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Singular equivalences arising from Morita rings

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Abstract. We obtain new classes of singular equivalences which are constructed from Gorenstein-projective modules.

1. Introduction

The singularity category $D_{sg}(A)$ of an algebra A over a field k, introduced by R.O. Buchweitz in [4], is defined as the Verdier quotient $D_{sg}(A)=D^b(A\text{-mod})/\text{per}(A)$ of the bounded derived category $D^b(A\text{-mod})$ by the category of perfect complexes. In recent years, D. Orlov ([14]) rediscovered the notion of singularity categories in his study of B-branes on Landau-Ginzburg models in the framework of the Homological Mirror Symmetry Conjecture. The singularity category measures the homological singularity of an algebra in the sense that an algebra A has finite global dimension if and only if its singularity category $D_{sg}(A)$ vanishes.

Two Artin algebras A and B are said to be singularly equivalent if there is a triangle equivalence between their singularity categories. In this case, the corresponding equivalence is called a singular equivalence between the two algebras. It is well known that derived equivalences can induce naturally singular equivalences. We recall that a derived equivalence between two algebras is a triangular equivalence between their bounded derived categories. J. Rickard ([15, Theorem 3.3]) proved that a tilting module T over an algebra A induces an equivalence between the derived category $D(A)$ and the derived category $D(B)$, where B is the endomorphism algebra of T. Through this point we can get many examples of singular equivalences. Inspired by stable equivalences of Morita type introduced by M.

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Broué ([3]), X.W. Chen and L.G. Sun ([10]) introduced a special singular equivalence between two k-algebras, which is called singularly equivalent of Morita type, as a generalization of stable equivalences of Morita type. Singularity categories and singular equivalences have drawn much attention. Their structural properties and construction were investigated among in e.g. [5], [6], [7], [8] and [9].

Recall that Morita rings are 2×2 matrix rings associated to Morita contexts ([2], [11]). A particular case of interest is the Morita ring with bimodule homomorphisms zero. Gao and Psaroudakis [13] investigated its Gorenstein homological properties.

The aim of this article is to construct new classes of singular equivalences arising from Morita rings.

2. Singular equivalences of Morita rings

We first recall the definition of a singular equivalence of Morita type.

Definition 2.1. ([10, Definition 3.1]) Let k be a field. Two finite-dimensional k-algebras A and B are singularly equivalent of Morita type if there exist an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that

(i) M and N are finitely generated projective as left and right modules;

(ii) $M\otimes_B N\cong A\oplus P$ as A-A-bimodules for some finitely generated A-A-bimodule P with finite projective dimension, and $N\otimes_A M\cong B\oplus Q$ as B-B-bimodules for some finitely generated B-B-bimodule Q with finite projective dimension.

Now we need recall the notion of a Gorenstein algebra and a Gorenstein-projective module. Let A be a finite-dimensional k-algebra over a field k. A is a d-Gorenstein algebra for some non-negative integer d if the injective dimension of A is finite and also equals d as left and right A-modules. Denote by A-mod the category of finitely generated left A-modules, and by A-proj the full subcategory of finitely generated projective A-modules. An A-module M in A-mod is called Gorenstein projective, if there exists an exact sequence $P^\bullet=\ldots\to P^{-1}\to P^0\xrightarrow{d^0}P^1\to\ldots$ in A-proj with $\mathrm{Hom}_\Lambda(P^\bullet,Q)$ exact for any $Q\in A$-proj, such that $M\cong\ker d^0$ (see [12]).

Lemma 2.2. *Let A and B be two finite-dimensional k-algebras which are singularly equivalent of Morita type induced by bimodules M and N. Then the following hold*:

(1) *The functors $M\otimes_B-:B\text{-mod}\to A\text{-mod}$ and $N\otimes_A-:A\text{-mod}\to B\text{-mod}$ are exact and take finitely generated projective modules to finitely generated projective modules.*

(2) *Suppose that A and B are Gorenstein. Then the functors $M\otimes_B -$ and $N\otimes_A -$ induce a one-to-one correspondence between the indecomposable non-projective objects of A-*Gproj *and B-*Gproj, *where A-*Gproj (*resp.* B-Gproj) *denotes the category of finitely generated Gorenstein-projective A-modules (resp. B-modules).*

Proof. (1) follows from the fact that ${}_AM$ and ${}_BN$ are finitely generated projective modules. (2) follows from [16, Proposition 3.7]. □

