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Injective objects in the category of finitely presented representations of an interval finite quiver

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Abstract. We characterize the indecomposable injective objects in the category of finitely presented representations of an interval finite quiver.

1. Introduction

Infinite quivers appear naturally in the covering theory of algebras; see such as [BG82], [Gab81]. The injective representations of an infinite quiver Q over an arbitrary ring R is studied in [EEGR09]. We are interested in the category $\mathrm{fp}(Q)$ of finitely presented representations when R is a field.

Recall that $\mathrm{fp}(Q)$ is studied in [RVdB02] when Q is locally finite of certain type. The result is used to classify the Noetherian Ext-finite hereditary abelian categories with Serre duality. More generally, when Q is strongly locally finite, the Auslander–Reiten quiver of $\mathrm{fp}(Q)$ is studied in [BLP13]. The result is used to study the bounded derived category of a finite dimensional algebra with radical square zero in [BL17].

In the study of Auslander–Reiten theory of $\mathrm{fp}(Q)$, a natural question is how about the injective objects. We find that we can deal with it when Q is *interval finite* (i.e., for any vertices a and b, the set of paths from a to b is finite).

For each vertex a, we denote by I_a the corresponding indecomposable injective representation. Let p be a left infinite path, i.e., an infinite sequence of arrows $\dots\alpha_i\dots\alpha_2\alpha_1$ with $s(\alpha_{i+1})=t(\alpha_i)$ for any $i\geq 1$. Denote by $[p]$ the equivalence class

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(see page 390 for the definition) of left infinite paths containing p. Consider the indecomposable representation $Y_{[p]}$ introduced in [Jia19, Section 5]. We have that if $Y_{[p]}$ lies in $\mathrm{fp}(Q)$, then it is an indecomposable injective object; see Proposition 3.10.

Moreover, we can classify the indecomposable injective objects in $\mathrm{fp}(Q)$.

Main Theorem (see Theorem 3.11) *Let Q be an interval finite quiver. Assume I is an indecomposable injective object in* $\mathrm{fp}(Q)$. *Then either $I \simeq I_a$ for certain vertex a, or $I \simeq Y_{[p]}$ for certain left infinite path p.*

Compared with [Jia19, Theorem 6.8], the difficulty here is to characterize when I_a and $Y_{[p]}$ are finitely presented. The result strengthens a description of finite dimensional indecomposable injective objects in $\mathrm{fp}(Q)$; see [BLP13, Proposition 1.16].

The paper is organized as follows. In Section 2, we recall some basic facts about quivers and representations. In Section 3, we study the injective objects in $\mathrm{fp}(Q)$ and give the classification theorem. Some examples are given in Section 4.

2. Quivers and representations

Let k be a field, and $Q=(Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. For each arrow $\alpha\colon a \to b$, we denote by $s(\alpha)=a$ its source and by $t(\alpha)=b$ its target.

A path p of length $l \geq 1$ is a sequence of arrows $\alpha_l \dots \alpha_2 \alpha_1$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for any $1 \leq i \leq l-1$. We set $s(p)=s(\alpha_1)$ and $t(p)=t(\alpha_l)$. We associate each vertex a with a trivial path (of length 0) e_a with $s(e_a)=a=t(e_a)$. A nontrivial path p is called an oriented cycle if $s(p)=t(p)$. For any $a, b \in Q_0$, we denote by $Q(a,b)$ the set of paths p from a to b, i.e., $s(p)=a$ and $t(p)=b$.

If $Q(a,b) \neq \varnothing$, then a is called a predecessor of b, and b is called a successor of a. For $a \in Q_0$, we denote by a^- the set of vertices b with some arrow $b \to a$; by a^+ the set of vertices b with some arrow $a \to b$.

A *right infinite path* p is an infinite sequence of arrows $\alpha_1 \alpha_2 \dots \alpha_n \dots$ such that $s(\alpha_i)=t(\alpha_{i+1})$ for any $i \geq 1$. We set $t(p)=t(\alpha_1)$. Dually, a *left infinite path* p is an infinite sequence of arrows $\dots \alpha_n \dots \alpha_2 \alpha_1$ such that $s(\alpha_{i+1})=t(\alpha_i)$ for any $i \geq 1$. We set $s(p)=s(\alpha_1)$. Here, we use the terminologies in [Che15, Section 2.1]. We mention that these are opposite to the corresponding notions in [BLP13, Section 1].

A representation $M=(M(a), M(\alpha))$ of Q over k means a collection of k-linear spaces $M(a)$ for every $a \in Q_0$, and a collection of k-linear maps $M(\alpha)\colon M(a) \to M(b)$ for every arrow $\alpha\colon a \to b$. For each nontrivial path $p=\alpha_l \dots \alpha_2 \alpha_1$, we denote $M(p) = M(\alpha_l) \circ \dots \circ M(\alpha_2) \circ M(\alpha_1)$. For each $a \in Q_0$, we set $M(e_a)=\mathbb{1}_{M(a)}$. A morphism $f\colon M \to N$ of representations is a collection of k-linear maps $f_a\colon M(a) \to N(a)$ for every $a \in Q_0$, such that $f_b \circ M(\alpha) = N(\alpha) \circ f_a$ for any arrow $\alpha\colon a \to b$.

Let Rep(Q) be the category of representation of Q over k. We denote by Hom(M,N) the set of morphisms from M to N in Rep(Q). It is well known that Rep(Q) is a hereditary abelian category; see [GR92, Section 8.2].

