

On Legendrian embeddings into open book decompositions

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Abstract. We study Legendrian embeddings of a compact Legendrian submanifold L sitting in a closed contact manifold (M, ξ) whose contact structure is supported by a (contact) open book \mathcal{OB} on M . We prove that if \mathcal{OB} has Weinstein pages, then there exist a contact structure ξ' on M , isotopic to ξ and supported by \mathcal{OB} , and a contactomorphism $f: (M, \xi) \rightarrow (M, \xi')$ such that the image $f(L)$ of any such submanifold can be Legendrian isotoped so that it becomes disjoint from the closure of a page of \mathcal{OB} .

1. Introduction

A *contact manifold* is a pair (M^{2n+1}, ξ) where M is a smooth manifold and $\xi \subset TM$ is a totally non-integrable $2n$ -plane field distribution on M . The distribution ξ is called a *contact structure* on M , and is said to be *co-oriented* if it is the kernel of a globally defined 1-form α with the property $\alpha \wedge (d\alpha)^n \neq 0$. Such a 1-form is called a *contact form* on M . Here we always assume that ξ is a co-oriented *positive* contact structure, that is, $\xi = \text{Ker}(\alpha)$ and $\alpha \wedge (d\alpha)^n > 0$ with respect to a given orientation on M . We say that two contact manifolds (M, ξ) and (M', ξ') are *contactomorphic* if there exists a diffeomorphism $f: M \rightarrow M'$ such that $f_*(\xi) = \xi'$. Two contact structures ξ_0, ξ_1 on a M are said to be *isotopic* if there exists a continuous 1-parameter family ξ_t ($0 \leq t \leq 1$) of contact structures joining them.

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A submanifold $L \subset (M^{2n+1}, \xi)$ is said to be *isotropic* if $TL \subset \xi$, and an isotropic submanifold is called *Legendrian* if $\dim(L) = n$. A *Legendrian embedding* is an embedding $\phi: \Sigma^n \hookrightarrow (M^{2n+1}, \xi)$ of a smooth manifold Σ^n such that the image $\phi(\Sigma^n)$ is Legendrian. A smooth 1-parameter family of Legendrian submanifolds is called a *Legendrian isotopy*. Equivalently, a Legendrian isotopy is a smooth 1-parameter family $\phi_t: \Sigma^n \hookrightarrow (M^{2n+1}, \xi)$ of Legendrian embeddings.

An *open book (decomposition)* \mathcal{OB} on a closed manifold M is determined by a pair (B, φ) where $B \hookrightarrow M$ is codimension 2 submanifold with trivial normal bundle, and $\varphi: M \setminus B \rightarrow S^1$ is a fiber bundle projection such that the normal bundle has a trivialization $B \times D^2$, where the angle coordinate on the disk agrees with the fibration map φ . The manifold B is called the *binding*, and for any $t_0 \in S^1$ a fiber $X = \varphi^{-1}(t_0)$ is called a *page* of the open book.

The following definition is due to Giroux [4]: A contact structure ξ on M is said to be *supported by* (or *carried by*, or *compatible with*) an open book $\mathcal{OB} = (B, \varphi)$ on M if there exists a contact form α for ξ such that

(i) $(B, \alpha|_{TB})$ is a contact manifold. (Here TB denotes the tangent bundle of B as a sub-bundle of the tangent bundle of TM .)

(ii) For any $t \in S^1$, the page $X = \varphi^{-1}(t)$ is a symplectic manifold with symplectic form $d\alpha$.

(iii) If \overline{X} denotes the closure of a page X in M , then the orientation of $B \cong \partial \overline{X}$ induced by its contact form $\alpha|_{TB}$ coincides with its orientation as the boundary of $(\overline{X}, d\alpha)$.

We will say that an open book \mathcal{OB} on M is called a *contact open book* if it carries a contact structure on M .

In the smooth category, given an n -dimensional submanifold L of a closed $(2n+1)$ -manifold M admitting an open book \mathcal{OB} , it is, in general, not possible to isotope L so that it becomes disjoint from a page \overline{X} of \mathcal{OB} . However, if \overline{X} deformation retracts onto a finite n -dimensional CW-subcomplex, the general position argument shows that we can make them disjoint. Moreover, it is known (see below for details) that any Weinstein domain of dimension $2n$ is homotopy equivalent to its core which is an n -dimensional CW-complex. Here we prove:

Theorem 1.1. *Let (M, ξ) be a contact manifold which admits a contact open book \mathcal{OB} supporting ξ . If the pages of \mathcal{OB} are Weinstein, then there exist a contact structure ξ' on M , which is isotopic to ξ and supported by \mathcal{OB} , and a contactomorphism $f: (M, \xi) \rightarrow (M, \xi')$ such that the image $f(L)$ of any compact embedded Legendrian submanifold L of (M, ξ) can be Legendrian isotoped until it becomes disjoint from the closure of a page of \mathcal{OB} .*

The proof of the theorem will be given in Section 3. One of the arguments used in the proof is making the given contact open book \mathcal{OB} “ β -standard” (see

the next definition) by altering the compatible contact structure ξ in its isotopy class.

Definition 1.2. Let $(X^{2n}, d\beta)$ an exact symplectic manifold, and M^{2n+1} be a manifold with a open book structure \mathcal{OB} with pages X , supporting a contact structure ξ . Then we say that the \mathcal{OB} is β -standard if there exists a contact form α for ξ , such that α restricts to β on every page.

We remark that any abstract contact open book is β -standard with respect to the contact form constructed by gluing the contact form $\beta+d\theta$ on the mapping torus $X \times \mathbb{R}/\sim$ with the one on $\partial X \times D^2$. This construction (which will be given in Remark 3.1) is originally due to Thurston and Winkelnkemper [6] for dimension three, and to Giroux [5] for higher dimensions.

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2. Preliminaries

A *Liouville domain* (or a *compact convex symplectic manifold*) is a pair (W, Λ) where W^{2n+2} is a compact manifold with boundary, together with a *Liouville structure* (or a *convex symplectic structure*), which means there is a 1-form Λ on W such that $\Omega=d\Lambda$ is symplectic and the Ω -dual vector field Z of Λ defined by $\iota_Z\Omega=\Lambda$ (or, equivalently, $\mathcal{L}_Z\Omega=\Omega$), where ι denotes the interior product and \mathcal{L} denotes the Lie derivative, should point strictly outwards along ∂W . Since Ω and Z (resp. Ω and Λ) together uniquely determine Λ (resp. Z), one can replace the notation with the triple (W, Ω, Z) (resp. (W, Ω, Λ)). Here Z is called *Liouville vector field*. The 1-form $\Lambda_\partial:=\Lambda|_{\partial W}$ is contact (i.e., $\Lambda_\partial \wedge (d\Lambda_\partial)^n > 0$), and the contact manifold $(\partial W, \text{Ker}(\Lambda_\partial))$ is called the *convex boundary* of (W, Λ) .

In the non-compact case (i.e., when W is an open manifold), if we further assume that Z is complete (i.e., its flow exists for all times), and also that there exists an exhaustion $W=\bigcup_{k=1}^\infty W^k$ by compact domains $W^k \subset W$ such that each $(W^k, \Lambda|_{W^k})$ is a Liouville domain with the convex boundary $(\partial W^k, \Lambda|_{\partial W^k})$ for all $k \geq 1$, then (W, Λ, Z) is called a *Liouville manifold* (see [1] for details). A *Liouville cobordism* (W, Λ, Z) is a compact cobordism W with a Liouville structure (Λ, Z) such that Z points outwards along $\partial_+ W$ and inwards along $\partial_- W$. Note that any Liouville domain is a Liouville cobordism with $\partial_- W = \emptyset$.

