A note on the Coifman-Fefferman and Fefferman-Stein inequalities

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Abstract. A condition on a Banach function space X is given under which the Coifman-Fefferman and Fefferman-Stein inequalities on X are equivalent.

1. Introduction

R. Coifman and C. Fefferman [2] and [3] proved that for $w \in A_\infty$,

$$(1.1) \qquad \|Tf\|_{L^p(w)} \le C\|Mf\|_{L^p(w)},$$

where T is a singular integral operator and M is the Hardy-Littlewood operator. Approximately in the same period, C. Fefferman and E. Stein [6] established that

$$(1.2) \qquad \|Mf\|_{L^p(w)} \le C\|f^{\#}\|_{L^p(w)},$$

where $f^{\#}$ is the sharp maximal operator. This result was established in [6] in the unweighted setting but the method of the proof is easily extended to $w \in A_\infty$.

Inequalities (1.1) and (1.2) play an important role in harmonic analysis. However, the problem of characterizing the weights w for which these inequalities hold is still open for both (1.1) and (1.2). It is known that the so-called C_p condition is necessary for (1.1) and (1.2), and $C_{p+\varepsilon}, \varepsilon > 0$, is sufficient, see [16] and [19].

The weak type versions of (1.1) and (1.2) (with the $L^p(w)$-norm replaced by the $L^{p,\infty}(w)$-norm on the left-hand side) have been recently characterized in [13]

by a uniform condition (denoted by SC_p). In particular, this means that the weak type versions of (1.1) and (1.2) are equivalent.

Therefore, it is natural to conjecture that inequalities (1.1) and (1.2) are equivalent as well (of course when (1.1) is considered on some natural subclass of non-degenerate singular integral operators). We investigate this question in a general context of Banach function spaces (BFS) X over $\mathbb{R}^n$ equipped with Lebesque measure.

Our main result says that the Coifman-Fefferman and Fefferman-Stein inequalities on X are equivalent if

$$(1.3) \qquad \|MMf\|_X \le C\|Mf\|_X.$$

This condition is probably superfluous if $X = L^p(w)$ (in the sense that the equivalence between (1.1) and (1.2) can be probably established without appealing to (1.3)). However, we do not know if it can be removed (or at least weakened), in general.

In what follows, it will be convenient to introduce the space MX equipped with norm

$$\|f\|_{MX} = \|Mf\|_X.$$

Then (1.3) means that the maximal operator M is bounded on MX.

Denote by $\mathfrak{M}$ a family of BFS X such that MX is also BFS. It is easy to show (see Lemma 2.3 below) that $\mathfrak{M}$ consists of those X for which $\varphi(x) = \frac{1}{1+|x|^n} \in X$.

Define the standard Riesz and the maximal Riesz transforms by

$$R_j f(x) = \lim_{\varepsilon \to 0} R_{j,\varepsilon} f(x) \quad \text{and} \quad R_j^{\star} f(x) = \sup_{\varepsilon > 0} |R_{j,\varepsilon} f(x)|,$$

respectively, where

$$R_{j,\varepsilon} f(x) = c_n \int_{|y| > \varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}} dy \quad (j = 1, ..., n).$$

Our main result is the following.

Theorem 1.1. *Let X be a BFS. Assume that $X \in \mathfrak{M}$ and that M is bounded on MX. Then the following statements are equivalent.*

(i) *The Fefferman-Stein inequality*

$$\|Mf\|_X \le C\|f^{\#}\|_X$$

holds.

(ii) *The Coifman-Fefferman inequality*

$$\|R_j^\star f\|_X \le C\|Mf\|_X$$

holds for each of the maximal Riesz transforms $R_j^\star, j = 1, ..., n$.

The implication (i)$\Rightarrow$(ii) here is easy; in fact, (i) along with the boundedness of M on MX implies that for every Calderón-Zygmund operator T,

$$\|T^\star f\|_X \le C\|Mf\|_X.$$

A non-trivial part of Theorem 1.1 is the implication (ii)$\Rightarrow$(i). Here we essentially use a recent interesting result of D. Rutsky [15] saying that for a BFS X, the boundedness of every Riesz transform R_j on X is equivalent to the boundedness of M on X and X', where X' is the space associate to X.