Recall that a subquiver Q' of Q is called *full* if any arrow α with $s(\alpha), t(\alpha) \in Q'_0$ lies in Q'. Let M be a representation of Q. The *support* $\operatorname{supp} M$ of M is the full subquiver of Q formed by vertices a with $M(a) \neq 0$. The *socle* $\operatorname{soc} M$ of M is the subrepresentation such that $(\operatorname{soc} M)(a) = \bigcap_{\alpha \in Q_1, s(\alpha)=a} \operatorname{Ker} M(\alpha)$ for any vertex a. The *radical* $\operatorname{rad} M$ of M is the subrepresentation such that $(\operatorname{rad} M)(a) = \sum_{\alpha \in Q_1, t(\alpha)=a} \operatorname{Im} M(\alpha)$ for any vertex a.

We mention the following fact; see [BLP13, Lemma 1.1].

Lemma 2.1. *If the support of a representation M contains no left infinite paths, then* $\operatorname{soc} M$ *is essential in M.*

Proof. Let N be a nonzero subrepresentation of M. Assume $x \in N(a)$ is nonzero for some vertex a. Since $\operatorname{supp} M$ contains no left infinite paths, there exists some path p in $\operatorname{supp} M$ with $s(p) = a$ such that $N(p)(x) \neq 0$ and $N(\alpha p)(x) = 0$ for any arrow α in Q. Then $N(p)(x) \in (N \cap \operatorname{soc} M)(t(p))$. It follows that $\operatorname{soc} M$ is essential in M. $\square$

Let a be a vertex in Q. We define a representation P_a as follows. For every vertex b, we let

$$P_a(b) = \bigoplus_{p \in Q(a,b)} kp.$$

For every arrow $\alpha\colon b \to b'$, we let

$$P_a(\alpha)\colon P_a(b) \longrightarrow P_a(b'), \;\; p \longmapsto \alpha p.$$

Similarly, we define a representation I_a as follows. For every vertex b, we let

$$I_a(b) = \operatorname{Hom}_k\Big(\bigoplus_{p \in Q(b,a)} kp, k\Big).$$

For every arrow $\alpha\colon b \to b'$, we let

$$I_a(\alpha)\colon I_a(b) \longrightarrow I_a(b'), \;\; f \longmapsto (p \longmapsto f(p\alpha)).$$

The following result is well known; see [GR92, Section 3.7]. It implies that P_a is a projective representation and I_a is an injective representation in Rep(Q).

Lemma 2.2. *Let $M \in \operatorname{Rep}(Q)$ and $a \in Q_0$.*

(1) *The k-linear map*

$$\eta_M\colon \operatorname{Hom}(P_a, M) \longrightarrow M(a), \;\; f \longmapsto f_a(e_a),$$

is an isomorphism natural in M.

(2) *The k-linear map*

$$\zeta_M\colon \operatorname{Hom}(M, I_a) \longrightarrow \operatorname{Hom}_k(M(a), k), \;\; f \longmapsto (x \longmapsto f_a(x)(e_a)),$$

is an isomorphism natural in M.

Proof. (1) Consider the k-linear map

$$\eta'_M\colon M(a) \longrightarrow \operatorname{Hom}(P_a, M)$$

given by $(\eta'_M(x))_b(p) = M(p)(x)$ for any $x \in M(a)$, $b \in Q_0$ and $p \in Q(a,b)$. We observe that $\eta'_M \circ \eta_M = \mathbb{1}_{\operatorname{Hom}(P_a,M)}$ and $\eta_M \circ \eta'_M = \mathbb{1}_{M(a)}$. Then η_M is an isomorphism.

(2) Consider the k-linear map

$$\zeta'_M\colon \operatorname{Hom}_k(M(a), k) \longrightarrow \operatorname{Hom}(M, I_a)$$

given by $(\zeta'_M(f))_b(x)(p) = f(M(p)(x))$ for any $f \in \operatorname{Hom}_k(M(a), k)$, $b \in Q_0$, $x \in M(b)$ and $p \in Q(b,a)$. We observe that $\zeta'_M \circ \zeta_M = \mathbb{1}_{\operatorname{Hom}(M,I_a)}$ and $\zeta_M \circ \zeta'_M = \mathbb{1}_{\operatorname{Hom}_k(M(a),k)}$. It follows that ζ_M is an isomorphism. □

An epimorphism $P \to M$ with projective P is called a projective cover of M if it is an essential epimorphism. A monomorphism $M \to I$ with injective I is called an injective envelope of M if it is an essential monomorphism. We mention that two injective envelopes of M are isomorphic.

Given a collection $\mathcal{A}$ of representations, we denote by $\operatorname{add}\mathcal{A}$ the full subcategory of $\operatorname{Rep}(Q)$ formed by direct summands of finite direct sums of representations in $\mathcal{A}$. We set $\operatorname{proj}(Q) = \operatorname{add}\{P_a \mid a \in Q_0\}$ and $\operatorname{inj}(Q) = \operatorname{add}\{I_a \mid a \in Q_0\}$.

A representation M is called *finitely generated* if there exists some epimorphism $f\colon \bigoplus_{i=1}^n P_{a_i} \to M$, and is called *finitely presented* if moreover $\operatorname{Ker} f$ is also finitely generated. We denote by $\operatorname{fp}(Q)$ the subcategory of $\operatorname{Rep}(Q)$ formed by finitely presented representations.

We have the following well-known fact.

Proposition 2.3. *The category* $\operatorname{fp}(Q)$ *is a hereditary abelian subcategory of* $\operatorname{Rep}(Q)$ *closed under extensions.*

Proof. Let $f\colon P\to P'$ be a morphism in proj(Q). We observe that $\operatorname{Im} f$ is projective since Rep(Q) is hereditary. Then the induced exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow P \longrightarrow \operatorname{Im} f \longrightarrow 0$$

splits. Therefore $\operatorname{Ker} f \in \operatorname{proj}(Q)$. It follows from [Aus66, Proposition 2.1] that fp(Q) is abelian. We observe by the horseshoe lemma that fp(Q) is closed under extensions in Rep(Q). In particular, it is hereditary. □

We mention the following observation.

