

Generalized Gorenstein Arf rings

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Abstract. In this paper we study generalized Gorenstein Arf rings; a class of one-dimensional Cohen-Macaulay local Arf rings that is strictly contained in the class of Gorenstein rings. We obtain new characterizations and examples of Arf rings, and give applications of our argument to numerical semigroup rings and certain idealizations. In particular, we generalize a beautiful result of Barucci and Fröberg concerning Arf numerical semigroup rings.

1. Introduction

In 1971 Lipman [30] proved that, if (R, \mathfrak{m}) is a complete, one-dimensional local domain with an algebraically closed field of characteristic zero, and R is saturated (as defined by Zariski in [40]), then R has minimal multiplicity, i.e., the embedding dimension of R equals the multiplicity of R . Lipman's proof based on the fact that such a ring R is an Arf ring, i.e., R satisfies a certain condition that was studied by Arf [2] in 1949 pertaining to a certain classification of curve singularities; see, for example, the survey papers [32], [33] and also Du Val's work [8] for details. As Lipman [30] pointed out, the defining condition of an Arf ring is technical, but it is convenient to work with, as well as, easy to state: if R is as above, then R is Arf provided that, $yz/x \in R$ whenever $0 \neq x \in \mathfrak{m}$ and $y/x, z/x \in \text{Frac}(R)$ (the field of fractions of R) are integral elements over R .

Examples of Arf rings are abundant. For example, R is Arf if the multiplicity of R is at most two. As Arf property is preserved by standard procedures in ring theory, such as completion, it is not difficult to construct examples of Arf rings; see, for example, [30, 2.5 and 2.7].

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Arf property for numerical semigroup rings, as well as algorithms to compute the Arf ring closure of various rings, such as the coordinate rings of curves, were already studied in the literature; see, for example, [3], [5], [36]. However, to our best knowledge, a homological characterization of Arf rings in local algebra – besides Lipman’s beautiful work – is yet to be given. The main purpose of this paper, rather than seeking fast algorithms to compute the Arf closure, is to initiate a homological investigation, and to attempt to motivate the interested reader to study further in this direction. Although it has seemingly a non-homological definition, Arf rings enjoy important homological properties: if R is Arf, then it has minimal multiplicity so that it is *Golod*, a class of (local) rings which is of active research interest; see, for example, [4, 5.2.8]. This particular property of Arf rings naturally raises the following question: if R has minimal multiplicity, then, under what conditions, R is an Arf ring? In this paper we are able to give an answer to this question and obtain new characterizations of a class of Arf rings. Our main result is:

Theorem 1.1. *Let R be a one-dimensional generalized Gorenstein local ring with a canonical ideal I which contains the parameter ideal (a) as a reduction. Set $S = \{\frac{x}{a} \mid x \in I\} \subseteq Q(R)$, where $Q(R)$ is the total quotient ring of R . Then R is Arf if and only if R has minimal multiplicity and the multiplicity of $S_{\mathcal{M}}$ is at most two for each maximal ideal \mathcal{M} of S .*

The notion of generalized Gorenstein ring introduced by [12] is one of the generalizations of that of Gorenstein ring, defined by a certain embedding of the ring into its canonical module; see 2.2 for a precise definition. Originally, the series of researches [7], [9], [12]–[14], [16]–[19], [22], [24], [25], [27], [37] aim, in order to stratify Cohen-Macaulay rings, to find new classes of them, measuring the distance or difference from Gorenstein rings as a barometer. The notion of generalized Gorenstein ring is one of the attempts, which beautifully extends the notion of almost Gorenstein ring, initially defined by Barucci and Fröberg [5] for one-dimensional analytically unramified local rings, and thereafter generalized by Goto, Matsuoka, and Phuong [14] for arbitrary Cohen-Macaulay local rings of dimension one.

The next two corollaries of Theorem 1.1 yield generalizations of a characterization of Barucci and Fröberg [5, 13] concerning Arf numerical semigroup rings; see Theorem 4.4, Corollary 4.5, and Proposition 4.6. We set $B = \mathfrak{m} :_{Q(R)} \mathfrak{m}$, the endomorphism algebra of \mathfrak{m} , where $Q(R)$ is the total quotient ring of R . Note that, if R is not Gorenstein but almost Gorenstein, then $B = S$; see [14, 3.16].

Corollary 1.2. *Let (R, \mathfrak{m}) be a one-dimensional Cohen–Macaulay local ring with canonical module. Then the following conditions are equivalent.*

- (i) R is an almost Gorenstein Arf ring.
- (ii) $e(B_{\mathcal{M}}) \leq 2$ for each maximal ideal \mathcal{M} of B .

Given a generalized Gorenstein numerical semigroup ring, our next result may be used to check whether it is Arf; see Proposition 4.6. We set $\mathfrak{c} = R :_{\mathbb{Q}(R)} S$.

Corollary 1.3. *Let $\ell > 0$ be an integer and let $0 < a_1 < a_2 < \dots < a_\ell$ be integers with $\gcd(a_1, a_2, \dots, a_\ell) = 1$. Assume k is a field, $R = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$ is the numerical semigroup ring over k , and $H = \langle a_1, a_2, \dots, a_\ell \rangle$ is the corresponding semigroup. For a generalized Gorenstein ring R , the following are equivalent.*

- (i) R is an Arf ring.
- (ii) R has minimal multiplicity, $2 + \ell_R(R/\mathfrak{c}) \cdot a_1 \in H$, and $2 + a_i \in H$ for each $i = 2, \dots, \ell$.

In Section 4 we consider idealizations of the form $A = R \times \mathfrak{c}$, where R is a one-dimensional Cohen-Macaulay local ring. We obtain a new criterion for A to be a generalized Gorenstein Arf ring in terms of the integral closure \overline{R} of R . A special case of our result is as follows; see Theorem 4.10.

Corollary 1.4. *Let R be a one-dimensional Cohen-Macaulay local ring R with a canonical module. If R is a generalized Gorenstein ring that has minimal multiplicity, and $S = \overline{R}$, then both R and A are generalized Gorenstein Arf rings.*

2. Preliminaries

This section is devoted to the definitions and some basic properties of Arf rings and generalized Gorenstein rings. Throughout the section, R denotes a d -dimensional Cohen-Macaulay local ring with unique maximal ideal \mathfrak{m} , residue field k and canonical module K_R .

We start by recalling the definition of an *Ulrich* module.