A sketch of the proof of (ii)$\Rightarrow$(i) is as follows. First we show that the Fefferman-Stein inequality on X is equivalent to the boundedness of M on $(MX)'$, which is a slight refinement of an earlier characterization established in [10]. Next, we obtain a pointwise estimate for the composition of M with the maximal Calderón-Zygmund operator, which along with (ii) and the boundedness of M on MX implies that $R_j^\star$, and so $R_j, j = 1, ...n$, are bounded on MX. It remains to apply the above-mentioned result of D. Rutsky [15] to MX instead of X.

Note that we will not need to use this result in full strength since the boundedness of M on MX is assumed. Therefore, in order to keep the paper essentially self-contained, we give a simple proof of the fact that if M and the Riesz transforms R_j are bounded on X, then M is bounded on X'.

Our method of the proof of (ii)$\Rightarrow$(i) does not allow to replace the maximal Riesz transforms $R_j^\star$ by the usual Riesz transforms R_j in the statement of (ii). Also, it is an interesting question how to extend the implication (ii)$\Rightarrow$(i) to a more general class of singular integrals.

The paper is organized as follows. In Section 2, we recall some notions related to Banach function spaces and consider the space MX. In Section 3, we obtain some pointwise estimates for Calderón-Zygmund operators. Section 4 contains a characterization of the Fefferman-Stein inequality. Finally, in Section 5, we prove Theorem 1.1.

2. Banach function spaces and the space MX

Denote by $\mathcal{M}^+$ the set of Lebesgue measurable non-negative functions on $\mathbb{R}^n$.

Definition 2.1. By a Banach function space (BFS) X over $\mathbb{R}^n$ equipped with Lebesque measure we mean a collection of functions f such that

$$\|f\|_X = \rho(|f|) < \infty,$$

where $\rho: \mathcal{M}^+ \to [0, \infty]$ is a mapping satisfying

(i) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e.; $\rho(\alpha f) = \alpha \rho(f), \alpha \geq 0$;
$\rho(f+g) \leq \rho(f) + \rho(g)$;

(ii) $g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$;

(iii) $f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(iv) if $E \subset \mathbb{R}^n$ is bounded, then $\rho(\chi_E) < \infty$;

(v) if $E \subset \mathbb{R}^n$ is bounded, then $\int_E f \, dx \leq c_E \rho(f)$.

A more common requirement is that E is a set of finite measure in (iv) and (v) (see, e.g., [1]). However, it is well known that all main elements of a general theory work with (iv) and (v) stated for bounded sets (see, e.g., [14]). In particular, if X is a BFS, then the associate space X' consisting of f such that

$$\|f\|_{X'} = \sup_{g \in X : \|g\|_X \leq 1} \int_{\mathbb{R}^n} |fg| \, dx < \infty$$

is also a BFS.

Recall that the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing x.

Definition 2.2. Assume that X is a BFS. Denote by MX the space of functions f such that

$$\|f\|_{MX} = \|Mf\|_X < \infty.$$

Lemma 2.3. *Assume that X is a BFS. Then MX is a BFS if and only if*

$$(2.1) \qquad \frac{1}{1+|x|^n} \in X.$$

Proof. We will use the standard fact that for every bounded set $E \subset \mathbb{R}^n$ of positive measure, there exist $C_1, C_2 > 0$ such that for all $x \in \mathbb{R}^n$,

$$(2.2) \qquad \frac{C_1}{1+|x|^n} \leq M\chi_E(x) \leq \frac{C_2}{1+|x|^n}.$$

Assume that MX is a BFS. Then by property (iv) of Definition 2.1, $\|M\chi_E\|_X < \infty$. Thus, (2.1) follows from the left-hand side of (2.2).

Assume now that (2.1) holds. Suppose that $\|f\|_X = \rho(|f|)$, where ρ satisfies properties (i)–(v) of Definition 2.1. Then MX is a BFS if

$$\rho'(f) = \rho(Mf)$$

satisfies these properties as well. Properties (i), (ii) and (v) are obvious. Next, (iv) follows from the right-hand side of (2.2).

Finally, (iii) follows from the fact that if $f_n \uparrow f$ a.e., then $Mf_n \uparrow Mf$ everywhere. Indeed, clearly Mf_n is increasing and $Mf_n \le Mf$. Assume, for example, that $Mf(x) < \infty$. Let $\varepsilon > 0$. Take a cube Q containing x such that $Mf(x) < f_Q + \varepsilon$, where $f_Q = \frac{1}{|Q|} \int_Q f$. By Fatou's lemma, there exists N such that for all $n \ge N$,

$$f_Q \le (f_n)_Q + \varepsilon \le Mf_n(x) + \varepsilon,$$

and hence $Mf(x) < Mf_n(x) + 2\varepsilon$, which proves that $Mf_n(x) \to Mf(x)$ as $n \to \infty$. The case when $Mf(x) = \infty$ is similar. $\square$

3. Calderón-Zygmund operators

Although Theorem 1.1 deals only with the Riesz transforms, some estimates we will use hold for general Calderón-Zygmund operators.