Lemma 2.4. *Let M be a finitely presented representation and N be a finitely generated subrepresentation. Then M/N is finitely presented.*

Proof. Let $f\colon P\to M$ be an epimorphism with $P\in\operatorname{proj}(Q)$. Then $\operatorname{Ker} f$ is finitely generated. Denote by g the composition of f and the canonical surjection $M\to M/N$. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} g & \longrightarrow & P & \xrightarrow{g} & M/N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0
\end{array}$$

We observe that the left square is a pushout and also a pullback. Then h is an epimorphism and $\operatorname{Ker} h \simeq \operatorname{Ker} f$. In particular, $\operatorname{Ker} h$ is finitely generated. Consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Ker} g \xrightarrow{h} N \longrightarrow 0.$$

It follows that $\operatorname{Ker} g$ is finitely generated. Then M/N is finitely presented. □

The injective objects in fp(Q) satisfy the following property.

Lemma 2.5. *Let I be an injective object in* fp(Q), *and let $a\in Q_0$. Assume p_i is a path from a to b_i for $1\leq i\leq n$ such that p_i is not of the form up_j with $u\in Q(b_j, b_i)$ for any $j\neq i$. Then the k-linear map*

$$\begin{pmatrix} I(p_1) \\ \vdots \\ I(p_n) \end{pmatrix} : I(a) \longrightarrow \bigoplus_{i=1}^{n} I(b_i)$$

is a surjection.

We mention that one can consider the special case that $p_i\colon a\to b_i$ for $1\leq i\leq n$ are pairwise different arrows.

Proof. We observe that the canonical morphism $\bigoplus_{i=1}^{n} P_{b_i} \to P_a$ induced by inclusions is a monomorphism. The injectivity of I gives a surjection $\mathrm{Hom}(P_a, I) \to \mathrm{Hom}(\bigoplus_{i=1}^{n} P_{b_i}, I)$. By identifying $\mathrm{Hom}(P_c, I)$ and $I(c)$ for any vertex c, we observe that the surjection is precisely the map needed. □

The following fact is a direct consequence.

Corollary 2.6. *The support of an injective object in* $\mathrm{fp}(Q)$ *is closed under predecessors.*

Proof. Let I be an injective object in $\mathrm{fp}(Q)$. Assume a is a vertex in $\operatorname{supp} I$ and $b \in a^-$. We can choose some arrow $\alpha\colon b \to a$. Lemma 2.5 implies that $I(\alpha)$ is a surjection. In particular, the vertex b lies in $\operatorname{supp} I$. Then the result follows. □

Recall that Q is called *interval finite* if $Q(a,b)$ is finite for any $a, b \in Q_0$. A quiver is called *top finite* if there exist finitely many vertices of which every vertex is a successor, and is called *socle finite* if there exist finitely many vertices of which every vertex is a predecessor.

We have the following observation.

Lemma 2.7. *A top finite interval finite quiver contains no right infinite paths; a socle finite interval finite quiver contains no left infinite paths.*

Proof. Let Q be a top finite interval finite quiver. Then there exist some vertices $b_1, b_2, \ldots, b_n$ such that any vertex is a successor of some b_i. Assume Q contains a right infinite path $\alpha_1\alpha_2\ldots\alpha_j\ldots$. For each $j \geq 0$, we set $a_j = t(\alpha_{j+1})$. Since $Q(b_i, a_0)$ is finite, there exists some nonnegative integer Z_i such that $Q(b_i, a_j) = \varnothing$ for any $j \geq Z_i$. Let $Z = \max_{1 \leq i \leq n} Z_i$. Then a_Z is not a successor of any b_i, which is a contradiction. It follows that Q contains no right infinite paths.

Similarly, a socle finite interval finite quiver contains no left infinite paths. □

3. Finitely presented representations

Let k be a field and Q be an interval finite quiver.

Recall that a representation M is called *pointwise finite dimensional* if $M(a)$ is finite dimensional for any vertex a, and is called *finite dimensional* if moreover $\operatorname{supp} M$ contains only finitely many vertices.

We mention the following fact.

Lemma 3.1. *The abelian category* $\mathrm{fp}(Q)$ *is Hom-finite Krull–Schmidt, and every object is pointwise finite dimensional.*

Proof. Assume $\bigoplus_{i=1}^{m} P_{a_i} \to M$ is an epimorphism. We observe that each P_{a_i} is pointwise finite dimensional, since Q is interval finite. Then so is M.

Moreover, assume $\bigoplus_{j=1}^{n} P_{b_j} \to N$ is an epimorphism. Consider the maps

$$\mathrm{Hom}(M,N) \hookrightarrow \mathrm{Hom}\Big(\bigoplus_{i=1}^{m} P_{a_i}, N\Big) \twoheadleftarrow \mathrm{Hom}\Big(\bigoplus_{i=1}^{m} P_{a_i}, \bigoplus_{j=1}^{n} P_{b_j}\Big).$$

We observe that $\mathrm{Hom}(\bigoplus_{i=1}^{m} P_{a_i}, \bigoplus_{j=1}^{n} P_{b_j})$ is finite dimensional since Q is interval finite. Then so is $\mathrm{Hom}(M,N)$. Therefore the abelian category fp(Q) is Hom-finite, and hence is Krull–Schmidt. □

We need the following properties of finitely presented representations.

Lemma 3.2. *Let M be a finitely presented representation.*
(1) $\mathrm{supp}\, M$ *is top finite.*
(2) $\bigcup_{a\in \mathrm{supp}\, M} a^+ \backslash \mathrm{supp}\, M$ *is finite.*

Proof. (1) Assume $f\colon \bigoplus_{i=1}^{m} P_{a_i} \to M$ is an epimorphism. Then every vertex in $\mathrm{supp}\, M$ is a successor of some a_i. In other words, $\mathrm{supp}\, M$ is top finite.

