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Optimal unions of scaled copies of domains and Pólya's conjecture

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Abstract. Given a bounded Euclidean domain Ω, we consider the sequence of optimisers of the k^{th} Laplacian eigenvalue within the family consisting of all possible disjoint unions of scaled copies of Ω with fixed total volume. We show that this sequence encodes information yielding conditions for Ω to satisfy Pólya's conjecture with either Dirichlet or Neumann boundary conditions. This is an extension of a result by Colbois and El Soufi which applies only to the case where the family of domains consists of *all* bounded domains. Furthermore, we fully classify the different possible behaviours for such sequences, depending on whether Pólya's conjecture holds for a given specific domain or not. This approach allows us to recover a stronger version of Pólya's original results for tiling domains satisfying some dynamical billiard conditions, and a strenghtening of Urakawa's bound in terms of packing density.

1. Introduction and main results

1.1. Pólya's conjecture for Laplace eigenvalues

For $d\geq 2$ let $\Omega\subset\mathbb{R}^d$ be a bounded open set with Lebesgue measure $|\Omega|$. We consider the Dirichlet eigenvalue problem

$$\begin{cases}\Delta u+\lambda u=0 & \text{in } \Omega\\ u\equiv 0 & \text{on } \partial\Omega.\end{cases}$$

It is well known that the eigenvalues of the above problem are discrete and form a sequence

$$0<\lambda_1(\Omega)\leq\lambda_2(\Omega)\leq\lambda_3(\Omega)\ldots\nearrow\infty$$

accumulating only at infinity. Moreover, if the boundary $\partial\Omega$ is Lipschitz, the Neumann problem

$$\begin{cases}\Delta u+\mu u=0 & \text{in } \Omega\\ \partial_\nu u\equiv 0 & \text{on } \partial\Omega,\end{cases}$$

where ν denotes the outer unit normal vector of Ω, also has discrete spectrum and forms a nondecreasing sequence

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega) \leq \ldots \nearrow \infty.$$

Note that we choose the convention to start numbering Neumann eigenvalues with 0 instead of with 1, which allows for a cleaner statement of our theorems. Both the Dirichlet and Neumann eigenvalues satisfy so-called *Weyl asymptotics*

$$\lambda_k = \mu_k + \mathrm{O}\left(k^{1/d}\right) = \frac{4\pi^2}{(\omega_d\,|\Omega|)^{2/d}} k^{2/d} + \mathrm{O}\left(k^{1/d}\right),$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. If Ω has smooth boundary and satisfies some dynamical conditions, namely that the measure of periodic trajectories in the billiard flow is zero, the eigenvalues also satisfy two-term Weyl asymptotics [21] and [35]

$$\lambda_k = \frac{4\pi^2}{(\omega_d|\Omega|)^{2/d}} k^{2/d} + \frac{2\pi^2}{d} \frac{\omega_{d-1}\,|\partial\Omega|}{(\omega_d\,|\Omega|)^{\frac{d+1}{d}}} k^{1/d} + \mathrm{o}\left(k^{1/d}\right) \tag{1}$$

and

$$\mu_k = \frac{4\pi^2}{(\omega_d|\Omega|)^{2/d}} k^{2/d} - \frac{2\pi^2}{d} \frac{\omega_{d-1}\,|\partial\Omega|}{(\omega_d\,|\Omega|)^{\frac{d+1}{d}}} k^{1/d} + \mathrm{o}\left(k^{1/d}\right). \tag{2}$$

The regularity assumption on the boundary can be weakened, see [22] for a precise description of the required conditions. From these asymptotic formulae it is clear that given a domain Ω for which (1) and (2) hold there exists $k^* = k^*(\Omega)$ such that for all $k \geq k^*$,

$$\mu_k(\Omega) < \frac{4\pi^2}{(\omega_d|\Omega|)^{2/d}} k^{2/d} < \lambda_k(\Omega). \tag{3}$$

Furthermore, the Rayleigh–Faber–Krahn [16] and [24] and the Hong–Krahn–Szegő [25] inequalities imply that the right-hand side inequality holds for λ_1 and λ_2, while the Szegő–Weinberger [37] and the Bucur–Henrot [12] inequalities ensure the inequality on the left-hand side for μ_1 and μ_2. In this paper, we investigate a conjecture of Pólya.

Open problem. (Pólya's conjecture) *For all $\Omega \subset \mathbb{R}^d$ and all $k \in \mathbb{N}$,*

$$\mu_k(\Omega) \leq \frac{4\pi^2}{(\omega_d|\Omega|)^{2/d}} k^{2/d} \leq \lambda_k(\Omega). \tag{4}$$

In 1961 Pólya proved that the above inequalities do hold for all domains which tile the plane, and conjectured that this would be true for general domains [34] – see [23] for the proof for general tiling domains with Neumann boundary conditions. Pólya's result was later extended to tiling domains in higher dimensions by Urakawa, who also obtained lower bounds for all Dirichlet eigenvalues of a domain based on its lattice packing density [36].

For general domains, the best results so far remain those by Berezin [5] and Li and Yau [30] in the Dirichlet case, while for Neumann eigenvalues the corresponding result was established by Kröger [26]. In either case, these are based on sharp bounds for the average of the first k eigenvalues of the Laplacian, namely,

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j(\Omega) \leq \frac{4\pi^2 d}{d+2}\left(\frac{k}{\omega_d|\Omega|}\right)^{2/d} \leq \frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Omega).$$

From these inequalities and an estimate in [26] it follows that, for individual eigenvalues,

$$\lambda_k(\Omega) \geq \frac{d}{d+2}\frac{4\pi^2}{(\omega_d|\Omega|)^{2/d}}k^{2/d},$$

and

$$\mu_k(\Omega) \leq \left(\frac{d+2}{2}\right)^{2/d}\frac{4\pi^2}{(\omega_d\,|\Omega|)^{2/d}}k^{2/d},$$

which both fall short of (4).

Note that inequalities (3) lead naturally to a strenghtening of Pólya's conjecture, which we also investigate.

Open problem. (Strong Pólya's conjecture) *For all* $\Omega\subset\mathbb{R}^d$ *and all* $k\in\mathbb{N}$,

$$\mu_k(\Omega) < \frac{4\pi^2}{(\omega_d|\Omega|)^{2/d}}k^{2/d} < \lambda_k(\Omega).$$

As mentioned above, the first two eigenvalues are known to satisfy the strong Pólya's inequalities since their extremal values are known. However, for higher eigenvalues and although some conjectures do exist, there are no other situations where the extremal values are known. Furthermore, numerical optimisations carried out within the last fifteen years by different researchers using different methods have made it clear that not much structure at this level is to be expected in the mid-frequency range, in the sense that extremal sets are not described in terms of known functions – see [1] and [32] for the Dirichlet and [1] for the Neumann problems respectively; see also [3] and [10] for the same problem but with a perimeter restriction. In the planar case, it has also been shown that, except for the first four eigenvalues, the Dirichlet extremal domains are never balls or unions of balls [9].

Recently, it has been shown that the Faber–Krahn inequality may be used to extend the range of low Dirichlet eigenvalues for which Pólya's conjecture holds [18]. For instance, in dimensions three and larger, eigenvalues up to λ_4 also satisfy Pólya's conjecture, with the number of eigenvalues which may be shown to do so by this method growing exponentially with the dimension.

These findings prompted the study of what happens at the other end of the spectrum, in the high-frequency regime, in the hope that some structure could be recovered there. The first of such results proved that, when restricted to the particular case of rectangles, extremal domains converge to the square as k goes to infinity [2]. In other words, they converge to the domain with minimal perimeter among all of those in the class of rectangles with fixed area, and indeed, just like with the first eigenvalue, the geometric isoperimetric inequality plays a role in the proof. This was followed by an extension of these results to higher-dimension rectangles in both the Dirichlet and Neumann cases [6], [7], [20] and [31]. In the case of general planar domains with a perimeter restriction, it was shown in [11] that extremal sets converge to the disk with the same perimeter as k goes to infinity, thus again displaying convergence to the geometric extremal set. Some results regarding existence of convergent subsequences within classes of convex domains and under a measure restriction were also obtained in [28].

The connection between the problem of determining extremal domains for the k^{th} eigenvalue and Pólya's conjecture was established in 2014 by Colbois and El Soufi [14]. There they showed that the sequences of extremal values $(\lambda_k^*)^{d/2}$ (Dirichlet) and $(\mu_k^*)^{d/2}$ (Neumann) are subadditive and superadditive, respectively. As a consequence of Fekete's lemma, both sequences $\lambda_k^*/k^{2/d}$ and $\mu_k^*/k^{2/d}$ are convergent as k goes to infinity and, furthermore, Pólya's conjecture is seen to be equivalent to

$$\lim_{k\to\infty} \frac{\lambda_k^*}{k^{2/d}} = \frac{4\pi^2}{\left(|\Omega|\omega_d\right)^{2/d}} \quad \text{and} \quad \lim_{k\to\infty} \frac{\mu_k^*}{k^{2/d}} = \frac{4\pi^2}{\left(|\Omega|\omega_d\right)^{2/d}},$$

in the Dirichlet and Neumann cases, respectively.

A major obstacle in attacking the general Pólya's conjecture is that it is not even known if there exists an open domain minimising λ_k or maximising μ_k for $k\geq 3$ under volume constraint. This prevents one from using properties of the minimisers to argue in favor of the conjecture. Our aim will be to restrict ourselves to the study of classes of domain within which we are able to show existence of extremisers, but within which the subadditivity and superadditivity results of Colbois and El Soufi still hold. Note that subadditivity or superadditivity for the optimal eigenvalues do not hold for all families of domains – if we take as a family of domains rectangles of unit area, the extremisers always exist but the optimal Dirichlet eigenvalues are $\lambda_1^*=2\pi^2$, $\lambda_3^*=5\pi^2$ and $\lambda_4^*=35\pi^2/(2\sqrt{6})\approx 7.144\pi^2$, see [2].

1.2. Suitable families of domains

Before stating our results, let us define precisely the class of domains under consideration in this paper. Given $r\in(0,\infty)$ and $\Omega\subset\mathbb{R}^d$, we denote by $r\Omega$ any subset of $\mathbb{R}^d$ obtained from Ω as a result of a homothety with scale factor r and an isometry.

Definition 1.1. Let $\Omega_1,...,\Omega_n$ be bounded, connected, open subsets of $\mathbb{R}^d$. We denote

$$\mathcal{R}:=\mathcal{R}(\Omega_1,...,\Omega_n):=\left\{\bigsqcup_{i=1}^{N} r_i\Omega_{n_i}\ :\ N\in\mathbb{N}, n_i\in\{1,...,n\}, r_i>0\right\}.$$

The sets $\Omega_1,...,\Omega_n$ are called the generators for $\mathcal{R}$. The above notation is to be understood in the sense that all sets $\Upsilon\in\mathcal{R}$ are subsets of $\mathbb{R}^d$ all of whose connected components are of the form $r_i\Omega_{n_i}$ for $1\le i\le N$. We denote by $\nu(\Upsilon)$ the number of connected components of Υ, by $|\Upsilon|$ its volume and we slightly abuse notation by denoting by $|\partial\Upsilon|$ the $(d-1)$-dimensional Hausdorff measure of the boundary. We also observe that the family $\mathcal{R}$ is closed under disjoint union and homothety, up to rearrangement. Whenever the Neumann eigenvalue problem is discussed, it is also assumed the generators have Lipschitz boundary.

One particular instance of this type of families, namely, those generated by rectangles, was used recently to study the possible asymptotic behaviour of extremal sets in the case of Robin boundary conditions [19]. We note that in the definition we could allow a countably infinite number of connected components. We are, however, interested in optimisers and it will be clear that sets with an infinite number connected components can never be one, see Lemmas 2.1 and 2.2.

The following elementary facts about scaling properties of volumes and eigenvalues will be used repeatedly in this paper:

- $|r\Upsilon|=r^d|\Upsilon|$;
- $|r\partial\Upsilon|=r^{d-1}|\partial\Upsilon|$;
- $\lambda_k(r\Upsilon)=r^{-2}\lambda_k(\Upsilon)$;
- $\mu_k(r\Upsilon)=r^{-2}\mu_k(\Upsilon)$;

It is easy to see from the first two points that the generator Ω_j minimising the isoperimetric ratio

$$I(\Upsilon):=\frac{|\partial\Upsilon|^d}{|\Upsilon|^{d-1}}$$

among $\Omega_1,...,\Omega_n$ also does so in $\mathcal{R}$. The first, third and four bullet points imply that the quantities $\lambda_k(\Upsilon)^{d/2}|\Upsilon|$ and $\mu_k(\Upsilon)^{d/2}|\Upsilon|$ are invariant by homothety.

Definition 1.2. We define the extremal eigenvalues

$$\lambda_k^*(\mathcal{R}) = \inf_{\substack{\Upsilon\in\mathcal{R}\\|\Upsilon|\le 1}} \lambda_k(\Upsilon)$$

and

$$\mu_k^*(\mathcal{R}) = \sup_{\substack{\Upsilon\in\mathcal{R}\\|\Upsilon|\ge 1}} \mu_k(\Upsilon).$$

We shall say that a domain $\Upsilon\in\mathcal{R}$ is a *minimiser for* $\lambda_k^*(\mathcal{R})$ or that it *realises* $\lambda_k^*(\mathcal{R})$ if $|\Upsilon|\le 1$ and if $\lambda_k(\Upsilon)=\lambda_k^*(\mathcal{R})$. Similarly, a domain can be a *maximiser* for $\mu_k^*(\mathcal{R})$ or it *realises* $\mu_k^*(\mathcal{R})$. Note that an extremiser necessarily verifies $|\Upsilon|=1$.

In Section 2, we show that these families $\mathcal{R}$ of domains are suitable for the study of asymptotic eigenvalue optimisation. By suitable, we understand that for every k, there exists $\Upsilon\in\mathcal{R}$ realising the extremal eigenvalues, and that the results of [14], [33] and [38] describing the extremal eigenvalues and their associated extremisers still hold within the families $\mathcal{R}$. Existence of the extremisers is proved in Lemmas 2.1 and 2.2.

The properties of extremal eigenvalues and their associated extremisers are the subject of Theorems 2.3–2.7. They rely on the fact that two properties are needed for the proofs of these theorems: closedness under homotheties, and under disjoint unions. Of specific use is Corollary 2.5, which says that it is sufficient to study the limit of the sequence of optimal eigenvalues if one wants to get universal bounds within a family $\mathcal{R}$.

