M_c$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. This result is one of the cornerstones of the harmonic analysis. While the absolute size of a maximal function is usually the principal interest, the applications in Sobolev-spaces and in the potential theory have motivated the active research of the regularity properties of maximal functions. The first observation was made by Kinnunen who verified $[Ki]$ that $M_c$ is bounded in Sobolev-space $W^{1,p}(\mathbb{R}^n)$ if $1 < p \leq \infty$, and inequality

\begin{equation}
|DM_c f(x)| \leq M_c(|Df|)(x)
\end{equation}

holds for all $x \in \mathbb{R}^n$. The proof is relatively simple and inequality (1.2) (and the boundedness) holds also for $M$ and many other variants.

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The most challenging open problem in this field is so-called ‘$W^{1,1}$-problem’: Does it hold for all $f \in W^{1,1}(\mathbb{R}^n)$, that $DMf \in L^1(\mathbb{R}^n)$ and

$$\|DMf\|_1 \leq C_n \|Df\|_1?$$

This problem has been discussed (and studied) for example in [AlPe], [CaHu], [CaMa], [HO], [HM], [Ku] and [Ta]. The fundamental obstacle is that $M$ is not bounded in $L^1$ and therefore inequality (1.2) is not enough to solve the problem. In the case $n=1$ the answer is known to be positive, as was proved by Tanaka [Ta]. For $M_c$ the problem turns out to be very complicated also when $n=1$; however, Kurka [Ku] managed to show that the answer is positive also in this case.

The goal of this paper is to develop technology for $W^{1,1}$-problem in higher dimensions, where the problem is still completely open. The known proofs in the one-dimensional case are strongly based on the simplicity of the topology: the crucial trick (in the non-centered case) is that $Mf$ does not have a strong local maximum (Definition 3.7) outside the set $\{Mf(x) = f(x)\}$. This fact is a strong tool when $n=1$ but is far from sufficient for higher dimensions.

The formula for the derivative of the maximal function (see Lemma 2.2 or [L]) has an important role in the paper. It says that if $Mf(x) = \int_B |f|$, $|f(x)| < Mf(x) < \infty$, and $Mf$ is differentiable at $x$, then

$$DMf(x) = \int_B Df(y) \, dy.$$  

From this formula one can see immediately the validity of the estimate (1.2) for $M$; however, since $B$ is exactly the ball which gives the maximal average (for $|f|$), it is expected that one can derive from (1.3) much more sophisticated estimates than (1.2). In Section 2 (Lemma 2.2), we perform basic analysis related to this issue. The key observation we make is that if $B$ is as above, then

$$\int_B Df(y) \cdot (y-x) \, dy = 0.$$  

In the backround of this equality stands a more general principle, concerning other maximal operators as well: if the value of the maximal function is attained to ball (or other permissible object) $B$, then the weighted integral of $|Df|$ over $B$ is zero for a set of weights depending on the maximal operator. We believe that the utilization of this principle is a key for a possible solution of $W^{1,1}$-problem.

As the main result of this paper, we employ equality (1.4) to show that in the case of radial functions the answer to $W^{1,1}$-problem is positive (Theorem 3.12). Even in this case, the problem is evidently non-trivial and truly differs from the one-dimensional case. To become convinced about this, consider the important special
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Case where $f$ is radially decreasing ($f(x) = g(|x|)$, where $g: [0, \infty) \rightarrow \mathbb{R}$ is decreasing). In this case, $Mf$ is radially decreasing as well and $Mf(0) = f(0)$. If $n = 1$, these facts immediately imply that $\|DMf\|_1 = \|Df\|_1$, but if $n \geq 2$ this is definitely not the case: the additional estimates are necessary. This type of estimate for radially decreasing functions can be derived from (1.3) and (1.4), saying that

$$|DMf(x)| \leq C_n \frac{1}{|x|} \int_{B(0, |x|)} |Df(y)||y| \, dy.$$  

By using this inequality, the positive answer to $W^{1,1}$-problem for radially decreasing functions follows straightforwardly by Fubini Theorem (Theorem 3.4).

For general radial functions, inequality (1.5) turns out to hold only if the maximal average is achieved in a ball with radius comparable to $|x|$. To overcome this problem, we study the auxiliary maximal function $M^I$, defined for $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$M^I f(x) = \sup_{x \in B(z, r), r \leq |z|/4} \int_{B(z, r)} |f(y)| \, dy,$$

and prove (Lemma 3.5) that for all radial $f \in W^{1,1}(\mathbb{R}^n)$ it holds that

$$\|DM^I f\|_1 \leq C_n \|Df\|_1.$$  

The proof of this auxiliary result resembles the proof of $W^{1,1}$-problem (for $M$) in the case $n = 1$. Recall again that in the case $n = 1$ the key is that $Mf$ does not have a strong local maximum in $\{Mf(x) > |f(x)|\}$. As a multidimensional counterpart for radial functions, we show that $M^I f$ does not have a strong local maximum in $\{M^I f(x) > |f(x)|\}$ and for every $k \in \mathbb{Z}$ it holds that

$$\int_{\{2^k \leq |y| \leq 2^{k+1}\}} |DM^I f(y)| \, dy \leq C_n \int_{\{2^{k-1} \leq |y| \leq 2^{k+2}\}} |Df||y| \, dy.$$  

Estimate (1.6) can be easily derived from this fact. The main result follows by combining (1.6) and exploiting the estimate (1.5) in $\{Mf(x) > M^I f(x)\}$.