(2) Denote $\Delta = \bigcup_{a\in \mathrm{supp}\, M} a^+ \backslash \mathrm{supp}\, M$. We observe that $\mathrm{Ker}\, f/\mathrm{rad}\,\mathrm{Ker}\, f$ is semisimple and $(\mathrm{Ker}\, f/\mathrm{rad}\,\mathrm{Ker}\, f)(b)\neq 0$ for any $b\in\Delta$. If Δ is not finite, then $\mathrm{Ker}\, f/\mathrm{rad}\,\mathrm{Ker}\, f$ is not finitely generated, which is a contradiction. It follows that Δ is finite. □

Corollary 3.3. *A finite dimensional representation M is finitely presented if and only if a^+ is finite for any vertex a in* $\mathrm{supp}\, M$.

Proof. For the necessary, we assume M is finitely presented and a is a vertex in $\mathrm{supp}\, M$. Lemma 3.2 implies that $a^+ \backslash \mathrm{supp}\, M$ is finite. Since $\mathrm{supp}\, M$ contains only finitely many vertices, then $a^+ \cap \mathrm{supp}\, M$ is finite. It follows that a^+ is finite.

For the sufficiency, we assume a^+ is finite for any vertex a in $\mathrm{supp}\, M$. Since M is finite dimensional, there exists some epimorphism $f\colon P \to M$ with $P\in \mathrm{proj}(Q)$.

Consider the subrepresentation N of P generated by $P(b)$, where b runs over $\bigcup_{a\in \mathrm{supp}\, M} a^+ \backslash \mathrm{supp}\, M$. Then N is contained in $\mathrm{Ker}\, f$. We observe by Lemma 3.1 each $P(b)$ is finite dimensional. Then N is finitely generated.

Consider the factor module $\mathrm{Ker}\, f/N$. Its support is contained in $\mathrm{supp}\, M$. Then it is finite dimensional and hence is finitely generated. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Ker}\, f \longrightarrow \mathrm{Ker}\, f/N \longrightarrow 0.$$

It follows that $\mathrm{Ker}\, f$ is finitely generated, and then M is finitely presented. □

Corollary 3.4. *Let a be a vertex. Then I_a is finitely presented if and only if a admits only finitely many predecessors b and each b^+ is finite.*

Proof. We observe that I_a is finitely generated if and only if $\operatorname{supp} I$ contains only finitely many vertices, since Q is interval finite. The vertices in $\operatorname{supp} I$ are precisely predecessors of a. Then the result follows from Corollary 3.3. □

The support of an injective object in $\operatorname{fp}(Q)$ satisfies the following conditions.

Lemma 3.5. *Let I be an injective object in* $\operatorname{fp}(Q)$.

(1) $a^- \cup a^+$ *is finite for any vertex a in* $\operatorname{supp} I$.

(2) *If* $\operatorname{supp} I$ *contains no left infinite paths, then it contains only finitely many vertices.*

Proof. Lemma 3.2 implies that $\operatorname{supp} I$ is top finite. Then there exist some vertices $b_1, b_2, ..., b_n$ such that any vertex in $\operatorname{supp} I$ is a successor of some b_i.

(1) We observe that $\operatorname{supp} I$ is closed under predecessors; see Corollary 2.6. Then any vertex in a^- is a successor of some b_i. If a^- is infinite, then at least one $Q(b_i, a)$ is infinite, which is a contradiction. It follows that a^- is finite.

We observe that $a^+ \cap \operatorname{supp} I$ is finite. Indeed, otherwise Lemma 2.5 implies that $I(a)$ is not finite dimensional, which is a contradiction. Since $a^+ \backslash \operatorname{supp} I$ is finite by Lemma 3.2, then a^+ is finite. It follows that $a^- \cup a^+$ is finite.

(2) Assume the vertices in $\operatorname{supp} I$ is infinite. Then there exists some b_i whose successors contained in $\operatorname{supp} I$ is infinite. Denote it by a_0. Since $a_0^+ \cap \operatorname{supp} I$ is finite, then there exists some $a_1 \in a_0^+ \cap \operatorname{supp} I$ whose successors contained in $\operatorname{supp} I$ is infinite. Choose some arrow $\alpha_1 : a_0 \to a_1$.

By induction, we obtain vertices a_i and arrows $\alpha_{i+1} : a_i \to a_{i+1}$ for $i \geq 0$ in $\operatorname{supp} M$. This is a contradiction, since $...\alpha_i...\alpha_2\alpha_1$ is a left infinite path in $\operatorname{supp} M$. It follows that $\operatorname{supp} M$ contains only finitely many vertices. □

For an injective object in $\operatorname{fp}(Q)$ whose support contains no left infinite paths, we have the following characterization.

Proposition 3.6. *Let I be an injective object in* $\operatorname{fp}(Q)$ *such that* $\operatorname{supp} I$ *contains no left infinite paths. Then*

$$I \simeq \bigoplus_{a \in Q_0} I_a^{\oplus \dim(\operatorname{soc} I)(a)}.$$

Proof. It follows from Lemma 3.5 that $\operatorname{supp} I$ contains only finitely many vertices. Let $J = \bigoplus_{a \in Q_0} I_a^{\oplus \dim(\operatorname{soc} I)(a)}$. This is a finite direct sum, since the vertices in $\operatorname{supp} I$ are finite and I is pointwise finite dimensional.

For any vertex a in $\operatorname{supp} I$, its predecessors are also contained in $\operatorname{supp} I$; see Corollary 2.6. It follows that $\operatorname{supp} J$ is a subquiver of $\operatorname{supp} I$. Then $\operatorname{soc} I$ and J are finite dimensional. Corollary 3.3 implies that they are finitely presented. We observe by Lemma 2.1 that the inclusion $\operatorname{soc} I \subseteq I$ and the injection $\operatorname{soc} I \to J$ are injective envelopes in $\operatorname{fp}(Q)$. Then the result follows. □

For an injective object in $\operatorname{fp}(Q)$ whose support contains some left infinite paths, we mention the following facts. They will be used technically in the proof of Theorem 3.11.