1.3. A trichotomy for Pólya's conjecture

In Section 3, we restrict our search to families $\mathcal{R}$ generated by a single domain Ω. There is no loss of generality here: we will first show that if Pólya's conjecture holds within two families $\mathcal{R}(\Omega_1)$ and $\mathcal{R}(\Omega_2)$ in either its standard or strong form, then it also holds in $\mathcal{R}(\Omega_1,\Omega_2)$.

Our aim is to characterise the structure of the set of optimisers in $\mathcal{R}(\Omega)$ depending on whether Pólya's conjecture holds or fails in $\mathcal{R}$. This gives, in principle, a way to investigate the conjecture for a given domain, since $\Omega\in\mathcal{R}(\Omega)$. We note that Pólya's conjecture remains open for except in very restrictive classes of domains. Indeed, it is only known in the following situations:

- Domains that tile $\mathbb{R}^d$ [23] and [34];
- For Dirichlet boundary conditions, domains of the form $\Omega_1\times\Omega_2\subset\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, $d_1+d_2=d$ where $d_1\ge 2$ and Ω_1 itself satisfies Pólya's conjecture (for instance by tiling $\mathbb{R}^{d_1}$) [27].

• For Dirichlet boundary conditions, domains of the form $\Omega=\Omega_1\times\Omega_2\subset\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, $d_1\geq 3$, Ω_1 itself satisfies Pólya's conjecture and Ω_2 is convex, then Ω satisfies the strong Pólya conjecture [29].

Notably, even in the case of the ball in $\mathbb{R}^d$ where we have explicit formulae for the eigenvalues the status of the conjecture is unknown.

Our main theorem is as follows.

Theorem 1.3. *The following trichotomy holds: either*

(1) *the generator Ω realises $\lambda_k^*(\mathcal{R})$ infinitely often and Pólya's conjecture for Dirichlet eigenvalues holds for all $\Upsilon\in\mathcal{R}(\Omega)$;*

(2) *the generator Ω realises λ_k^* only finitely many times, Pólya's conjecture for Dirichlet eigenvalues holds for all $\Upsilon\in\mathcal{R}(\Omega)$ and, for infinitely many $k\in\mathbb{N}$,*

$$\frac{\lambda_k^*(\mathcal{R})^{d/2}}{k}=\frac{(2\pi)^d}{\omega_d},$$

or

(3) *the generator Ω realises λ_k^* only finitely many times, Pólya's conjecture for Dirichlet eigenvalues does not hold for Ω and, for infinitely many $k\in\mathbb{N}$,*

$$\frac{\lambda_k^*(\mathcal{R})^{d/2}}{k}=\inf_j\frac{\lambda_j^*(\mathcal{R})^{d/2}}{j}.$$

The same trichotomy holds replacing all instances of Dirichlet with Neumann, of λ with μ, and inf *with* sup.

In Theorem 3.4, we furthermore obtain an indication of when Ω can realise $\lambda_k^*(\mathcal{R})$ or $\mu_k^*(\mathcal{R})$ infinitely often. Namely, we show that as soon as there exists a subsequence $\{k_n\}$ such that the number of connected components of the domain realising $\lambda_{k_n}^*$, respectively $\mu_{k_n}^*$ has slower than linear growth, then Ω realises λ_k^*, respectively μ_k^* infinitely often. This, in combination with Lemmas 2.6 and 2.7 allows us to understand the propagation of extremal domains in $\mathcal{R}$ as $k\to\infty$.

Finally, when the generator Ω satisfies the two-term Weyl law (1) or (2), we obtain the following list of equivalences with the strong Pólya conjecture

Theorem 1.4. *Suppose that $\Omega\subset\mathbb{R}^d$ is such that the two-term Weyl law (1) holds. Let $\Omega_k^*=\bigsqcup_{i\leq N_k} r_{i,k}\Omega$ be a sequence of domains realising $\lambda_k^*(\mathcal{R})$. Suppose that $|\Omega_k^*|=1$ and $r_{i,k}\geq r_{j,k}$ whenever $i<j$. The following are equivalent:*

(1) *The strong Pólya conjecture for Dirichlet eigenvalues holds in $\mathcal{R}(\Omega)$.*

(2) *The largest coefficient $r_{1,k}\to 1$ as $k\to\infty$.*

(3) *The largest coefficient $r_{1,k}\to 1$ along a subsequence.*

The same equivalence hold replacing all instances of Dirichlet with Neumann, λ with μ, and the two-term Weyl law (1) with (2).

Comparing those equivalent statements to the trichotomy in Theorem 1.3, it is clear that if the strong Pólya conjecture holds, Ω realises $\lambda_k^*(\mathcal{R})$ infinitely often. On the other hand, if Ω does realise $\lambda_k^*(\mathcal{R})$ infinitely often, it is the case that $r_{1,k}\to 1$ along a subsequence. Theorem 1.4 indicates that for domains Ω that satisfy a two-term Weyl law the strong Pólya conjecture for $\mathcal{R}(\Omega)$ is equivalent to weaker statements than those needed to imply Pólya's conjecture in Theorem 1.3.

1.4. Density lower bounds for Dirichlet eigenvalues

In the paper [36], Urakawa obtained a lower bound for Dirichlet eigenvalues in terms of the lattice packing density of a domain Ω. As an application of our construction, we obtain in Section 4 similar results for the asymptotic packing density defined as follows.

Given a set Ω and $n\in N$, we define the *n-th propagation of* Ω as the set

$$\Omega^{(n)} = \bigsqcup_{\ell=1}^{n} \frac{1}{n^{1/d}}\,\Omega.$$

Definition 1.5. Given two bounded domains Ω and V with volume 1, an integer $n\in\mathbb{N}$ and a real number $\rho\in(0,1]$, a *packing of* $\Omega^{(n)}$ *into* V *of density* ρ is an isometric quasi-embedding $f:\overline{\Omega^{(n)}}\to\rho^{-1/d}V$. Here, we call a map a *quasi-embedding* if it is injective on the interior of its domain. Note furthermore that Ω, and hence any element in $\mathcal{R}$, is canonically equipped with a Riemannian metric. The term *isometry* is to be understood as "preserving Riemannian metrics".

An *asymptotic packing of* Ω *into* V is a triple $P=\{(n_i,\rho_i,f_i)\}_{i\in\mathbb{N}}$ where $\{n_i\}_{i\in\mathbb{N}}$ is a strictly increasing sequence of integers, $\{\rho_i\}_{i\in\mathbb{N}}\subset(0,1]$ converges to the *asymptotic density* $\rho_P\in(0,1]$ and each f_i is a packing of $\Omega^{(n_i)}$ into V of density ρ_i.

The *packing number or packing density of* Ω *into* V is

$$\rho_{\Omega,V} = \sup\{\rho_P \mid P \text{ is an asymptotic packing of } \Omega \text{ into } V\}.$$

The *packing number or packing density of* Ω is

$$\rho_\Omega = \sup\{\rho_{\Omega,V} \mid V \text{ is a bounded domain with volume } 1\}.$$

Definition 1.6. A domain $D\subset\mathbb{R}^d$ is a *tile* or is said *to tile* $\mathbb{R}^d$ if there is an isometric quasi-embedding $F:\sqcup_{i\in\mathbb{N}}\overline{D}\to\mathbb{R}^d$, called the *tiling*, which is surjective.

Remark 1.7. The lattice packing density of Urakawa [36] is always smaller or equal to this packing density, as it is equivalent to considering only V that are parallelepipeds, as well as having P constrained more strictly. It is not hard to find examples of concave, simply connected domains that have a higher asymptotic packing density than their lattice packing density.

We obtain the following theorem for a lower bound on Dirichlet eigenvalues in terms of this asymptotic density.

Theorem 1.8. *For every $\Omega\subset\mathbb{R}^d$ open and bounded, with $|\Omega|=1$, the lower bound*

$$\inf_k \frac{\lambda_k^*(\mathcal{R}(\Omega))^{d/2}}{k} \geq \rho_\Omega \frac{(2\pi)^d}{\omega_d}$$

holds.

Obviously, the previous Theorem allows us to recover Pólya's theorem as a corollary.

Corollary 1.9. (Pólya [34]) *If Ω tiles $\mathbb{R}^d$, then Pólya's conjecture holds for any domain in $\mathcal{R}(\Omega)$.*

Proof. If Ω tiles $\mathbb{R}^d$, then $\rho_\Omega=1$ (see Proposition 4.2). Then Theorem 1.8 implies the result. □

We also obtain the following strengthening of Pólya's theorem for domains that are said to simply tile $\mathbb{R}^d$ and for which the two-term Weyl law (1) holds, in which case the strong Pólya conjecture holds.

Definition 1.10. Let $V\subset\mathbb{R}^d$ be a domain of volume 1. A domain Ω is a V*-tile* or is said *tile V* if there is an asymptotic packing $P=\{(n_i,1,f_i)\}_{i\in\mathbb{N}}$ of Ω into V with constant packing density 1.

Theorem 1.11. *Let V be a domain in $\mathbb{R}^d$ of unit volume satisfying the two-term Weyl law* (1)*. If Ω tiles V, then Ω realises $\lambda_k^*(\mathcal{R}(\Omega))$ infinitely often and satisfies the strong Pólya conjecture. The same holds for Neumann eigenvalues, if V satisfies* (2) *instead.*

1.5. Computational results

In Section 5, we investigate numerically the set of extremisers for Dirichlet eigenvalues within families $\mathcal{R}$ generated by the disk, the square, and a rectangle with aspect ratio 5. We chose these domains to see if the markers for the Pólya conjecture differed between the rectangles, for which the conjecture is known to hold, and the disk, for which it's not. In all four cases, we look for extremisers up to eigenvalue rank 66 000.

We investigate the number of connected components of the extremising set, in view of Theorem 3.4. In all the cases we are studying, we see that this number is bounded by 5, up to rank 66 000. Recall that for Pólya's conjecture to hold, we only need for a subsequence of the extremisers to have a strictly sublinear growth for their number of connected components.

We also investigate the asymptotic log-density of the number of times the generator can be Ω_k^*. For a set $J \subset \mathbb{N}$, we define its counting function as

$$N_J(x) := \# \{ j \in J : j \le x \}$$

and its log-density as

$$F_J(x) := \frac{\log(N_J(x))}{\log x}. \tag{5}$$

We have that for every $\varepsilon > 0$,

$$\lim_{x \to \infty} F_J(x) = \alpha > 0 \quad \Longleftrightarrow \quad N_J(x) \ge x^{\alpha - \varepsilon}$$

for x large enough. In particular, for J the set of ranks k for which the generator realises λ_k^*, $\lim_{x \to \infty} F_J(x) = \alpha > 0$ implies that the cardinality of J is infinite. The log-density in all cases we investigated seemed to converge quite quickly to a constant greater than 0.8, albeit not the same constant for the disk and the various rectangles. It would be an interesting line of investigation to understand the geometric properties that influence the value of this constant.

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2. Eigenvalue optimisation within a family

Recall that for $\Omega_1, \ldots, \Omega_n$, each of volume 1, we investigate the family of domains

$$\mathcal{R}(\Omega_1, \ldots, \Omega_n) := \left\{ \bigsqcup_{i \in I} r_i \Omega_{n_i} \, : \, I \text{ countable}, n_i \in \{1, \ldots, n\}, \sum_{i \in I} r_i^d < \infty \right\}.$$

Our first two results concern the existence of eigenvalue extremisers in this restricted collection $\mathcal{R}$. We recall that for any domain Υ, we denote its number of connected components by $\nu(\Upsilon)$.

Lemma 2.1. *For all k, there exists a domain $\Omega_k^* \in \mathcal{R}$ of volume 1 such that*

$$\lambda_k(\Omega_k^*) = \lambda_k^*(\mathcal{R}).$$

For any minimising domain Υ for $\lambda_k^*(\mathcal{R})$, $\nu(\Upsilon) \le k$.

Proof. Fix $k\geq 1$. For any $j\in\mathbb{N}\cup\{\infty\}$, denote

$$\lambda_k^{(j)}=\inf\left\{\lambda_k(\Upsilon)\,:\,\Upsilon\in\mathcal{R},\,|\Upsilon|\leq 1,\,\nu(\Upsilon)=j\right\}.$$

Of course, $\lambda_k^*(\mathcal{R})=\inf_j\,\lambda_k^{(j)}$.

Our first step is to show that if $j>k$, then $\lambda_k^{(j)}\geq\lambda_k^{(l)}$ for some $l\leq k$; It follows in particular that the previous infimum is a minimum.

The argument for this first step will follow the proof of [8, Lemma 8]. Indeed, consider $\Upsilon=\bigsqcup_{i\in I}r_i\Omega_{n_i}\in\mathcal{R}$ with $|\Upsilon|=1$ and $\nu(\Upsilon)=j$. Suppose without loss of generality that

$$\lambda_1(r_j\Omega_{n_j})\leq\lambda_1(r_{j'}\Omega_{n_{j'}})\quad\text{whenever } j\leq j'$$

Let

$$l=\min\left\{k,\max\left\{m:\lambda_1(r_m\Omega_{n_m})\leq\lambda_k(\Upsilon)\right\}\right\}\leq k,$$

and

$$\widetilde{\Upsilon}=r_1\Omega_{n_1}\sqcup\ldots\sqcup r_l\Omega_{n_l}.$$

Note that if $\nu(\Upsilon)=\infty$, m is still finite since $r_j\to 0$ as $j\to\infty$, and observe that $\lambda_k(\widetilde{\Upsilon})\leq\lambda_k(\Upsilon)$. Since $|\widetilde{\Upsilon}|\leq 1$, we can dilate it to a set $\widehat{\Upsilon}$ of volume 1 whose eigenvalues are all smaller than the ones of $\widetilde{\Upsilon}$, so that $\lambda_k(\widehat{\Upsilon})\leq\lambda_k(\Upsilon)$. Taking the infimum of this inequality over all appropriate sets Υ and recalling that $\nu(\widehat{\Upsilon})=l\leq k<j=\nu(\Upsilon)$, we get indeed

$$\lambda_k^{(l)}\leq\lambda_k^{(j)}.$$

We therefore deduce that

$$\lambda_k^*=\min_{1\leq j\leq k}\lambda_k^{(j)}.$$

Our second step is to show that for every $1\leq j\leq k$, either there exists a minimiser $\Upsilon^{(j)}\in\mathcal{R}$ for $\lambda_k^{(j)}$ or $\lambda_k^{(j)}\geq\lambda_k^{(j-1)}$.