If $Z^{-t}:W \rightarrow W$ ($t > 0$) denotes the contracting flow of Z , then the *core* (or *skeleton*) of the Liouville manifold (W, Λ) is defined to be the set

$$\text{Core}(W, \Lambda) := \bigcup_{k=1}^\infty \bigcap_{t>0} Z^{-t}(W^k).$$

The interior of the core of any Liouville manifold is empty (Lemma 11.1, [1]), and so the core of any Liouville domain (W, Λ, Z) is compact. If M denotes the convex boundary ∂W , then one can see that the negative half of the symplectization $(M \times \mathbb{R}, d(e^t \Lambda|_M))$ symplectically embeds into W (as a collar neighborhood of M in W) so that its complement in W is $\text{Core}(W, \Lambda)$ and the embedding matches the positive t -direction of \mathbb{R} with Z . The *completion* $(\widehat{W}, \widehat{\Lambda})$ of a Liouville domain (W, Λ) is obtained from W by gluing the positive part $\partial W \times [0, \infty)$ of the *symplectization* $(\partial W \times \mathbb{R}, d(e^t \Lambda_\partial))$ of its convex boundary. Two Liouville domains (W_i, Λ_i) are said to be *Liouville isomorphic* if there exists a diffeomorphism $\phi: (\widehat{W}_0, \widehat{\Lambda}_0) \rightarrow (\widehat{W}_1, \widehat{\Lambda}_1)$, called a *Liouville isomorphism*, between their completions $(\widehat{W}_i, \widehat{\Lambda}_i)$ such that $\phi^*(\widehat{\Lambda}_1) = \widehat{\Lambda}_0 + df$ where f is some compactly supported smooth function on \widehat{W}_0 .

To define Weinstein manifolds and domains, we need three preliminary definitions:

Definition 2.1. (i) A vector field Z on a smooth manifold W is said to be *gradient-like* for a smooth function $\Psi: W \rightarrow \mathbb{R}$ if $Z \cdot \Psi = \mathcal{L}_Z \Psi > 0$ away from the critical points of Ψ .

(ii) A real-valued function is said to be *exhausting* if it is proper and bounded from below.

(iii) An exhausting function $\Psi: W \rightarrow \mathbb{R}$ on a symplectic manifold (W, Ω) is said to be Ω -*convex* if there exists a complete Liouville vector field Z which is gradient-like for Ψ .

Definition 2.2. A *Weinstein manifold* (W, Ω, Z, Ψ) is a symplectic manifold (W, Ω) which admits an Ω -convex Morse function $\Psi: W \rightarrow \mathbb{R}$ whose complete gradient-like Liouville vector field is Z . The triple (Ω, Z, Ψ) is called a *Weinstein structure* on W . A *Weinstein cobordism* (W, Ω, Z, Ψ) is a Liouville cobordism (W, Ω, Z) whose Liouville vector field Z is gradient-like for a Morse function $\Psi: W \rightarrow \mathbb{R}$ which is constant on the boundary ∂W . A Weinstein cobordism with $\partial_- W = \emptyset$ is called *Weinstein domain*.

Remark 2.3. Any Weinstein manifold (W, Ω, Z, Ψ) can be exhausted by Weinstein domains $W_k = \{\Psi^{-1}(-\infty, d_k]\} \subset W$ where $\{d_k\}$ is an increasing sequence of regular values of Ψ , and therefore, any Weinstein manifold is a Liouville manifold. In particular, any Weinstein domain is a Liouville domain. Also note that any Weinstein domain (W, Ω, Z, Ψ) has the convex boundary $(\partial W, \text{Ker}(\iota_Z \Omega|_{\partial W}))$, and the completion of a Weinstein domain is a Weinstein manifold.

The following topologically characterizes Weinstein domains and will be used later. Note that any Weinstein domain has a compact core and so admits a finite handle decomposition.

Theorem 2.4. ([7], see also Lemma 11.13 in [1]) *Any Weinstein domain of dimension $2n$ admits a finite handle decomposition whose handles have indices at most n .*

3. Legendrian embeddings into contact open books

In this section we will prove Theorem 1.1. Let us start with by recalling how an open book $\mathcal{OB}=(B, \varphi)$ on a closed manifold M determines an abstract open book (X, h) : By definition B is a codimension-two submanifold of M with a trivial normal bundle $B \times D^2$ and $\varphi: M \setminus B \rightarrow S^1$ is a fiber bundle map agreeing with the angular coordinate on the D^2 -factor. To describe an abstract open book, we first pick a page $X = \varphi^{-1}(p)$ for some $p \in S^1$. Then the monodromy $h: X \rightarrow X$ can be read from the first return map of a vector field on M transversal to the fibers of φ . Now using the resulting abstract open book we can construct a closed manifold

$$M(X, h) := \Sigma(X, h) \cup_{\partial} (\partial X \times D^2)$$

where $\Sigma(X, h)$ is the mapping torus determined by h . Observe that the construction defines an open book decomposition $\mathcal{OB}_{(X, h)}$ on $M(X, h)$, and also that M can be identified with $M(X, h)$ via some diffeomorphism Υ respecting the fibration maps on $M \setminus B$ and $M(X, h) \setminus (\partial X \times \{0\})$.

Next, from a collection of results from [4] and [5] (see also Section 7.3 of [3] for a detailed explanation) we recall the construction of a contact form (under suitable assumptions) on the manifold $M(X, h)$ in the following remark.

Remark 3.1. Given an abstract open book (X, h) , consider the closed manifold $M(X, h) = \Sigma(X, h) \cup_{\partial} (\partial X \times D^2)$. Suppose that there exists a Liouville form β on X and $h \in \text{Symp}(X, d\beta)$. Then one can construct a contact structure ξ_{β} on $M(X, h)$ so that $\mathcal{OB}_{(X, h)}$ becomes β -standard. Here we recall the explicit construction of the contact form α_{β} defining ξ_{β} : Since $h \in \text{Symp}(X, d\beta)$, the form $h^*(\beta) - \beta$ is closed, and it can be made exact by deforming h through symplectomorphisms which are identity near ∂X . Such a deformation of h changes $M(X, h)$ in its diffeomorphism class, and so we may assume that $h^*(\beta) - \beta = -d\rho$ for some smooth function $\rho: X \rightarrow \mathbb{R}$. Adding a large enough constant, one can assume that ρ is strictly positive everywhere on X , and so we can use ρ to construct a smooth mapping torus

$$\Sigma(X, h)_{\rho} := X \times \mathbb{R} / \sim_{\rho} \quad \text{where} \quad (x, z) \sim_{\rho} (h(x), z + \rho(x)).$$

Consider the contact form $\alpha = \beta + dz$ on $X \times \mathbb{R}$, and let $\sigma_{\rho}: X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ be the diffeomorphism defining \sim_{ρ} , that is, $\sigma_{\rho}(x, z) = (h(x), z + \rho(x))$. Then we compute

$$\sigma_{\rho}^*(\beta + dz) = h^*(\beta) + d(z + \rho) = \beta - d\rho + dz + d\rho = \beta + dz$$

which shows that $\alpha = \beta + dz$ descends to a contact form, say $\tilde{\alpha}$, on $\Sigma(X, h)_\rho$. Then using appropriate cut-off functions, one can construct a contact form α_β on

$$\Sigma(X, h)_\rho \cup_\partial (\partial X \times D^2) \approx M(X, h)$$

by smoothly gluing $\tilde{\alpha}$ with the contact form $\beta|_{\partial X} + \frac{1}{2}r^2 d\theta$ on the normal bundle $\partial X \times D^2$.