Definition 2.1. ([12]) Let M be a finitely generated R -module of dimension $s \geq 0$, and let \mathfrak{a} be an \mathfrak{m} -primary ideal of R . Then M is said to be an Ulrich R -module with respect to \mathfrak{a} provided the following conditions hold:

- (a) M is a Cohen-Macaulay R -module.
- (b) $e_{\mathfrak{a}}^0(M) = \ell_R(M/\mathfrak{a}M)$.
- (c) $M/\mathfrak{a}M$ is a free R/\mathfrak{a} -module.

Here $\ell_R(M)$ and $e_{\mathfrak{a}}^0(M)$ denote the length of M as an R -module and the multiplicity of M with respect to \mathfrak{a} , respectively.

Ulrich modules, with respect to the unique maximal ideal, were originally defined in [6] as maximally generated maximal Cohen-Macaulay modules. This definition was then generalized in the article [20] of the third author, Ozeki, Takahashi, Watanabe, and Yoshida. If R is a non-regular and has minimal multiplicity, then it follows from the definition that \mathfrak{m} is an Ulrich ideal. Recall that R is said to

have *minimal multiplicity* if $e(R) = \text{embdim}(R) - \dim A + 1$, where $\text{embdim}(R)$ denotes the embedding dimension of R . Moreover higher syzygies of Ulrich ideals are Ulrich modules; see [20, 3.2]. We refer the reader to [12], [20], [21], [24] for further information on Ulrich modules.

Definition 2.2. ([12]) R is said to be a *generalized Gorenstein ring*, if either R is Gorenstein, or R is not Gorenstein, but there exists an \mathfrak{m} -primary ideal \mathfrak{a} of R and an exact sequence of R -modules

$$0 \longrightarrow R \xrightarrow{\varphi} K_R \longrightarrow C \longrightarrow 0,$$

where C is an Ulrich R -module with respect to \mathfrak{a} , and the induced homomorphism

$$R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \longrightarrow K_R/\mathfrak{a}K_R$$

is injective. If the latter case occurs, then R is called a *generalized Gorenstein ring with respect to \mathfrak{a}* .

Definition 2.3. ([24]) R is said to be an *almost Gorenstein ring* if it is Gorenstein, or not Gorenstein but is a generalized Gorenstein ring with respect to \mathfrak{m} .

Next we record some preliminary results pertaining to *Arf rings*.

On Arf rings

For this subsection on Arf rings, A denotes a commutative Noetherian *semi-local* ring satisfying the following condition:

(#) $A_{\mathcal{M}}$ is a one-dimensional Cohen-Macaulay ring for each maximal ideal \mathcal{M} of A .

Let \mathcal{F}_A be the set of ideals of A that contain a non-zero-divisor on A . Then, for each $I \in \mathcal{F}_A$, there is a filtration of endomorphism algebras as follows:

$$A \subseteq I :_{Q(A)} I \subseteq I^2 :_{Q(A)} I^2 \subseteq \dots \subseteq I^n :_{Q(A)} I^n \subseteq \dots \subseteq \overline{A}.$$

Here \overline{A} and $Q(A)$ denote the integral closure and the total quotient ring of A , respectively. We set

$$A^I = \bigcup_{n \geq 1} [I^n :_{Q(A)} I^n].$$

The ring A^I , a module-finite extension over A , is called the *blowup of A at I* . Notice, if $a \in I$ is a reduction of I , then one has:

$$A^I = A \left[\frac{I}{a} \right], \text{ where } \frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\} \subseteq Q(A).$$

Definition 2.4. An ideal $I \in \mathcal{F}_A$ is called a *stable ideal* provided $A^I = I :_{Q(A)} I$.

Note that, for each $I \in \mathcal{F}_A$, I^n is stable for some $n \geq 1$. Moreover, an ideal I is stable if and only if $I^2 = xI$ for some $x \in I$; see [30] for details.

Definition 2.5. ([2], [30]) A is called an *Arf ring* provided every integrally closed ideal $I \in \mathcal{F}_A$ is stable.

Notation 2.6. For each nonnegative integer n , we set:

$$A_n = \begin{cases} A & \text{if } n=0, \\ A_{n-1}^{J(A_{n-1})} & \text{if } n \geq 1, \end{cases}$$

where $J(A_{n-1})$ stands for the Jacobson radical of the ring A_{n-1} .

Notice $A_1 = A^{J(A)}$ is a one-dimensional Noetherian semi-local ring which is a module-finite extension over A . Moreover, A_1 satisfies the condition (#), namely the localization $(A_1)_{\mathcal{N}}$ is a Cohen–Macaulay local ring of dimension one for each $\mathcal{N} \in \text{Max}(A_1)$, i.e., for each maximal ideal \mathcal{N} of A_1 .

The following characterization of Arf rings plays an important role for our argument; see, for example, the proof of Proposition 2.8 and that of Theorem 1.1. One can deduce it from the results of Lipman [30], but we include here a short and a different proof for the sake of completeness.

Proposition 2.7. (see [30, 2.2]) *The following conditions are equivalent.*

- (i) A is an Arf ring.
- (ii) $\text{embdim}((A_n)_{\mathcal{M}}) = e((A_n)_{\mathcal{M}})$ for each $n \geq 0$ and maximal ideal \mathcal{M} of A_n .

Proof. (i) \Rightarrow (ii): Let $B = A^J$ be the blowup of A at its Jacobson radical $J = J(A)$. By the induction argument, it suffices to prove B is Arf and $\text{embdim}(A_{\mathcal{M}}) = e(A_{\mathcal{M}})$ for each $\mathcal{M} \in \text{Max}(A)$. Since $\mathcal{M} \in \mathcal{F}_A$ is integrally closed and A is Arf, \mathcal{M} is stable, i.e., there exists an element $f \in \mathcal{M}$ satisfying $\mathcal{M}^2 = f\mathcal{M}$, which yields $\text{embdim}(A_{\mathcal{M}}) = e(A_{\mathcal{M}})$. Let us make sure of the Arf property for B . Note that the Jacobson radical $J = J(A)$ is an integrally closed open ideal of A . Thus we choose $x \in J$ with $J^2 = xJ$. Therefore we have

$$B = A^J = A \left[\frac{J}{x} \right] = \frac{J}{x}.$$

Let $L \in \mathcal{F}_B$ be an integrally closed ideal of B and we will show that L is stable. By setting $I = xL$, we get

$$I \subseteq J \subseteq A$$

so that I is an open ideal of A , i.e., $I \in \mathcal{F}_A$. It is straightforward to show that I is integrally closed, whence $I^2 = \xi I$ for some $\xi \in I$. Hence

$$L^2 = \frac{\xi}{x} L \quad \text{and} \quad \frac{\xi}{x} \in L$$

which imply that B is Arf, as desired.