The statement is obvious for $\lambda_k^{(1)}$, as there is only a finite number of set, namely $\Omega_1,...,\Omega_n$ to verify. For $j>1$, consider a minimising sequence

$$\Upsilon_p^{(j)}=\bigsqcup_{i=1}^{j}r_{i,p}\Omega_{n_{i,p}}$$

of sets in $\mathcal{R}$ which can all be taken to have volume 1, *i.e.*

$$\lambda_k^{(j)}=\lim_{p\to\infty}\lambda_k(\Upsilon_p^{(j)}).$$

Assume without loss of generality $1>r_{1,p}\geq...\geq r_{j,p}$ for each p. If $r_{j,p}\to 0$ as $p\to\infty$, then for p large enough, $\lambda_1(r_{j,p}\Omega_{n_{j,p}})\geq\lambda_k^{(j)}$. This implies that $\lambda_k(\Upsilon_p^{(j)}\setminus r_{j,p}\Omega_{n_{j,p}})=\lambda_k(\Upsilon_p^{(j)})$ but $\nu(\Upsilon_p^{(j)}\setminus r_{j,p}\Omega)=j-1$, hence $\lambda_k^{(j)}\geq\lambda_k^{(j-1)}$. If $r_{j,p}\not\to 0$ as $p\to\infty$, then the set $\{r_{i,p}\}_{1\le i\le j, p\in\mathbb{N}}$ belongs to a compact interval $[\varepsilon, 1-\varepsilon]\subset(0,1)$. For every $1\le i\le j$ let $(r_i^{(j)}, n_i^{(j)})$ be an accumulation point of $\{(r_{i,p}, n_{i,p})\}_{p\in\mathbb{N}}$, then set

$$\Upsilon^{(j)} = \bigsqcup_{i=1}^{j} r_i^{(j)} \Omega_{n_i^{(j)}} \in \mathcal{R}.$$

By continuity of the k-th eigenvalue and of the volume as functions of the variables $r_1, ..., r_j$, the set $\Upsilon^{(j)}$ has volume 1 and verifies $\lambda_k(\Upsilon^{(j)}) = \lambda_k^{(j)}$.

We proved that there is a set of indices $J\subseteq\{1,...,k\}$ such that for all $j\in J$, there exists a minimiser $\Upsilon^{(j)}$ of $\lambda_k^{(j)}$, whereas $\lambda_k^{(i)} \ge \min_{j\in J} \lambda_k^j$ for all $i\notin J$. Therefore,

$$\lambda_k^* = \min_{1\le j\le k} \lambda_k^{(j)},$$

is realised by the set $\Omega_k^* := \Upsilon^{(j)}$ for any (say, the smallest) index j realising the previous minimum, thus completing the proof. $\square$

We now show the equivalent lemma for Neumann eigenvalues.

Lemma 2.2. *For all $k\ge 1$, there exists a domain $\Omega_k^*\in\mathcal{R}$ such that*

$$\mu_k(\Omega_k^*) = \mu_k^*(\mathcal{R}).$$

For any maximising domain Υ for $\mu_k^*(\mathcal{R})$, $\nu(\Upsilon)\le k$.

Proof. The first step of this proof is easier in the setting of Neumann eigenvalues. Indeed, no maximising sequence $\{\Upsilon_n\}$ for $\mu_k^*(\mathcal{R})$ can have $\nu(\Upsilon_n) > k$ infinitely often, since $\nu(\Upsilon) > k$ implies immediately $\mu_k(\Upsilon) = 0$.

For the second step, since the supremum for $\mu_k^*(\mathcal{R})$ is taken over domains of volume larger or equal to 1, we need to verify both that no connected component of a maximising sequence converges to 0 and that none grows unbounded. This last possibility is easily excluded by restricting our attention to maximising sequences of domains which all have volume 1.

Suppose that there is a maximising sequence with the volume of a connected component converging to 0. In other words, there is a maximising sequence

$$\Upsilon_p = \bigsqcup_{i=1}^{q} r_{i,p} \Omega_{n_{i,p}}$$

with the following properties.

- For all p, the number of connected components q is smaller than k.
- Arranging $r_{1,p} \le r_{2,p} \le \ldots \le r_{q,p}$, we have that $r_{1,p}\to 0$ as $p\to\infty$.
- The eigenvalues $\mu_k(\Upsilon_p)$ increase and converge to $\mu_k^*(\mathcal{R})$ as $p\to\infty$.

We will write $\Upsilon_p = r_{1,p}\Omega_{n_1,p} \cup \Xi_p$, each of them having volume $r_{1,p}^d$ and $1-r_{1,p}^d$ respectively.

From [26], we know that there is a constant C_k such that for all k and all domains Υ, $\mu_k(\Upsilon) < C_k$. There is an r_0 such that for all l, $r_0^{-2}\mu_1(\Omega_l) \geq C_k$. For p large enough so that $r_{1,p} < r_0$, we have $r_{1,p}^{-2}\mu_1(\Omega_{n_{1,p}}) > C_k$, hence $\mu_k(\Upsilon_p) = \mu_{k-1}(\Xi_p)$.

For any $\eta \in (0, r_0)$, consider the following sequence of domains of volume 1 in $\mathcal{R}$:

$$\widetilde{\Upsilon}_p^{(\eta)} = \eta\Omega_1 \sqcup \left(\frac{1-\eta^d}{1-r_{1,p}^d}\right)^{1/d} \Xi_p.$$

Without loss of generality, we have supposed $\eta < 1$.

For p large and since $\eta < r_0$,

$$\begin{aligned}
\mu_k(\widetilde{\Upsilon}_p) &= \mu_{k-1}\left(\left(\frac{1-\eta^d}{1-r_{1,p}^d}\right)^{1/d} \Xi_p\right) \\
&= \left(\frac{1-r_{1,p}^d}{1-\eta^d}\right)^{2/d} \mu_k(\Upsilon_p) \\
&= \frac{1}{(1-\eta^d)^{2/d}} \mu_k(\Upsilon_p)\left(1+\mathrm{O}\left(r_{1,p}\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_k(\widetilde{\Upsilon}_p) - \mu_k(\Upsilon_p) &= \left(\frac{1+\mathrm{O}\left(r_{1,p}\right)}{(1-\eta^d)^{2/d}} - 1\right)\mu_k(\Upsilon_p) \\
&\geq \frac{2\eta^d \mu_k(\Upsilon_1)}{d}\left(1+\mathrm{O}\left(r_{1,p}\right)\right)
\end{aligned}$$

Since $\mu_k^*(\mathcal{R}) > \mu_k(\widetilde{\Upsilon}_p)$ this implies that for p large enough, $\mu_k(\Upsilon_p) \leq \mu_k^*(\mathcal{R}) - d^{-1}\eta^d\mu_k(\Upsilon_1)$, contradicting the fact that it was a maximising sequence.

The same compactness argument as in the Dirichlet case then implies the existence of a maximisers. □

Note that both of these proofs show existence but say nothing about uniqueness. Despite this possible lack of uniqueness, in this paper we shall write Ω_k^* to denote any extremiser of λ_k or of μ_k on $\mathcal{R}$.

Lemma 2.3. *The sequence*

$$\left\{\lambda_k^*(\mathcal{R})^{d/2}\right\}_{k\in\mathbb{N}}$$

is subadditive, that is for every $j_1, \ldots, j_p$ such that $j_1 + \ldots j_p = k$, we have

$$\lambda_k^*(\mathcal{R})^{d/2} \leq \lambda_{j_1}^*(\mathcal{R})^{d/2} + \ldots + \lambda_{j_p}^*(\mathcal{R})^{d/2}.$$

Proof. The proof here follows that of [14, Theorem 2.1]. Fix $k\geq 1$ and let $j_1,\ldots,j_p\in\mathbb{N}$ be such that $j_1+\ldots+j_p=k$. By Lemma 2.1, for each $1\leq q\leq p$, there exists $\Omega^*_{j_q}\in\mathcal{R}$ with volume 1 such that

$$\lambda^*_{j_q}(\mathcal{R})=\lambda_{j_q}(\Omega^*_{j_q}).$$

Let

$$\Upsilon_q:=\left(\frac{\lambda^*_{j_q}(\mathcal{R})}{\lambda^*_k(\mathcal{R})}\right)^{1/2}\Omega^*_{j_q},$$

which implies that $\lambda_{j_q}(\Upsilon_q)=\lambda^*_k(\mathcal{R})$ and that

$$|\Upsilon_q|=\left(\frac{\lambda^*_{j_q}(\mathcal{R})}{\lambda^*_k(\mathcal{R})}\right)^{d/2}.$$

Define the domain

$$\Upsilon=\bigsqcup_{q=1}^{p}\Upsilon_q.$$

Since the spectrum of a disjoint union is the union of the spectra, we have

$$N(\lambda^*_k(\mathcal{R});\Upsilon)=\sum_{q=1}^{p}N(\lambda^*_k(\mathcal{R});\Upsilon_q)=\sum_{q=1}^{p}N(\lambda^*_{j_q}(\Upsilon_q);\Upsilon_q)\geq\sum_{q=1}^{p}j_q=k$$

where N is the eigenvalue counting function

$$N(\lambda;\Upsilon):=\#\left\{k:\lambda_k(\Upsilon)\leq\lambda\right\}. \tag{6}$$

It follows that $\lambda_k(\Upsilon)\leq\lambda^*_k(\mathcal{R})$. Since $|\Upsilon|^{-1/d}\Upsilon$ has volume 1 we have $\lambda^*_k(\mathcal{R})\leq\lambda_k(|\Upsilon|^{-1/d}\Upsilon)=\lambda_k(\Upsilon)|\Upsilon|^{2/d}$, thus

$$|\Upsilon|\geq\left(\frac{\lambda^*_k(\mathcal{R})}{\lambda_k(\Upsilon)}\right)^{d/2}\geq 1,$$

whence

$$1\leq\sum_{q=1}^{p}|\Upsilon_n|=\frac{1}{\lambda^*_k(\mathcal{R})^{d/2}}\sum_{q=1}^{p}\lambda^*_{j_q}(\mathcal{R})^{d/2}.$$

Multiplying both sides of this inequality by $\lambda^*_k(\mathcal{R})^{d/2}$ finishes the proof. □

Lemma 2.4. *The sequence*

$$\left\{\mu^*_k(\mathcal{R})^{d/2}\right\}_{k\in\mathbb{N}}$$

is super-additive, that is for every $j_1,\ldots,j_p$ such that $j_1+\ldots j_p=k$, we have

$$\mu^*_k(\mathcal{R})^{d/2}\geq\mu^*_{j_1}(\mathcal{R})^{d/2}+\ldots+\mu^*_{j_p}(\mathcal{R})^{d/2}.$$

Proof. Suppose on the contrary that there exist $j_1, ..., j_p, k \in \mathbb{N}$ such that $j_1 + ... + j_p = k$ and

$$\mu_k^*(\mathcal{R})^{d/2} < \mu_{j_1}^*(\mathcal{R})^{d/2} + \ldots + \mu_{j_p}^*(\mathcal{R})^{d/2},$$

that is

$$1 < \sum_{q=1}^{p} \left(\frac{\mu_{j_q}^*(\mathcal{R})}{\mu_k^*(\mathcal{R})} \right)^{d/2}.$$

From Lemma 2.2, for every $1 \leq q \leq p$ there exists $\Omega_{j_q}^* \in \mathcal{R}$ with volume 1 such that $\mu_{j_n}(\Omega_{j_q}^*) = \mu_{j_q}^*(\mathcal{R})$. We set

$$\Upsilon = \bigsqcup_{q=1}^{p} \Upsilon_q \quad \text{where } \Upsilon_q = \left(\frac{\mu_{j_q}^*(\mathcal{R})}{\mu_k^*(\mathcal{R})} \right)^{1/2} \Omega_{j_q}^*.$$

It follows that $\mu_{j_q}(\Upsilon_q) = \mu_k^*(\mathcal{R})$ and that

$$|\Upsilon| = \sum_{q=1}^{p} |\Upsilon_q| = \sum_{q=1}^{p} \left(\frac{\mu_{j_q}^*(\mathcal{R})}{\mu_k^*(\mathcal{R})} \right)^{d/2} > 1.$$

From this and since $|\Upsilon|^{-1/d}\Upsilon$ has volume 1, we have

$$\mu_k(\Upsilon) < |\Upsilon|^{2/d} \mu_k(\Upsilon) = \mu_k \left(|\Upsilon|^{-1/d} \Upsilon \right) \leq \mu_k^*(\mathcal{R}).$$

Consequently $\mu_k(\Upsilon) < \mu_{j_q}(\Upsilon_n)$ for each q and we deduce, recalling that the spectrum of Υ is the union of the spectra of the Υ_q's,

$$k + 1 \leq N(\mu_k(\Upsilon); \Upsilon) = \sum_{q=1}^{p} N(\mu_k(\Upsilon); \Upsilon_q) \leq \sum_{q=1}^{p} j_q = k,$$

where the counting function is defined as in (6) but for Neumann eigenvalues. This contradiction yields the claim. □

Corollary 2.5. *We have*

$$L := \lim_{k \to \infty} \frac{\lambda_k^*(\mathcal{R})^{d/2}}{k} = \inf_k \frac{\lambda_k^*(\mathcal{R})^{d/2}}{k} > 0$$

and

$$+\infty > M := \lim_{k \to \infty} \frac{\mu_k^*(\mathcal{R})^{d/2}}{k} = \sup_k \frac{\mu_k^*(\mathcal{R})^{d/2}}{k} > 0.$$

Proof. For the Dirichlet case, that the limit exists and is equal to the infimum follows from Fekete's lemma applied to the subadditive and nonnegative sequence $a_k = \lambda_k^*(\mathcal{R})^{d/2}$. That the limit is positive is a consequence of the works of Berezin [5] and Li and Yau [30] proving that

$$\frac{\lambda_k^{*\,d/2}}{k} \geq \left(\frac{d}{d+2}\right)^{d/2} \frac{(2\pi)^d}{\omega_d}.$$

For the Neumann case, that the limit exists in $\mathbb{R}$ and is equal to the supremum follows from Fekete's lemma applied to the super-additive and linearly bounded sequence $a_k = \mu_k^*(\mathcal{R})^{d/2}$, where the linear boundedness results from Kröger's estimate [26](1)

$$\frac{\mu_k^{*\,d/2}}{k} \leq \frac{d+2}{2} \frac{(2\pi)^d}{\omega_d}.$$

That the limit is positive follows from $\mu_k(\Omega) \leq \mu_k^*$ and from Weyl's asymptotic law

$$\lim_{k\to\infty} \frac{\mu_k(\Omega)^{d/2}}{k} = \frac{(2\pi)^d}{\omega_d}. \quad \square$$

Pólya's conjecture therefore holds in $\mathcal{R}(\Omega)$ if and only if

$$L = \frac{(2\pi)^d}{\omega_d} = M$$

and thus reduces to finding a subsequence of extremisers Ω_k^* such that

$$\lim_{k\to\infty} \frac{\lambda_k(\Omega_k^*)^{d/2}}{k} = \frac{(2\pi)^d}{\omega_d} = \lim_{k\to\infty} \frac{\mu_k(\Omega_k^*)^{d/2}}{k}.$$

The following lemma is an adaptation of a famous result of Wolf and Keller [38] to the class $\mathcal{R}$. Our proof however differs somewhat from the original proof.