The key observation for the proof of Theorem 1.1 is the following:

Theorem 3.2. *Let X be a Weinstein domain of dimension $2n \geq 2$ whose underlying Liouville structure is given by the Liouville form β . For $h \in \text{Symp}(X, d\beta)$, consider the contact manifold $(M(X, h), \xi_\beta := \text{Ker}(\alpha_\beta))$ as in Remark 3.1. Let $\phi: \Sigma^n \hookrightarrow (M(X, h), \xi_\beta)$ be a Legendrian embedding of a compact Legendrian submanifold $S := \phi(\Sigma) \subset M(X, h)$. Then S can be Legendrian isotoped (through Legendrian embeddings) to another embedded Legendrian submanifold S' which is disjoint from the closure of a page of $\mathcal{OB}_{(X, h)}$ on $M(X, h)$.*

In order to prove this theorem we will make use of contact vector fields. A vector field on a contact manifold is said to be *contact* if its flow preserves the contact distribution. The following fundamental lemma in contact geometry characterizes contact vector fields on a given contact manifold. More details can be found, for instance, in [3].

Lemma 3.3. *Let $(M, \text{Ker}(\alpha))$ be any contact manifold and R_α denote the Reeb vector field of α . Then there is a one-to-one correspondence between the set $\{Z \in \Gamma(M) \mid Z \text{ is contact}\}$ of all contact vector fields on M and the set $\{H: M \rightarrow \mathbb{R} \mid H \text{ is smooth}\}$ of all smooth functions on M . The correspondence is given by $Z \rightarrow H_Z := \alpha(Z)$ (H_Z is called the “contact Hamiltonian” of the contact vector field Z), and $H \rightarrow Z_H$ where Z_H is the contact vector field uniquely determined by the equations $\alpha(Z_H) = H$ and $\iota_{Z_H} d\alpha = dH(R_\alpha)\alpha - dH$. \square*

We note that a similar statement also holds between locally defined contact vector fields and locally defined smooth functions.

Proof of Theorem 3.2. For simplicity we will write $M(X, h) = Y$ and $\alpha_\beta = \alpha$. We may assume that the Legendrian submanifold $S = \phi(\Sigma^n) \subset (Y, \text{Ker}(\alpha))$ is connected since all the arguments used below can be easily adapted to the case where S has more than one connected component. Assume thus that we have a Legendrian embedding of a compact connected manifold Σ in the open book $\mathcal{OB}_{(X, h)} = (B, \varphi)$ on Y corresponding to the abstract open book (X, h) . So we have

$$\phi: \Sigma^n \hookrightarrow Y = Y_1 \cup Y_2 = \Sigma(X, h)_\rho \cup (B \times D^2)$$

where $B = B \times \{0\} \subset B \times D^2 \subset Y$ is the binding, $Y_2 := B \times D^2$, and $Y_1 := \Sigma(X, h)_\rho$ is the mapping torus as above. We may consider $(Y_2, \text{Ker}(\alpha|_{Y_2}))$ as the contact manifold

$$(B \times D^2, \text{Ker}(\beta + (x/2) dy - (y/2) dx))$$

where (x, y) are the coordinates on the unit disk D^2 . Consider the following vector fields:

$$(1) \quad Z_1 = \partial x - (y/2)R_\beta, \quad Z_2 = \partial y + (x/2)R_\beta, \quad Z_3 = \chi + \theta \partial \theta, \quad Z_4 = R_\alpha$$

where χ is the $d\beta$ -dual vector field of β , R_α (resp. R_β) denotes the Reeb vector field of α (resp. $\beta|_B$), and θ is the S^1 -coordinate in the fibration $\varphi: Y \setminus B \rightarrow S^1$ determined by $\mathcal{OB}_{(X,h)}$. Here $\partial x = \partial/\partial x, \partial y = \partial/\partial y$, etc. Note that the first two are defined on $Y_2 = B \times D^2$, and the third is defined on the contact manifold $(Y_1 \setminus X, \text{Ker}(\alpha|_{Y_1 \setminus X}))$ where X is any fixed page of $\mathcal{OB}_{(X,h)}$. Here we note that α restricts to β on every page of $\mathcal{OB}_{(X,h)}$ (from its construction described in Remark 3.1) and $Y_1 \setminus X = X \times \mathbb{R}$, and so, in particular, we may also write $Z_3 = \chi + z \partial z$ where z is the \mathbb{R} -coordinate on $X \times \mathbb{R}$. It is easy to check that:

$$\mathcal{L}_{Z_i} \alpha = \begin{cases} 0 & \text{if } i=1, 2, 4 \\ \alpha & \text{if } i=3 \end{cases}$$

So each Z_i is a contact vector field on the region where it is defined. In fact, if H_i denotes the contact Hamiltonian function corresponding to Z_i as in Lemma 3.3, then we have

$$H_1 = -y, \quad H_2 = x, \quad H_3 = \theta, \quad H_4 \equiv 1.$$

We first see that one can make S transverse to the binding B :

Lemma 3.4. *In any given ε -neighborhood of S , one can Legendrian isotope S (in a small neighborhood of the binding B) so that it becomes everywhere transverse to B .*

Proof. Let N_ε be any ε -neighborhood of S . We will use the fact that Legendrian (more generally, isotropic) submanifolds stays Legendrian (isotropic) under the flows of contact vector fields. Note that for any constants $a_1, a_2 \in \mathbb{R}$ the vector field $Z = a_1 Z_1 + a_2 Z_2$ is contact with the contact Hamiltonian $H_Z = a_1 H_1 + a_2 H_2$. If S intersects B transversally, then there is nothing to prove. If not, let $K \subset S \cap (B \times \{(0, 0)\})$ be the region where they do not intersect transversally. Consider $Z_1|_{B \times \{(0,0)\}} = \partial x, Z_2|_{B \times \{(0,0)\}} = \partial y$ on $B = B \times \{(0, 0)\}$. For any $p \in K$, the tangent space

$$(\{\mathbf{0}\} \times TD^2)|_p \subset (TB \times TD^2)|_p$$

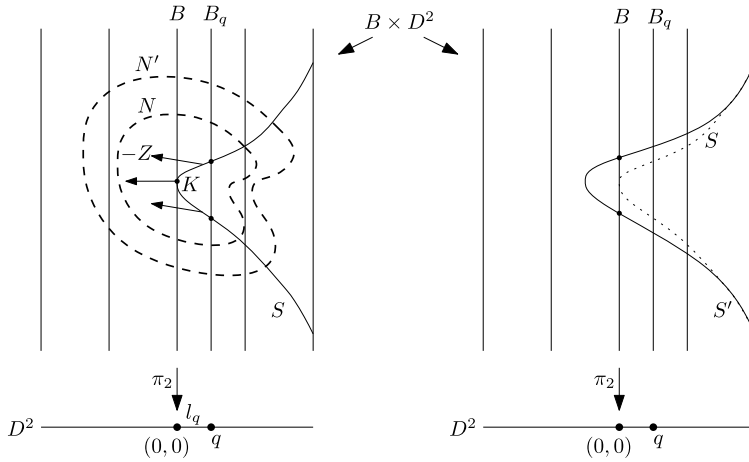


Figure 1. Isotoping S to another Legendrian submanifold S' which is transverse to the binding $B=B \times \{(0,0)\}$.