Lemma 3.7. *Let I be an injective object in $\operatorname{fp}(Q)$, whose support contains some left infinite paths. Denote by Δ the set of left infinite paths p contained in $\operatorname{supp} I$ with $s(p)^- \cap \operatorname{supp} I = \varnothing$. Then Δ is finite, and every left infinite path p in $\operatorname{supp} I$ admits some path u with $pu \in \Delta$.*

Proof. Lemma 3.2 implies that $\operatorname{supp} I$ is top finite. Assume vertices $b_1, b_2, ..., b_n$ satisfy that any vertex in $\operatorname{supp} I$ is a successor of some b_i. We can assume each $b_i^- \cap \operatorname{supp} I = \varnothing$.

Let p be a left infinite path in $\operatorname{supp} I$. We observe that $s(p)$ is a successor of some b_i. Choose some $u \in Q(b_i, s(p))$. Then $pu \in \Delta$. In particular, if $p \in \Delta$ then $b_i = s(p)$ since $s(p)^- \cap \operatorname{supp} I = \varnothing$.

Assume Δ is infinite. Then for $Z = \max_{1 \le i \le n} \dim I(b_i)$, One can find $nZ+1$ paths u_j from some b_i such that each u_j is not of the form $vu_{j'}$ for any $j' \neq j$. We observe that at least one $1 \le i \le n$ such that the number of u_j from b_i is greater than Z. Then Lemma 2.5 implies that $\dim I(b_i) > Z$, which is a contradiction. Then the result follows. □

Lemma 3.8. *Let I be an injective object in $\operatorname{fp}(Q)$, whose support contains some left infinite path $...\alpha_i...\alpha_2\alpha_1$. Set $a_i = s(\alpha_{i+1})$ for any $i \ge 0$. Then there exists some nonnegative integer Z such that $a_i^+ = \{a_{i+1}\}$, $a_{i+1}^- = \{a_i\}$ and $I(\alpha_{i+1})$ is a bijection for any $i \ge Z$.*

Proof. We observe by Lemma 3.2 that $\operatorname{supp} I$ is top finite and there exists some nonnegative integer Z_1 such that a_i^+ is contained in $\operatorname{supp} I$ for any $i \ge Z_1$.

It follows from Lemma 2.5 that $\dim I(a_i) \ge \dim I(a_{i+1})$ for any $i \ge 0$. Then there exists some nonnegative integer $Z_2 \ge Z_1$ such that $\dim I(a_i) = \dim I(a_{Z_2})$ for any $i \ge Z_2$. Since $I(\alpha_{i+1})$ is a surjection by Lemma 2.5, then it is a bijection.

We claim that $a_i^+ = \{a_{i+1}\}$ and $Q(a_i, a_{i+1}) = \{\alpha_{i+1}\}$ for any $i \ge Z_2$. Indeed, otherwise there exist some arrow $\beta\colon a_i \to b$ in $\operatorname{supp} I$ with $i \ge Z_2$ and $\beta \neq \alpha_{i+1}$. Then Lemma 2.5 implies that $\dim I(a_i) \ge \dim I(a_{i+1}) + \dim I(b) > \dim I(a_{i+1})$, which is a contradiction.

Assume vertices $b_1, b_2, ..., b_n$ satisfy that every vertex in $\operatorname{supp} I$ is a successor of some b_j. We observe that $|Q(b_j, a_{i+1})| \geq |Q(b_j, a_i)|$ for any $1 \leq j \leq n$ and $i \geq 0$. If moreover $|a_{i+1}^-| > 1$ for some $i \geq 0$, then there exists some j such that $|Q(b_j, a_{i+1})| > |Q(b_j, a_i)|$.

We claim the existence of nonnegative integer Z_3 such that $a_{i+1}^- = \{a_i\}$ for any $i \geq Z_3$. Indeed, otherwise there exists some $1 \leq j \leq n$ such that $\{|Q(b_j, a_i)| \,|\, i \geq 0\}$ is unbounded. Then Lemma 2.5 implies that $I(b_j)$ is not finite dimensional, which is a contradiction.

Let $Z = \max\{Z_2, Z_3\}$. Then the result follows. □

Following [Che15, Subsection 2.1], we define an equivalence relation on left infinite paths. Two left infinite paths $...\alpha_i...\alpha_2\alpha_1$ and $...\beta_i...\beta_2\beta_1$ are equivalent if there exist some positive integers m and n such that

$$...\alpha_i...\alpha_{m+1}\alpha_m = ...\beta_i...\beta_{n+1}\beta_n.$$

Let p be a left infinite path. We denote by $[p]$ the equivalence class containing p. We mention that $[p]$ is a set. For any vertex a, we denote by $[p]_a$ the subset of $[p]$ formed by left infinite paths u with $s(u) = a$.

Considering [Jia19, Section 5] and [Che15, Subsection 3.1], we introduce a representation $Y_{[p]}$ as follows. For every vertex a, we let

$$Y_{[p]}(a) = \operatorname{Hom}_k\Big(\bigoplus_{u \in [p]_a} ku, k\Big).$$

For every arrow $\alpha\colon a \to b$, we let

$$Y_{[p]}(\alpha)\colon Y_{[p]}(a) \longrightarrow Y_{[p]}(b), \quad f \longmapsto (u \longmapsto f(u\alpha)).$$

We mention that these $Y_{[p]}$ are indecomposable and pairwise non-isomorphic; see the dual of [Jia19, Proposition 5.4].

Recall that a quiver is called *uniformly interval finite*, if there exists some positive integer Z such that for any vertices a and b, the number of paths p from a to b is less than or equal to Z; see [Jia19, Definition 2.3].