Lemma 2.6. *For every $k \in \mathbb{N}$,*

$$\lambda_k^*(\mathcal{R})^{d/2} = \min\left\{\min_j \lambda_k(\Omega_j)^{d/2}, \min_{j_1+\ldots+j_p=k} \sum_{q=1}^{p} \lambda_{j_q}^*(\mathcal{R})^{d/2}\right\}.$$

Furthermore, for any Ω_k^* realising $\lambda_k^*(\mathcal{R})$, there exists a partition $j_1+\ldots+j_p=k$ such that

$$\Omega_k^* = \bigsqcup_{q=1}^{p} \alpha_q \Omega_{j_q}^* := \bigsqcup_{q=1}^{p} \sqrt{\frac{\lambda_k^*(\mathcal{R})}{\lambda_{j_q}^*(\mathcal{R})}}\,\Omega_{j_q}^*.$$

(1) In Kröger's article, Neumann eigenvalues are numbered starting with 1 so that $\mu_1 = 0$.

Proof. If λ_k^* is realised by one of the Ω_j, we are done. Suppose it is not. By Lemma 2.1, any minimiser for λ_k has at most k connected components. One also sees that the largest eigenvalue smaller or equal to $\lambda_k^*(\mathcal{R})$ of each component has to be equal to $\lambda_k^*(\mathcal{R})$. If not it would be possible to decrease $\lambda_k^*(\mathcal{R})$ by shrinking slightly a component for which that's not the case. This means slightly expanding the other components, thus decreasing the eigenvalue.

In other words, if Ω_k^* is an optimal domain for $\lambda_k^*(\mathcal{R})$, then each of its p components ($p \leq k$) will have some eigenvalue rank j_q such that

$$\Omega_k^* = \sqcup_{q=1}^{p} \Upsilon_q, \quad \Upsilon_q = \alpha_q \Omega_{n_q},$$

where

$$\sum_{q=1}^{p} \alpha_q^d = 1, \quad \sum_{q=1}^{p} j_q = k,$$

and

$$\lambda_{j_1}(\Upsilon_1) = \ldots = \lambda_{j_p}(\Upsilon_p) = \lambda_k^*(\mathcal{R}).$$

Furthermore, each of these Υ_q realises $\lambda_{j_q}^*$, otherwise it could be replaced by a domain who does while improving the eigenvalue. The identities between the eigenvalues of the different components may now be written as

$$\alpha_q^2 \lambda_{j_p}(\Omega_{n_p}) = \alpha_p^2 \lambda_{j_q}(\Omega_{n_q}), \quad q = 1, ..., p-1,$$

or

$$\alpha_q^d \lambda_{j_p}^{d/2}(\Omega_{n_p}) = \alpha_p^d \lambda_{j_q}^{d/2}(\Omega_{n_q}), \quad q = 1, ..., p-1, .$$

Summing up these identities for j from 1 to $p-1$,

$$\left(\sum_{q=1}^{p-1} \alpha_q^d\right) \lambda_{j_p}^{d/2}(\Omega_{n_p}) = \alpha_p^d \sum_{q=1}^{p-1} \lambda_{j_q}^{d/2}(\Omega_{n_1}).$$

Hence

$$(1 - \alpha_p^d) \lambda_{j_p}^{d/2}(\Omega) = \alpha_p^d \sum_{q=1}^{p-1} \lambda_{j_q}^{d/2}(\Omega_{n_q})$$

and

$$\alpha_p^d = \frac{\lambda_{j_p}^{d/2}(\Omega_{n_p})}{\sum_{q=1}^{p} \lambda_{j_q}^{d/2}(\Omega_{n_q})}.$$

We finally obtain

$$\begin{aligned}\lambda_k(\Omega_k^*) &= \alpha_p^{-2}\lambda_{j_p}(\Omega_{n_p})\\ &= \left(\sum_{q=1}^{p}\lambda_{j_q}^{d/2}(\Omega_{n_q})\right)^{2/d},\end{aligned}$$

yielding the desired result. □

A corresponding statement for Neumann eigenvalues is proved by Poliquin and Roy-Fortin [33] by closely mirroring Wolf and Keller's proof, and the result is recollected and somewhat generalised by Colbois and El Soufi [14]. We include their proof in our formalism for completeness.

Lemma 2.7. *For every* $k\in\mathbb{N}$,

$$\mu_k^*(\mathcal{R})^{d/2} = \max\left\{\max_j \mu_k(\Omega_j)^{d/2}, \max_{j_1+\ldots+j_p=k}\sum_{q=1}^{p}\mu_{j_q}^*(\mathcal{R})^{d/2}\right\}.$$

Furthermore, for any Ω_k^* *realising* $\mu_k^*(\mathcal{R})$*, there exists a partition* $j_1+\ldots+j_p=k$ *such that*

$$\Omega_k^* = \bigsqcup_{q=1}^{p}\alpha_q\Omega_{j_q}^* := \bigsqcup_{q=1}^{p}\sqrt{\frac{\mu_k^*(\mathcal{R})}{\mu_{j_q}^*(\mathcal{R})}}\,\Omega_{j_q}^*.$$

Proof. Once again, if μ_k is realised by one of the Ω_j, we are done. A rather simple induction argument reduces the problem to the case $p=2$ and $\Omega_k^*=\Upsilon_1\sqcup\Upsilon_2$ into two nonempty unions of connected components, so that $|\Upsilon_1|,|\Upsilon_2|>0$ and $|\Upsilon_1|+|\Upsilon_2|=|\Upsilon_k^*|=1$.

Choose $k+1$ of the $N(\mu_k^*(\mathcal{R}),\Omega_k^*)$ lowest and linearly independent eigenfunctions on Ω_k^*, say $u_0,\ldots,u_k$ ordered according to their eigenvalues, in such a way that every eigenfunction with eigenvalue strictly smaller than $\mu_k^*(\mathcal{R})$ is chosen and that every eigenfunction is supported in either Υ_1 or Υ_2.(2) We have in particular $\mu_k(u_k)=\mu_k^*(\mathcal{R})\geq\mu_k(\Omega)>0$, where the last inequality follows since Ω is connected. For every $0\leq l\leq k$, the function u_l is not identically zero on at least one of the two Υ_q's; without loss of generality, assume that u_k is not identically zero on Υ_1. Notice that if the number of u_l's which are not identically zero on Υ_1 is j_1+1, then $\mu_{j_1}(\Upsilon_1)=\mu_k(u_k)$.

Since the spectrum of $\Omega_k^*=\Upsilon_1\sqcup\Upsilon_2$ is the (ordered) union of the spectra of Υ_1 and Υ_2, and since the u_l's span any eigenfunction on Ω_k^* with eigenvalue strictly smaller than $\mu_k^*(\mathcal{R})$, the number of u_l's which are not identically zero on Υ_2 is $j_2=k-j_1$. Considering the (j_2+1)-th eigenfunction on Υ_2 we get $\mu_{j_2}(\Upsilon_2)\geq\mu_k^*(\mathcal{R})>0$; in particular $j_2\geq1$. We claim that in fact $\mu_k^*(\mathcal{R})=\mu_{j_2}(\Upsilon_2)$; to see this, suppose on

(2) Recall that u_0 is necessarily a locally constant function.

the contrary that $\mu_k^*(\mathcal{R})<\mu_{j_2}(\Upsilon_2)$. Then consider any sufficiently small deformation Ω' (with volume 1) of Ω_k^* obtained by contracting Υ_1 to Υ_1' and dilating Υ_2 to Υ_2', so as to have

$$\mu_{j_2-1}(\Upsilon_2') < \mu_{j_2-1}(\Upsilon_2) \leq \mu_{j_1}(\Upsilon_1) < \mu_{j_1}(\Upsilon_1') < \mu_{j_2}(\Upsilon_2') < \mu_{j_2}(\Upsilon_2).$$

Hence $\mu_k(\Omega')=\mu_{j_1}(\Upsilon_1')$ and thus $\mu_k(\Omega')>\mu_k^*(\mathcal{R})$. This contradicts the maximality of Ω_k^*. As a result $\mu_{j_1}(\Upsilon_1)=\mu_{j_2}(\Upsilon_2)=\mu_k^*(\mathcal{R})>0$. Since $\mu_j(D)>0$ if and only if $j\geq\nu(D)$, we deduce $j_i\geq\nu(\Upsilon_i)\geq 1$. That we have a partition follows from $j_2:=k-j_1$.

We claim that the normalised domain $|\Upsilon_1|^{-1/d}\Upsilon_1$ realises $\mu_{j_1}^*(\mathcal{R})$. Suppose differently: There exists a maximiser $\Omega_{j_1}^*$ (with volume 1) such that $\mu_{j_1}(|\Upsilon_1|^{-1/d}\Upsilon_1)<\mu_{j_1}(\Omega_{j_1}^*)=\mu_{j_1}^*(\mathcal{R})$, from which it follows that

$$\mu_k^*(\mathcal{R}) = \mu_{j_1}(\Upsilon_1) = |\Upsilon_1|^{-2/d}\mu_{j_1}(|\Upsilon_1|^{-1/d}\Upsilon_1) < |\Upsilon_1|^{-2/d}\mu_{j_1}^*(\mathcal{R}). \tag{7}$$

Consider the domain

$$\widetilde{\Omega} = \widetilde{\Upsilon}_1 \sqcup \Upsilon_2 = \left(\frac{\mu_{j_1}^*(\mathcal{R})}{\mu_k^*(\mathcal{R})}\right)^{1/2} \Omega_{j_1}^* \sqcup \Upsilon_2.$$

Equation (7) implies that its volume is strictly greater than $|\Upsilon_1||\Omega_{j_1}^*|+|\Upsilon_2|=1$. The j_1+1 first eigenvalues coming from $\widetilde{\Upsilon}_1$ have eigenvalue at most $\mu_k^*(\mathcal{R})$, the (j_1+1)-th eigenvalue $\mu_{j_1}(\widetilde{\Upsilon}_1)$ being equal to this value. Together with the same $j_2=k-j_1$ eigenfunctions on Υ_2 as before, we deduce that $\mu_k(\widetilde{\Omega})=\mu_k^*(\mathcal{R})$. Therefore the $(k+1)$-th eigenvalue of the normalised domain $|\widetilde{\Omega}|^{-1/d}\widetilde{\Omega}$ is strictly larger than $\mu_k^*(\mathcal{R})$, which is a contradiction to the maximality of Ω_k^*. A similar argument implies that the normalised domain $|\Upsilon_2|^{-1/d}\Upsilon_2$ realises $\mu_{j_2}^*(\mathcal{R})$. Incidentally, $|\Upsilon_i|=(\mu_{j_i}^*(\mathcal{R})/\mu_k^*(\mathcal{R}))^{d/2}$. □

3. A trichotomy

In this section, we set out to prove Theorem 1.3. Note that all of the results of the previous sections have a Dirichlet and Neumann version, where the only difference is that the inequalities are reversed. As such, we will only prove the Dirichlet case of Theorem 1.3, and only state the corollaries in term of the Dirichlet eigenvalues. However, since we rely only on the formal properties obtained in the previous section, all the results also apply for Neumann eigenvalues, reversing the inequalities when needed and changing the proofs *mutatis mutandis*.

We start with the following proposition, allowing us to consider classes of domains generated by a single domain Ω. As such, when no confusion arises we may write $\mathcal{R}$ for $\mathcal{R}(\Omega)$ once a generating domain is fixed.

Proposition 3.1. *If Pólya's conjecture holds within $\mathcal{R}(\Omega_1)$ and $\mathcal{R}(\Omega_2)$, then it holds within $\mathcal{R}(\Omega_1,\Omega_2)$. The same is true of the strong Pólya conjecture.*

It is clear that it is sufficient to show that if Pólya's conjecture holds for two domains $\Upsilon_1 \in \mathcal{R}(\Omega_1)$ and $\Upsilon_2 \in \mathcal{R}(\Omega_2)$, then it holds for the disjoint union of these two domains $\Upsilon_1 \sqcup \Upsilon_2$. This will rely on the following abstract lemma about superlinear sequences.

Lemma 3.2. *Let $\{a_k : k \in \mathbb{N}\}$ and $\{b_k : k \in \mathbb{N}\}$ be two increasing sequences satisfying*

$$a_k \geq \frac{k}{A} \quad \text{and} \quad b_k \geq \frac{k}{B}$$

for some $A, B > 0$. Denote c_k the sequence obtained as the arrangement in increasing order of all elements in $\{a_k\} \sqcup \{b_k\}$, repeated with multiplicity. Then,

$$c_k \geq \frac{k}{A+B}.$$

The same holds when all inequalities are replaced with strict inequalities.