does not lie in $TS|_p$ (otherwise S and B would intersect transversally at $p \in K$). Therefore, there exists a vector $v = a_1 \partial x + a_2 \partial y \in (\{\mathbf{0}\} \times TD^2)|_{B \times \{(0,0)\}}$ (for some constants $a_1, a_2 \in \mathbb{R}$) which is everywhere transverse to $S \cap B$. We consider $Z = a_1 Z_1 + a_2 Z_2$ as the smooth extension of v to the whole $B \times D^2$. Note that Z will stay transverse to S in a small neighborhood $N_\delta := B \times \{(x, y) \in D^2 \mid x^2 + y^2 < \delta\}$ for some $\delta > 0$. Let $N_S K \subset N_\delta$ be a small neighborhood of K in S . Now choose a regular value $q \in \{(x, y) \in D^2 \mid x^2 + y^2 < \delta\} \subset D^2$ of the composition

$$\Sigma^n \supset U \xrightarrow{\phi} B \times D^2 \xrightarrow{\pi_2} D^2, \text{ where } \phi(\Sigma) = S$$

where π_2 is the projection and $U = \phi^{-1}(B \times \text{int}(D^2))$, such that $q \in \pi_2(N_S K)$ lies on the line segment joining $(0, 0)$ and (a_1, a_2) , and the above composition has no critical value other than $(0, 0)$ on the line segment, say l_q , joining $(0, 0)$ and q . Note that, by construction, S intersects the identical copy $B_q = \pi_2^{-1}(q)$ of B transversally, and Z is everywhere transverse to $\pi_2^{-1}(l_q)$. Let $\mu: B \times D^2 \rightarrow \mathbb{R}$ be a cut-off function such that $\mu \equiv 1$ on a neighbourhood $N \subset N_\varepsilon$ of $\pi_2^{-1}(l_q)$ in N_δ , and $\mu \equiv 0$ on outside of a slightly larger neighbourhood $N_\varepsilon \subset N'$ (see Figure 1).

Next consider the contact vector field Z_μ corresponding to the contact Hamiltonian $H_\mu := \mu H_Z$. By the choice of μ , Z_μ agrees with Z on N and it is identically zero outside N' . Using the backward flow Z_μ^{-t} of Z_μ we define the following 1-parameter smooth family:

$$\Phi_t : \Sigma^n \longrightarrow Y, \quad \Phi_t(p) = Z_\mu^{-t}(\phi(p)), \quad t \in [0, T]$$

where T is the time elapsed during the points of $S \cap B_q$ are moved to their final images in $B = B \times \{(0, 0)\}$ under the backward flow of Z_μ . (Note that all the points of $S \cap B_q \subset N$ reach B at the same time because the “horizontal” components of $Z_\mu|_N = Z|_N$ is defined by the constants a_1, a_2 .) Observe that $\Phi_0 = \phi$, for each $t \in [0, T]$ we have $\Phi_t = \phi$ outside N' and $\Phi_t(\Sigma^n)$ is Legendrian, and $S' := \Phi_1(\Sigma^n) \subset Y$ is everywhere transverse to the binding B as depicted in Figure 1. Finally, by choosing δ small enough, one can guarantee that the isotopy Φ_t stays in the given ε -neighborhood N_ε of S . \square

By the above lemma we may assume that the Legendrian submanifold $S = \phi(\Sigma^n)$ is transverse to the binding B . Next, by picking a regular value $p \in S^1$ for the projection $\pi_2: \phi(\Sigma^n) \cap \Sigma(X, h)_\rho \rightarrow S^1$ we can assume that for the page $\varphi^{-1}(p) \approx X$ of $\mathcal{OB}_{(X, h)}$, the intersection

$$L := \phi(\Sigma^n) \cap \varphi^{-1}(p) = S \cap X$$

is a properly imbedded $n - 1$ dimensional submanifold $(L, \partial L) \subset (X, B = \partial X)$ meeting the binding along an $n - 2$ dimensional submanifold ∂L (for simplicity we will write X for $\varphi^{-1}(p)$).

Lemma 3.5. *In any given ε -neighborhood of S , one can Legendrian isotope S (in a small neighborhood of the page X) so that $L = S \cap X$ becomes disjoint from $\Delta = \text{Core}(X, \beta)$.*

Proof. Since X^{2n} is Weinstein (by assumption), $\dim(\Delta) = n$ by Theorem 2.4. Also we have $\dim(\Delta) + \dim(L^{n-1}) = 2n - 1 < 2n = \dim(X)$. Hence, by the general position in X , we can (topologically) isotope L^{n-1} to a nearby copy which is disjoint from Δ . This means that there exists a vector field Z on X which is transverse to both L and Δ along their intersection $L \cap \Delta$. In what follows, using contact vector fields which are compactly supported near $L \cap \Delta$ (and which are generated from Z), we will construct an isotopy which transforms L to some nearby copy L' (disjoint from Δ), and recognize this isotopy (in X) as the restriction of a local Legendrian isotopy (in Y) moving S to another Legendrian submanifold, say S' .

Recall that there is a canonical contact model (Legendrian Neighborhood Theorem) for the tubular neighborhood $N_Y(S)$ of S in Y . That is, there exists a contactomorphism

$$\Upsilon : (T^*\Sigma^n \times \mathbb{R}, \text{Ker}(\mathbf{q}d\mathbf{p} + dz)) \longrightarrow (N_Y(S), \text{Ker}(\bar{\beta}^{st}|_{N_Y(S)}))$$

from the 1-jet bundle $(T^*\Sigma^n \times \mathbb{R}, \text{Ker}(\mathbf{q}d\mathbf{p} + dz))$ where $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n)$ are the standard coordinates on $T^*\Sigma^n$ and z is the real coordinate. (Here Υ maps

the zero section $\Sigma_0 = \{\mathbf{q} = \mathbf{0}\} \times \{0\} \subset T^*\Sigma^n \times \mathbb{R}$ onto S .) Observe that, on $T^*\Sigma^n \times \mathbb{R}$, there are $2n+1$ linearly independent contact vector fields:

(2)

$$Z'_1 = \partial p_1, \dots, Z'_n = \partial p_n, \quad Z'_{n+1} = \partial q_1 - p_1 \partial z, \dots, Z'_{2n} = \partial q_n - p_n \partial z, \quad Z'_{2n+1} = \partial z$$

The corresponding contact Hamiltonian functions (as in Lemma 3.3), respectively, are

(3)
$$H'_1 = q_1, \dots, H'_n = q_n, \quad H'_{n+1} = -p_1, \dots, H'_{2n} = -p_n, \quad H'_{2n+1} = 1.$$

Note that these contact vector fields are defined locally near S . We will use them for local Legendrian isotopies in $N_Y(S) \cong T^*\Sigma^n \times \mathbb{R}$ that we need for our purpose.

Let $L \cap \Delta = \sqcup_{i=1}^s L_i$ where L_i 's are (disjoint) connected components. Note that L is compact as both $S = \phi(\Sigma^n)$ and X are compact. Moreover, the core Δ is compact, and so $L \cap \Delta$ is also compact from which we conclude that each L_i is a compact CW-complex of finite type. Denote by L_i^K the K -skeleton of L_i for $K=0, 1, \dots, n-1$. In particular, we have $L_i = L_i^{n-1}$ (Figure 2-a). Using the isotopies mentioned above, we will first make the closure of every $(n-1)$ -cell in each L_i disjoint from Δ , and then do the same for $(n-2)$ -cells, etc.

Let $\{E_{i,1}^K, \dots, E_{i,l_i^K}^K\}$ be the set of all K -cells in L_i^K . For any $1 \leq j \leq l_i^K$, consider the following open cover for the closure $\overline{E_{i,j}^K}$ (recall the vector field Z in the proof of Lemma 3.5):

$$\mathcal{U}_{i,j}^K := \{U_x \mid x \in E_{i,j}^K, U_x \text{ is a nbhd of } x \text{ in } \overline{E_{i,j}^K} \text{ s.t. } Z(x) \pitchfork \overline{U_x}\}.$$

Clearly, $\mathcal{U}_{i,j}^K$ covers $\overline{E_{i,j}^K}$ which is compact. So, $\mathcal{U}_{i,j}^K$ has a finite subcover

$$\{U_{x_{i,j,1}^K}, U_{x_{i,j,2}^K}, \dots, U_{x_{i,j,m_{i,j}^K}^K}\}$$

for some finite number of points $\{x_{i,j,1}^K, x_{i,j,2}^K, \dots, x_{i,j,m_{i,j}^K}^K\}$ in $E_{i,j}^K$. We label these points in such a way that the neighborhood of any point has nonempty intersection with the union of the neighborhoods of the preceding points in the list as depicted in Figure 2-b (for the case $K=n-1$).