Lemma 2.3. *Let A and B be two finite-dimensional Gorenstein k-algebras which are singularly equivalent of Morita type induced by bimodules M and N. Let P be a finitely generated A-A-bimodule satisfying Definition 2.1(ii) with the minimal projective resolution as A-A-bimodule*

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow {}_AP_A \longrightarrow 0 \qquad (*)$$

Let C be a k-algebra and V(resp. W) an A-C-bimodule (resp. C-A-bimodule) such that V(resp. W) is a finitely generated non-projective Gorenstein-projective left (resp. right) A-module. Then we have the following:

(1) *$P_i\otimes_A V=0$ for all $1\le i\le n$, and $P_0\otimes_A V\cong P\otimes_A V$ as a left A-module.*

(2) *$W\otimes_A P_i=0$ for all $1\le i\le n$, and $W\otimes_A P_0\cong W\otimes_A P$ as a right A-module.*

(3) *Let $\Lambda=\begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C \end{pmatrix}$ be the Morita ring which is an Artin algebra such that $V\otimes_C W=0$ and $W\otimes_A V=0$. If V and W are indecomposable, then $P\otimes_A V=0$ and $W\otimes_A P=0$, and moreover, P has finite projective dimension as a Λ-Λ-bimodule.*

Proof. (1). By assumption P has finite projective dimension as a right A-module. Since A is d-Gorenstein for some non-negative integer d, it follows that Ω^dP is Gorenstein-projective with finite projective dimension. So Ω^dP is either zero or projective. On the other hand, since ${}_AV$ is Gorenstein-projective, it follows that there exists a left A-module V_1 such that $V\cong\Omega^dV_1$. Then for all $i\ge 1$,

$$\mathrm{Tor}_i^A(P,V)\cong\mathrm{Tor}_i^A(P,\Omega^dV_1)\cong\mathrm{Tor}_i^A(\Omega^dP,V_1)=0$$

So we obtain a long exact sequence of left A-modules

$$0 \longrightarrow P_n\otimes_A V \xrightarrow{d_n\otimes \mathrm{Id}_V} P_{n-1}\otimes_A V \longrightarrow \dots \longrightarrow P_1\otimes_A V \xrightarrow{d_1\otimes \mathrm{Id}_V} P_0\otimes_A V \longrightarrow P\otimes_A V \longrightarrow 0 \qquad (**)$$

Since ${}_A(P_i\otimes_A V)$ is projective for all $0\le i\le n$ and $(*)$ is a minimal, then $(**)$ is a minimal projective resolution of the left A-module $P\otimes_A V$. On the other hand, since V and $M\otimes_B N\otimes_A V$ are both Gorenstein-projective left A-modules, and $M\otimes_B N\cong A\oplus P$, it follows that $P\otimes_A V$ is also Gorenstein-projective. It follows that $M\otimes_B$

$N\otimes_A V\cong V\oplus P\otimes_A V$ and so $P\otimes_A V$ is a projective left A-module. Therefore, by the minimality of $(**)$, we obtain that $P_i\otimes_A V=0$ for all $1\leq i\leq n$, and $P_0\otimes_A V\cong P\otimes_A V$.

(2). By assumption P has finite projective dimension as a left A-module. Since A is d-Gorenstein for some non-negative integer d, it follows that $\Omega^d P$ is Gorenstein-projective with finite projective dimension. So $\Omega^d P$ is either zero or projective. On the other hand, since W_A is Gorenstein-projective, it follows that there exists a right A-module W_1 such that $W\cong\Omega^d W_1$. Then for all $i\geq 1$,

$$\mathrm{Tor}_i^A(W,P)\cong\mathrm{Tor}_i^A(\Omega^d W_1,P)\cong\mathrm{Tor}_i^A(W_1,\Omega^d P)=0$$

So we obtain a long exact sequence of right A-modules

$$0\longrightarrow W\otimes_A P_n\xrightarrow{\mathrm{Id}_W\otimes d_n} W\otimes_A P_{n-1}\longrightarrow \dots\longrightarrow W\otimes_A P_1\xrightarrow{\mathrm{Id}_W\otimes d_1} W\otimes_A P_0$$
$$\longrightarrow W\otimes_A P\longrightarrow 0 \qquad (**)$$

Since $(W\otimes_A P_i)_A$ is projective for all $0\leq i\leq n$ and $(*)$ is a minimal, then $(**)$ is a minimal projective resolution of the right A-module $W\otimes_A P$. On the other hand, since W and $W\otimes_A M\otimes_B N$ are both Gorenstein-projective right A-modules, and $M\otimes_B N\cong A\oplus P$, it follows that $W\otimes_A P$ is also Gorenstein-projective. It follows that $W\otimes_A M\otimes_B N\cong W\oplus W\otimes_A P$ and so $W\otimes_A P$ is a projective right A-module. Therefore, by the minimality of $(**)$, we obtain that $W\otimes_A P_i=0$ for all $1\leq i\leq n$, and $W\otimes_A P_0\cong W\otimes_A P$.