**Question**

The analysis presented in this paper raises the interest towards the study of the integrability properties of some conditional maximal operators. As an example, (1.3) and (1.4) yield that $|DMf(x)| \leq \widetilde{M}(D|f|)(x)$, where $\widetilde{M}$ is defined for all locally integrable gradient fields $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\widetilde{M}F(x) = \sup \left\{ \left| \int_{B(z, r)} F \right| : x \in B(z, r), \int_{B(z, r)} F(y) \cdot (y-x) \, dy = 0 \right\}.$$
It is clear that $\tilde{M}F$ is bounded by $M(|F|)$, but does it hold that $\tilde{M}$ has even better integrability properties than $M$? What about the boundedness in the Hardy-space $H^1$ or even in $L^1$? Notice that the boundedness of $\tilde{M}$ in $L^1$ would imply the solution to $W^{1,1}$-problem. This problem is almost completely open, even in the case $n=1$. Counterexamples would be highly interesting as well.

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2. Preliminaries and general results

Let us introduce some notation. The boundary of the $n$-dimensional unit ball is denoted by $S^{n-1}$. The $s$-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$. The volume of the $n$-dimensional unit ball is denoted by $\omega_n$ and the $\mathcal{H}^{n-1}$-measure of $S^{n-1}$ by $\sigma_n$. The weak derivative of $f$ (if exists) is denoted by $Df$. $Df(x)$ may also denote the classical derivative of $f$ at $x$, in the case it is known to exist. If $v \in S^{n-1}$, then

$$D_v f(x) := \lim_{h \to 0} \frac{1}{h} (f(x+hv) - f(x)),$$

in the case the limit exists.

Definition 2.1. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ let

$$B_x := \{B(z,r) : x \in \overline{B}(z,r), r > 0, \int_B |f| = Mf(x)\}.$$

It is easy to see that if $f \in L^1(\mathbb{R}^n)$, $x$ is a Lebesgue point for $f$, and $|f(x)| < Mf(x) < \infty$, then $B_x \neq \emptyset$.

The following lemma is the main result of this section. We point out that below (6) is especially useful in the case of radial functions.

Lemma 2.2. If $f \in W^{1,1}(\mathbb{R}^n)$, $Mf(x) > |f(x)|$ and $Mf$ is differentiable at $x$, then

1. For all $v \in S^{n-1}$ and $B \in B_x$, it holds that

$$DMf(x) = \int_B D|f|(y) \, dy \quad \text{and} \quad D_v Mf(x) = \int_B D_v |f|(y) \, dy.$$

2. If $x \in B$ for some $B \in B_x$, then $DMf(x) = 0$.
3. If $x \in \partial B$, $B = B(z,r) \in B_x$ and $DMf(x) \neq 0$, then

$$\frac{DMf(x)}{|DMf(x)|} = \frac{z-x}{|z-x|}.$$
1. If $B \in B_x$, then

$$\int_B D|f|(y) \cdot (y-x) \, dy = 0.$$  

(2.7)

(4) If $x \in \partial B$, $B=B(z,r) \in B_x$, then

$$|DMf(x)| = \frac{1}{r} \int_B D|f|(y) \cdot (z-y) \, dy.$$  

(5) If $B \in B_x$, then

$$DMf(x) \cdot \frac{x}{|x|} = \frac{1}{|x|} \int_B D|f|(y) \cdot y \, dy.$$  

(2.8)

The proof of Lemma 2.2 is essentially based on the following auxiliary propositions.

**Proposition 2.3.** Suppose that $f \in W^{1,1}(\mathbb{R}^n)$, $B$ is a ball, $h_i \in \mathbb{R}$ such that $h_i \to 0$ as $i \to \infty$, and $B_i=L_i(B)$, where $L_i$ are affine mappings and

$$\lim_{i \to \infty} \frac{L_i(y)-y}{h_i} = g(y).$$

Then

$$\lim_{i \to \infty} \frac{1}{h_i} \left( \int_{B_i} f(y) \, dy - \int_B f(y) \, dy \right) = \int_B Df(y) \cdot g(y) \, dy.$$  

(2.9)

**Proof.** The proof is a simple calculation:

$$\frac{1}{h_i} \left( \int_{B_i} f(y) \, dy - \int_B f(y) \, dy \right) = \frac{1}{h_i} \left( \int_{L_i(B)} f(y) \, dy - \int_B f(y) \, dy \right)$$

$$= \frac{1}{h_i} \left( \int_B f(L_i(y)) - f(y) \, dy \right) = \int_B \frac{f(y+(L_i(y)-y)) - f(y)}{h_i} \, dy$$

$$\approx \int_B \frac{Df(y) \cdot (L_i(y)-y)}{h_i} \, dy \to \int_B Df(y) \cdot g(y) \, dy,$$

if $i \to \infty$.  