We will first make $\overline{U_{x_{i,j,1}^{n-1}}}$ disjoint from Δ . From the definition of $\mathcal{U}_{i,j}^{n-1}$, we have a constant vector field $Z(x_{i,j,1}^{n-1}) \in TX|_{\overline{U_{x_{i,j,1}^{n-1}}}}$ which is everywhere transverse to $\overline{U_{x_{i,j,1}^{n-1}}}$.

(Here at every point of $\overline{U_{x_{i,j,1}^{n-1}}}$, we have the same vector $Z(x_{i,j,1}^{n-1})$.) Observe that, since $\{Z'_i\}_{i=1}^{2n+1}$ is a linearly independent set, the pull-back vector field $\Upsilon^*(Z(x_{i,j,1}^{n-1}))$ can be written as the unique linear combination

$$\Upsilon^*(Z(x_{i,j,1}^{n-1})) = c_1 Z'_1 + \dots + c_{2n+1} Z'_{2n+1}$$

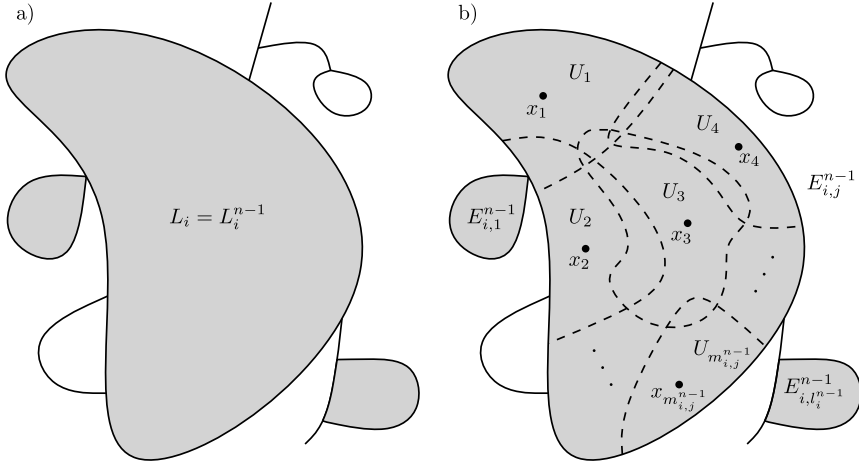


Figure 2. a) A typical connected component $L_i = L_i^{n-1}$ of $L \cap \Delta$, b) Finite subcover of $U_{i,j}^{n-1}$ (where we write $x_{i,j,1}^{n-1} = x_1, \dots, x_{i,j,m_{i,j}^{n-1}}^{n-1} = x_{m_{i,j}^{n-1}}^{n-1}$, and similarly $U_{x_{i,j,1}^{n-1}}^{n-1} = U_1, \dots, U_{x_{i,j,m_{i,j}^{n-1}}^{n-1}}^{n-1} = U_{m_{i,j}^{n-1}}^{n-1}$ for simplicity).

for some unique constants $c_1, \dots, c_{2n+1} \in \mathbb{R}$. Since these constants depend on the point $x_{i,j,1}^{n-1}$, we set the notation

$$\mathbf{Z}_{x_{i,j,1}^{n-1}} := c_1 Z'_1 + \dots + c_{2n+1} Z'_{2n+1}.$$

Let us write $V_{x_{i,j,1}^{n-1}}^{n-1}$ for the pre-image $\Upsilon^{-1}(U_{x_{i,j,1}^{n-1}})$. Observe that $\mathbf{Z}_{x_{i,j,1}^{n-1}}$ is a contact vector field on $T^*\Sigma^n \times \mathbb{R}$ with the contact Hamiltonian

$$\mathbf{H}_{x_{i,j,1}^{n-1}} := c_1 H'_1 + \dots + c_{2n+1} H'_{2n+1},$$

and also that it is everywhere transverse to $\overline{V_{x_{i,j,1}^{n-1}}} = \Upsilon^{-1}(\overline{U_{x_{i,j,1}^{n-1}}}) \subset \Upsilon^{-1}(L) \subset \Sigma_0$.

Denote by $\widetilde{V_{x_{i,j,1}^{n-1}}}$ a small neighborhood of $\overline{V_{x_{i,j,1}^{n-1}}}$ in $\Upsilon^{-1}(N_Y(S) \cap X) \subset T^*\Sigma^n \times \mathbb{R}$. Let $\mu_{x_{i,j,1}^{n-1}} : \Upsilon^{-1}(N_Y(S) \cap X) \rightarrow \mathbb{R}$ be a smooth cut-off function such that $\mu_{x_{i,j,1}^{n-1}} \equiv 1$ near $\overline{V_{x_{i,j,1}^{n-1}}}$, and $\mu_{x_{i,j,1}^{n-1}} \equiv 0$ on the complement $\Upsilon^{-1}(N_Y(S) \cap X) \setminus \widetilde{V_{x_{i,j,1}^{n-1}}}$. Now consider the contact vector field $\mathbf{Z}_{\mu_{x_{i,j,1}^{n-1}}}$ whose corresponding contact Hamiltonian is equal to $\mu_{x_{i,j,1}^{n-1}} \mathbf{H}_{x_{i,j,1}^{n-1}}$. By the choice of the cut-off function $\mathbf{Z}_{\mu_{x_{i,j,1}^{n-1}}} \Big|_{\overline{V_{x_{i,j,1}^{n-1}}}} = \mathbf{Z}_{x_{i,j,1}^{n-1}}$, and so $\mathbf{Z}_{\mu_{x_{i,j,1}^{n-1}}}$ is also transverse to $\overline{V_{x_{i,j,1}^{n-1}}}$. Using the flow $\mathbf{Z}_{\mu_{x_{i,j,1}^{n-1}}}^t$ we isotope $\overline{V_{x_{i,j,1}^{n-1}}}$ to its nearby copy $\mathbf{Z}_{\mu_{x_{i,j,1}^{n-1}}}^t(\overline{V_{x_{i,j,1}^{n-1}}})$ for some fixed time t . Note that pushing along a

transverse contact vector field implies that $\mathbf{Z}^t_{\mu_{x_{i,j,1}}^{n-1}}(\widehat{V_{x_{i,j,1}}^{n-1}})$ is disjoint from Δ , and is still isotropic in $(T^*\Sigma^n \times \mathbb{R}, \text{Ker}(\mathbf{qdp} + dz))$. Indeed, since the transversality is an open condition, we know that the isotropic image $\mathbf{Z}^t_{\mu_{x_{i,j,1}}^{n-1}}(\widehat{V_{x_{i,j,1}}^{n-1}})$ is disjoint from Δ where $\widehat{V_{x_{i,j,1}}^{n-1}} \supset \overline{V_{x_{i,j,1}}^{n-1}}$ is a neighborhood of $\overline{V_{x_{i,j,1}}^{n-1}}$ in L_i such that

$$\overline{V_{x_{i,j,1}}^{n-1}} \subset \widehat{V_{x_{i,j,1}}^{n-1}} \subset \widetilde{V_{x_{i,j,1}}^{n-1}} \cap L_i.$$