(ii) \Rightarrow (i) Let $I \in \mathcal{F}_A$ be an integrally closed ideal of A . We may assume that I is a proper ideal of A . Localizing A at its maximal ideal, we may also assume A is a local ring with maximal ideal \mathfrak{m} . Since A has minimal multiplicity, there exists $x \in \mathfrak{m}$ such that $\mathfrak{m}^2 = x\mathfrak{m}$. Let

$$B := A^{\mathfrak{m}} = A \left[\frac{\mathfrak{m}}{x} \right] = \frac{\mathfrak{m}}{x}$$

be the blowup of A at \mathfrak{m} . As $I \subseteq \mathfrak{m}$, we see that $L := \frac{I}{x} \subseteq B$ is an ideal of B . One can show that L is integrally closed. We now assume that L is not stable in B . Then $L \subsetneq B$ and, for each $\mathcal{N} \in \text{Max}(B)$, one has:

$$\ell_{B_{\mathcal{N}}}(B_{\mathcal{N}}/LB_{\mathcal{N}}) \leq \ell_B(B/L) \leq \ell_A(B/L) < \ell_A(A/I).$$

Repeating the same process for $B_{\mathcal{N}}$ recursively contradicts the fact that $\ell_A(A/I)$ is finite. Therefore L is stable, so that we can choose $\xi \in L$ satisfying $L^2 = \xi L$. Since $I = xL$, we have

$$I^2 = (x\xi)I,$$

which yields that A is an Arf ring. \square

We finish this section with a few more observations on the Arf property. For commutative rings A, B , and C , we say that C is an *intermediate ring between A and B* , if A, C are subrings of B and $A \subseteq C \subseteq B$.

Proposition 2.8. *Assume A is a one-dimensional, Cohen–Macaulay and local ring with unique maximal ideal \mathfrak{m} . Let C be an intermediate ring between A and $\mathbb{Q}(A)$ such that C is a finitely generated A -module. Assume $e(C_{\mathcal{M}}) \leq 2$ for each maximal ideal \mathcal{M} of C . If $C \subseteq D \subseteq \mathbb{Q}(C)$ is an intermediate ring that is a finitely generated C -module, then $e(D_{\mathcal{N}}) \leq 2$ for each maximal ideal \mathcal{N} of D . In particular, C is an Arf ring.*

Proof. Let $C \subseteq D \subseteq \mathbb{Q}(C)$ be an intermediate ring such that D is a module-finite extension over C . Let $\mathcal{N} \in \text{Max}(D)$, and set $\mathcal{M} = \mathcal{N} \cap C$. Then \mathcal{M} is a maximal ideal of C . Note that $D_{\mathcal{M}} \cong D_{\mathcal{N}}$ and $\mathbb{Q}(C)_{\mathcal{M}} \cong \mathbb{Q}(C_{\mathcal{M}})$. Therefore, $C_{\mathcal{M}} \subseteq D_{\mathcal{M}} \subseteq \mathbb{Q}(C_{\mathcal{M}})$, where $D_{\mathcal{M}}$ is a module-finite extension over $C_{\mathcal{M}}$. Since $e(C_{\mathcal{M}}) \leq 2$, [24, 12.2] shows $e(D_{\mathcal{N}}) \leq 2$.

Now, to see C is Arf, let $D = C_n$, a blowup of C ; see Notation 2.6. Then, $e(D_{\mathcal{N}}) \leq 2$, which implies $1 \leq \text{embdim}(D_{\mathcal{N}}) = e(D_{\mathcal{N}})$. Therefore, C is an Arf ring by Proposition 2.7. \square

3. A Proof of the main result

Setup 3.1. Let (R, \mathfrak{m}) be a one-dimensional Cohen–Macaulay local ring with canonical module K_R . We set $B = \mathfrak{m} :_{Q(R)} \mathfrak{m}$, the endomorphism algebra of the maximal ideal \mathfrak{m} , where $Q(R)$ denotes the total quotient ring of R .

We fix a canonical ideal I of R , and assume I contains a parameter ideal aR of R as a reduction. Let

$$S = R[K], \text{ where } K = \frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\} \subseteq Q(R).$$

Furthermore we define the conductor as $\mathfrak{c} = R :_{Q(R)} S$. Notice $R \subseteq K \subseteq \overline{R}$, where \overline{R} is the integral closure of R in $Q(R)$.

Note $K \cong I \cong K_R$, and both B and S , being module finite extensions of R , are one-dimensional semi-local rings. Note also that $e(R) = e_{\mathfrak{m}}^0(S) = e_{\mathfrak{m}}^0(B)$. \square

We will make use of the following theorems for our proof of Theorem 3.4, which is the key ingredient in our proof of Theorem 1.1.

Theorem 3.2. ([14, 5.1]) *The following conditions are equivalent.*

- (i) R is an almost Gorenstein ring and $\text{embdim}(R) = e(R)$.
- (ii) B is a Gorenstein ring.

Theorem 3.3. ([12, 4.18]) *Assume there is an element $x \in \mathfrak{m}$ such that $\mathfrak{m}^2 = x\mathfrak{m}$. Then the following conditions are equivalent.*

- (i) R is a generalized Gorenstein ring, but R is not an almost Gorenstein ring.
- (ii) B is not a Gorenstein ring, but B is a generalized Gorenstein local ring with maximal ideal \mathfrak{n} such that $\mathfrak{n}^2 = x\mathfrak{n}$.

When one of these equivalent conditions hold, we have $R/\mathfrak{m} \cong B/\mathfrak{n}$, and

$$\ell_B(B/(B : B[L])) = \ell_R(R/\mathfrak{c}) - 1,$$

where $L = BK$.

Theorem 3.4. *Assume $e(R) \geq 3$. Then the following conditions are equivalent.*

- (i) R is a generalized Gorenstein ring with minimal multiplicity.
- (ii) S is Gorenstein, and there is an integer $N > 0$ such that the following hold:
 - (a) $S = R_N$.
 - (b) For each integer $n = 0, \dots, N - 1$, it follows R_n is a local ring such that $\text{embdim}(R_n) = e(R_n) = e(R)$.

Furthermore, if condition (ii) holds, then we have $N = \ell_R(R/\mathfrak{c})$.

Proof. We set $\ell = \ell_R(R/\mathfrak{c})$. By [14, 3.5, 3.7], let us remark that R is Gorenstein if and only if $\ell = 0$. Besides, by [14, 3.5, 3.16], the condition that R is a non-Gorenstein almost Gorenstein ring is equivalent to the case where $\ell = 1$.