(3). Since $M\otimes_B N\cong A\oplus P$ as A-A-bimodule, it follows that $M\otimes_B N\otimes_A V\cong V\oplus (P\otimes_A V)$ as left A-modules. Then we have that $P\otimes_A V=0$. Since P_l is a projective A-A-module for all $0\leq l\leq n$, it follows that there exists a finite index set I and pairs (e_i,e_j) of idempotents of A such that $P_l=\oplus_{(i,j)\in I}Ae_i\otimes_k e_jA$. Since $P_l\otimes_A V=0$, we have $e_jV\cong e_jA\otimes_A V=0$, and moreover, $e_jA\cong e_jA\oplus e_jV$ is a projective right Λ-module. On the other hand, $W\otimes_A M\otimes_B N\cong W\oplus(W\otimes_A P)$ as right A-modules. Then we have that $W\otimes_A P=0$. Since P_l is a projective A-A-module for all $0\leq l\leq n$, $P_l=\oplus_{(i,j)\in I}Ae_i\otimes_k e_jA$. Since $W\otimes_A P_l=0$, we have $We_i\cong W\otimes_A Ae_i=0$, and moreover, $Ae_i\cong Ae_i\oplus We_i$ is a projective left Λ-module. Thus we get that $Ae_i\otimes_k e_jA$ is a projective Λ-Λ-bimodule. This means that P has finite projective dimension as a Λ-Λ-bimodule. □

Remark 2.4. Use the notation in Lemma 2.3. Let $\Gamma=\begin{pmatrix} B & N\otimes_A V\\ W\otimes_A M & C\end{pmatrix}$ be the Morita ring which is an Artin algebra. Since $N\otimes_A V$ and $W\otimes_A M$ are indecomposable non-projective Gorenstein-projective B-modules by Lemma 2.2, we can adapt the proof of Lemma 2.3(3) to obtain that Q is also a Γ-Γ-bimodule with finite projective dimension such that $Q\otimes_B N\otimes_A V=0$ and $W\otimes_A M\otimes_B Q=0$.

Theorem 2.5. *Let A and B be Gorenstein k-algebras which are singularly equivalent of Morita type induced by bimodules M and N. Let C be a k-algebra. Let V (resp. W) be an A-C-bimodule (resp. C-A-bimodule) such that V (resp. W) is a non-projective Gorenstein-projective left (resp. right) A-module and $V\otimes_C W=0$ and $W\otimes_A V=0$. Let $\Lambda=\begin{pmatrix} A & {}_AV_C \\ {}_CW_A & C\end{pmatrix}$ and $\Gamma=\begin{pmatrix} B & N\otimes_A V \\ W\otimes_A M & C\end{pmatrix}$ be the Morita rings which are Artin algebras. Suppose that $\mathrm{End}_{B\otimes_k C^{\mathrm{op}}}(N\otimes_A V)=k\mathrm{Id}$ and $\mathrm{End}_{C\otimes_k B^{\mathrm{op}}}(W\otimes_A M)=k\mathrm{Id}$. Then Λ and Γ are singularly equivalent of Morita type.*