Proof. Without loss of generality, we assume that $c_k = a_p$ for some $1 \leq p \leq k$. We distinguish two cases: $p = k$ and $1 \leq p < k$. In the former situation, we have that

$$c_k = a_k \geq \frac{k}{A} > \frac{k}{A+B}.$$

In the second case, it follows that $a_p \geq b_j$ for all j, $1 \leq j \leq k-p$. We then have

$$\begin{aligned} \frac{k}{A+B} &= \frac{p + (k-p)}{A+B} \\ &\leq \frac{A a_p}{A+B} + \frac{B b_{k-p}}{A+B} \\ &\leq a_p = c_k, \end{aligned} \tag{8}$$

where the last line holds from the fact that $a_p \geq \max\{a_p, b_{k-p}\}$, hence it is also greater than any convex combination of both. This concludes the proof, and it is readily seen that if the inequalities in the statement of the lemma were strict, then the second line in (8) would be a strict inequality. □

To prove Proposition 3.1, apply the previous lemma with $a_k = \lambda_k(\Omega_1)^{d/2}$, $b_k = \lambda_k(\Omega_2)^{d/2}$, $A = \frac{\omega_d |\Omega_1|}{(2\pi)^d}$, and $B = \frac{\omega_d |\Omega_2|}{(2\pi)^d}$.

Let us now define the set $J := J(\Omega) \subset \mathbb{N}$ of indices where the generator Ω realises $\lambda_k^*(\mathcal{R}(\Omega))$, that is

$$J(\Omega) := \{k \in \mathbb{N} : \lambda_k(\Omega) = \lambda_k^*(\mathcal{R}(\Omega))\}.$$

Clearly, J is never empty since $1 \in J(\Omega)$ for any Ω.

Proposition 3.3. *Suppose $J(\Omega)$ is infinite, so that there exists a sequence $j_1 < j_2 < \ldots \nearrow +\infty$ such that $\Omega = \Omega_{j_n}^*(\mathcal{R}(\Omega))$ for all n. Then Pólya's conjecture is true for every $\Upsilon \in \mathcal{R}(\Omega)$.*

Proof. On the one hand, Weyl's law implies

$$\lim_{n\to\infty} \frac{\lambda_{j_n}(\Omega)}{j_n^{2/d}} = \frac{4\pi^2}{\omega_d^{2/d}}.$$

On the other hand, since Ω realises $\lambda_{j_n}^*(\mathcal{R})$ for every n, it follows from Corollary 2.5

$$\lim_{n\to\infty} \frac{\lambda_{j_n}(\Omega)}{j_n^{2/d}} = \inf_k \frac{\lambda_k^*(\mathcal{R})}{k^{2/d}} = \frac{4\pi^2}{\omega_d^{2/d}}.$$

We therefore conclude that $\lambda_k(\Upsilon)^{d/2}\, k^{-1} \geq (2\pi)^d \omega_d^{-1}$ for every $\Upsilon \in \mathcal{R}$ with volume 1, which is Pólya's conjecture. $\square$

The following theorem characterises when J is finite.

Theorem 3.4. *The set $J(\Omega)$ is finite if and only if there exists a constant c such that for all k, $\nu(\Omega_k^*) \geq ck$.*

Proof. If J is infinite, it is clear that such a constant c does not exist. Conversely, suppose that the set $J = \{k \in \mathbb{N} : \lambda_k(\Omega) = \lambda_k^*(\mathcal{R})\}$ is finite. This implies that any minimiser realising $\lambda_k^*(\mathcal{R})$ is of the form

$$\Omega_k^* = \bigsqcup_{j\in J} \bigsqcup_{m=1}^{n_{k,j}} r_{k,j}\Omega_j^*.$$

The number of connected components of Ω_k^* is

$$\nu(\Omega_k^*) = \sum_{j\in J} n_{k,j}, \tag{9}$$

and referring to Lemma 2.6 we get

$$\lambda_k^*(\mathcal{R})^{d/2} = \sum_{j\in J} n_{k,j} \lambda_j^*(\mathcal{R})^{d/2}. \tag{10}$$

Corollary 2.5 states that there is a constant c such that $\lambda_k^*(\mathcal{R})^{d/2} \geq c'k$. Let $j' = \max J$, combining (9) and (10) we obtain

$$\begin{aligned} \nu(\Omega_k^*) &\geq \frac{1}{\lambda_{j'}(\Omega)^{d/2}} \sum_{j\in J} n_{k,j} \lambda_j^*(\mathcal{R})^{d/2} \\ &\geq \frac{c'}{\lambda_{j'}(\Omega)^{d/2}} k. \end{aligned}$$

The proof is completed by taking $c = c'\lambda_{j'}(\Omega)^{-d/2}$. $\square$

Considering that all known results in the literature point to the validity of Pólya's conjecture, we are thus naturally led to the following, stronger, conjecture.

Open problem. *For every domain* $\Omega \subset \mathbb{R}^d$ *there exists a subsequence* $\lambda^*_{k_n}(\mathcal{R}(\Omega))$, *with minimisers* $\Omega^*_{k_n}$ *such that*

$$\nu(\Omega_{k_n}) = o\,(k_n)\,.$$

That this open problem is a potentially strictly stronger statement than Pólya's conjecture follows from this partial converse to Proposition 3.3.

Proposition 3.5. *Suppose* $J(\Omega) \subset \mathbb{N}$ *is finite. Then,*

$$\inf_k \frac{\lambda^*_k(\mathcal{R}(\Omega))^{d/2}}{k} = \min_{j\in J} \frac{\lambda_j(\Omega)^{d/2}}{j} \leq \frac{(2\pi)^d}{\omega_d}.$$

Furthermore, for infinitely many $j \in \mathbb{N}$

$$\frac{\lambda^*_j(\mathcal{R}(\Omega))}{j} = \inf_k \frac{\lambda^*_k(\mathcal{R}(\Omega))}{k}.$$

Before starting with the proof, let us observe two things about this statement. First, it means that $\inf_k \lambda^*_k(\mathcal{R}(\Omega))^{d/2} k^{-1}$ is realised. Second, it means that if Ω is a minimiser in $\mathcal{R}(\Omega)$ only for finitely many k's and if Pólya's conjecture holds, then Pólya's bound is attained since the realised minimum of $\lambda^*_k(\mathcal{R}(\Omega))^{d/2}$ would be exactly $(2\pi)^d k \omega_d^{-1}$.

Proof. Let

$$L' = \min_{j\in J} \frac{\lambda_j(\Omega)^{d/2}}{j}.$$

It exists as J is finite, and $L' \geq L$. For any $k \notin J$, a set which realises $\lambda^*_k(\mathcal{R})$ necessarily has several connected components. It results from Lemma 2.6 that

$$\lambda^*_k(\mathcal{R})^{d/2} = \sum_{j\in J} n_j \lambda_j(\Omega)^{d/2}$$

where $\{n_j : j \in J\}$ are nonnegative integers such that

$$\sum_{j\in J} n_j j = k.$$

Therefore

$$\frac{\lambda^*_k(\mathcal{R})^{d/2}}{k} = \frac{1}{k} \sum_{j\in J} n_j \lambda_j^{d/2} \geq \frac{1}{k} \sum_{j\in J} n_j j L' = L',$$

which immediately implies

$$L = \inf_k \frac{\lambda^*_k(\mathcal{R})^{d/2}}{k} \geq L' = \min_{j\in J} \frac{\lambda_j(\Omega)^{d/2}}{j} \geq L.$$

Furthermore, since $\lambda_k(\Omega) \geq \lambda_k^*(\mathcal{R})$ for every $k \in \mathbb{N}$ and since Weyl's law implies that

$$\lim_{k\to\infty} \frac{\lambda_k(\Omega)^{d/2}}{k} = \frac{(2\pi)^d}{\omega_d},$$

we get from Corollary 2.5 that indeed

$$\min_{j\in J} \frac{\lambda_j(\Omega)^{d/2}}{j} = \lim_{k\to\infty} \frac{\lambda_k^*(\mathcal{R})^{d/2}}{k} \leq \frac{(2\pi)^d}{\omega_d}$$

Finally, recall that for any set Υ we defined $\Upsilon^{(n)} := n^{1/d} \bigsqcup_{\ell=1}^n \Upsilon$. We see that $|\Upsilon| = |\Upsilon^{(n)}|$ and that for all j, $\lambda_j(\Upsilon)^{d/2} j^{-1} = \lambda_{nj}(\Upsilon)^{d/2}(nj)^{-1}$. Therefore, if j is an eigenvalue rank such that

$$\frac{\lambda_j(\Omega)^{d/2}}{j} = \inf_k \frac{\lambda_k^*(\mathcal{R})^{d/2}}{k},$$

then for all $n \in \mathbb{N}$

$$\frac{\lambda_{nj}(\Omega^{(n)})^{d/2}}{nj} = \inf_k \frac{\lambda_k^*(\mathcal{R})^{d/2}}{k}$$

so that the infimum is attained infinitely often. □

Proof of Theorem 1.3. We have proved in Proposition 3.3 that if J is infinite, then Pólya's conjecture holds. The two other parts of the trichotomy are proved by Proposition 3.5. □

We now turn our attention to the proof of Theorem 1.4, in the case where the domain Ω satisfies the two-term Weyl law (1).

Proof of Theorem 1.4. In all generality, clearly (2) implies (3), and (1) implies (3). Indeed, (1) places us in the first possibility of the trichotomy Theorem 1.3, which implies (3). We shall show that the assumption that a two-term Weyl law holds can be used to infer that (1) implies (2) and that (3) implies (1).

Proof of (1) implies (2). Write the sequence of minimisers, all of volume 1, as

$$\Omega_k^* = \bigsqcup_{q=1}^{\nu_k} r_{k,q}\Omega,$$

where $\nu_k := \nu(\Omega_k^*) < \infty$ by Lemma 2.1. Suppose that the $r_{k,q}$ coefficients are in decreasing order,

$$r_{k,1} \geq \ldots \geq r_{k,\nu_k}.$$

It follows from Lemma 2.6 that for every $1 \leq q \leq \nu_k$ there is $j_q := j_q(k) \in J$ such that

$$r_{k,q} = \left(\frac{\lambda_{j_q}(\Omega)}{\lambda_k^*(\mathcal{R})}\right)^{1/2},$$

and $j_1+\ldots+j_{\nu_k}=k$. It follows from Weyl's law that

$$\lim_{k\to\infty} r_{k,1}=1 \iff \lim_{k\to\infty}\frac{j_1(k)}{k}=1.$$

Suppose that the righthand side of the previous equivalence does not hold, i.e. that there exists $\delta>0$ and a subsequence, that we still label with k, such that for all k, $j_1(k)\le(1-\delta)k$. For all $\varepsilon>0$, it follows from the two-term Weyl law that there exists a rank N such that for all $j>N$,

$$\lambda_j(\Omega)^{d/2}\ge\frac{(2\pi)^d}{\omega_d}j+\underbrace{\left(\frac{(2\pi)\omega_{d-1}}{4\omega_d^{\frac{2d-1}{d}}}|\partial\Omega|-\varepsilon\right)}_{:=A-\varepsilon}j^{\frac{d-1}{d}}. \tag{11}$$

For all k, let $Q:=Q(k)$ be defined as

$$Q:=\begin{cases}0 & \text{if } j_q\le N \text{ for all } 1\le q\le\nu_k,\\ \max\{q:j_q>N\} & \text{otherwise.}\end{cases}$$

We define

$$\Upsilon_k:=\bigsqcup_{q=1}^{Q} r_{k,q}\Omega \quad\text{and}\quad \Xi_k:=\bigsqcup_{q=Q+1}^{\nu_k} r_{k,q}\Omega.$$

We claim that $\nu(\Xi_k)$ is bounded in k. Indeed, it follows from the strong Pólya conjecture that there exists M such that for all $j>M$,

$$\frac{\lambda_j(\Omega)^{d/2}}{j}<\frac{\lambda_{j_q}(\Omega)^{d/2}}{j_q}$$

for all $q>Q$. Writing

$$j_{Q+1}+\ldots+j_{\nu_k}=j'>\nu(\Xi_k),$$

it follows from Lemma 2.6 that if $j'\ge M$, then

$$\begin{aligned}\lambda_{j'}(\Xi_k)^{d/2}&=\lambda^*_{j'}(\mathcal{R})^{d/2}\\ &\le\sum_{q=Q+1}^{\nu_k} j_q\frac{\lambda_{j'}(\Omega)^{d/2}}{j'}\\ &<\sum_{q=Q+1}^{\nu_k}\lambda_{j_q}(\Omega)^{d/2}\\ &=\lambda_{j'}(\Xi_k)^{d/2},\end{aligned}$$

a contradiction. Hence, $\nu(\Xi_k)\le j'<M$, and

$$k_0:=\sum_{q=1}^{Q} j_q\ge k-M \tag{12}$$

Recall that we assumed that there is $\delta>0$ such that $j_1<(1-\delta)k$, and it follows from (12) that, up to choosing δ a bit smaller, $j_1<(1-\delta)k_0$. Let $R:=R(k)$ be defined as

$$R:=\max\left\{r:2\le r\le Q \text{ and } \frac{1}{k_0}\sum_{q=r}^{Q} j_q>\delta\right\},$$

and we denote

$$\delta_k:=\frac{1}{k_0}\sum_{q=R}^{Q} j_q.$$

There is no loss of generality in assuming $\delta<1/3$. That the j_q are in decreasing order ensures that in that case $\delta<\delta_k<1-\delta$. Recall that for all $1\le q\le Q$, $j_q>N$ hence (11) holds. It is a consequence again of Lemma 2.6 that

$$\begin{aligned}\lambda_k^*(\mathcal{R})^{d/2}&\ge\sum_{q=1}^{Q}\lambda_{j_q}(\Omega)^{d/2}\\&\ge\sum_{q=1}^{Q}\left[\frac{(2\pi)^d}{\omega_d}j_q+(A-\varepsilon)\,j_q^{\frac{d-1}{d}}\right]\\&\ge\frac{(2\pi)^d}{\omega_d}k+(A-\varepsilon)\sum_{q=1}^{Q}j_q^{\frac{d-1}{d}}+\mathrm{O}\left(1\right).\end{aligned}\tag{13}$$

We study the sum in the last line of the previous display. It follows from subadditivity of the function $x\mapsto x^\alpha$ for $\alpha<1$, and from $k_0^\alpha=k^\alpha+\mathrm{O}(k^{\alpha-1})$ that

$$\begin{aligned}\sum_{q=1}^{Q}j_q^{\frac{d-1}{d}}&\ge\left(\sum_{q=1}^{R-1}j_q\right)^{\frac{d-1}{d}}+\left(\sum_{q=R}^{Q}j_q\right)^{\frac{d-1}{d}}\\&\ge\left((1-\delta_k)^{\frac{d-1}{d}}+\delta_k^{\frac{d-1}{d}}\right)k^{\frac{d-1}{d}}+\mathrm{O}\left(k^{-1/d}\right)\end{aligned}$$

It is a simple exercise to see that the function $x\mapsto x^\alpha+(1-x)^\alpha$, $\alpha<1$ being concave and symmetric on $[0,1]$ and $\delta<\delta_k<1-\delta$ imply that

$$(1-\delta_k)^{\frac{d-1}{d}}+\delta_k^{\frac{d-1}{d}}\ge(1-\delta)^{\frac{d-1}{d}}+\delta^{\frac{d-1}{d}}\ge1+\left(2^{1/d}-1\right)\delta=:1+c_d\delta,$$

and $c_d>0$. Putting this back into (13), it follows that

$$\lambda_k^*(\mathcal{R})^{d/2} \geq \frac{(2\pi)^d}{\omega_d}k+(A-\varepsilon)\left(1+c_d\delta\right)k^{\frac{d-1}{d}}+\mathrm{O}\left(1\right).$$

Choosing

$$\varepsilon=\frac{c_d\delta A}{2(1+c_d\delta)}$$

gives, for k large enough, that

$$\lambda_k^*(\mathcal{R})^{d/2} \geq \frac{(2\pi)^d}{\omega_d}k+\left(A+\frac{Ac_d\delta}{3}\right)k^{\frac{d-1}{d}}.$$

However, since

$$\lambda_k(\Omega)^{d/2}=\frac{(2\pi)d}{\omega_d}k+Ak^{\frac{d-1}{d}}+o\left(k^{\frac{d-1}{d}}\right),$$

we have that for k large enough, $\lambda_k(\Omega)^{d/2}<\lambda_k^*(\mathcal{R})^{d/2}$, a contradiction. Hence, for any $\delta>0$ there are no subsequences along which $j_1(k)<(1-\delta)k$ for all k. It is readily seen that $r_{k,1}$ converges to 1.