□

**Lemma 2.4.** Let $f \in W^{1,1}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $B \in B_x$, $\delta > 0$, and let $L_h$, $h \in [-\delta, \delta]$, be affine mappings such that $x \in L_h(B)$ and

$$\lim_{h \to 0} \frac{L_h(y)-y}{h} = g(y).$$

Then

$$\int_B D|f|(y) \cdot g(y) \, dy = 0.$$  

(2.10)
Proof. Let us denote $B_h := L_h(B)$. By Proposition 2.3 it holds that
\[
\int_B D|f|(y) \cdot g(y) \, dy = \lim_{h \to 0} \frac{1}{h} \left( \int_{B_h} |f|(y) \, dy - \int_B |f|(y) \, dy \right).
\]
Since $B \in \mathcal{B}_x$ and $x \in \overline{B}_h$, the sign of the quantity inside the large parentheses is non-positive for all $h \in [-\delta, \delta]$; however, the sign of $1/h$ depends on the sign of $h$. The conclusion is that the above equality is possible only if (2.11) is valid. □

Proof of Lemma 2.2

(1) The claim is counterpart for the formula for $DM_c f$, which was first time proved in [L]. Suppose that $B = B(z, r) \in \mathcal{B}_x$ and let $B_h := B(z + hv, r)$. Then it holds that
\[
D_v Mf(x) = \lim_{h \to 0} \frac{1}{h} (Mf(x + hv) - Mf(x)) \\
\geq \lim_{h \to 0} \frac{1}{h} \left( \int_B |f(y)| \, dy - \int_{B_h} |f(y)| \, dy \right) \\
= \lim_{h \to 0} \frac{1}{h} \left( \int_B |f(y + hv)| - |f(y)| \, dy \right) = \int_B D_v |f|(y) \, dy.
\]
On the other hand, if $B_h := B(z - hv, r)$, then
\[
D_v Mf(x) = \lim_{h \to 0} \frac{1}{h} (Mf(x) - Mf(x - hv)) \\
\leq \lim_{h \to 0} \frac{1}{h} \left( \int_B |f(y)| \, dy - \int_{B_h} |f(y)| \, dy \right) \\
= \lim_{h \to 0} \frac{1}{h} \left( \int_B |f(y)| - |f(y + hv)| \, dy \right) = \int_B D_v |f|(y) \, dy.
\]
These inequalities imply the claim.

(2) If $B \in \mathcal{B}_x$ and $x \in B$, then $y \in B$ if $|y - x|$ is small enough, and thus $Mf(y) \geq Mf(x)$.

(3) Let $B = B(z, r) \in \mathcal{B}_x$, $v \in S^{n-1}$ such that $v \cdot (z - x) = 0$, and let us denote for all $h \in (0, \infty)$ that $x_h := x + hv$, $r_h := |z - x_h|$, and $B_h := B(z, r_h)$. These definitions guarantee that $x_h \in \overline{B}_h \setminus B$ for all $h$, and $B \subset B_h$. Moreover, since $v \cdot (z - x) = 0$, it is elementary fact that
\[
r_h = |z - x - hv| \leq |z - x| + \frac{h^2}{2r}.
\]
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Therefore, \( r/r_h \geq 1 - (\frac{h}{r})^2 \), and

\[
Mf(x_h) \geq \int_{B_h} |f(z)| \, dz \geq \frac{|B|}{|B_h|} \int_B |f(z)| \, dz = \left( \frac{r}{r_h} \right)^n \int_B |f(z)| \, dz
\]

\[ \geq \left( 1 - \frac{h^2}{r^2} \right)^n \ Mf(x). \]

This implies that \( D_v Mf(x) \geq 0 \) for all \( v \in S^{n-1} \) such that \( v \cdot (z-x) = 0 \). Since we assumed that \( Mf \) is differentiable at \( x \), it follows that

\[ D_v Mf(x) = 0 \quad \text{if} \quad v \in S^{n-1}, \, v \cdot (z-x) = 0. \]

In particular, it follows that \( DMf(x) \) is parallel to \( z-x \) or \( x-z \). The final claim follows easily by the fact that \( Mf(x+h(z-x)) \geq Mf(x) \) if \( 0 \leq h \leq 2 \).