Similarly, we can make the closure of all the other open sets in the above finite subcover of $\mathcal{U}_{i,j}^{n-1}$ disjoint from Δ (this will isotope the whole closed $(n-1)$ -cell $\overline{E_{i,j}^{n-1}}$ to some isotropic copy which is disjoint from Δ). However, for each such closure, the choice of how much we push it (using the flow of the corresponding contact vector field) needs a little bit of more care: Let us discuss this in an inductive way: Suppose that we have already isotoped the union

$$\overline{V_{x_{i,j,1}}^{n-1}} \cup \overline{V_{x_{i,j,2}}^{n-1}} \cup \dots \cup \overline{V_{x_{i,j,k}}^{n-1}}, \quad \text{for some } k \in \{1, \dots, m_{i,j}^{n-1} - 1\}$$

along the contact vector fields $\{\mathbf{Z}_{\mu_{x_{i,j,m}}^{n-1}}\}_{m=1}^k$ (where the smooth cut-off functions $\mu_{x_{i,j,m}}^{n-1} : \Upsilon^{-1}(N_Y(S) \cap X) \rightarrow \mathbb{R}$ are constructed in the same way as above) so that the image of the union

$$\bigcup_{m=1}^k \widehat{V_{x_{i,j,m}}^{n-1}}$$

is isotropic in $(T^*\Sigma^n \times \mathbb{R}, \text{Ker}(\mathbf{qdp} + dz))$ and is disjoint from Δ where $\widehat{V_{x_{i,j,m}}^{n-1}} \subset \widetilde{V_{x_{i,j,m}}^{n-1}} \cap L_i$ is a small neighborhood of $\overline{V_{x_{i,j,m}}^{n-1}}$ in L_i . Now we would like to push (i.e., isotope) $\overline{V_{x_{i,j,k+1}}^{n-1}}$ using $\mathbf{Z}_{\mu_{x_{i,j,k+1}}^{n-1}}$. Observe that the region

$$\overline{V_{x_{i,j,k+1}}^{n-1}} \cap (\widehat{V_{x_{i,j,1}}^{n-1}} \cup \dots \cup \widehat{V_{x_{i,j,k}}^{n-1}})$$

has been already made disjoint from Δ , and also that $\mathbf{Z}_{\mu_{x_{i,j,k+1}}^{n-1}}$ might be tangent to the image of this region at some points, or even its flow might transform some points in the region back to Δ (if we let them flow too much). On the other hand, since transversality is an open condition there exists a codim-0 subset $\overline{V_{x_{i,j,k+1}}^{n-1}} \subset \overline{V_{x_{i,j,k+1}}^{n-1}}$ with a codim-0 nonempty intersection

$$\overline{\overline{V_{x_{i,j,k+1}}^{n-1}}} \cap (\widehat{V_{x_{i,j,1}}^{n-1}} \cup \dots \cup \widehat{V_{x_{i,j,k}}^{n-1}})$$

to which $\mathbf{Z}_{\mu_{x_{i,j,k+1}}^{n-1}}$ is everywhere transverse. Therefore, we can make the union

$$\overline{V_{x_{i,j,1}}^{n-1}} \cup \dots \cup \overline{V_{x_{i,j,k+1}}^{n-1}}$$

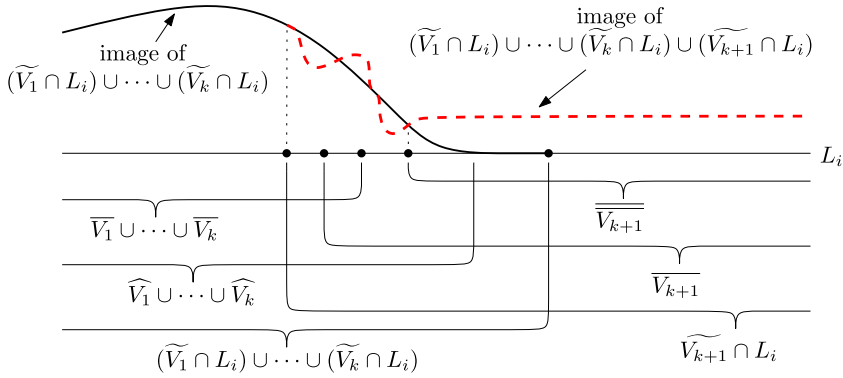


Figure 3. Making $\overline{V_{x_{i,j,k+1}}^{n-1}}$ disjoint from Δ (where for each $m=1, \dots, k+1$ we write $\overline{V_{x_{i,j,m}}^{n-1}} = \overline{V_m}$, $\widehat{V_{x_{i,j,m}}^{n-1}} = \widehat{V_m}$, $\widetilde{V_{x_{i,j,m}}^{n-1}} = \widetilde{V_m}$, and also $\overline{V_{x_{i,j,k+1}}^{n-1}} = \overline{V_{k+1}}$ for simplicity).

disjoint from Δ by pushing (in an appropriate amount) along $\mathbf{Z}_{\mu_{x_{i,j,k+1}}^{n-1}}$ as shown in Figure 3.

Repeating the above process we can make the union $\overline{E_{i,1}^{n-1}} \cup \dots \cup \overline{E_{i,l_i^{n-1}}^{n-1}}$ of the closures of all $(n-1)$ -cells disjoint from Δ by pushing to a nearby isotropic copy in $\Upsilon^{-1}(N_Y(S) \cap X)$. Note that for any particular closure in the union if some part of it has been already pushed (this might happen if it has a common boundary part with another $(n-1)$ -cell which has been pushed earlier), then we isotope it in an appropriate amount (as in the above discussion) so that previously pushed regions in the cell would not be moved back to Δ .

Similarly, we can deal with the union $\{\overline{E_{i,1}^K} \cup \dots \cup \overline{E_{i,l_i^K}^K}\}$ of all closed K -cells in L_i (for $0 \leq K \leq n-2$) in the same way (under the assumption that all closed $(K+1)$ -cells have been already made disjoint from Δ). Note that we do not need to push any K -cell which appears as a part of the boundary of some $(K+1)$ -cell(s) because such K -cells have been already made disjoint from Δ in the previous step. This process deforms the connected component L_i to its image L'_i which is isotropic and disjoint from Δ . Repeating this for each connected component, we conclude that there exists an isotopy of the embeddings in X from L to its nearby isotropic copy L' which is disjoint from Δ . This isotopy is generated by contact vector fields $\mathbf{Z}_{\mu_{x_{i,j,k}}^K}$ and compactly supported in the neighborhood

$$\widetilde{V} := \bigcup_{0 \leq i \leq s} \bigcup_{0 \leq K \leq n-1} \bigcup_{0 \leq j \leq l_i^K} \bigcup_{0 \leq k \leq m_{i,j}^K} \widetilde{V_{x_{i,j,k}}^K} \subset \Upsilon^{-1}(N_Y(S) \cap X)$$

Next, we want to extend this local isotopy of L in $\Upsilon^{-1}(N_Y(S) \cap X) \times \{0\}$ to a local Legendrian isotopy of S compactly supported in $\tilde{V} \times [-1, 1] \subset \Upsilon^{-1}(N_Y(S) \cap X) \times [-1, 1] \subset T^*\Sigma^n \times \mathbb{R}$. (Here since L is codimension 1 submanifold of S , we can consider the tubular neighborhood of L in S as the product $L \times [-1, 1]$ such that L corresponds to $L \times \{0\}$.) Consider the smooth cut-off function

$$f : \Upsilon^{-1}(N_Y(S) \cap X) \times [-1, 1] \longrightarrow \mathbb{R}, \quad f(x, t) = \mu(t)$$

where $\mu : [-1, 1] \rightarrow \mathbb{R}$ is a smooth cut-off function which is equal to 1 near $t=0$, and 0 near $t=\pm 1$. Also let $\tilde{\mu}_{x_{i,j,k}^K} : \Upsilon^{-1}(N_Y(S) \cap X) \times [-1, 1] \rightarrow \mathbb{R}$ be the extension of $\mu_{x_{i,j,k}^K}$ given by