Proof of (3) implies (1). Assume that the Strong Pólya conjecture doesn't hold for $\mathcal{R}(\Omega)$. It follows from Lemma 2.6 that it cannot hold for Ω, so that there is a rank j such that

$$\lambda_j(\Omega)^{d/2} \leq \frac{(2\pi)^d}{\omega_d}j. \tag{14}$$

By assumption there is a subsequence, labeled by k, such that

$$\Omega_k^*=(1-\varepsilon_k)\Omega\sqcup\Upsilon_k,$$

with $\varepsilon_k\to 0$. From Lemma 2.6, for every k there exists a rank j_k such that

$$\lambda_k^*(\mathcal{R})^{d/2}=(1-\varepsilon_k)^{-d}\lambda_{j_k}(\Omega)^{d/2}, \tag{15}$$

and that $\Omega=\Omega_{j_k}^*$. It follows from Corollary 2.5 and equation (15) that $j_k\to\infty$. By the two-term Weyl law (1), there is $A>0$ such that

$$\lambda_{j_k}(\Omega)^{d/2}=\frac{(2\pi)^d}{\omega_d}j_k+Aj_k^{\frac{d-1}{d}}+o\left(j_k^{\frac{d-1}{d}}\right),$$

hence there exists a constant $C>0$ such that for every k large enough,

$$\lambda_{j_k}(\Omega)^{d/2}-\frac{(2\pi)^d}{\omega_d}j_k \geq Cj_k^{\frac{d-1}{d}}. \tag{16}$$

We now show that for large enough k, Ω is in fact not a minimiser for λ_{j_k} amongst $\mathfrak{R}$. Write $j_k = n_k j + r$, with j as in (14) and $0 \le r < j$. Consider the domain Ω' defined as

$$\Omega' = \left(\frac{\lambda_r(\Omega)}{\lambda_j(\Omega)}\right)^{1/2} n_k^{-1/d}\Omega \sqcup \left(\bigsqcup_{q=1}^{n_k} n_k^{-1/d}\Omega\right). \tag{17}$$

We have constructed Ω' explicitly so that the first component in (17) has $n_k^{2/d}\lambda_j(\Omega)$ as its r^{th} eigenvalue, and all the other components have $n_k^{2/d}\lambda_j(\Omega)$ as its j^{th} eigenvalue, it then follows that

$$\lambda_{j_k}(\Omega')^{d/2} = n_k \lambda_j(\Omega)^{d/2}.$$

Furthermore,

$$|\Omega'| = \left(1 + \left(\frac{\lambda_r(\Omega)}{\lambda_j(\Omega)}\right)^{d/2} \frac{1}{n_k}\right).$$

Combining these equalities with (14), we deduce that

$$\begin{aligned}
|\Omega'|\,\lambda_{j_k}(\Omega')^{d/2} &\le \left(1 + \left(\frac{\lambda_r(\Omega)}{\lambda_j(\Omega)}\right)^{d/2} \frac{1}{n_k}\right) \frac{(2\pi)^d}{\omega_d} n_k j \\
&= \left(1 + \left(\frac{\lambda_r(\Omega)}{\lambda_j(\Omega)}\right)^{d/2} \frac{j}{j_k - r}\right) \frac{(2\pi)^d}{\omega_d}(j_k - r) \\
&= \frac{(2\pi)^d}{\omega_d} j_k + \mathrm{O}(1)
\end{aligned}$$

This combined with estimate (16) implies that for k large enough, $|\Omega'|\lambda_{j_k}(\Omega')^{d/2} < \lambda_{j_k}(\Omega)^{d/2}$, contradicting optimality of Ω for λ_{j_k}. $\square$

For the next few results we shall assume that Ω is a minimiser only finitely many times, namely $\Omega = \Omega_k^*$ if and only if $k \in J = \{j_1, \ldots, j_p\} \subset \mathbb{N}$. Our goal is to investigate what the minimisers can be in such a case. We shall continue to write simply $L = \inf_k k^{-1}\lambda_k^*(\mathfrak{R}^{d/2})$. We shall say that a *minimiser* Ω_k^* *realises* L if $\lambda_k(\Omega_k^*)^{d/2} k^{-1} = L$.

Recall that for a set $\Upsilon \in \mathfrak{R}$ and $n \in N$, the n-th propagation of Υ is the set

$$\Upsilon^{(n)} = \bigsqcup_{\ell=1}^{n} \frac{1}{n^{1/d}} \Upsilon.$$

Observe that $|\Upsilon| = |\Upsilon^{(n)}|$ and that $\lambda_k(\Upsilon)^{d/2} k^{-1} = \lambda_{nk}(\Upsilon^{(n)})^{d/2} (nk)^{-1}$ for any $n \in \mathbb{N}$.

Definition 3.6. A minimiser Ω_k^* *propagates as a minimiser in* $\mathfrak{R}$ if for every $n \in \mathbb{N}$ we have $\Omega_k^{*(n)} = \Omega_{nk}^*$. A minimiser Ω_k^* *weakly propagates as a minimiser in* $\mathfrak{R}$ if there exist a sequence of integers $n_1 < n_2 < \ldots \nearrow +\infty$ and a corresponding sequence of minimisers in $\mathcal{R}$ of the form

$$\Omega^*_{k'_i} = r_i \Omega_k^{*\,(n_i)} \sqcup \Upsilon_i. \tag{18}$$

Proposition 3.7. *A minimiser Ω^*_k realises L if and only if it propagates as a minimiser in $\mathcal{R}(\Omega)$.*

Proof. Fix $k\in\mathbb{N}$ and a minimiser Ω^*_k. We have $\lambda^*_{nk}(\mathcal{R})\le\lambda_{nk}(\Omega_k^{(n)})=n^{2/d}\lambda_k(\Omega^*_k)$. Fix $n>1$. Whether or not nk belongs to J, there exist nonnegative integers $n_1, ..., n_p$ such that $nk=\sum_{i=1}^p n_i j_i$ and

$$\lambda^*_{nk}(\mathcal{R})^{d/2} = \sum_{i=1}^p n_i \lambda_{j_i}^{d/2} \ge \sum_{i=1}^p n_i j_i L = nkL.$$

Therefore we have

$$L \le \frac{\lambda^*_{nk}(\mathcal{R})^{d/2}}{nk} \le \frac{\lambda_{nk}\left(\Omega_k^{(n)}\right)^{d/2}}{nk} = \frac{\lambda_k(\Omega^*_k)^{d/2}}{k}. \tag{19}$$

In view of this, it follows that Ω^*_k realises L if and only if for every $n\in\mathbb{N}$ both inequalities in (19) are equalities.

In turn, this is equivalent to only the second inequality being an equality for every n. Indeed, the latter would imply that the sequence $n\mapsto\lambda^*_{nk}(\mathcal{R})^{d/2}(nk)^{-1}$ is constant, but we know that it converges to L as $n\to\infty$ hence the first inequality being an equality too.

Now for any fixed n, the equality $\lambda^*_{nk}(\mathcal{R})^{d/2}(nk)^{-1}=\lambda_{nk}(\Omega_k^{(n)})^{d/2}(nk)^{-1}$ is equivalent to the claim that $\Omega_k^{(n)}$ realises $\lambda^*_{nk}(\mathcal{R})$. Consequently, the second inequality in (19) being an equality for every $n\in N$ means precisely that Ω^*_k propagates as a minimiser. □

Lemma 3.8. *A minimiser Ω^*_k propagates as a minimiser in $\mathcal{R}(\Omega)$ if and only if it weakly propagates as a minimiser in $\mathcal{R}(\Omega)$.*

Proof. The "only if" part is trivial. For the "if" part, consider a sequence of minimisers $\Omega^*_{k_j}$ as in equation (18). It follows from Lemma 2.6 that for each $j\in\mathbb{N}$, the set $\Omega_k^{*\,(n_j)}$ realises $\lambda^*_{n_j k}(\mathcal{R})$. As a consequence of this and of Corollary 2.5, we compute

$$\frac{\lambda_k(\Omega^*_k)^{d/2}}{k} = \frac{\lambda_k\left(\Omega_k^{*\,(n_j)}\right)^{d/2}}{n_j k} = \frac{\lambda^*_{n_j k}(\mathcal{R})^{d/2}}{n_j k} \underset{j\to+\infty}{\longrightarrow} L.$$

This means that Ω^*_k realises L. Proposition 3.7 thus implies that Ω^*_k propagates as a minimiser in $\mathcal{R}$. □

Let us consider the sets

$$K_L := K_L(\Omega) := \{k \in \mathbb{N} : \lambda_k^*(\mathcal{R})^{d/2}\, k^{-1} = L\} \quad \text{and} \quad J_L := J_L(\Omega) = J(\Omega) \cap K_L(\Omega).$$

We observe that K_L is closed under finite sums.

Continuing with the assumption of finite J, Proposition 3.5 implies that the set is not empty. Set $j_L = \max J_L$. Proposition 3.7 implies that the minimiser $\Omega = \Omega_j^*$ associated to $j \in J_L$ propagates as a minimiser in $\mathcal{R}$. One might expect these minimisers to be special, for instance to have a minimum numbers of connected components among minimisers of a given eigenvalue functional, if not to be unique. These expectations are even more vivid for the propagations of $\Omega_{j_L}^*$. The next result investigates these possibilities.

Lemma 3.9. *Assume $J_L(\Omega)$ is finite and set $j_L := \max J_L(\Omega)$. Let $j \in J_L(\Omega)$. If there exist $n \in \mathbb{N}$ and a minimiser $\Omega_{nj}^* \neq \Omega_j^{*\,(n)}$, then $\{j\} \subsetneq J_L(\Omega)$. If furthermore $\nu(\Omega_{nj}^*) \leq \nu(\Omega_j^{*\,(n)})$, then $j < j_L$. If instead $\nu(\Omega_{nj}^*) > \nu(\Omega_j^{*\,(n)})$, then there exists $j' \in J_L(\Omega)$ such that $j' < j$.*

Proof. Both $\Omega_j^{*\,(n)}$ and Ω_{nj}^* realises $\lambda_{nj}^*(\mathcal{R})$. As a result of Lemma 2.6 we have a decomposition

$$\Omega_{nj}^* = \bigsqcup_{i=1}^{p} \bigsqcup_{m=1}^{n_i} r_i \Omega_{j_i}^* \quad \text{with} \quad \sum_{i=1}^{p} n_i j_i = nj \tag{20}$$

which induces the equality

$$\lambda_{nj}^*(\mathcal{R})^{d/2} = \sum_{i=1}^{p} n_i \lambda_{j_i}^*(\mathcal{R})^{d/2}.$$

We claim that there is an index h such that $j_h \neq j$ and $n_h > 0$. Otherwise the only positive n_i would be n_l where $j_l = j$; It would follow from (20) that $n_l = n$ and that $r_i = n_l^{-1/d}$, hence $\Omega_{nj}^* = \Omega_j^{(n)}$. This is a contradiction with our assumptions, hence the claim.

Since $j \in J_L$, $\Omega_j^{*\,(n)}$ realises L and so does Ω_{nj}^*. We compute

$$\begin{aligned} L = \frac{\lambda_{nj}(\Omega_{nj}^*)^{d/2}}{nj} &= \frac{1}{nj} \sum_{i=1}^{p} n_i \lambda_{j_i}^*(\mathcal{R})^{d/2} \\ &= \frac{1}{nj} \sum_{i=1}^{p} n_i j_i \frac{\lambda_{j_i}^*(\mathcal{R})^{d/2}}{j_i} \geq \frac{1}{nj} \sum_{i=1}^{p} n_i j_i L = L, \end{aligned}$$

which implies that $\lambda_{j_i}^*(\mathcal{R})^{d/2} j_i^{-1} = L$ for every i such that $n_i > 0$, so in particular for $i = h$. This means $j_h \neq j$ satisfies $j_h \in J_L$, hence $\{j\} \subsetneq J_L$.

Assume now moreover $\nu(\Omega^*_{nj}) \leq \nu(\Omega^{*\,(n)}_j) = n$. By the pigeonhole principle and – in case the previous inequality is an equality – by $\Omega^*_{nj} \neq \Omega^{*\,(n)}_j$, at least one of the connected components of $\Omega^{*\,(n)}_j$ has volume strictly greater than n^{-1}. Put differently, if h is the index of such a component then $r_h > n^{-1/d}$. We compute

$$\begin{aligned} L &= \frac{\lambda^*_{j_h}(\mathcal{R})^{d/2}}{j_h} = \frac{r_h^d\,\lambda_{j_h}(r_h\Omega^*_{j_h})^{d/2}}{j_h} \\ &= \frac{r_h^d\,\lambda^*_{nj}(\mathcal{R})^{d/2}}{j_h} > \frac{n^{-1}\lambda^*_{nj}(\mathcal{R})^{d/2}}{j_h} = \frac{j}{j_h}\,\frac{\lambda^*_{nj}(\mathcal{R})^{d/2}}{nj} = \frac{j}{j_h}\,L, \end{aligned}$$

which means that $j < j_h$ and *a fortiori* that $j < j_L$.