Proof. By assumption, we have an A-A-bimodule isomorphism $\rho=(\rho_1,\rho_2)$: $M\otimes_B N\cong A\oplus P$ and a B-B-bimodule isomorphism $\sigma=(\sigma_1,\sigma_2)$: $N\otimes_A M\cong B\oplus Q$, where P and Q have finite projective dimension. From the A-C-bimodule isomorphism $M\otimes_B N\otimes_A V\cong V$, we have two B-C-bimodule isomorphisms $\mathrm{Id}_N\otimes\mu(\rho_1\otimes\mathrm{Id}_V): N\otimes_A M\otimes_B N\otimes_A V\cong N\otimes_A V$ and $\mu'(\sigma_1\otimes\mathrm{Id}_{N\otimes_A V}): N\otimes_A M\otimes_B N\otimes_A V\cong N\otimes_A V$, where $\mu: A\otimes_A V\to V$ and $\mu': B\otimes_B(N\otimes_A V)\to N\otimes_A V$ are the multiplication maps. Since $\mathrm{End}_{B\otimes_k C^{\mathrm{op}}}(N\otimes_A V)=k\mathrm{Id}$, there exists a non-zero element $k_0\in k$ such that $\mathrm{Id}_N\otimes\mu(\rho_1\otimes\mathrm{Id}_V)=k_0(\mu'(\sigma_1\otimes\mathrm{Id}_{N\otimes_A V}))$. Without loss of generality, we may assume that $k_0=1$. On the other hand, from the C-A-bimodule isomorphism $W\otimes_A M\otimes_B N\cong W$, we have two A-C-bimodule isomorphisms $\mu''(\mathrm{Id}_W\otimes\rho_1)\otimes\mathrm{Id}_M$: $W\otimes_A M\otimes_B N\otimes_A M\cong W\otimes_A M$ and $\mu'''(\mathrm{Id}_{W\otimes_A M}\otimes\sigma_1): W\otimes_A M\otimes_B N\otimes_A M\cong W\otimes_A M$, where $\mu'': W\otimes_A A\to W$ and $\mu''': (W\otimes_A M)\otimes_B B\to W\otimes_A M$ are the multiplication maps. Since $\mathrm{End}_{C\otimes_k B^{\mathrm{op}}}(W\otimes_A M)=k\mathrm{Id}$, there exists a non-zero element $k_0'\in k$ such that $\mu''(\mathrm{Id}_W\otimes\rho_1)\otimes\mathrm{Id}_M=k_0'(\mu'''(\mathrm{Id}_{W\otimes_A M}\otimes\sigma_1))$. Without loss of generality, we may assume that $k_0'=1$.

Recall that each finitely generated Λ-module X can be described as a tuple $X=(X_0,X_\omega,f,g)$, where X_0 is in A-mod, X_ω is in C-mod, and $f: V\otimes_C X_\omega\to X_0$ is an A-homomorphism, $g: W\otimes_A X_0\to X_\omega$ is a C-homomorphism, by the way that $X=X_0\oplus X_\omega$ with Λ-module structure given by $\begin{pmatrix} a & v\\ w & c\end{pmatrix}(x,y)=(ax+f(v\otimes y), g(w\otimes x)+cy)$. Now given a Λ-module $X=(X_0,X_\omega,f,g)$, we put $F(X):=(N\otimes_A X_0, X_\omega, f(\mu''\otimes\mathrm{Id}_{X_0})(\mathrm{Id}_W\otimes\rho_1\otimes\mathrm{Id}_{X_0}), \mathrm{Id}_N\otimes g)$. Then $F:\Lambda\text{-mod}\to\Gamma\text{-mod}$ is a well-defined exact functor preserving finitely generated projective modules. By Watt's Theorem (e.g. [17, Theorem 3.3.16]), $F\cong{}_\Gamma F(\Lambda)\otimes_\Lambda -$. Now let us define the functor $G:\Gamma\text{-mod}\to\Lambda\text{-mod}$.

Recall that each finitely generated Γ-module Y can be described as a tuple $Y=(U,T,s,t)$, where U is in B-mod, T is in C-mod, and $s: N\otimes_A V\otimes_C T\to U$ is a B-homomorphism, $t: W\otimes_A M\otimes_C U\to T$ is a C-homomorphism, by the way that $Y=U\oplus T$ with Γ-module structure given by $\begin{pmatrix} b & n\otimes v\\ w\otimes m & c\end{pmatrix}(u,p)=(bu+s(n\otimes v\otimes p), t(w\otimes m\otimes u)+cp)$. Now for $(U,T,s,t)\in\Gamma\text{-mod}$ with the C-morphism $s: W\otimes_A$

$M\otimes_B U\to T$ and the B-morphism $t\colon N\otimes_A V\otimes_C T\to U$, we define

$$G(U,T,s,t):=(M\otimes_B U,T,s,(\mathrm{Id}_M\otimes t)((\mu(\rho_1\otimes\mathrm{Id}_V))^{-1}\otimes\mathrm{Id}_T)$$