Definition 3.1. We say that T is a Calderón-Zygmund operator with Dini-continuous kernel if T is linear, L^2 bounded, represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad \text{for all } x \notin \operatorname{supp} f$$

with kernel K satisfying the size condition $|K(x,y)| \le \frac{C_K}{|x-y|^n}, x \ne y$, and the smoothness condition

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le \omega\left(\frac{|x-x'|}{|x-y|}\right)\frac{1}{|x-y|^n}$$

for $|x-y| > 2|x-x'|$, where ω is an increasing and subadditive on $[0,1]$ function such that $\int_0^1 \omega(t)\frac{dt}{t} < \infty$.

We associate with T the grand maximal truncated operator $\mathcal{M}_T$ and the usual maximal truncated operator $T^\star$ defined by

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \|T(f\chi_{\mathbb{R}^n \setminus 3Q})\|_{L^\infty(Q)}$$

and

$$T^\star f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|,$$

respectively, where $T_\varepsilon f(x) = \int_{|y-x|>\varepsilon} K(x,y)f(y)dy$.

It was shown in [11] that for every Calderón-Zygmund operator T with Dini-continuous kernel,

$$(3.1) \qquad \mathcal{M}_T f(x) \le C_{n,T} M f(x) + T^\star f(x)$$

for all $x \in \mathbb{R}^n$. Exactly the same proof (replacing first T by T_ε) shows that T on the left-hand side of (3.1) can be replaced by $T^\star$, namely for all $x \in \mathbb{R}^n$,

$$(3.2) \qquad \mathcal{M}_{T^\star} f(x) \le C_{n,T} M f(x) + T^\star f(x).$$

Lemma 3.2. *For every Calderón-Zygmund operator T with Dini-continuous kernel and for all $x \in \mathbb{R}^n$,*

$$M(T^\star f)(x) \le C_{n,T} M M f(x) + T^\star f(x).$$

Proof. A variant of this estimate was obtained in [12], and therefore we outline the proof briefly. For every cube Q containing the point x,

$$(3.3) \qquad \frac{1}{|Q|} \int_Q T^\star f \le \frac{1}{|Q|} \int_Q T^\star(f\chi_{3Q}) + \|T^\star(f\chi_{\mathbb{R}^n \setminus 3Q})\|_{L^\infty(Q)}.$$

Since $T^\star$ is of weak type $(1,1)$ and L^2 bounded, interpolation along with Yano's extrapolation shows that the first part on the right-hand side of (3.3) is controlled by $\|f\|_{L\log L(3Q)}$, which in turn is controlled by $MMf(x)$. For the second part of (3.3) we use (3.2). $\square$

Remark 3.3. Lemma 3.2 implies the well-known fact [9] that for every $f \in L^p \cap L^\infty$,

$$\|T^\star f\|_{BLO} \le C_{n,T} \|f\|_{L^\infty},$$

where $\|f\|_{BLO} = \|Mf - f\|_{L^\infty}$. For usual (non-maximal) Calderón-Zygmund operators T only a weaker property that $\|Tf\|_{BMO} \le C\|f\|_{L^\infty}$ holds, and by this reason $T^\star$ cannot be replaced by T in the statement of Lemma 3.2.

Define the sharp maximal function $f^\#$ and the local sharp maximal function $M_\lambda^\# f$ respectively by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q| dy$$

and

$$M_\lambda^\# f(x) = \sup_{Q \ni x} \inf_c \left((f-c)\chi_Q \right)^* \left(\lambda|Q| \right) \quad (0 < \lambda < 1),$$

where f^* denotes the non-increasing rearrangement of f.