Assume now instead $\nu(\Omega^*_{nj}) > \nu(\Omega^{*\,(n)}_j) = n$. The pigeonhole principle now implies that at least one connected component has volume strictly less than n^{-1}. The same argument as before with the direction of inequalities inverted yields the existence of $j' = j_h \in J_L$ such that $j > j'$. □

A consequence of this last lemma is that for $n \in \mathbb{N}$, the domain with the least number of connected components realising the eigenvalue $\lambda^*_{nj_L}(\mathcal{R})$ is unique and is given by the propagation $\Omega^{*\,(n)}_{j_L}$.

Another consequence of the proof is that K_L is generated by J_L, that is any $k \in K_L$ is a finite sum of elements in J_L. Indeed, given $k \in K_L \setminus J_L$ and a minimiser Ω^*_k, the propagation $\Omega^{*\,(j_L)}_k$ realises $\lambda^*_{j_L k}(\mathcal{R})$. The connected components of this propagation are thus contracted copies of minimisers canonically associated with J_L and so are the ones of Ω^*_k, hence the result. The minimisers $\Omega^*_{j_i} = \Omega$ with $j_i \in J_L$ are thus the building blocks of any minimiser realising L.

4. Bounds from packings

We have just seen that the failure of Pólya's conjecture for a domain Ω implies that infinitely many minimisers in $\mathcal{R}(\Omega)$ are realised by propagators $\Omega^{(n)} = \cup_{j=1}^n n^{-1/d}\Omega$. It is thus natural to study the spectrum of those propagators, notably by geometrically realising them as subsets of other domains, that is by packing the $\Omega^{(n)}$s into others domains. This packing idea leads to the main result in this section, to wit an estimate from below on $L = \inf_{k\in\mathbb{N}} \lambda^*_k(\mathcal{R})^{d/2}\, k^{-1}$ in term of the "packing density" of Ω. Recall that this packing density was defined in Definition 1.5.

We start by proving a few properties of this packing density.

Lemma 4.1. *Given three bounded domains Ω, V and W,*

$$\rho_{\Omega,W} \geq \rho_{\Omega,V}\,\rho_{V,W}.$$

Proof. Given any $\varepsilon>0$, there exist a packing g of $\Omega^{(m)}$ into V of density $\rho_g>\rho_{\Omega,V}-\varepsilon$ and an asymptotic packing $P=\{(n_i,\rho_i,f_i)\}_{i\in\mathbb{N}}$ of V into W with asymptotic density $\rho_P>\rho_{V,W}-\varepsilon$. It is very clear how g and P can be "composed" to yield an asymptotic packing of Ω into W with asymptotic density $\rho_g\rho_P>\rho_{\Omega,V}\rho_{V,W}-O(\varepsilon)$. The lemma readily follows. □

Proposition 4.2. *Let Ω and V be two bounded domains in $\mathbb{R}^d$ with volume* 1. *Suppose that Ω tiles $\mathbb{R}^d$ and that the upper Minkowski dimension of ∂V is strictly smaller than d. Then $\rho_{\Omega,V}=1$ and thus $\rho_\Omega=1$.*

Remark 4.3. We recall that the upperbox dimension or upper Minkowski dimension of a set $S\subset\mathbb{R}^d$ could be defined as

$$d_{\text{up}}(S):=d-\liminf_{r\to 0^+}\frac{\log|S(r)|}{\log r}$$

where $S(r):=\{y\in\mathbb{R}^d : \|y-S\|<r\}$ is the r-neighbourhood of S.

Proof. For simplicity, suppose $0\in\text{int}(V)\subset\mathbb{R}^d$ and consider that any homothety to be performed below is with respect to 0. We shall also think of the tiling F as a mere quasi-inclusion and we will not use F in our notations.

Since V is bounded, there exists $R>0$ such that $V\subset rV$ for all $r\geq R$. Consequently, we have the sequence of inclusions

$$V\subset RV\subset R^2V\subset R^3V\subset ...$$

Without lost of generality, take $R\in\mathbb{N}$.

Denote Ω_i the i-th component Ω in the disjoint union $\sqcup_{i\in\mathbb{N}}\Omega$. For $n\in\mathbb{N}$, let $I_n\subset\mathbb{N}$ be the largest set such that $\overline{\Omega_i}\subset nV$ for every $i\in I_n$. This set is finite as its cardinality is at most $|nV|/|\overline{\Omega}|=n$. Because of the previous paragraph, $I_{R^i}\subset I_{R^{i+1}}$ for every $i\in\mathbb{N}$. For $i\in\mathbb{N}$, set $n_i=\#I_{R^i}$.

Because Ω and hence $\overline{\Omega}$ are bounded, the latter is contained in an open ball B of diameter D. Let

$$(nV)_{2D}=\{p\in nV : \text{dist}(p,(nV)^c)\geq 2D\}.$$

We claim that the set $(nV)_{2D}\setminus\cup_{i\in I_n}\overline{\Omega_i}$ is empty. Suppose otherwise; then there exist a point x in this nonempty set and, since Ω is a tile, an index $i\in(I_n)^c$ such that $x\in\overline{\Omega_i}\subset B_i$. The definition of I_n implies $\overline{\Omega_i}\cap(nV)^c\neq\varnothing$, so there exists y in this latter intersection and thus in B_i. It follows that $\text{dist}(x,y)<2D$, which is a contradiction. This proves the claim, and consequently $(nV)_{2D}\subset\cup_{i\in I_n}\overline{\Omega_i}\subset nV$.

By assumption on $\partial(nV)$, the volume of the $2D$-neighbourhood of $\partial(nV)$ grows like $o(n^d)$, so that the volume of $(nV)_{2D}$ grows like $n^d - o(n^d)$. From the set inclusions obtained in the previous paragraph, the same asymptotic is true for the growth of the volume of $\cup_{i\in I_n}\overline{\Omega_i}$, that is of $\sharp\, I_n$.

Consider the asymptotic packing $P=\{(n_i,\rho_i,f_i)\}_{i\in\mathbb{N}}$ given by $n_i=\sharp\, I_{R^i}$, $\rho_i=n_i/n^d$ and

$$f_i : \overline{\Omega^{(n_i)}} \cong \sqcup_{i\in I_{R^i}} n_i^{-1/d}\,\overline{\Omega_i} \hookrightarrow n_i^{-1/d}\, nV = \rho_i^{-1/d}\, V.$$

From the previous paragraph we get $\rho_P=\lim_{i\to\infty}\rho_i=1$, thus $\rho_{\Omega,V}=1$. □

The previous result suggests to define another, *a priori* smaller notion of packing density, namely the *lower packing number or lower packing density of* Ω is

$$\underline{\rho}_\Omega = \inf\,\{\rho_{\Omega,V}\ |\ V \text{ bounded domain, } |V|=1,\ d_{\text{upperbox}}(\partial V)<d\ \}\,.$$

Corollary 4.4. *For any bounded domain* $\Omega\subset\mathbb{R}^d$,

$$\underline{\rho}_\Omega = \rho_{\Omega,V} > 0$$

for any bounded tile $V\subset\mathbb{R}^d$ *whose boundary has upperbox dimension strictly less than* d.

Proof. Let $W\subset\mathbb{R}^d$ be any bounded domain whose boundary has upper Minkowski dimension strictly less than d. Then from the two previous results we get $\rho_{\Omega,W}\geq\rho_{\Omega,V}\,\rho_{V,W}=\rho_{\Omega,V}$. Taking the infimum over all W yields the equality claimed in the statement.

To prove the inequality, let's take $V=[0,1]^d$. Since Ω is bounded, there clearly is some $\rho\in(0,1]$ such that Ω can be packed in $\rho^{-1/d}V$. Since $V^{(i^d)}$ fully pack V for each integer i, by "composing" packings we deduce that there is at least one asymptotic packing of Ω into V with constant density $\rho>0$, and *a fortiori* we get $\rho_{\Omega,V}>0$. □

Remark 4.5. In the few last results, the assumption on the upperbox dimension – which guaranteed that the boundary had vanishing Lebesgue measure – was not superfluous. Indeed, given any $\varepsilon>0$, it is possible to find a bounded tile $V_\varepsilon\subset\mathbb{R}^d$ with volume 1 such that $|\text{int}(V_\varepsilon)|<\varepsilon$, for instance by applying a suitable symmetric adaptation of Knopp's construction of a Osgood "surface" on the sides of a cube; the packing density of a typical domain Ω into V_ε would thus be smaller than ε. We leave the details to the industrious reader.

We are now in a position to prove a lower bound on

$$L := \inf_k \lambda_k^*(\mathcal{R})^{d/2}\, k^{-1}$$

for the Dirichlet Laplacian eigenvalue problem in the class $\mathcal{R}(\Omega)$.

Lemma 4.6. *Assume that* $K_L=\{k\in\mathbb{N} : \lambda_k^*(\mathcal{R})^{d/2}\, k^{-1}=L\}$ *is non-empty. Then*

$$L\geq \rho_\Omega\, \frac{(2\pi)^d}{\omega_d}. \tag{21}$$

Proof. Using Lemma 2.6 in a way we already repeatedly used it before, we deduce from the assumption $K_L\neq\varnothing$ that $J_L=\{k\in K_L : \lambda_k(\Omega)=\lambda_k^*(\mathcal{R})\}\neq\varnothing$. Pick some $j\in J_L$.

Let $\varepsilon>0$ and consider an open bounded domain V with volume 1 such that $\rho_{\Omega,V}\geq\rho_\Omega-\varepsilon/2$. Consequently, there exist an asymptotic packing $P=\{(n_i,\rho_i,f_i)\}_{i\in\mathbb{N}}$ of Ω into V such that $\rho_P=\lim_{i\to\infty}\rho_i\geq\rho_\Omega-\varepsilon$.

The isometric quasi-embedding $f_i:\Omega_j^{*\,(n_i)}\to\rho_i^{-1/d}\,V$ allows us to view $\Omega_j^{*\,(n_i)}$ as a genuine subset of $\rho_i^{-1/d}\,V$. Considering the well-known fact that any Dirichlet eigenvalue functional $\Upsilon\mapsto\lambda_k(\Upsilon)$ is decreasing with respect to inclusion, namely that $\Upsilon_1\subset\Upsilon_2$ implies $\lambda_k(\Upsilon_1)\geq\lambda_k(\Upsilon_2)$, it follows that

$$\lambda_{n_ij}\left(\Omega_j^{*\,(n_i)}\right)^{d/2}\geq\lambda_{n_ij}\left(\rho_i^{-1/d}\,V\right)^{d/2}.$$

The left-hand side is equal to $n_i\lambda_j(\Omega_j^*)=n_ijL$, whereas the right-hand side equals $\rho_i\,\lambda_{n_ij}(V)^{d/2}$. Therefore

$$L\geq\rho_i\,\frac{\lambda_{n_ij}\,(V)^{d/2}}{n_ij}.$$

Since $\lim_{i\to\infty}\rho_i=\rho_P\geq\rho_\Omega-\varepsilon$ and because of Weyl's asymptotic law, taking the limit $i\to+\infty$ on the right-hand side yields

$$L\geq(\rho_\Omega-\varepsilon)\,\frac{(2\pi)^d}{\omega_d}.$$

As this is true for any $\varepsilon>0$, the result follows. □

Theorem 1.8 follows as a corollary of the previous Lemma.

Proof of Theorem 1.8. The set $J=\{k\in\mathbb{N} : \Omega \text{ realises } \lambda_k^*(\mathcal{R})\}$ is either infinite or finite. If it is infinite, Proposition 3.3 implies Pólya's conjecture and *a fortiori* (21) as $\rho_\Omega\leq 1$. If instead it is finite, then Proposition 3.5 implies that K_L is non-empty and the claim follows from the previous lemma. □

We now prove that if a domain V of unit volume satisfies a two-term Weyl law for Dirichlet eigenvalues, then all V-tiles satisfy the strong Pólya conjecture.

Proof of Theorem 1.11 for Dirichlet eigenvalues. Fix a rank j for which Ω realises λ_j^*. Since Ω is a V-tile, there is an asymptotic packing $P=\{(n_i,1,f_i)\}_{i\in\mathbb{N}}$

of Ω into V with constant packing density 1. Since V satisfies the two-term Weyl law (1), there is $M\in\mathbb{N}$ such that

$$\frac{\lambda_m(V)^{d/2}}{m} > \frac{(2\pi)^d}{\omega_d} \quad \forall m \geq M.$$

Consider $i\in\mathbb{N}$ sufficiently large so that $n_i j\geq M$, and consider the (full) packing $f_i:\Omega^{(n_i)}\to V$. Invoking the monotonicity of Dirichlet eigenvalues we thus get

$$\frac{\lambda_j(\Omega)^{d/2}}{j} = \frac{\lambda_{n_i j}(\Omega^{(n_i)})^{d/2}}{n_i j} \geq \frac{\lambda_{n_i j}(V)^{d/2}}{n_i j} > \frac{(2\pi)^d}{\omega_d}.$$

Considering Proposition 3.5, this implies that Ω realises λ_k^* infinitely often. Using Corollary 2.5, we deduce

$$L := \inf_k \frac{\lambda_k^*(\mathcal{R})^{d/2}}{k} = \inf_{j\in J} \frac{\lambda_j^*(\mathcal{R})^{d/2}}{j}.$$

It also means that L is not attained among the indices in J. We claim that L is not attained in $\mathcal{R}$ at all, from which the last part of the theorem readily results. Suppose otherwise, so that there exist $k\in\mathbb{N}$ and $\Omega_k^*\in\mathcal{R}$ such that $\lambda_k(\Omega_k^*)k^{-1}=L$. By Lemma 2.6, any connected component of Ω_k^* is a (contracted copy of some) minimiser Ω_m^*. Note in particular that $m\in J$. From Proposition 3.7 follows that Ω_k^* propagates as a minimiser, hence Ω_m^* weakly propagates as a minimiser by definition. Lemma 3.8 implies that Ω_m^* propagates as a minimiser, and so Ω_m^* realises L by Proposition 3.7. This is a contradiction. □

The proof of Theorem 1.11 for Neumann eigenvalues is a bit more subtle and this is due to the fact that Neumann eigenvalues do not behave in any simple way under inclusion. This is also why Theorem 1.8 or modifications of it fail in that situation: the behaviour under inclusion depends on the eigenfunctions of the Laplacian. When the quasi-embeddings are actually surjective, however, we can adapt [13, Theorem 63] to our needs.