(4) Let \( B \in B_x \) and \( L_h(y) := y + h(y-x), \, h \in \mathbb{R} \). Then it holds that \( L_h \) is affine mapping, \( L_h(x) = x \), and so \( x \in L_h(B) = :B_h \), and \( (L_h(y)-y)/h = y-x \) for all \( h \in \mathbb{R} \). Therefore, Lemma 2.4 implies that

\[ \int_B D|f|(y) \cdot (y-x) \, dy = 0. \]

(5) By combining (1), (3) and (4) the claim follows by

\[ |DMf(x)| = DMf(x) \cdot \left( \frac{z-x}{|z-x|} \right) = \int_B D|f|(y) \cdot \left( \frac{z-x}{|z-x|} \right) \, dy \]

\[ = \int_B D|f|(y) \cdot \left( \frac{z-y}{|z-x|} \right) \, dy. \]

(6) The claim follows from (1) and (4). \( \square \)

3. \( W^{1,1} \)-problem for radial functions

Radial functions and notation

In what follows, we will interpret a radial function on \( \mathbb{R}^n \) as a function on \( (0, \infty) \) in a natural way. To be more precise, if \( f \in W^{1,1}_{loc}(\mathbb{R}^n) \) is radial, it is well known fact that there exists continuous function \( \tilde{f} : (0, \infty) \rightarrow \mathbb{R} \) such that \( \tilde{f} \) is weakly differentiable,

\[ \int_0^\infty |\tilde{f}'(t)| t^{n-1} \, dt < \infty, \]

and (by a possible redefinition of \( f \) in a set of measure zero) for all \( t \in (0, \infty) \) it holds that \( f(x) = \tilde{f}(t) \) and \( D_{x/|x|} f(x) = \tilde{f}'(t) \) if \( |x| = t \). In what follows, we will simplify the
notation and use $f$ to denote $\tilde{f}$ as well. To avoid the possibility of misunderstanding, we usually use variable $t$ and notation $f'$ (instead of $Df$) when we are actually working with $\tilde{f}$. We also say that $f$ is radially decreasing if $f$ is radial and $f(t_1) < f(t_2)$ if $t_1 > t_2$. Notice also that if $f$ is radial then $Mf$ is also radial.

We begin with establishing couple of auxiliary lemmas. The following auxiliary result is repeatedly utilized in the proof. The proof is well known, see for example [HKM, Theorem 1.20].

**Lemma 3.1.** Suppose that $\Omega \subset \mathbb{R}^n$, $f \in W^{1,1}(\Omega)$ is continuous, $g: \Omega \to \mathbb{R}$ is continuous and weakly differentiable in $E := \{ x \in \Omega : g(x) > f(x) \}$, and $\int_E |Dg| < \infty$. Then $\max\{ f, g \}$ is weakly differentiable in $\Omega$ and

$$D(\max\{ f, g \}) = \chi_E Dg + \chi_{\Omega \cap E^c} Df.$$ 

Let us define an auxiliary maximal operator $M_\lambda$ for $\lambda > 0$ by

$$M_\lambda f(x) = \sup_{x \in B(z, r), \lambda \leq r} \int_{B(z, r)} |f(y)| \, dy.$$ 

**Proposition 3.2.** If $f \in L^1(\mathbb{R}^n)$, then $M_\lambda$ is Lipschitz.

**Proof.** The result is well known, but we give a proof for readers convenience. Suppose that $x, y \in \mathbb{R}^n$ such that $M_\lambda f(x) > M_\lambda f(y)$. Clearly there exists $r \geq \lambda$ and $x_0 \in \mathbb{R}^n$ such that $x \in \overline{B(x_0, r)}$ and $M_\lambda f(x) = \int_{B(x_0, r)} |f|$. The claim follows by

$$M_\lambda f(x) - M_\lambda f(y) \leq \int_{B(x_0, r)} |f(z)| \, dz - \int_{B(x_0, r+|x-y|)} |f(z)| \, dz$$

$$\leq \frac{1}{\omega_n} \left( \frac{1}{r^n} - \frac{1}{(r+|x-y|)^n} \right) \int_{B(x_0, r)} |f(z)| \, dz \leq C(n, \lambda) |x-y| \|f\|_1. \quad \Box$$

The following result is especially related to the assumption ‘$Mf(x)$ is differentiable at $x$’ in Lemma 2.2.

**Proposition 3.3.** Suppose that $f \in W^{1,1}(\mathbb{R}^n)$ is radial and

$$E := \{ x \in \mathbb{R}^n \setminus \{0\} : Mf(x) > |f(x)| \}.$$ 

Then $E$ is open, $DMf$ exists in $E$ and $Mf$ is differentiable almost everywhere in $E$.

**Proof.** The first claim ($E$ is open) follows by the fact that $f$ is continuous outside the origin. The claims concerning the differentiability (weak and classical) follow if we can show that $Mf$ is locally Lipschitz in $E$. But this follows rather easily from Proposition 3.2 and the fact that $f$ is continuous in $\mathbb{R}^n \setminus \{0\}. \quad \Box$
The following result is a straightforward consequence of Lemma 2.2 and the above auxiliary results.

**Theorem 3.4.** If $f \in W^{1,1}(\mathbb{R}^n)$ is radially decreasing, then $DfMf \in W^{1,1}(\mathbb{R}^n)$ and $\|DfMf\|_1 \leq C_n \|Df\|_1$.