Then G is a well-defined exact functor preserving finitely generated projective modules. By Watts' Theorem (e.g. [17, Theorem 3.3.16]), $G\cong{}_\Lambda G(\Gamma)\otimes_\Gamma -$. Denote by $\mu''''\colon(N\otimes_A V)\otimes_C C\to N\otimes_A V$ and $\mu'''''\colon V\otimes_C C\to V$ the multiplication maps. Since there are the Λ-Λ-bimodule isomorphisms

$$\begin{aligned}
G(\Gamma)\otimes_\Gamma F(\Lambda) &\cong G(F(\Lambda))=G(F((A,W,\mu'',0)\oplus(V,C,0,\mu'''')))\\
&=G((N\otimes_A A,W,\mu''(\mu''\otimes\mathrm{Id}_A)(\mathrm{Id}_W\otimes\rho_1\otimes\mathrm{Id}_A),0)\\
&\oplus(N\otimes_A V,C,0,\mathrm{Id}_N\otimes\mu'''''))\\
&=(M\otimes_B N\otimes_A A,W,\mu''(\mu''\otimes\mathrm{Id}_A)(\mathrm{Id}_W\otimes\rho_1\otimes\mathrm{Id}_A),0)\\
&\oplus(M\otimes_B N\otimes_A V,C,0,(\mu(\rho_1\otimes\mathrm{Id}_V))^{-1}\mu''''')\\
&\cong\Lambda\oplus P
\end{aligned}$$

and the Γ-Γ-bimodule isomorphisms

$$\begin{aligned}
F(\Lambda)\otimes_\Lambda G(\Gamma) &\cong F(G(\Gamma))=F(G((B,W\otimes_B M,\mu''',0)\oplus(N\otimes_A V,C,0,\mu'''')))\\
&=F((M\otimes_B B,W\otimes_B M,\mu''',0)\\
&\oplus(M\otimes_B N\otimes_A V,C,0,(\mu(\rho_1\otimes\mathrm{Id}_V))^{-1}\mu''''')\\
&=(N\otimes_A M\otimes_B B,W\otimes_B M,\mu'''(\mu''\otimes\mathrm{Id}_B)(\mathrm{Id}_W\otimes\rho_1\otimes\mathrm{Id}_B),0)\\
&\oplus(N\otimes_A M\otimes_B N\otimes_A V,C,0,(\mathrm{Id}_N\otimes(\mu(\rho_1\otimes\mathrm{Id}_V))^{-1})\mu'''')\\
&\cong\Gamma\oplus Q,
\end{aligned}$$

it follows from Lemma 2.3 and Remark 2.4 that Λ and Γ are singularly equivalent of Morita type. □

Corollary 2.6. *Let A and B be Gorenstein k-algebras which are singularly equivalent of Morita type induced by bimodules M and N. Let C be a k-algebra and V an A-C-bimodule such that V is an indecomposable non-projective Gorenstein-projective left A-module such that $\mathrm{End}_{B\otimes_k C^{op}}(N\otimes_A V)=k\mathrm{Id}$. Let $\Lambda=\begin{pmatrix} A & {}_AV_C\\ 0 & C\end{pmatrix}$ and $\Gamma=\begin{pmatrix} B & N\otimes_A V\\ 0 & C\end{pmatrix}$. Then Λ and Γ are singularly equivalent of Morita type.*

Hochschild homology was introduced by Hochschild as a tool for studying the structure of associative algebras. Hochschild homology of a finite dimensional algebra has a very rich structure. It has a cup of product making it into a graded commutative algebra; it also has a Lie bracket of degree -1, making it into a graded

Lie algebra. Zhou-Zimmermann ([18]) proved that a singular equivalence of Morita type between finite dimensional k-algebras preserve the Hochschild homology group. We recall the notion of the Hochschild homology group.

Let A be an Artin algebra and $A^e = A\otimes_k A^{\mathrm{op}}$ be the enveloping algebra. Let M be an A-bimodule. Recall that the Hochschild homology group of A with coefficients in M is defined as $\mathrm{HH}_n(A,M) = \mathrm{Tor}_n^{A^e}(A,M)$.

Corollary 2.7. *Let A and B be Gorenstein k-algebras which are singularly equivalent of Morita type. Suppose that C, W, V, Λ and Γ be as above in Theorem 2.5. Then Λ and Γ have isomorphic Hochschild homology groups for each $n>0$,*

$$\mathrm{HH}_n(\Lambda) \cong \mathrm{HH}_n(\Gamma)$$

Proof. By Theorem 2.5, we know that Λ and Γ are singularly equivalent of the Morita type. Thus by [18, Theorem 4.1] there are isomorphisms of the Hochschild homology groups for each $n>0$,

$$\mathrm{HH}_n(\Lambda) \cong \mathrm{HH}_n(\Gamma) \quad \square$$

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