$$\tilde{\mu}_{x_{i,j,k}^K}(x, t) = \mu_{x_{i,j,k}^K}(x).$$

Denote by $\tilde{\mathbf{Z}}_{\mu_{x_{i,j,k}^K}}$ the contact vector field on $\Upsilon^{-1}(N_Y(S) \cap X) \times [-1, 1]$ whose corresponding contact Hamiltonian (as in Lemma 3.3) is equal to $f \tilde{\mu}_{x_{i,j,k}^K} \mathbf{H}_{x_{i,j,k}^K}$. Now we isotope S to its nearby Legendrian copy S' by applying the flow maps of $\tilde{\mathbf{Z}}_{\mu_{x_{i,j,k}^K}}$ in the same order and amount that we apply the flow maps of $\mathbf{Z}_{\mu_{x_{i,j,k}^K}}$ to isotope L to L' . By construction, this isotopy is compactly supported in $\tilde{V} \times [-1, 1]$, and its restriction to $\Upsilon^{-1}(N_Y(S) \cap X) \times \{0\}$ is the isotopy taking L to L' (constructed above) as $\tilde{\mathbf{Z}}_{\mu_{x_{i,j,k}^K}}|_{\Upsilon^{-1}(N_Y(S) \cap X) \times \{0\}} = \mathbf{Z}_{\mu_{x_{i,j,k}^K}}$. In particular, for the new Legendrian submanifold S' , its intersection $L' = S' \cap X$ is disjoint from the core Δ of X .

Finally, we note that by working on a small enough neighborhood $N_Y(S)$ one can guarantee that S' lies in any given ε -neighborhood of S in Y . \square

By the last lemma we may assume that the transverse intersection $L = S \cap X$ of the Legendrian submanifold $S = \phi(\Sigma^n)$ is disjoint from the core Δ of $X = \varphi^{-1}(p)$. To finish the proof of Theorem 3.2, we will construct an isotopy of Legendrian embeddings of S which will be compactly supported in a neighborhood of X in Y and will push L completely outside X . To this end, we will first isotope S along a contact vector field generated from Z_3 given in the list (1) above so that the part $L \cap Y_1$ of L (recall $Y_1 = \Sigma(X, h)_\rho$ is the mapping torus of the open book $\mathcal{OB}_{(X,h)}$) is completely pushed into the interior of $Y_2 = B \times D^2$ (tubular neighborhood of the binding $B = \partial X$). Then using Z_1, Z_2 of the list (1) we will isotope S until $L \cap Y_2$ completely crosses the binding $B = B \times \{0\}$.

Remark 3.6. So far, when we write X we meant the whole page (in particular, ∂X was the binding B). However, for what follows it is better to use the abstract open book description as in the previous paragraph. Therefore, from now on X will denote the complement of the collar neighborhood of the binding B in the corresponding page.

From its construction the contact manifold $(Y_1 = \Sigma(X, h)_\rho, \alpha|_{Y_1})$ is obtained as the quotient space of $(X \times \mathbb{R}, \beta + dz)$ using the equivalence relation \sim_ρ . Recall that we have

$$Y_1 = X \times \mathbb{R} / \sim_\rho \quad \text{where} \quad (x, z) \sim_\rho (h(x), z + \rho(x)).$$

By translating (i.e., isotoping) S along the Reeb direction $Z_4 = R_\alpha$ (which corresponds to ∂_z on $X \times \mathbb{R}$), we may assume that $X = \varphi^{-1}(p)$ corresponds to $X \times \{0\}$ under this identification. Therefore, we can identify a neighborhood of X in Y_1 with $X \times [-a, a] \subset X \times \mathbb{R}$ for some real number $0 < a < K$ where $K > 0$ is a constant satisfying

$$K < \rho(x), \quad \forall x \in X.$$

(Note that such K exists since X is compact and ρ is a strictly positive continuous function). The contact form α on Y is equal to $\beta + dz$ on $X \times [-a, a]$ where Z_3 takes the form $Z := \chi + z\partial_z$ as mentioned earlier. By choosing a small enough, we may guarantee that the intersection $S \cap (X \times [-a, a])$ is equal to $L \times [-a, a]$ (S and $X = X \times \{0\}$ intersect transversally), and also that $L \times [-a, a]$ is disjoint from $\Delta \times [-a, a]$ (this is because all cut-off functions which we used to isotope S to S' in the proof of Lemma 3.5 are all equal to 1 near $(L \times \{0\}) \cap (\Delta \times \{0\})$).

One can think of Y_1 slightly larger by considering $Y_2 = B \times D^2$ slightly smaller. More precisely, let $N = B \times D \subset Y_2$ be a smaller neighborhood of B where $D \subset D^2$ is a smaller disk in \mathbb{R}^2 around the origin. By expanding each (X, β) in Y_1 to a larger domain $(\tilde{X}, \tilde{\beta})$, we get another decomposition $Y = Y'_1 \cup N$ where $Y'_1 = Y \setminus N$. Note that extending the above identification, a neighborhood of \tilde{X} in Y'_1 can be identified with $\tilde{X} \times [-a, a]$ ($\tilde{X} = \tilde{X} \times \{0\}$) on which the contact form α is given as $\alpha = \tilde{\beta} + dz$ and we have the extension

$$\tilde{Z} = \tilde{\chi} + z\partial_z$$

of Z where $\tilde{\chi}$ is the $d\tilde{\beta}$ -dual of $\tilde{\beta}$. We remark that \tilde{Z} is contact with the contact Hamiltonian $H = z$. Let $\mu_1 : Y \rightarrow \mathbb{R}$ be a smooth cut-off function such that $\mu_1 \equiv 1$ near $\tilde{X} \times \{0\}$ and $\mu_1 \equiv 0$ in the complement $Y \setminus \tilde{X} \times (-\varepsilon, \varepsilon)$ for some $0 < \varepsilon < a$ which will be determined later. Denote by Z_{μ_1} the contact vector field on Y which corresponds to the contact Hamiltonian $\mu_1 H_3$.

Now we first push S using the flow $Z_{\mu_1}^t$ until the image of L lies completely in the interior $\text{int}(Y_2)$ of Y_2 as follows: First note that $Z_{\mu_1} = \tilde{\chi}$ on $\tilde{X} = \tilde{X} \times \{0\}$ (as $z = 0$ there). Since L is disjoint from the core of \tilde{X} (which is the same as that of X), for every point $x \in L \cap \tilde{X}$ there exists a unique flow line of $\tilde{\chi}$ (i.e., of $Z_{\mu_1}|_{\tilde{X}}$) passing through x . All such flow lines reach the region $\tilde{X} \cap \text{int}(Y_2 \setminus N)$. Consider

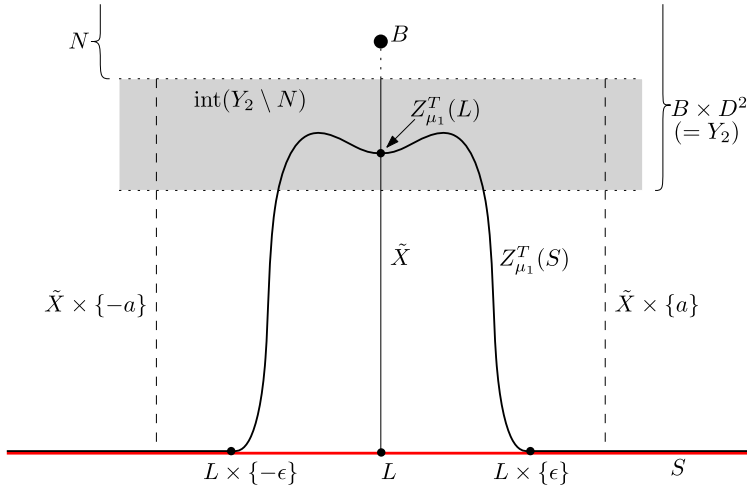


Figure 4. Making L closer to the binding via a Legendrian isotopy of S .