Lemma 4.7. *Let $V_1,\ldots,V_N,W\subset\mathbb{R}^d$ be domains with Lipschitz boundaries. Assume that $F:V:=\sqcup_{j=1}^N \overline{V_j}\to W$ is an isometric quasi-embedding, which induces a pullback map $F^*:H^1(W)\to H^1(V)$ between Sobolev spaces. Denote $E_V(k)\subset H^1(V)$ and $E_W(k)\subset H^1(W)$ the subspaces generated by the first k Neumann eigenfunctions on V and W, respectively. Then for any fixed $k\in\mathbb{N}$, there is a nonzero $\varphi\in E_W(k)$ such that $F^*\varphi$ is L^2-orthogonal to $E_V(k-1)$ and*

$$\mu_k(V) \leq \frac{\|\varphi\|^2_{L^2(W)}}{\|F^*\varphi\|^2_{L^2(V)}}\,\mu_k(W).$$

Proof. Let $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ be L^2-orthonormal bases of Neumann eigenfunctions on V and W respectively, numbered in increasing order of their eigenvalue.

Since F is an isometric quasi-embedding, we can define pushforwards $F_* f_k \in L^2(W)$ by extension by 0 outside the image of F, and $\{F_* f_k\}_{k\in\mathbb{N}}$ are still L^2-orthonormal. Given $k\in\mathbb{N}$, consider a nonzero linear combination $\varphi=\sum_{j=0}^{k} a_j g_j$, so that $\|\varphi\|^2_{L^2(W)}=\sum_{j=0}^{k} a_j^2$. The requirement that it be L^2-orthogonal to the first k functions $F_* f_j$ uniquely specifies φ up to a multiplicative constant; we note that $F^*\varphi$ is then L^2-orthogonal to the first k functions f_j. On the one hand, we have

$$\int_W \|\nabla\varphi\|^2\, dm = \sum_{i,j=0}^{k} a_i a_j \int_W \langle \nabla g_i, \nabla g_j\rangle\, dm = \sum_{j=0}^{k} a_j^2 \mu_j(W) \le \mu_k(W)\|\varphi\|^2_{L^2(W)},$$

while on the other hand

$$\int_W \|\nabla\varphi\|^2\, dm = \int_V \|\nabla F^*\varphi\|^2\, dm \ge \mu_k(V)\|F^*\varphi\|^2_{L^2(V)}$$

due to the fact that $f_0, \ldots, f_{k-1}, \varphi \in H^1(V)$ generates a k-dimensional subspaces and the variational characterisation of $\mu_K(V)$ as the infimum over such subspaces of the maximum of the Rayleigh quotient over elements of the subspace. The claim readily follows. □

Corollary 4.8. *In the context of the previous lemma, if we further assume that F is surjective, then $\mu_k(V)\le\mu_k(W)$ for all k.*

Proof. Since the boundary of V has vanishing Lebesgue measure (being Lipschitz) and since F is an isometry, it follows that $\|\varphi\|^2_{L^2(W)}=\|F^*\varphi\|^2_{L^2(V)}$. □

We now have all the necessary ingredients to prove Theorem 1.11.

Proof of Theorem 1.11 for Neumann eigenvalues. The proof follows the same scheme as the proof of Theorem 1.11 for Dirichlet eigenvalues, using everywhere the corresponding results; notably, monotonicity is replaced by the Corollary 4.8 and Lemma 2.6 is replaced by Lemma 2.7. □

5. Computational results

The proposed way of approaching Pólya's conjecture for a given domain Ω generates a sequence of extremal sets made up of copies of Ω. As we have seen, this sequence encodes information as to whether the generator set Ω satisfies the conjecture, which goes beyond whether the corresponding eigenvalues satisfy inequalities (4). These include the behaviour of the number of connected components of the sequence of extremal sets and the behaviour of the largest scaling coefficient $r_{1,k}$, for instance.

In this section, we present an investigation of the set of ranks for which the generator is a minimiser for the Dirichlet eigenvalues, and how the above indicators evolve. We chose as generators the disk, the square, and a rectangle of aspect ratio 1:5. The reasons for choosing these generators are as follows.

• The exact values of the eigenvalues are known, and can be computed to high accuracy even at high ranks. This would not necessarily be the case if we had to approximate eigenvalues using, say, finite element methods.

• It is not known whether or not the disk satisfies Pólya's conjecture, as opposed to rectangles. This means that we can compare the evolution of the indicators in comparison for those two settings.

To generate the set of minimisers, we proceed in two steps. The first one consists in creating a list of eigenvalues for the generators; for the square and the rectangle this is not a problem since eigenvalues are given by sum of squares of integers. For the disk the first step consists in generating the zeros of Bessel functions. We denote by $j_{\nu,k}$ the k-th zero of the Bessel function J_ν. The generation of the list of $j_{\nu,k}$ was done using the Chebfun MATLAB package [15]. Two things were important to consider:

• Bessel functions of high rank ν are very small (under machine precision) but strictly positive for a large interval starting at 0. Root finding algorithms would nevertheless find zeros in that range.

• All zeros have to be accounted for under a given value.

The first point is addressed by using the well-known fact that the first zero of the Bessel functions J_ν is always located at some $x>\nu$, and J_ν is sufficiently large within that range that no spurious zeros are found. The second point is addressed by using the property that if $\nu'>\nu$, then for all $k\in\mathbb{N}$, we have that $j_{\nu',k}>j_{\nu,k}$. Hence, we can choose as natural stopping points the first zero of a Bessel function of rank N. We then find all $j_{\nu,k}\in[\nu, j_{N,1}]$ and we can be assured that no zeros have been skipped. The following pseudo-code will generate the list of Dirichlet eigenvalues of the disk (keeping in mind that the multiplicity of the eigenvalues coming from Bessel zeros of rank $\nu\geq 1$ is 2, and those coming from J_0 have multiplicity 1).

Algorithm 5.1: GenerateDiskEigenvalues(N)

bound=First root of J_N above N
evalues=All roots $j_{0,k}^2$ between 0 and *bound*
for $\nu=1$ **to** N
  do *evalues*=*evalues*+2 copies of all roots $j_{\nu,k}^2$ between ν and *bound*
return (*evalues*)

To find the roots, we used the routine associated with the *chebfun* type of the aforementioned Chebfun package.

To find the minimisers, we used an approach based on Theorem 2.6. For some eigenvalue rank, say k, the minimiser is either the generator, or, for any partition of the set of connected components into two subsets, these two subsets themselves realise $\lambda_j^*(\mathcal{R})$ and $\lambda_{j'}^*(\mathcal{R})$ for some $j+j'=k$. Furthermore, in any such case $\lambda_k^*=\lambda_j^*+\lambda_{j'}^*$. We therefore can find the minimisers recursively, if we have a list of the eigenvalues of the generator, and a list of previous minimisers. The following pseudocode will generate such a list under these conditions; it is defined recursively and outputs a pair consisting of the list of minimal eigenvalues and a list of the ranks each connected component making up the minimiser at rank k minimises themselves, according to Theorem 2.6.

Algorithm 5.2: $\{\textsc{minevs},\textsc{ranks}\}(generatorevs, k)$

$min = generatorevs[k]$
$minrank = k$
for $j=1$ **to** $k/2$
  do if $minevs[j]+minevs[k-j]<min$
  then $\begin{cases} min = minevs[j]+minevs[k-j] \\ minrank = j \end{cases}$
$minevs[k]=min$
if $minrank==k$
  then $ranks[k]=\{k\}$
  else $ranks[k]=ranks[minrank]\cup ranks[k-minrank]$

The trichotomy in Theorem 1.3 indicates that if the generator itself is a minimiser infinitely often in $\mathcal{R}$, then Pólya's conjecture holds in this case, as well as for any disjoint union of it. As such, we investigate the log-density of the number ranks for which the generator itself is a minimiser, that is, the function defined in (5).

Theorem 1.4 tells us that another indicator to verify is the largest homothety coefficient r_k of the minimiser, and that the strong Pólya conjecture is equivalent to this coefficient converging to 1 as $k\to\infty$. As seen in the proof of Theorem 1.4, this is implied also by the rank of the maximal eigenvalue supported by one of the connected component growing asymptotically like k.

We show these relevant quantities for the case of the disk in Figure 1, with the corresponding values for the square being shown in Figure 2 for comparison. At a first glance, the qualitative behaviour for these two examples appears to be similar, with the only major difference that is visible is that the logarithmic density for the disk as a minimiser in the corresponding sequence appears to be approaching a value somewhat below that of the square.

In view of Theorem 3.4, another interesting indicator is the number of connected components of the minimisers. As we have proved, if it grows at $o(k)$ rate, k being the eigenvalue rank, then Pólya's conjecture holds. In the range of eigenvalues that we investigated, Figure 3 shows that for the disk, square and a rectangle with

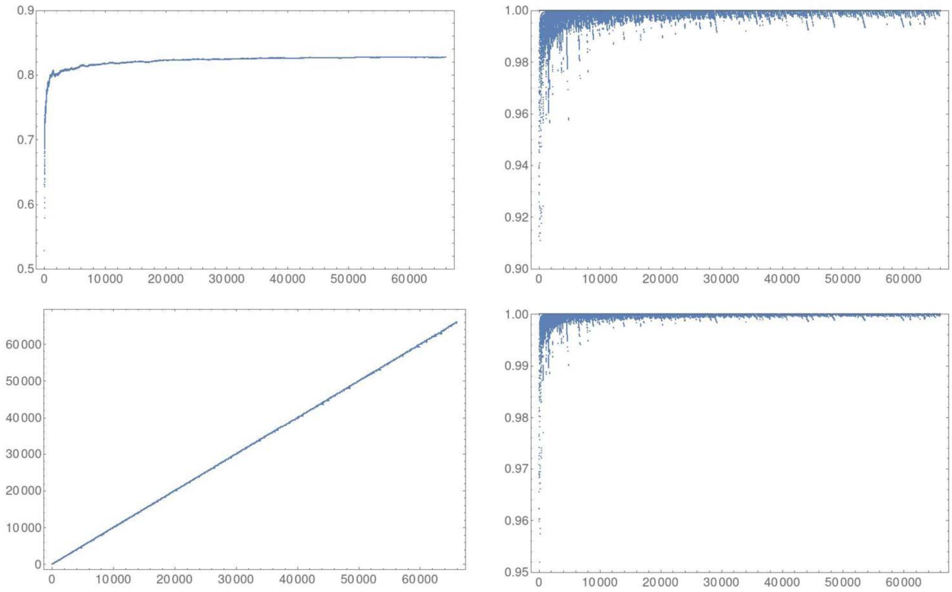

Figure 1. Logarithmic density, largest value of coefficient r_k, largest rank of an eigenvalue on one connected component and the corresponding logarithmic plot, in the case of the disk.

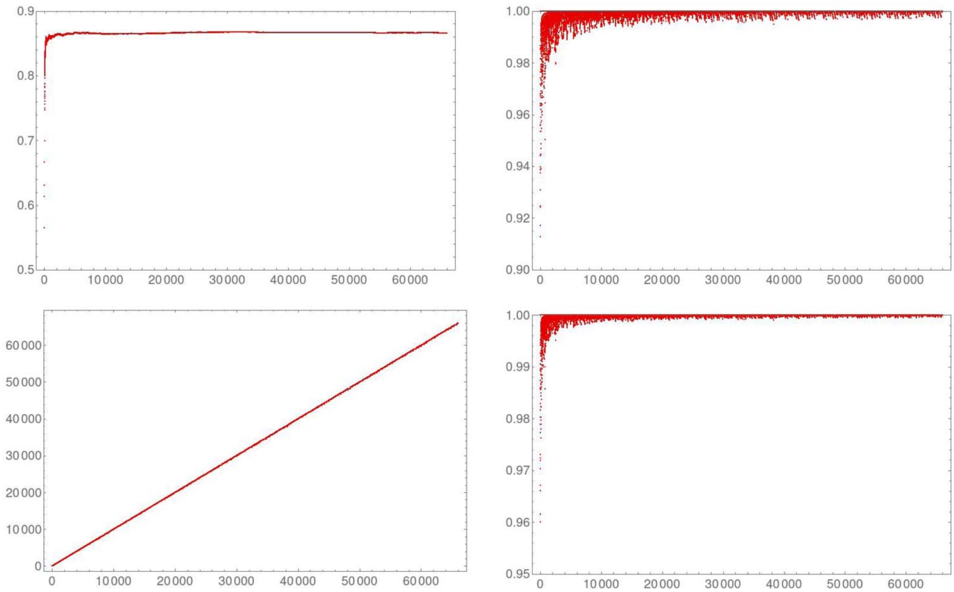

Figure 2. Same as in Figure 1, now in the case of the square.

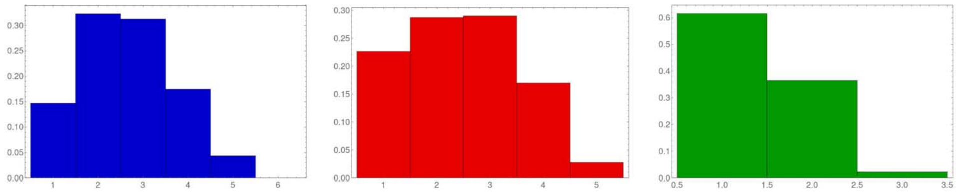

Figure 3. Histograms of the number of components: from left to right, disk, square, and rectangle with sides in the proportion of 1:5.

side ratio 1:5, the number of connected components of the minimisers keeps quite small, both the disk and the square having a maximum of five components, while the elongated rectangle exhibits at most only three.

Of course, one cannot deduce Pólya's conjecture from these experiments. However, they show that from the perspective of the quantities introduced in this paper the behaviour of the disk up to the range considered is not that dissimilar from that of the square, for instance, which is known to satisfy Pólya's conjecture. Furthermore, seeing that the behaviour of these indicators is in line with Pólya's conjecture holding, one might hope that it would be easier to prove indirectly results about the number of connected components of an extremiser, or about convergence to the generator.

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