We characterize when $Y_{[p]}$ is finitely presented.

Lemma 3.9. *Let p be a left infinite path. Then $Y_{[p]}$ is finitely presented if and only if $\operatorname{supp} Y_{[p]}$ is top finite uniformly interval finite and $\bigcup_{a \in \operatorname{supp} Y_{[p]}} a^+ \setminus \operatorname{supp} Y_{[p]}$ is finite.*

Proof. For the necessary, we assume $Y_{[p]}$ is finitely presented. It follows from Lemma 3.2 that $\operatorname{supp} Y_{[p]}$ is top finite and $\bigcup_{a\in \operatorname{supp} Y_{[p]}} a^+ \backslash \operatorname{supp} Y_{[p]}$ is finite.

Assume vertices $b_1, b_2, ..., b_n$ satisfy that any vertex in $\operatorname{supp} Y_{[p]}$ is a successor of some b_i. Let a and a' be a pair of vertices in $\operatorname{supp} Y_{[p]}$. Then there exists some $Q(b_i, a) \neq \varnothing$. Set $Z = \max_{1\leq i\leq n} \dim Y_{[p]}(b_i)$. Since Q contains no oriented cycles, we have that

$$|Q(a,a')| \leq |Q(b_i,a')| \leq |[p]_{b_i}| = \dim Y_{[p]}(b_i) \leq Z.$$

It follows that $\operatorname{supp} Y_{[p]}$ is uniformly interval finite.

For the sufficiency, we assume $p = ...\alpha_i ... \alpha_2\alpha_1$. Set $a_i = s(\alpha_{i+1})$ for any $i \geq 0$. Assume vertices $b_1, b_2, ..., b_n$ satisfy that any vertex in $\operatorname{supp} Y_{[p]}$ is a successor of some b_i. Since $\operatorname{supp} Y_{[p]}$ is uniformly interval finite, then

$$\{|Q(b_j, a_i)| \,|\, i \geq 0, 1 \leq j \leq n\}$$

is bounded. We observe that $|Q(b_j, a_i)| \leq |Q(b_j, a_{i+1})|$. Then there exists some nonnegative integer Z_1 such that $|Q(b_j, a_i)| = |Q(b_j, a_{Z_1})|$ for any $i \geq Z_1$ and $1 \leq j \leq n$. In particular, $a_{i+1}^- = \{a_i\}$ and $Q(a_i, a_{i+1}) = \{\alpha_{i+1}\}$ for any $i \geq Z_1$.

Since $\bigcup_{a\in \operatorname{supp} Y_{[p]}} a^+ \backslash \operatorname{supp} Y_{[p]}$ is finite, there exists some nonnegative integer Z_2 such that a_i^+ is contained in $\operatorname{supp} Y_{[p]}$ for any $i \geq Z_2$. Let $Z = \max\{Z_1, Z_2\}$. Then $a_{i+1}^- = \{a_i\}$, $a_i^+ = \{a_{i+1}\}$ and $Y_{[p]}(\alpha_{i+1})$ is a bijection.

Consider the subrepresentation N of $Y_{[p]}$ generated by $Y_{[p]}(a_Z)$. We observe that $N \simeq P_{a_Z}^{\oplus \dim Y_{[p]}(a_Z)}$ and hence is finitely presented. Moreover, $\operatorname{supp}(Y_{[p]}/N)$ contains only finitely many vertices b and b^+ is finite. Then $Y_{[p]}/N$ is finitely presented by Corollary 3.3. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow Y_{[p]} \longrightarrow Y_{[p]}/N \longrightarrow 0.$$

It follows that $Y_{[p]}$ is finitely presented. $\square$

We show the injectivity of finitely presented $Y_{[p]}$ in fp(Q); compare the dual of [Jia19, Proposition 6.2].

Proposition 3.10. *Let p be a left infinite path such that $Y_{[p]}$ is finitely presented. Then $Y_{[p]}$ is an indecomposable injective object in* fp(Q).

Proof. Assume $p = ...\alpha_i ... \alpha_2\alpha_1$. For each $i \geq 0$, we set $a_i = s(\alpha_{i+1})$. Consider the morphism $\psi_{i+1} \colon I_{a_{i+1}} \to I_{a_i}$ given by $(\psi_{i+1})_b(f)(u) = f(\alpha_{i+1}u)$ for any $f \in I_{a_{i+1}}(b)$ and $u \in Q(b, a_i)$. We observe that $(I_{a_i})_{i\geq 0}$ forms an inverse system, and $Y_{[p]}$ is the inverse limit in Rep(Q); see also [Jia19, Lemma 5.7].

Given any exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathrm{fp}(Q)$, it is also an exact sequence in $\mathrm{Rep}(Q)$. Applying $\mathrm{Hom}(-,I_{a_i})$, we obtain an exact sequence of inverse systems of k-linear spaces

$$0\longrightarrow(\mathrm{Hom}(N,I_{a_i}))\longrightarrow(\mathrm{Hom}(M,I_{a_i}))\longrightarrow(\mathrm{Hom}(L,I_{a_i}))\longrightarrow 0.$$

Lemma 2.2 implies that $\mathrm{Hom}(N,I_{a_i})\simeq\mathrm{Hom}_k(N(a_i),k)$. We observe by Lemma 3.1 that it is finite dimensional. Then $(\mathrm{Hom}(N,I_{a_i}))$ satisfies the Mittag–Leffler condition naturally. It follows from [Wei94, Proposition 3.5.7] the exact sequence

$$0\longrightarrow\varprojlim\mathrm{Hom}(N,I_{a_i})\longrightarrow\varprojlim\mathrm{Hom}(M,I_{a_i})\longrightarrow\varprojlim\mathrm{Hom}(L,I_{a_i})\longrightarrow 0.$$

For any $X\in\mathrm{fp}(Q)$, there exist natural isomorphisms

$$\varprojlim\mathrm{Hom}(X,I_{a_i})\simeq\mathrm{Hom}(X,\varprojlim I_{a_i})\simeq\mathrm{Hom}(X,Y_{[p]}).$$

Then we obtain the exact sequence

$$0\longrightarrow\mathrm{Hom}(N,Y_{[p]})\longrightarrow\mathrm{Hom}(M,Y_{[p]})\longrightarrow\mathrm{Hom}(L,Y_{[p]})\longrightarrow 0.$$

It follows that $Y_{[p]}$ is an indecomposable injective object in $\mathrm{fp}(Q)$. $\square$

Now, we can classify the indecomposable injective objects in $\mathrm{fp}(Q)$.