the set

$$A = \{t \in (0, \infty) \mid Z_{\mu_1}^t(x) \in \tilde{X} \cap \text{int}(Y_2 \setminus N) \text{ for all } x \in L \cap \tilde{X}\}.$$

Since $L \cap \tilde{X}$ is a compact, the set A is non-empty. Choose a finite number $T > 0$ from A . We isotope the Legendrian sphere $S \subset Y$ using the flow maps $\{Z_{\mu_1}^t \mid 0 \leq t \leq T\}$. Indeed, by the construction of Z_{μ_1} , we only push the region $(L \cap \tilde{X}) \times (-\epsilon, \epsilon) \subset S$ during the isotopy. Observe that by choosing $\epsilon > 0$ small enough, we can guarantee that the image $Z_{\mu_1}^T(L \times (-\epsilon, \epsilon))$ completely lies in $\tilde{X} \times [-a, a]$. Therefore, $Z_{\mu_1}^T(S)$ is an embedded Legendrian which is Legendrian isotopic to S (see Figure 4).

By the choice of the flow parameter T above, we know that $Z_{\mu_1}^T(L)$ is completely lie in the region $\text{int}(Y_2)$. For $0 < r < 1$ consider the disk

$$D_r = \{(x, y) \in D^2 \mid x^2 + y^2 \leq r^2\} \subset D^2$$

and the neighborhood $N_r = B \times D_r \subset \text{int}(B \times D^2) = \text{int}(Y_2)$ of the binding. Then after the above isotopy we know that $Z_{\mu_1}^T(L) \subset N_r$ for some r . Let $(a_1, a_2) \in D^2$ be a point which corresponds to the angular coordinate $p \in S^1$ (recall that $\tilde{X} = \tilde{X} \times \{p\}$). Recall the contact vector fields Z_1, Z_2 from the list (1) and their contact Hamiltonians H_1, H_2 . Then the vector field $Z' = a_1 Z_1 + a_2 Z_2$ is also contact whose contact Hamiltonian is given by $H' = a_1 H_1 + a_2 H_2$. Let $\mu_2: Y \rightarrow \mathbb{R}$ be a smooth cut-off function such that $\mu_2 \equiv 1$ near N_r and $\mu_2 \equiv 0$ in the complement $Y \setminus Y_2$. Denote by Z_{μ_2} the contact vector field on Y which corresponds to the contact Hamiltonian $\mu_2 H'$.

Now we isotope $Z_{\mu_1}^T(S)$ using the flow maps of the contact vector field Z_{μ_2} until $Z_{\mu_1}^T(L)$ completely crosses the binding. Say for $T' > 0$ the image of $Z_{\mu_1}^T(L)$ under the flow map $Z_{\mu_2}^{T'}$ completely crosses the binding. As a result, the closure of the page $\varphi^{-1}(p)$ of the open book $\mathcal{OB}_{(X,h)}$ is disjoint from the final image $S' := Z_{\mu_2}^{T'}(Z_{\mu_1}^T(S))$ which is Legendrian isotopic to $S = \phi(\Sigma^n)$ as claimed. This finishes the proof of Theorem 3.2. \square

Proof of Theorem 1.1. Let \mathcal{OB} be an open book decomposition carrying a contact structure ξ on a (closed) manifold M of dimension $2n+1 \geq 3$. In particular, there exists a contact form α for ξ such that α restricts to a Liouville form on every page of \mathcal{OB} . By assumption, for any page X of \mathcal{OB} , the restriction $\alpha|_X$ is, indeed, the underlying Liouville form of a Weinstein structure on X .

Pick a page X equipped with the Liouville form $\beta := \alpha|_X$ which is, by assumption, the underlying Liouville form of a Weinstein structure on X . Denote by $h \in \text{Symp}(X, d\beta)$ the monodromy of \mathcal{OB} and consider the manifold $M(X, h)$ equipped with the contact structure $\xi_\beta = \text{Ker}(\alpha_\beta)$ where α_β is the contact form on $M(X, h)$ constructed as in Remark 3.1. Let

$$\xi_X := (\Upsilon_X^{-1})_*(\xi_\beta)$$

be the contact structure on M obtained by pushing forward ξ_β using the inverse of the identification map $\Upsilon_X : M \rightarrow M(X, h)$. As explained at the beginning of this section, Υ_X respects the fibration maps on $M \setminus B$ and $M(X, h) \setminus (\partial X \times \{0\})$ associated to the open books \mathcal{OB} and $\mathcal{OB}_{(X,h)}$, respectively. Therefore, we have a contactomorphism

$$\Upsilon_X : (M, \xi_X) \longrightarrow (M(X, h), \xi_\beta)$$

mapping pages of \mathcal{OB} to those of $\mathcal{OB}_{(X,h)}$.

Next, observe that ξ and ξ_X are supported by the same open book \mathcal{OB} (by construction of ξ_X), so we know, by Giroux’s work, that there exists an isotopy ξ_t ($t \in [0, 1]$) of contact structures on M connecting $\xi_0 = \xi$ and $\xi_1 = \xi_X$. Then Gray’s Stability (see, for instance, Theorem 2.2.2 of [3]) implies that there is a diffeotopy

$$\Xi_t : M \longrightarrow M, \quad t \in [0, 1],$$

such that $(\Xi_t)_*(\xi_0) = \xi_t$ for each t . In particular, $(\Xi_1)_*(\xi) = \xi_X$, and hence we obtain two contactomorphisms

$$f_X := \Xi_1 : (M, \xi) \longrightarrow (M, \xi_X), \quad \Upsilon_X \circ f_X : (M, \xi) \longrightarrow (M(X, h), \xi_\beta).$$

Suppose now we are given a Legendrian embedding $\psi : \Sigma^n \hookrightarrow (M, \xi)$ of a compact Legendrian submanifold $L = \psi(\Sigma^n)$. By pushing forward ψ using the above

contactomorphisms, we obtain two Legendrian embeddings

$$f_X \circ \psi : \Sigma^n \hookrightarrow (M, \xi_X), \quad \phi := \Upsilon_X \circ f_X \circ \psi : \Sigma^n \hookrightarrow (M(X, h), \xi_\beta).$$

We set $S := \phi(\Sigma^n)$. By Theorem 3.2 we have a smooth 1-parameter family

$$\Phi_t : \Sigma^n \hookrightarrow (M(X, h), \xi_\beta), \quad t \in [0, 1]$$

of Legendrian embeddings such that $\Phi_0(\Sigma^n) = S$ and the Legendrian submanifold $S' := \Phi_1(\Sigma^n)$ is disjoint from the closure of a page of the open book $\mathcal{OB}_{(X, h)}$ on $M(X, h)$ associated to (X, h) . By composing Φ_t with the contactomorphism Υ_X^{-1} , we obtain a smooth 1-parameter family

$$\Psi_t := \Upsilon_X^{-1} \circ \Phi_t : \Sigma^n \hookrightarrow (M, \xi_X), \quad t \in [0, 1]$$

of Legendrian embeddings such that $\Psi_0(\Sigma^n) = f_X(L)$. Finally, using this isotopy and the fact that Υ_X^{-1} is mapping pages of $\mathcal{OB}_{(X, h)}$ to the corresponding pages of \mathcal{OB} , we conclude that the Legendrian submanifold

$$\Psi_1(\Sigma^n) = \Upsilon_X^{-1}(S')$$

is Legendrian isotopic to $f_X(L)$ and disjoint from a page of the open book \mathcal{OB} . Thus, setting $f := f_X$ and $\xi' := \xi_X$ finishes the proof of Theorem 1.1. \square

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