(i) \Rightarrow (ii) Since $\text{embdim}(R)=e(R)$, we choose $x\in\mathfrak{m}$ such that $\mathfrak{m}^2=x\mathfrak{m}$. Set $N=\ell=\ell_R(R/\mathfrak{c})$. As $e(R)\geq 3$, R is not a Gorenstein ring, so that $\ell>0$. If $\ell=1$, then R is an almost Gorenstein ring and $S=\mathfrak{m}:\mathbb{Q}(R)\mathfrak{m}$ is a Gorenstein ring by Theorem 3.2. Suppose that $\ell>1$ and the assertion holds for $\ell-1$. We then have R is not an almost Gorenstein ring. By Theorem 3.3, B is a generalized Gorenstein ring, but not Gorenstein. Furthermore, $\mathfrak{n}^2=x\mathfrak{n}$ and $R/\mathfrak{m}\cong B/\mathfrak{n}$, where \mathfrak{n} denotes the maximal ideal of B . Hence we have:

$$e(R)=e_{\mathfrak{m}}^0(B)=\ell_R(B/xB)=\ell_B(B/xB)=e_{\mathfrak{n}}^0(B).$$

Note that $L=KB$ is a B -submodule of $\mathbb{Q}(B)$ such that $B\subseteq L\subseteq\overline{B}$, $L\cong K_B$, and $S=B[L]$; see [7, 5.1] for the details. Therefore, we get

$$\ell_B(B/\mathfrak{a})=\ell_R(R/\mathfrak{a})-1=\ell-1,$$

where $\mathfrak{a}=B:\mathbb{Q}(R)B[L]$ denotes the conductor of $B[L]$. Hence, by induction hypothesis, $S=B[L]$ is a Gorenstein ring and $R_n=B_{n-1}$ is a local ring with minimal multiplicity $e(R)$ for every $1\leq n<\ell$.

(ii) \Rightarrow (i) If $N=1$, then $S=\mathfrak{m}:\mathbb{Q}(R)\mathfrak{m}$, since \mathfrak{m} is stable. Hence, by Theorem 3.2, the Gorensteinness of S implies that R is an almost Gorenstein ring and $\ell=N$. Suppose that $N>1$ and the assertion holds for $N-1$. Since $R=R_0$ has minimal multiplicity, there exists $x\in\mathfrak{m}$ such that $\mathfrak{m}^2=x\mathfrak{m}$. Then, since $N\geq 2$, it follows

$$R_1=\mathfrak{m}:\mathbb{Q}(R)\mathfrak{m}=B$$

is a local ring with minimal multiplicity $e(R)$. Note xB is a parameter ideal of B . Moreover we have

$$e(R)=e_{\mathfrak{m}}^0(R)=e_{\mathfrak{m}}^0(B)=\ell_R(B/xB)\geq\ell_B(B/xB)\geq e_{\mathfrak{n}}^0(B)=e(R),$$

where \mathfrak{n} denotes the maximal ideal of B . It follows that $R/\mathfrak{m}\cong B/\mathfrak{n}$. Thus, by [34], xB is a reduction of \mathfrak{n} , whence $\mathfrak{n}^2=x\mathfrak{n}$. Therefore the induction arguments shows B is a generalized Gorenstein ring. Since B is not a Gorenstein ring, by Theorem 3.3, we see R is a generalized Gorenstein ring, but not an almost Gorenstein ring, and also

$$\ell_B(B/\mathfrak{a})=\ell-1,$$

where $\mathfrak{a}=B:B[KB]$. Note that R_n has minimal multiplicity $e(R)$ for every $1\leq n<N$, so is B_{n-1} . By the induction hypothesis, we conclude that $N-1=\ell-1$, as desired. \square

We are now ready to prove our main result, namely Theorem 1.1 advertised in the introduction.

Proof of Theorem 1.1. Notice, if R is Arf, then the maximal ideal \mathfrak{m} is stable so that R has minimal multiplicity; see Definition 2.4. Therefore, throughout, we may assume R has minimal multiplicity. We may also assume $e(R) \geq 3$ by Proposition 2.8. So, by Theorem 3.4, we conclude S is Gorenstein, $S = R_N$ for some positive integer N , and the blowup R_n of R is a local ring with minimal multiplicity for each integer $n = 0, \dots, N-1$.

(i) \Rightarrow (ii): Assume R is Arf. Then, for each nonnegative integer n and each maximal ideal \mathcal{M} of R_n , it follows from Proposition 2.7 that $\text{embdim}((R_n)_{\mathcal{M}}) = e((R_n)_{\mathcal{M}})$. Since $S_v = R_{N+v}$ for each nonnegative integer v , we conclude, using Proposition 2.7 once more, that S is Arf. One can now observe that $S_{\mathcal{M}}$ has minimal multiplicity for all maximal ideals \mathcal{M} of S . Thus the Cohen-Macaulay type of $S_{\mathcal{M}}$ equals $e(S_{\mathcal{M}}) - 1$, provided $S_{\mathcal{M}}$ is not regular. In particular, since S is Gorenstein, we obtain $e(S_{\mathcal{M}}) \leq 2$.

(ii) \Rightarrow (i): Assume R has minimal multiplicity and $e(S_{\mathcal{M}}) \leq 2$ for each maximal ideal \mathcal{M} of S . Letting $C = S$, we see from Proposition 2.8 that S is an Arf ring. Letting $A = S$ in Proposition 2.7, we conclude that all the localizations of the blowups of S have minimal multiplicity, i.e., for each nonnegative integer l , and for each maximal ideal \mathcal{N} of S_l , the ring $(S_l)_{\mathcal{N}}$ has minimal multiplicity. Since $S = R_N$, this property is also true for the blowups of R , which are local rings. Therefore $\text{embdim}(R_n) = e(R_n)$ for each integer $n = 0, \dots, N-1$. Finally we deduce from Proposition 2.7 that R is Arf. \square

4. Corollaries of the main argument

In this section we maintain the notations of Setup 3.1. We give applications of our argument and obtain new characterizations of Arf rings. In particular, we extend a result of Barucci and Fröberg [5] and determine certain conditions that make the idealization $R \bowtie \mathfrak{c}$ to be a generalized Gorenstein Arf ring; see Corollaries 4.5, 4.6 and Theorem 4.10.

We start by giving two examples that show Arf and generalized Gorenstein properties are independent of each other, in general.