Theorem 3.11. *Let Q be an interval finite quiver. Assume I is an indecomposable injective object in $\mathrm{fp}(Q)$. Then either $I\simeq I_a$ where a admits only finitely many predecessors b and each b^+ is finite, or $I\simeq Y_{[p]}$ where $[p]$ is an equivalence class of left infinite paths such that $\mathrm{supp}\,Y_{[p]}$ is top finite uniformly interval finite and $\bigcup_{a\in\mathrm{supp}\,Y_{[p]}}a^+\backslash\mathrm{supp}\,Y_{[p]}$ is finite.*

Proof. If $\mathrm{supp}\,I$ contains no left infinite paths, Proposition 3.6 implies that $I\simeq I_a$ for some vertex a. Corollary 3.4 implies that a admits only finitely many predecessors b and each b^+ is finite.

Now, we assume $\mathrm{supp}\,I$ contains some left infinite paths. Let Δ be the set of left infinite paths p contained in $\mathrm{supp}\,I$ with $s(p)^-\cap\mathrm{supp}\,I=\varnothing$. It follows from Lemma 3.7 that Δ is finite.

For every $p\in\Delta$, we assume $p=\ldots\alpha_{p,j}\ldots\alpha_{p,2}\alpha_{p,1}$. Set $a_{p,j}=s(\alpha_{p,j+1})$ for each $j\geq 0$. By Lemma 3.8, there exists some nonnegative integer Z_p such that $a_{p,j}^+=\{a_{p,j+1}\}$, $a_{p,j+1}^-=\{a_{p,j}\}$ and $I(\alpha_{p,j+1})$ is a bijection for any $j\geq Z_p$.

Consider the subrepresentation M of I generated by $I(a_{p,Z_p+1})$ for all $p\in\Delta$. It follows from Lemma 2.4 that I/M is finitely presented. It is an injective object in $\mathrm{fp}(Q)$, since $\mathrm{fp}(Q)$ is hereditary.

We observe by Lemma 3.7 that $\mathrm{supp}(I/M)$ contains no left infinite paths. Indeed, assume $\ldots\alpha_i\ldots\alpha_2\alpha_1$ is a left infinite path in $\mathrm{supp}(I/M)$. It also lies in supp I. Then these α_i for i large enough lie in supp M. Therefore, they do not lie in supp(I/M), which is a contradiction.

It follows from Proposition 3.6 that

$$I/M \simeq \bigoplus_{b\in Q_0} I_b^{\oplus \dim(\operatorname{soc}(I/M))(b)}.$$

For any $p\in\Delta$, we observe that $(\operatorname{soc}(I/M))(a_{p,Z_p})=I(a_{p,Z_p})\neq 0$ and $I(\alpha_{p,j+1})$ is a bijection for any $j\geq Z_p$. The previous isomorphism can be extended as

$$I \simeq \left(\bigoplus_{b\in Q_0} I_b^{\oplus \dim(\operatorname{soc} I)(b)}\right) \oplus \left(\bigoplus_{p\in\Delta} Y_{[p]}^{\oplus \dim I(a_{p,Z_p})}\right).$$

Since I is indecomposable, then Δ contains only one left infinite path p and $\dim I(a_{p,Z_p})=1$. Then $\operatorname{soc} I=0$ and $I\simeq Y_{[p]}$. It follows from Lemma 3.9 that supp $Y_{[p]}$ is top finite and uniformly interval finite, and $\bigcup_{a\in \operatorname{supp} Y_{[p]}} a^{+}\setminus \operatorname{supp} Y_{[p]}$ is finite. □

Remark 3.12. Let $\mathcal{C}$ be a k-linear *spectroid*, i.e., a Hom-finite category whose objects are pairwise non-isomorphic with local endomorphism rings. Assume k is algebraically closed, and the infinite radical of $\mathcal{C}$ vanishes, and the category of modules over $\mathcal{C}$ is hereditary. Then the quiver of $\mathcal{C}$ is interval finite. It can be viewed as a category naturally, and its k-linearization is precisely $\mathcal{C}$; see [GR92, Sections 8.1 and 8.2] for more details. Therefore, Theorem 3.11 can be applied to the category of finitely presented modules over $\mathcal{C}$.

4. Examples

Let k be a field. We will give some examples.

Example 4.1. Assume Q is the following quiver.

$$\cdots \xleftarrow{\alpha_4} \underset{3}{\circ} \xleftarrow{\alpha_3} \underset{2}{\circ} \xleftarrow{\alpha_2} \underset{1}{\circ} \xleftarrow{\alpha_1} \underset{0}{\circ}$$

For each $n\geq 0$, we consider the representation I_n. We observe that the predecessors of n are i for $0\leq i\leq n$, and i^+ is finite. Corollary 3.4 implies that I_n is finitely presented.

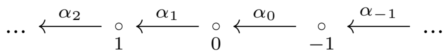
Let $p=\ldots\alpha_i\ldots\alpha_2\alpha_1$. Then supp $Y_{[p]}=Q$. We observe that supp $Y_{[p]}$ is top finite uniformly interval finite and $\bigcup_{a\in Q_0} a^{+}\setminus Q_0$ is the empty set. Lemma 3.9 implies that $Y_{[p]}$ is finitely presented.