It was shown in [8] that for all $x \in \mathbb{R}^n$,

$$C_1 M M_\lambda^\# f(x) \leq f^\#(x) \leq C_2 M M_\lambda^\# f(x) \quad (0 < \lambda \leq 1/2)$$

and for every Calderón-Zygmund operator T with Dini-continuous kernel,

$$M_\lambda^\#(T^\star f)(x) \leq C M f(x)$$

(actually the latter estimate was proved in [8] for T instead of $T^\star$ but the proof for $T^\star$ is essentially the same). A combination of these two estimates yields

$$(4.4) \qquad (T^\star f)^\#(x) \leq C M M f(x).$$

4. A characterization of the Fefferman-Stein inequality

Consider the Fefferman-Stein inequality

$$(4.1) \qquad \|Mf\|_X \leq C \|f^\#\|_X.$$

It was shown in [10] that (4.1) is actually equivalent to the same estimate but with Mf replaced by f on the left-hand side, namely, (4.1) holds if and only if

$$(4.2) \qquad \|f\|_X \leq C \|f^\#\|_X.$$

Also, it was shown in [10] that (4.1) holds if and only if

$$\int_{\mathbb{R}^n} (Mf) g \, dx \leq C \|f\|_{X'} \|Mg\|_X.$$

This estimate can be rewritten in the form

$$(4.3) \qquad \|Mf\|_{(MX)'} \leq C \|f\|_{X'}.$$

Here we show that essentially the same proof as in [10] yields a bit more precise version of (4.3) with $\|f\|_{X'}$ on the right-hand side replaced by a smaller expression $\|f\|_{(MX)'}$.

Theorem 4.1. *Let X be a Banach function space such that $X \in \mathfrak{M}$. The Fefferman-Stein inequality (4.1) holds if and only if the maximal operator M is bounded on $(MX)'$.*

Remark 4.2. Theorem 4.1 says that exactly as (4.2) can be self-improved to (4.1), the estimate (4.3) can be self-improved to

$$\|Mf\|_{(MX)'} \leq C \|f\|_{(MX)'}.$$

Proof of Theorem 4.1. Since $\|f\|_{(MX)'} \leq \|f\|_{X'}$, the boundedness of M on $(MX)'$ implies (4.3), which in turns implies (4.1) by a characterization obtained in [10].

To show the converse direction, we use a result by A. de la Torre [17] saying that for every locally integrable f, there is a linear operator $\mathcal{M}_f$ such that Mf is pointwise equivalent to $\mathcal{M}_f f$ and for every locally integrable g,

$$(\mathcal{M}_f^* g)^{\#}(x) \leq CMg(x),$$

where $\mathcal{M}_f^*$ is the adjoint of $\mathcal{M}_f$. From this, for every $f, g \geq 0$ we obtain

$$\int_{\mathbb{R}^n} (Mf)g \leq C \int_{\mathbb{R}^n} (\mathcal{M}_f f)g = C \int_{\mathbb{R}^n} f\mathcal{M}_f^{\star} g$$
$$\leq C\|f\|_{(MX)'}\|M\mathcal{M}_f^{\star}g\|_X \leq C\|f\|_{(MX)'}\|(\mathcal{M}_f^* g)^{\#}\|_X$$
$$\leq C\|f\|_{(MX)'}\|Mg\|_X = C\|f\|_{(MX)'}\|g\|_{MX},$$

which, by duality, implies that M is bounded on $(MX)'$. $\square$

5. Proof of Theorem 1.1

As we have mentioned in the Introduction, Theorem 1.1 can be deduced by combining the ingredients from Sections 3 and 4 with the result of D. Rutsky [15]. We give a simplified proof of a weaker version of this result, which is enough for our purposes. We start with the following lemma.

Lemma 5.1. *Assume that for all $x \in \mathbb{R}^n$,*

(5.1) $$|R_j \varphi(x)| \leq K\varphi(x) \quad (j = 1, ..., n).$$

Then there exists $0 < q < 1$ depending only on n such that

$$M\varphi(x) \leq c_n K M(\varphi^q)(x)^{1/q}$$

for all $x \in \mathbb{R}^n$.

Remark 5.2. A conclusion of this lemma can be refined. In fact, it was shown by I. Vasilyev [18] that (5.1) implies $\log \varphi \in BMO$. Moreover, D. Rutsky [15] observed that (5.1) implies $\varphi \in A_{\infty}$.

We also note that M. Cotlar and C. Sadosky [4] showed that (5.1) for the Hilbert transform on the unit circle implies $\varphi \in A_2$. We give a simple proof of this fact for the Hilbert transform H on the real line.