Example 4.1. Let k be a field and set $R = k[[t^3, t^7, t^{11}]]$. Then R is a Cohen-Macaulay, non-Gorenstein ring of Cohen-Macaulay type two. We proceed and prove that R is almost Gorenstein, and hence generalized Gorenstein, but is not Arf; see Definitions 2.3 and 2.5.

Note that $K_R = R + Rt^4$ by [28, 2.1.9] (see also [1, Section 2] and [29, Section 6]). As $\mathfrak{m}K_R = \mathfrak{m}t^4 \subseteq R$, we conclude from [14, 3.11] that R is almost Gorenstein.

To show R is not Arf, we compute the blowup of R at \mathfrak{m} :

$$(4.1.1) \quad R_1 = R^{\mathfrak{m}} = R \left[\frac{\mathfrak{m}}{t^3} \right] = k[[t^3, t^4]]$$

Here, in (4.1.1), the second equality holds since t^3 is a reduction of \mathfrak{m} ; see the discussion preceding Definition 2.4. As R_1 does not have minimal multiplicity, R is not Arf by Proposition 2.7.

Example 4.2. Let k be a field and set $R=k[[t^4, t^7, t^9, t^{10}]]$. Then R is not generalized Gorenstein ring; see [12, 4.27]. To see R is Arf, we proceed as in Example 4.1.

$$(4.2.1) \quad R_1 = R^{\mathfrak{m}} = R \left[\frac{\mathfrak{m}}{t^4} \right] = k[[t^3, t^4, t^5]]$$

Here, in (4.2.1), the second equality holds since t^4 is a reduction of \mathfrak{m} . Letting \mathfrak{m}_1 to be the unique maximal ideal of R_1 , we get:

$$(4.2.2) \quad R_2 = R_1^{\mathfrak{m}} = R \left[\frac{\mathfrak{m}_1}{t^4} \right] = k[[t]].$$

The second equality in (4.2.2) holds since $\mathfrak{m}_1^2 = t^3 \mathfrak{m}_1$. As each blowup of R is contained in the integral closure $\overline{R} = k[[t]]$ of R , we conclude that $R = R_0$, R_1 and $R_2 = \overline{R}$ are the only distinct blowups of R . Since each of these blowups have minimal multiplicities, we see that R is Arf by Proposition 2.7. \square

Note that it follows from the definition that S is a local ring in case R is a numerical semigroup ring. Hence the ring S in Example 4.1 is local, whilst R is a non-Arf ring with minimal multiplicity three. The next corollary of Theorem 1.1 shows that such a ring R must be Arf in case S is not local.

Corollary 4.3. *Assume $\text{embdim}(R) = e(R) = 3$, i.e., R has minimal multiplicity three. If S is not a local ring, then R is an Arf ring.*

Proof. Note that, since R has minimal multiplicity, there exists an element $x \in \mathfrak{m}$ such that $\mathfrak{m}^2 = x\mathfrak{m}$. We conclude by [12, 4.8] that R is a generalized Gorenstein ring. Therefore we have:

$$\begin{aligned} 3 = e(R) &= e_{\mathfrak{m}}^0(S) = e_{xR}^0(S) = \ell_R(S/xS) \geq \ell_S(S/xS) \\ &= \sum_{\mathcal{M} \in \text{Max}(S)} \ell_{S_{\mathcal{M}}}(S_{\mathcal{M}}/xS_{\mathcal{M}}) \\ &\geq \sum_{\mathcal{M} \in \text{Max}(S)} e(S_{\mathcal{M}}), \end{aligned}$$

where $\text{Max}(S)$ denotes the set of all maximal ideals of S . Since S is not local, there are at least two distinct ideals in $\text{Max}(S)$. This implies that $e(S_{\mathcal{M}}) \leq 2$ for each $\mathcal{M} \in \text{Max}(S)$. Now, since R is a generalized Gorenstein ring with minimal multiplicity, it follows from Theorem 1.1 that R is Arf. \square

Next we recall a beautiful result of Barucci and Fröberg [5, 13] that gives a characterization of almost Gorenstein Arf numerical semigroup rings.

Theorem 4.4. (see [5, 13]) *Let $\ell > 0$ be an integer and $0 < a_1 < a_2 < \dots < a_\ell$ be integers such that $\gcd(a_1, a_2, \dots, a_\ell) = 1$. Assume k is a field, $R = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$ is the numerical semigroup ring over k , and $H = \langle a_1, a_2, \dots, a_\ell \rangle$ is the corresponding semigroup. Then the following are equivalent.*

- (i) R is an almost Gorenstein Arf ring.
- (ii) $2 + a_i \in H$ for each $i = 1, \dots, \ell$.

In Corollary 4.5 and Proposition 4.6, we will obtain natural generalizations of Theorem 4.4. These results will be useful to construct new examples of almost Gorenstein Arf rings; see Examples 4.7 and 4.8.

Recall that, in the following, and unless otherwise stated, we maintain the notations of Setup 3.1.

Corollary 4.5. *The following conditions are equivalent.*

- (i) R is an almost Gorenstein Arf ring.
- (ii) $e(B_{\mathcal{N}}) \leq 2$ for each maximal ideal \mathcal{N} of B .

Proof. First assume $e(R) \leq 2$. Then it follows from Proposition 2.8 that R is Arf. Hence, since R is Gorenstein, (i) follows. If $e(R) = 1$, then R is regular so that $B = R = \overline{R}$; in particular (ii) holds. Moreover, if $e(R) = 2$, then (ii) follows from [24, 12.2]. Consequently we may assume $e(R) \geq 3$.

$(i) \Rightarrow (ii)$ Assume R is Arf. Then R has minimal multiplicity so that \mathfrak{m} is stable and $B = R_1 = R^{\mathfrak{m}}$; see Definition 2.4 and Notation 2.6. Therefore, by Proposition 2.7, $B_{\mathcal{N}}$ has minimal multiplicity for each maximal ideal \mathcal{N} of B . This implies that B is Gorenstein. Consequently, given a maximal ideal \mathcal{N} of B , since $B_{\mathcal{N}}$ is a Gorenstein ring with minimal multiplicity, we conclude that $e(B_{\mathcal{N}}) \leq 2$.

$(ii) \Rightarrow (i)$ Assume $e(B_{\mathcal{N}}) \leq 2$ for each maximal ideal \mathcal{N} of B . Then B is Gorenstein. Hence, by Theorem 3.2, R is an almost Gorenstein ring with minimal multiplicity.