We observe by Theorem 3.11 that

$$\{I_n \mid n\geq 0\}\cup\{Y_{[p]}\}$$

is a complete set of indecomposable injective objects in fp(Q).

Example 4.2. Assume Q is the following quiver.

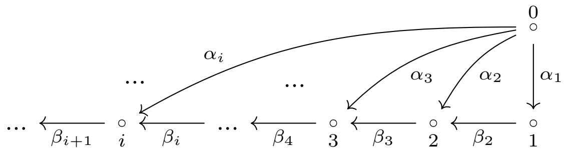


For each integer n, we consider the representation I_n. We observe that all $i \le n$ are predecessors of n. Corollary 3.4 implies that I_n is not finitely presented.

Let $p = \dots\alpha_i\dots\alpha_2\alpha_1$. Then $\operatorname{supp} Y_{[p]} = Q$, which contains a right infinite path $\alpha_{-1}\alpha_{-2}\dots\alpha_{-i}\dots$. Then it is not top finite by Lemma 2.7. It follows from Lemma 3.9 that $Y_{[p]}$ is not finitely presented.

We observe by Theorem 3.11 that $\mathrm{fp}(Q)$ contains no nonzero injective objects.

Example 4.3. Assume Q is the following quiver.



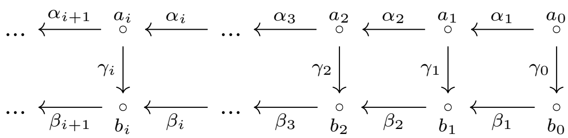
We mention that Q is interval finite, but not *locally finite* (i.e., for any vertex a, the set of arrows α with $s(\alpha) = a$ or $t(\alpha) = a$ is finite).

For each $n \ge 0$, we consider the representation I_n. We observe that the set of predecessors of n is $\{0 \le i \le n\}$, but 0^+ is not finite. Then Corollary 3.4 implies that I_n is not finitely presented.

Let $p = \dots\beta_i\dots\beta_3\beta_2$. Then $\operatorname{supp} Y_{[p]} = Q$, which is not uniformly interval finite. Lemma 3.9 implies that $Y_{[p]}$ is not finitely presented.

We observe by Theorem 3.11 that $\mathrm{fp}(Q)$ contains no nonzero injective objects.

Example 4.4. Assume Q is the following quiver.



For each $n \ge 0$, we consider the representations I_{a_n} and I_{b_n}. We observe that the set of predecessors of a_n is $\{a_i | 0 \le i \le n\}$, and the one of b_n is $\{a_i | 0 \le i \le n\} \cup \{b_i | 0 \le i \le n\}$. Since each a_i^+ and b_i^+ are both finite, Corollary 3.4 implies that I_{a_n} and I_{b_n} are finitely presented.

Let $p = \dots\alpha_i\dots\alpha_2\alpha_1$. Then $\operatorname{supp} Y_{[p]}$ is the full subquiver of Q formed by a_i for all $i \ge 0$. We observe that $\bigcup_{a \in \operatorname{supp} Y_{[p]}} a^+ \setminus \operatorname{supp} Y_{[p]}$ contains all b_i and then is infinite. Lemma 3.9 implies that $Y_{[p]}$ is not finitely presented.

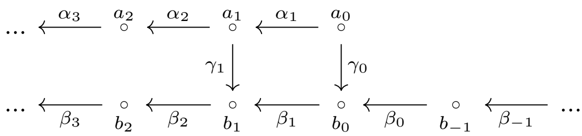
Let $q=...\beta_i...\beta_2\beta_1$. Then $\operatorname{supp} Y_{[q]}=Q$. Since Q is not uniformly interval finite, then $Y_{[q]}$ is not finitely presented by Lemma 3.9.

We observe that $\{[p],[q]\}$ is the set of equivalence classes of left infinite paths. It follows from Theorem 3.11 that

$$\{I_{a_i} | i \geq 0\} \cup \{I_{b_i} | i \geq 0\}$$

is a complete set of indecomposable injective objects in fp(Q).

Example 4.5. Assume Q is the following quiver.



We observe that a^+ is finite for any $a \in Q_0$. Consider the representations I_{a_i} for $i \geq 0$ and I_{b_j} for $j \in \mathbb{Z}$. The set of predecessors of a_i is finite and the one of b_j is not. It follows from Corollary 3.4 that I_{a_i} is finitely presented, while I_{b_j} is not.

Let $p=...\alpha_i...\alpha_2\alpha_1$. Then $\operatorname{supp} Y_{[p]}$ is the full subquiver of Q formed by a_i for all $i \geq 0$. We observe that $\operatorname{supp} Y_{[p]}$ is top finite uniformly interval finite, and $\bigcup_{a \in \operatorname{supp} Y_{[p]}} a^+ \backslash \operatorname{supp} Y_{[p]} = \{b_0, b_1\}$. It follows from Lemma 3.9 that $Y_{[p]}$ is finitely presented.

Let $q=...\beta_i...\beta_2\beta_1$. Then $\operatorname{supp} Y_{[q]}$ is the full subquiver of Q formed by a_0, a_1 and b_j for all $j \in \mathbb{Z}$. We observe that $\operatorname{supp} Y_{[q]}$ contains a right infinite path $\beta_{-1}\beta_{-2}...\beta_{-i}...$. Then it is not top finite by Lemma 2.7. It follows from Lemma 3.9 that $Y_{[q]}$ is not finitely presented.

We observe that $\{[p],[q]\}$ is the set of equivalence classes of left infinite paths. It follows from Theorem 3.11 that

$$\{I_{a_i} | i \geq 0\} \cup \{Y_{[p]}\}$$

is a complete set of indecomposable injective objects in fp(Q).

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