Assume that $|H\varphi|\leq K\varphi$. Applying the "magic identity" $(Hf)^2=f^2+2H(fHf)$, we obtain

(5.2) $$\int_{\mathbb{R}}(Hf)^2\varphi=\int_{\mathbb{R}}f^2\varphi+2\int_{\mathbb{R}}H(fHf)\varphi.$$

Further,

$$\Big|\int_{\mathbb{R}}H(fHf)\varphi\Big|=\Big|\int_{\mathbb{R}}f(Hf)H(\varphi)\Big|\leq K\int_{\mathbb{R}}|f||Hf|\varphi$$
$$\leq K\|f\|_{L^2(\varphi)}\|Hf\|_{L^2(\varphi)}.$$

Combining this with (5.2) yields

$$\|Hf\|_{L^2(\varphi)}\leq C_K\|f\|_{L^2(\varphi)},$$

which implies $\varphi\in A_2$.

Proof of Lemma 5.1. Let P_t be the Poisson kernel. Denote

$$u_j(x,t)=(R_j(\varphi)*P_t)(x), j=1,...,n,\quad u_{n+1}(x,t)=(\varphi*P_t)(x)$$

and $F=(u_1,...,u_{n+1})$. It is well known [7, p. 143] that $|F|^q$ is subharmonic when $q\geq\frac{n-1}{n}$. In particular, this implies (see [7, p. 145]) that for all $x\in\mathbb{R}^n$ and $t,\varepsilon>0$,

$$|F(x,t+\varepsilon)|^q\leq(|F(\cdot,\varepsilon)|^q*P_t)(x).$$

Passing to the limit when $\varepsilon\to0$ and applying (5.1), we obtain

$$|\varphi*P_t(x)|^q\leq|F(x,t)|^q\leq K^q(n+1)^{q/2}(\varphi^q)*P_t(x).$$

Taking here the supremum over $t>0$ completes the proof. $\square$

The following lemma is the above mentioned weaker version of the result in [15].

Lemma 5.3. *Let X be a BFS. Assume that the maximal operator M and that every Riesz transform $R_j, j=1,...,n$ are bounded on X. Then M is bounded on X'.*

Proof. By duality, the Riesz transforms R_j are bounded on X'. Denote

$$Rf(x)=\sum_{j=1}^{n}|R_jf(x)|,$$

and consider the following Rubio de Francia type operator

$$S_Rf(x)=\sum_{k=0}^{\infty}\frac{1}{(2\gamma)^k}\varphi_k(x),$$

where $\gamma = \|R\|_{X' \to X'}, \varphi_0 = |f|$ and $\varphi_k = R(\varphi_{k-1}), k \in \mathbb{N}$. Then

$$(5.3) \qquad\qquad R(S_R f)(x) \le 2\gamma S_R f(x)$$

and also

$$(5.4) \qquad\qquad |f| \le S_R f \quad \text{and} \quad \|S_R f\|_{X'} \le 2\|f\|_{X'}.$$

Combining (5.3) with Lemma 5.1 yields

$$M(S_R f)(x) \le c_n \gamma M((S_R f)^q)^{1/q}(x),$$

where $q = q(n) < 1$. Therefore, by the Fefferman-Stein inequality [5] along with (5.4),

$$\begin{aligned}
\int_{\mathbb{R}^n} (Mf)g \, dx &\le \int_{\mathbb{R}^n} (M(S_R f))g \, dx \le c_n \gamma \int_{\mathbb{R}^n} M((S_R f)^q)^{1/q} g \, dx \\
&\le c_n' \gamma \int_{\mathbb{R}^n} (S_R f)(Mg) dx \le c_n' \gamma \|S_R f\|_{X'} \|Mg\|_X \\
&\le C\|f\|_{X'} \|g\|_X,
\end{aligned}$$

which implies that M is bounded on X'. $\square$

   Proof of Theorem 1.1. The implication (i)$\Rightarrow$(ii) follows from (3.4) and from the assumption that M is bounded on MX.

   Turn to the implication (ii)$\Rightarrow$(i). Applying Lemma 3.2 and using the boundedness of M on MX, we obtain

$$\|M(R_j^\star f)\|_X \le C\|MMf\|_X + \|R_j^\star f\|_X \le C\|Mf\|_X \quad (j = 1, ..., n).$$

This means that every $R_j^\star$ is bounded on MX, and therefore R_j is bounded on MX as well. By Lemma 5.3 we obtain that M is bounded on $(MX)'$, which, by Theorem 4.1, is equivalent to that the Fefferman-Stein inequality holds on X. $\square$

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