Now, if R is Gorenstein, then R is a hypersurface with $e(R) \leq 2$. In particular, R is Arf by Proposition 2.8. If R is not Gorenstein, since it is almost Gorenstein, we have from [14, 3.16] that $B = S$. Hence, by the hypothesis, $e(S_{\mathcal{N}}) \leq 2$ for each maximal ideal \mathcal{N} of S . So R is Arf by Theorem 1.1. \square

In passing, we give a short proof of Theorem 4.4 which is different from the argument of Barucci and Fröberg [5]. We will use Corollary 4.5 and the fact that, if R is a numerical semigroup ring as in Theorem 4.4, then $B = \mathfrak{m}_{:Q(R)} \mathfrak{m}$ is also a numerical semigroup ring (in particular, B is local); see, for example, [23, 2.13].

A proof of Theorem 4.4. Assume R is an almost Gorenstein Arf ring. Then it follows from Corollary 4.5 that $e(B) \leq 2$. Therefore $t^2 \in B$ and $t^2 \mathfrak{m} \subseteq \mathfrak{m}$. This shows $2 + a_i \in H$ for each $i = 1, \dots, \ell$.

Now assume $2 + a_i \in H$ for each $i = 1, \dots, \ell$. Hence $t^2 \mathfrak{m} \subseteq \mathfrak{m}$ and $t^2 \in B$. Let V and \mathfrak{n} denote \overline{R} and the unique maximal ideal of B , respectively. Note that $\mathfrak{n}V = xV$ for some $x \in \mathfrak{n}$. Then, setting $\mathfrak{q} = (x)$, we have:

$$e(B) = e_{\mathfrak{n}}^0(B) = e_{\mathfrak{n}}^0(V) = e_{\mathfrak{q}}^0(V) = \ell_B(V/\mathfrak{q}V) = \ell_B(V/\mathfrak{n}V).$$

Here the third equality holds since \mathfrak{q} is a reduction of \mathfrak{n} . Therefore it follows that:

$$e(B) = \ell_B(V/\mathfrak{n}V) \leq \ell_B(V/t^2V) = \ell_V(V/t^2V) = 2.$$

Consequently, by Corollary 4.5, R is almost Gorenstein and Arf. \square

Our next result yields an extension of Theorem 4.4.

Proposition 4.6. *Assume R is a numerical semigroup ring as in Theorem 4.4. Assume further R is a generalized Gorenstein ring. Then the following conditions are equivalent.*

- (i) R is an Arf ring.
- (ii) R has minimal multiplicity, $2 + \ell_R(R/\mathfrak{c}) \cdot a_1 \in H$, and $2 + a_i \in H$ for each $i = 2, \dots, \ell$.

Proof. Recall that an Arf ring has minimal multiplicity. Hence we may assume R has minimal multiplicity throughout the proof.

Assume $a_1 = e(R) \leq 2$. Then R is Arf by Proposition 2.8. Hence (i) holds. Since $a_1 \in H$, we see that $2 + a_i \in H$ for all $i = 1, \dots, \ell$. Moreover, as R is Gorenstein, $\ell_R(R/\mathfrak{c}) = 0$ so that $2 + \ell_R(R/\mathfrak{c}) \cdot a_1 \in H$. In particular (ii) holds. Consequently we may assume $e(R) \geq 3$. So it follows from Theorem 3.4 that $S = R_N$ is Gorenstein and R_i is a local ring with minimal multiplicity $e(R)$ for each $i = 0, \dots, N - 1$, where $N = \ell_R(R/\mathfrak{c})$.

Fix i with $1 \leq i \leq N - 1$, and let \mathfrak{m}_i denote the unique maximal ideal of the local ring R_i . Then, since \mathfrak{m}_i is stable, we have $R_i = \mathfrak{m}_{i-1} :_{Q(R_i)} \mathfrak{m}_{i-1}$; see Definition 2.4. (Here $\mathfrak{m}_0 = \mathfrak{m}$).

Note, by Proposition 2.7, R is Arf if and only if R_{N-1} is Arf. Also, since $S = R_N = \mathfrak{m}_{N-1} :_{Q(R_N)} \mathfrak{m}_{N-1}$ is Gorenstein, we see from Theorem 3.2 that R_{N-1} is almost Gorenstein. So R is Arf if and only if R_{N-1} is almost Gorenstein Arf.

The ring R_i , since it has minimal multiplicity, is a numerical semigroup ring; see [7, Section 6]. More precisely, $R_i = k[[H_i]]$, where H_i is the numerical semigroup generated by $\{a_1, a_2 - ia_1, \dots, a_\ell - ia_1\}$. Therefore, by Theorem 4.4, for each $j = 2, \dots, N$, we have:

$$(4.6.1) \quad R \text{ is Arf if and only if } 2 + a_1 \in H_{N-1} \text{ and } a_j - (N-1) \cdot a_1 + 2 \in H_{N-1}.$$

Now we claim that $x \in H_1$ if and only if $x + a_1 \in H$ for each nonnegative integer x . First we proceed by assuming the claim.

It follows from the claim, for a nonnegative integer x , we have that:

$$(4.6.2) \quad x \in H_{N-1} \text{ if and only if } x + (N-1)a_1 \in H.$$

Now, letting $x = a_1 + 2$, we obtain from (4.6.2) that:

$$(4.6.3) \quad 2 + a_1 \in H_{N-1} \text{ if and only if } 2 + a_1 \cdot N = 2 + \ell_R(R/\mathfrak{c}) \cdot a_1 \in H$$

Moreover, for each $j = 2, \dots, N$, it follows from (4.6.2) that:

$$(4.6.4) \quad \begin{aligned} & a_j - (N-1) \cdot a_1 + 2 \in H_{N-1} \quad \text{if and only if} \\ & [a_j - (N-1) \cdot a_1 + 2 \in H_{N-1}] + (N-1) \cdot a_1 \in H. \end{aligned}$$

Therefore, by (4.6.1), (4.6.3) and (4.6.4), we conclude that R is Arf if and only if $2 + \ell_R(R/\mathfrak{c}) \cdot a_1 \in H$, and $2 + a_j \in H$ for each $j = 2, \dots, \ell$.

Next, to complete the proof, we justify the above claim. Let $x \in H_1$. Then $t^x \in R_1 = \mathfrak{m} :_{\mathbb{Q}(R)} \mathfrak{m}$, i.e., $t^x \mathfrak{m} \subseteq \mathfrak{m}$. This implies $t^{x+a_1} \in \mathfrak{m}$, or equivalently, $x + a_1 \in H$.