**Proof.** Since $f$ is radially decreasing, it follows that $Mf(x) > |f(x)|$ for all $x \neq 0$. Especially, this guarantees the existence of a weak derivative in $\mathbb{R}^n \setminus \{0\}$, and the classical differentiability almost everywhere (by the above auxiliary results).

If $B \in B_x$, $x \neq 0$, it is easy to show (the proof is left to the reader) that $B \subset B(0, |x|)$. It also follows that $0 \in \bar{B}$. To see this, observe (e.g.) that whenever $0 \not\in B$, $B \subset B(0, |x|)$, then $B$ is of type $B = B(cx, |c-1||x|)$, where $\frac{1}{2} < c < 1$. By choosing

$$L(y) = x + 2(1-c)(y-x) \quad \text{and} \quad B^* = B\left(\frac{1}{2}x, \frac{1}{2}|x|\right),$$

it is easy to check that $L(B^*) = B$ and, especially, $|L(y)| > |y|$ for all $y \in B^*$. Therefore,

\[
\int_B |f(z)| \, dz = \int_{L(B^*)} |f(z)| \, dz = \int_{B^*} |f(L(z))| \, dz < \int_{B^*} |f(z)| \, dz.
\]

This proves that $0 \in \bar{B}(x)$, whenever $B \in B_x$, $x \neq 0$. Especially, we get by Lemma 2.2, (6) that

\[
|DfMf(x)| \leq \frac{C_n}{|x|} \int_{B(0,|x|)} |Df(y)||y| \, dy \quad \text{for a.e. } x.
\]

Then the claim follows by Fubini theorem:

\[
\begin{align*}
\int_{\mathbb{R}^n} \left( \frac{1}{|x|} \int_{B(0,|x|)} |Df(y)||y| \, dy \right) \, dx \\
= \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{\mathbb{R}^n} \frac{1}{\omega_n |x|^{n+1}} \, dx \right) \, dy \\
= \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{\{x:|x| \geq |y|\}} \frac{1}{\omega_n |x|^{n+1}} \, dx \right) \, dy \\
= \int_{\mathbb{R}^n} |Df(y)||y| \left( \int_{S^{n-1}} \int_{|y|}^{\infty} \frac{1}{\omega_n t^{n+1}} \, dt \, d\mathcal{H}^{n-1} \right) \, dy \\
= n \int_{\mathbb{R}^n} |Df(y)| \left( \int_{|y|}^{\infty} \frac{1}{t^2} \, dt \right) \, dy \\
= n \int_{\mathbb{R}^n} |Df(y)| \, dy. \quad \Box
\end{align*}
\]
In the case of general radial functions, (1.5) is in general valid (and useful) only for those \( x \) for which the radius of \( B \in B_x \) is comparable to \( |x| \). As it was explained in the introduction, the main auxiliary tool in the case of general radial functions is the following result (recall the definition of \( M \) in the introduction):

**Lemma 3.5.** If \( f \in W^{1,1}(\mathbb{R}^n) \) is radial, then \( M^1 f \in W^{1,1}(\mathbb{R}^n) \) and \( \|DM^1 f\|_1 \leq C_n \|Df\|_1 \).

Before the actual proof of this result, we prove several auxiliary results. The first of them is well known.

**Proposition 3.6.** Suppose that \( E \subset \mathbb{R} \) is open. Then there exist disjoint intervals \((a_i, b_i)\) such that \( E = \bigcup_{i=1}^\infty (a_i, b_i) \) and \( a_i, b_i \in \partial E \cup \{-\infty, \infty\} \) for all \( i \in \mathbb{N} \).

**Definition 3.7.** Let \( f: \Omega \to \mathbb{R} \), where \( \Omega \subset \mathbb{R} \) is open. We say that \( x \) is a strong local maximum of \( f \) in \((a, b) \subset \Omega\), \(-\infty \leq a < b \leq \infty\), if there exist \( a', b' \in (a, b) \) such that \( a' < x < b' \), \( f(t) \leq f(x) \) if \( t \in (a', b') \), and \( \max\{f(a'), f(b')\} < f(x) \).

**Proposition 3.8.** Suppose that \( f:[a, b] \to \mathbb{R} \) is continuous and \( c \in (a, b) \) such that \( f(c) > \max\{f(a), f(b)\} \). Then \( f \) has a strong local maximum on \((a, c)\).

**Proof.** It is easy to see that now any maximum point \( c \) (\( f(c) = \max f \)), which is known to exist, is also a strong local maximum of \( f \). \( \Box \)

**Proposition 3.9.** Suppose that \( f:[a, b] \to \mathbb{R} \) is continuous and does not have a strong local maximum on \((a, b)\). Then there exists \( c \in [a, b] \) such that \( f \) is non-increasing on \([a, c]\) and non-decreasing on \([c, b]\).