Now assume $x + a_1 \in H$ for each nonnegative integer x . Since R has minimal multiplicity, it follows that $\mathfrak{m}^2 = t^{a_1} \mathfrak{m}$. Therefore,

$$(4.6.5) \quad R_1 = R^{\mathfrak{m}} = R \left[\frac{\mathfrak{m}}{t^{a_1}} \right] = \bigcup_{i \geq 0} \frac{\mathfrak{m}^i}{(t^{a_1})^i} = \frac{\mathfrak{m}}{t^{a_1}}$$

Here, in (4.6.5), the second equality holds since t^{a_1} is a reduction of \mathfrak{m} ; see the discussion preceding Definition 2.4. Moreover the fourth equality is due to the fact that $\frac{\mathfrak{m}^i}{(t^{a_1})^i} = \frac{\mathfrak{m}}{t^{a_1}}$ for each $i \geq 2$. Hence, since $x + a_1 \in H$, we have:

$$t^x = \frac{t^{x+a_1}}{t^{a_1}} \in \frac{\mathfrak{m}}{t^{a_1}} = R_1 = \mathfrak{m} :_{\mathbb{Q}(R)} \mathfrak{m}$$

This implies that $x \in H_1$. \square

We are now ready to construct several examples. The rings in Examples 4.7 and 4.8 are almost Gorenstein Arf rings. On the other hand, the one in Example 4.9 is a generalized Gorenstein Arf ring which is not almost Gorenstein.

Example 4.7. Let k be a field, $e \geq 2$ an integer, and $R = k[[t^e, t^{e+1}, \dots, t^{2e-1}]]$ be the numerical semigroup ring. Since the conductor of the corresponding semigroup is e , we see $(e+i)+2 \in H$ for all $i=0, \dots, e-1$. Therefore, by Theorem 4.4, R is an almost Gorenstein Arf ring.

Notice $\mathfrak{m}^2 = t^e \mathfrak{m}$. So it follows that

$$B = \mathfrak{m} :_{\mathbb{Q}(R)} \mathfrak{m} = R \left[\frac{\mathfrak{m}}{t^e} \right] = k[[t, t^2, \dots, t^{e-1}]] = k[[t]] = \overline{R}.$$

Hence $e(B) = 1 \leq 2$, cf., Corollary 4.5. \square

Example 4.8. Let k be a field, $e \geq 3$ an integer, and $R = k[[t^e, t^{e+2}, \dots, t^{2e-1}, t^{2e+1}]]$ be the numerical semigroup ring. Since the conductor of the corresponding semigroup is $e+2$, we see that $2+e$, $2+(2e+1)$, as well as $2+(e+i)$, for all $i=2, \dots, e-1$, belong to H . Therefore, by Theorem 4.4, R is an almost Gorenstein Arf ring.

Notice $\mathfrak{m}^2 = t^e \mathfrak{m}$. So it follows that

$$B = \mathfrak{m} :_{\mathbb{Q}(R)} \mathfrak{m} = R \left[\frac{\mathfrak{m}}{t^e} \right] = k[[1, t^2, t^3, \dots, t^{e-1}, t^{e+1}]] = k[[t^2, t^3]].$$

Hence $e(B) = 2 \leq 2$, cf., Corollary 4.5. \square

Example 4.9. Let k be a field and let $R = k[[t^5, t^{16}, t^{17}, t^{18}, t^{19}]]$ be the numerical semigroup ring. It was proved in [12, 4.27] that R is a generalized Gorenstein ring.

The canonical module K_R of R is $R + Rt + Rt^2 + Rt^3$. As $t \in K_R$, we have that $k[[t]] \subseteq R[K_R] = S$. Since the conductor of the corresponding semigroup H is 15, setting $V = \overline{R}$, we see that $\mathfrak{c} = R :_{\mathbb{Q}(R)} S = (tV)^{15}$. Therefore we have $\ell_R(R/\mathfrak{c}) = \ell_R(V/\mathfrak{c}) - \ell_R(V/R) = 15 - 12 = 3$. This implies $2 + \ell_R(R/\mathfrak{c}) \cdot a_1 = 17 \in H$. Moreover $2 + a_i \in \{18, 19, 20, 21\} \subseteq H$ for each $i=2, \dots, 5$. So R is Arf by Proposition 4.6. Note $\mathfrak{m}K_R \not\subseteq R$ since $t^5 \cdot t \in \mathfrak{m}K_R - R$. Consequently, R is not almost Gorenstein; see [14, 3.11]. \square

Next we move to another application somewhat different in nature.

An application of Theorem 1.1 on a certain idealization

The aim of this subsection is to give a necessary and sufficient condition for the ring $A = R \times \mathfrak{c}$ to be generalized Gorenstein and Arf, where $A = R \times \mathfrak{c}$ is the idealization of $\mathfrak{c} = R :_{\mathbb{Q}(R)} S$ (Recall we follow the notations of Setup 3.1). Note that, when R is a generalized Gorenstein, A is always generalized Gorenstein; see [12, 4.15].

Theorem 4.10. *Assume R is a generalized Gorenstein ring and set $A = R \times \mathfrak{c}$, where $\mathfrak{c} = R :_{\mathbb{Q}(R)} S$. Then the following conditions are equivalent.*

- (i) A is an Arf ring.
- (ii) R has minimal multiplicity and $S = \overline{R}$.

Moreover, if (i) or (ii) holds, then R is an Arf ring.

Proof. We start by noting that $K_A \cong \text{Hom}_R(\mathfrak{c}, K) \times K \cong S \times K$; see [12, Section 4]. Set $L = S \times K$. Then it follows that $A[L] = L^2 = S \times S$; see [12, 4.14]. Notice A is a generalized Gorenstein ring; this follows from [12, 4.15] in case A is not Gorenstein. Note also that $\mathfrak{n} = \mathfrak{m} \times \mathfrak{c}$ is the unique maximal ideal of A

(1) \Rightarrow (2) Assume A is an Arf ring. Then, by Theorem 1.1, $e(A[L]_{\mathcal{N}}) \leq 2$ for each maximal ideal \mathcal{N} of $A[L]$. Thus $2 \cdot e(S_{\mathcal{M}}) = e(S_{\mathcal{M}} \times S_{\mathcal{M}}) \leq 2$, i.e., $e(S_{\mathcal{M}}) \leq 1$. So S is a (semi-local) regular ring, and this implies that $S = \overline{R}$ (recall $R \subseteq S \subseteq \overline{R}$).