**Proof.** Since \( f \) is continuous, we can choose \( c \in [a, b] \) such that \( f(c) = \min f \). To show that \( f \) is non-decreasing on \([c, b]\), let \( c < y_1 < y_2 < b \) and assume, on the contrary, that \( f(y_2) < f(y_1) \). This implies that \( f(y_1) > \max\{f(c), f(y_2)\} \), and thus \( f \) has a strong local maximum on \((c, y_2)\) by Proposition 3.8. This is the desired contradiction. The first claim, \( f \) is non-increasing on \([a, c]\), follows by a similar argument. \( \Box \)

Let us define for \( 0 < a \leq b < \infty \) the annular domains

\[
A_n(a, b) := A(a, b) := \{x \in \mathbb{R}^n : a < |x| < b\} \quad \text{and} \\
A_n[a, b] := A[a, b] := \{x \in \mathbb{R}^n : a \leq |x| \leq b\}.
\]

**Lemma 3.10.** If \( f \in W^{1,1}(\mathbb{R}^n) \) is radial, then \( M^1 f \) does not have a strong local maximum in \( \{t \in (0, \infty) : M^1 f(t) > |f(t)|\} \).
Proof. Suppose, on the contrary, that $t_0 \in (0, \infty)$ is a strong local maximum of $M^f$ and $M^f(t_0) > |f(t_0)|$. Let us choose

$$t^- := \sup \{ t < t_0 : M^f(t) < M^f(t_0) \} \quad \text{and} \quad t^+ := \inf \{ t > t_0 : M^f(t) < M^f(t_0) \}.$$ 

By the definition of the strong local maximum, it follows that $t_0 \in [t^-, t^+]$ and

$$M^f(t) = M^f(t_0) \quad \text{for all} \quad t \in [t^-, t^+]. \tag{3.13}$$

Suppose that $|x| = t_0$. Since $M^f(t_0) > |f(t_0)|$, it follows that there exists a ball $B = B(z, r)$ such that $x \in B$, $r \leq |z|/4$, and $M^f(t_0) = \int_B |f|$. Suppose first that $B \not\subset A[t^-, t^+]$. In this case, there exists $\varepsilon > 0$ such that $[t^- - \varepsilon, t^-] \subset \{|y| : y \in B\}$ or $[t^+, t^+ + \varepsilon] \subset \{|y| : y \in B\}$. Especially, it follows by the definition of $M^f$ that $M^f(t) \geq \int_B |f| = M^f(t_0)$ if $t \in [t^- - \varepsilon, t^-]$ or $t \in [t^+, t^+ + \varepsilon]$, respectively. Obviously, this contradicts with the choice of $t^-$ and $t^+$. This verifies that $B \subset A[t^-, t^+]$. Therefore, it holds by (3.13) that

$$M^f(y) = M^f(t_0) \quad \text{for all} \quad y \in B. \tag{3.14}$$

However, $|f(t_0)| < M^f(t_0)$ also implies that there exists a ball $B'$ with positive radius such that $B' \subset B$ and $|f| < M^f(t_0)$ in $B'$. Combining this with (3.14) yields the desired contradiction by

$$M^f(t_0) = \int_B |f| \leq \frac{1}{|B|} \left( \int_{B \setminus B'} |f| + \int_{B'} |f| \right) 
< \frac{1}{|B|} \left( \int_{B \setminus B'} M^f + \int_{B'} M^f(t_0) \right) = M^f(t_0). \quad \square$$

The following estimate is well known.

**Proposition 3.11.** If $f \in W^{1,1}(\mathbb{R}^n)$ is radial and $0 < a < b < \infty$, then

$$\sigma_n a^{n-1} \int_a^b |f'(t)| \, dt \leq \int_{A(a,b)} |Df(y)| \, dy \leq \sigma_n b^{n-1} \int_a^b |f'(t)| \, dt.$$

The proof of Lemma 3.5

Let

$$E := \{ x \in \mathbb{R}^n \setminus \{0\} : M^f(x) > |f(x)| \} \quad \text{and} \quad E_k := E \cap A[2^{-k}, 2^{-k+1}], \quad k \in \mathbb{N}.$$ 

Then $E$ is open, since $M^f$ and $f$ are continuous in $\mathbb{R}^n \setminus \{0\}$. A standard argument (see the proof of Proposition 3.2) shows that mapping $M^f$ is locally Lipschitz in
and, especially, $D(M^f) \text{ exists in } E$. By Lemma 3.1, it suffices to show that 
\[ \int_E |DM^f| \leq C_n \|Df\|_1. \]

First, observe that since $|f|$ is radial, it follows that $M^f$ is radial as well. In particular, if 
\[ E_k^k := \{|x| : x \in E_k\}, \]
then $x \in E_k$ if and only if $|x| \in E_k^k$. Since $E_k^k$ is open in $[2^{-k}, 2^{-k+1}]$, we can write 
\[ E_k^k = \bigcup_{i=1}^{\infty} (a_i, b_i), \]
such that $a_i < b_i$, $(a_i, b_i)$ are pairwise disjoint and $a_i, b_i \in \partial E_k^k$. In the other words, 
\[ E_k = \bigcup_{i=1}^{\infty} A(a_i, b_i), \]
and (by the definition of $E_k$) for all $i \in \mathbb{N}$ it holds that 
\[ (3.15) \]
\[ M^f(x) = |f(x)| \text{ if } |x| = a_i > 2^{-k} \quad \text{and} \quad M^f(x) = |f(x)| \text{ if } |x| = b_i < 2^{-k+1}. \]
Moreover, since $M^f > |f|$ in $E_k$, Lemma 3.10 says that $M^f$ does not have a strong local maximum in $E_k^k$. In particular, by Proposition 3.9 there exist $c_i \in (a_i, b_i)$ such that 
\[ \int_{A(a_i, b_i)} |DM^f(y)| dy \leq \sigma_n b_i^{n-1} \int_{a_i}^{b_i} |(M^f)'(t)| dt \]
\[ = \sigma_n b_i^{n-1} (M^f(a_i) - M^f(c_i) + M^f(b_i) - M^f(c_i)) \]
\[ \leq \sigma_n b_i^{n-1} (M^f(a_i) - |f|(c_i) + M^f(b_i) - |f|(c_i)). \]
Combining this with (3.15) implies that if $2^{-k} < a_i < b_i < 2^{-k+1}$, then 
\[ \int_{A(a_i, b_i)} |DM^f(y)| dy \leq \sigma_n b_i^{n-1} (|f|(a_i) - |f|(c_i) + |f|(b_i) - |f|(c_i)) \]
\[ \leq \sigma_n b_i^{n-1} \int_{a_i}^{b_i} |f'(t)| dt \leq \left( \frac{b_i}{a_i} \right)^{n-1} \int_{A(a_i, b_i)} |Df(y)| dy \]
\[ \leq 2^{n-1} \int_{A(a_i, b_i)} |Df(y)| dy. \]
For the case $a_i = 2^{-k}$ or $b_i = 2^{-k+1}$, we employ the fact 
\[ M^f(2^{-k}), M^f(2^{-k+1}) \leq \sup_{y \in A(2^{-k-1}, 2^{-k+2})} |f(y)| \]
to obtain the estimates \((a_i = 2^{-k} \text{ or } b_i = 2^{-k+1})\)

\[
\int_{A(a_i, b_i)} |DM^I f(y)| \, dy \leq \sigma_n b_i^{n-1} (M^I f(a_i) - |f|(c_i) + M^I f(b_i) - |f|(c_i)) \\
\leq \sigma_n b_i^{n-1} \int_{2^{-k-1}}^{2^{-k+2}} |f'(t)| \, dt \\
\leq 2^{3(n-1)} \int_{A(2^{-k-1}, 2^{-k+2})} |Df(y)| \, dy.
\]

Combining these estimates implies that

\[
\int_{E_k} |DM^I f(y)| \, dy = \sum_{i=1}^{\infty} \int_{A(a_i, b_i)} |DM^I f(y)| \, dy \\
\leq 2^{n-1} \sum_{i=1}^{\infty} \left[ \int_{A(a_i, b_i)} |Df(y)| \, dy \right] + 2(2^{3(n-1)}) \int_{A(2^{-k-1}, 2^{-k+2})} |Df(y)| \, dy \\
\leq 2^{3n} \int_{A(2^{-k-1}, 2^{-k+2})} |Df(y)| \, dy.
\]

Therefore,

\[
\int_E |DM^I f(y)| \, dy \leq \sum_{k \in \mathbb{Z}} \int_{E_k} |DM^I f(y)| \, dy \\
\leq 2^{3n} \sum_{k \in \mathbb{Z}} \int_{A(2^{-k-1}, 2^{-k+2})} |Df(y)| \, dy \\
= 3(2^{3n}) \sum_{k \in \mathbb{Z}} \int_{A(2^{-k}, 2^{-k+1})} |Df(y)| \, dy = 3(2^{3n}) \|Df\|_1.
\]

This completes the proof. \( \square \)

Then we are ready to prove our main theorem.

**Theorem 3.12.** If \( f \in W^{1,1}(\mathbb{R}^n) \) is radial, then \( Mf \in W^{1,1}(\mathbb{R}^n) \) and \( \|DMf\|_1 \leq C_n \|Df\|_1 \).