It follows, since A has minimal multiplicity, that $\mathfrak{n}^2 = \zeta \mathfrak{n}$. Setting $\zeta = (\alpha, x)$ with $\alpha \in \mathfrak{m}$ and $x \in \mathfrak{c}$, and using the natural projection $p: A \rightarrow R$, $(\alpha, x) \mapsto \alpha$, we deduce that $\mathfrak{m}^2 = \alpha \mathfrak{m}$, i.e., R has minimal multiplicity. In particular, by Theorem 1.1, we see that R is an Arf ring.

(2) \Rightarrow (1) Assume R has minimal multiplicity and $S = \overline{R}$. Notice, for each maximal ideal \mathcal{M} of S , it follows that $e(S_{\mathcal{M}} \times S_{\mathcal{M}}) = 2$ since $S_{\mathcal{M}}$ is regular. Therefore it suffices to prove A has minimal multiplicity; see Theorem 1.1 and recall $A[L] = S \times S$.

As R has minimal multiplicity, we can pick an element $\alpha \in \mathfrak{m}$ such that $\mathfrak{m}^2 = \alpha \mathfrak{m}$. Then, since \mathfrak{m} is stable, we have $\mathfrak{c} \subseteq R \subseteq R[\frac{\mathfrak{m}}{\alpha}] = R^{\mathfrak{m}} = \mathfrak{m}_{:Q(R)} \mathfrak{m} = B$. Thus \mathfrak{c} is an ideal of B . As $R[\frac{\mathfrak{m}}{\alpha}] = \frac{\mathfrak{m}}{\alpha} \in B$, it follows that $\frac{\mathfrak{m}}{\alpha} \cdot \mathfrak{c} \subseteq \mathfrak{c}$, i.e., $\mathfrak{m} \cdot \mathfrak{c} = \alpha \cdot \mathfrak{c}$. Therefore we have $\mathfrak{n}^2 = \mathfrak{m}^2 \times \mathfrak{m} \mathfrak{c} = \alpha \mathfrak{m} \times \alpha \mathfrak{m} = (\alpha, 0) \cdot (\mathfrak{m} \times \mathfrak{c})$, i.e., A has minimal multiplicity. Now, by Theorem 1.1, A is Arf. \square

Here is a consequence of Theorem 4.10 that gives a useful criterion for the idealization $R \rtimes \mathfrak{m}$ to be almost Gorenstein Arf. Recall, if R is an almost Gorenstein ring that is not Gorenstein, then $\mathfrak{c} = R_{:Q(R)} S = \mathfrak{m}$; see 3.1 and [14, 3.16].

Corollary 4.11. $R \rtimes \mathfrak{m}$ is an almost Gorenstein Arf ring if and only if $\mathfrak{m} \overline{R} \subseteq R$.

Proof. Let $A = R \rtimes \mathfrak{m}$ and $\mathfrak{n} = \mathfrak{m} \times \mathfrak{m}$.

Assume first A is an almost Gorenstein Arf ring. Then, by Theorem 3.2, the endomorphism algebra $\mathfrak{n}_{:Q(A)} \mathfrak{n}$ is Gorenstein, where $\mathfrak{n} = \mathfrak{m} \times \mathfrak{m}$. Let \mathcal{N} be a maximal ideal of $B = \mathfrak{m}_{:Q(R)} \mathfrak{m}$. Then, since $\mathfrak{n}_{:Q(A)} \mathfrak{n} = B \times B$, and $(\mathfrak{n}_{:Q(A)} \mathfrak{n}) / (\mathcal{N} \times B) = B / \mathfrak{n}$, we see $\mathcal{N} \times B$ is a maximal ideal of $\mathfrak{n}_{:Q(A)} \mathfrak{n}$. Moreover, the localization $(\mathfrak{n}_{:Q(A)} \mathfrak{n})_{\mathcal{N} \times B} = B_{\mathfrak{n}} \times B_{\mathfrak{n}}$ has multiplicity two; see Corollary 4.5. Therefore B is regular, and $B = \overline{R}$.

Conversely, assume $\mathfrak{m} \overline{R} \subseteq R$. If R is regular, then $\mathfrak{m} \cong R \cong K_R$ so that $R \rtimes \mathfrak{m}$ is Gorenstein [35]. So we may assume $R \neq \overline{R}$. Then $\mathfrak{m} \overline{R} \neq R$. Since $\overline{R} \subseteq \mathfrak{m}_{:Q(R)} \mathfrak{m} = B$, we have $B = \overline{R}$. Then it follows that $\mathfrak{n}_{:Q(A)} \mathfrak{n} = B \times B$. In particular, $e((\mathfrak{n}_{:Q(A)} \mathfrak{n})_{\mathcal{N}}) = 2$

for each maximal ideal \mathcal{N} of $B \times B$. So, by Corollary 4.5, we conclude A is an almost Gorenstein Arf ring. \square

Here is an application of Corollary 4.11.

Example 4.12. Let (S, \mathfrak{n}) be a regular local ring of dimension d with $d \geq 3$. Set $R = S/I$ where $I = \bigcap_{i=1}^d (x_1, \dots, \widehat{x}_i, \dots, x_d)$, where \widehat{x}_i is used to remove x_i from the ideal. Then R is an almost Gorenstein local ring with $B = \overline{R}$; see [14, 5.3]. Since B is Gorenstein, it follows from Theorem 3.2 that R has minimal multiplicity. Hence the maximal ideal \mathfrak{m} of R is stable, so that $R_1 = \overline{R}$. This implies all the blowups of R equal to \overline{R} , and have multiplicity one. Now, by Proposition 2.8, R is Arf. Since $\mathfrak{m}\overline{R} = \mathfrak{m}B \subseteq \mathfrak{m}$, we conclude that $R \rtimes \mathfrak{m}$ is an almost Gorenstein Arf ring.

In the next example we find out an idealization ring of the form $R \rtimes \mathfrak{m}$ which is almost Gorenstein, but neither R nor $R \rtimes \mathfrak{m}$ is Arf.

Example 4.13. Let $R = k[[t^4, t^5, t^6]]$. Then $R \rtimes \mathfrak{m} \cong k[[X, Y, Z, U, V, W]]/I$, where I is the sum of the ideals $(YU - XV, ZU - XW, ZU - YV, ZV - YW, X^2U - ZW)$, $(X^3 - Z^2, Y^2 - ZX)$ and $(U, V, W)^2$. Since R is Gorenstein, we know from [14, 6.5] that $R \rtimes \mathfrak{m}$ is almost Gorenstein. Furthermore, since $t^7 \in \mathfrak{m}\overline{R} \not\subseteq R$, we conclude from Corollary 4.11 that $R \rtimes \mathfrak{m}$ is not an Arf ring.

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