**Proof.** Let

\[
E := \{ x \in \mathbb{R}^n : Mf(x) > M^I f(x), \ DMf(x) \neq 0 \}.
\]

It is well known that \( Mf \) is locally Lipschitz in \( \{ Mf(x) > |f(x)| \} \) (combine e.g. Proposition 3.2 and the fact that \( f \) is continuous in \( \mathbb{R}^n \setminus \{0\} \)), implying the existence of \( DMf \) in \( \{ Mf(x) > |f(x)| \} \). Since \( Mf \geq M^I \), it holds that \( Mf(x) = \)
max\{Mf(x), M^f f(x)\}. Therefore, the theorem follows by Lemmas 3.1 and 3.5, if we can show that

\[
(3.16) \quad \int_E |DMf(y)| \, dy \leq C_n \|Df\|_1.
\]

To show this, observe first that for all \(x \in E\) there exist \(z_x \in \mathbb{R}^n\) and \(r_x > \frac{|x|}{4}\) such that \(x \in \overline{B(z_x, r_x)} \subseteq B_x\). Moreover, since \(DMf(x) \neq 0\), Lemma 2.2 ((2) and (3)) says that \(x \in \partial B(z_x, r_x)\) and \(DMf(x)/|DMf(x)| = (z_x - x)/|z_x - x|\). On the other hand, \(Mf\) is radial and so \(DMf(x)/|DMf(x)| = \pm x/|x|\). We conclude that

\[B_x = B(c_x x, |c_x x - x|)\quad \text{for some } c_x \in \mathbb{R}.\]

Firstly, it holds that \(c_x \geq 0\) for all \(x \in E\). To see this, observe that if \(c_x < 0\), then \(-x \in B_x\) and, since \(Mf\) is radial, \(B_x \subseteq B_{-x}\), implying by Lemma 2.2 that \(0 = DMf(-x) = DMf(x)\), which contradicts the assumption \(x \in E\). Moreover, \(r_x = |c_x x - x| = |c_x - 1||x|/|c_x x|/4\) by the assumption, implying that \(c_x < 4/5\) or \(c_x > 4/3\). Summing up, we can write \(E = E_+ \cup E_-\), where

\[E_+ = \{x \in E : c_x > 4/3\} \quad \text{and} \quad E_- = \{x \in E : 0 \leq c_x < 4/5\}.\]

We are going to use different estimates for \(DMf(x)\) in \(E_+\) and \(E_-\). Since \(|DMf(x)| = |DMf(x) \cdot \frac{x}{|x|}|\), it follows from Lemma 2.2 (2.8) that

\[|DMf(x)| \leq \frac{1}{|x|} \int_{B_x} |Df(y)||y| \, dy.\]

This estimate will be used in \(E_-\), while in \(E_+\) we will use (easier) estimate \(|DMf(x)| \leq \int_{B_x} |Df|\) (Lemma 2.2, (1)). We get that

\[
\int_E |DMf(x)| \, dx \leq \int_E \chi_{E_+}(x) |DMf(x)| + \chi_{E_-}(x) |DMf(x)| \, dx
\]

\[
\leq \int_E \chi_{E_+}(x) \left( \int_{B_x} |Df(y)||y| \, dy \right) + \chi_{E_-}(x) \left( \int_{B_x} |Df(y)||y|/|x| \, dy \right) \, dx
\]

\[
= \int_{\mathbb{R}^n} \chi_{E_+}(x) \chi_{B_x}(y) |Df(y)| \, dy + \chi_{E_-}(x) \chi_{B_x}(y) |Df(y)|/|x| \, dy \, dx
\]

\[
= \int_{\mathbb{R}^n} |Df(y)| \left( \int_{E_+} \frac{\chi_{B_x}(y)}{|B_x|} \, dx + \int_{E_-} \frac{\chi_{B_x}(y)/|y|}{|B_x|/|x|} \, dx \right) \, dy.
\]

If \(y \in B_x\) and \(x \in E_+\), it follows from the definition of \(E_+\) that \(|x| \leq |y|\). Moreover, \(y \in B_x\) and \(x \in E\) imply also that \(r_x \geq \max\{\frac{|y - x|}{2}, \frac{|x|}{3}\} \geq \frac{|y|}{5}\). This implies the estimate

\[
\int_{E_+} \frac{\chi_{B_x}(y)}{|B_x|} \, dx \leq \int_{B(0, |y|)} \frac{dx}{\omega_n(|y|/5)^n} \leq C_n, \quad \text{for all } y \in \mathbb{R}^n.
\]
On the other hand, if \( x \in E_- \), then \( 0 \leq c_x < 4/5 \) especially implies that \( B_x \subset B(0, |x|) \). Therefore, if \( x \in E_- \) and \( y \in B_x \), then \( y \in B(0, |x|) \), and thus \( |x| \geq |y| \). Recall also that \( r_x \geq \frac{|x|}{5} \). Combining these yields that

\[
\int_{E_-} \frac{\chi_{B_x}(y) |y|}{|B_x||x|} \, dx \leq |y| \int_{\mathbb{R}^n \setminus B(0, |y|)} \frac{dx}{\omega_n(|x|/5)^{n+1}} = C'_n |y| \int_{|y|}^{\infty} \frac{dt}{t^{2n+1}} = C'_n,
\]

for all \( y \in \mathbb{R}^n \). This completes the proof. □

References


Hannes Luiro
Department of Mathematics and Statistics
University of Jyväskylä
P.O.Box 35 (MaD)
FI-40014 University of Jyväskylä
Finland
hannes.luiro@gmail.com

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