Faces of polyhedra associated with relation modules

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Abstract. Relation Gelfand-Tsetlin $\mathfrak{gl}_n$-modules were introduced in [FRZ19], and are determined by some special directed graphs and Gelfand-Tsetlin characters. In this work, we constructed polyhedra associated with the class of relation modules, which includes as a particular case, any classical Gelfand-Tsetlin polytope. Following the ideas presented in [LM04], we give a characterization of d-faces of the associated polyhedra in terms of certain tilings and matrices related to the corresponding graph.

1. Introduction

The Gelfand-Tsetlin construction [GT50], is one of the most remarkable results in representation theory and gives an explicit realization of each simple finite-dimensional $\mathfrak{gl}_n$-module. This construction includes an explicit basis formed by tableaux with entries satisfying certain betweenness conditions. Using Gelfand-Tsetlin basis and their defining betweenness conditions, Gelfand and Zelevinsky [GZ85] introduced classes of convex polytopes as a tool for a geometric interpretation of the multiplicities of simple modules in the tensor product of two simple finite-dimensional $\mathfrak{gl}_n$-modules, and weight spaces of simple finite-dimensional $\mathfrak{gl}_n$-modules. That was the first construction to provide an effective way to relate representation theory of the general Lie algebra $\mathfrak{gl}_n$ and polyhedra. The polytopes whose number of integral points measure the weight multiplicities are called *Gelfand-Tsetlin polytopes.* In 2004, De Loera and McAllister in [LM04] get to characterize the points of a d-face (points in a face of dimension d) of the Gelfand-Tsetlin polytopes.

In [FRZ19], Futorny, Ramirez and Zhang introduced *sets of relations* as an attempt to generalize the betweenness conditions that characterize basis elements in the Gelfand-Tsetlin construction [GT50]. Associated with sets of relations the authors constructed explicit modules called *relation Gelfand-Tsetlin* $\mathfrak{gl}_n$-*modules,* *relation* $\mathfrak{gl}_n$-*modules*, or *relation modules.* Previously, several classes of relation modules were constructed by Gelfand and Graev [GG65], Lemire and Patera [LP79], and Mazorchuk [Maz03]. Using as a motivation the results in [LM04], and [FRZ19], the concept of Gelfand-Tsetlin polyhedra associated with relation modules was introduced in [Car19], and some well behave cases were studied.

Construction of polyhedra associated to a directed graph is well known, however, in this paper we focus on special graphs such that the associated polyhedra preserves the relationship between the number of "integral points" and the dimensions of the weight spaces of the corresponding relation module. Our approach includes a combinatorial characterization for the dimension of the minimal face associated to a point of the polyhedron.

In the literature, Gelfand-Tsetlin patterns appear in connection with polyhedra in [GZ85], [BZ89], [KB95], [LM04], [Pos09], [ABS11], and probability in [WW09]. In [BZ89], [KB95], [LM04], Gelfand-Tsetlin patterns are defined to be triangular arrangements of non-negative integers subject to some order restrictions, namely $\mathcal{C}_1$ (see Example 3.1). There are many different ways to define Gelfand-Tsetlin polytopes and polyhedra. In [KB95, p. 92], and [LM04, Definition 1.2., p. 460] are certain subsets of $\mathbb{R}_{\geq 0}^{n(n+1)/2}$. In [ABS11], [GKT13] the authors fixed the nth row and do not require the entries to be non-negative. In contrast, Danilov, Karzanov and Koshevoy, in [DKK05], study certain polyhedra related to an extension of Gelfand-Tsetlin patterns, in the sense that it is not necessary a triangular configuration. In this paper we fix $\mathfrak{gl}_n$ and we will define $\mathcal{C}$-pattern to be a triangular arrangement of complex numbers subject to some order restrictions $\mathcal{C}$. Our definition of $\mathcal{C}$-pattern is also an extension of Gelfand-Tsetlin patterns associated with $\mathfrak{gl}_n$.

This paper is organized as follows. In Section 2, we discuss some preliminaries on highest weight modules and Gelfand-Tsetlin modules. Section 3 is dedicated to relation modules, to this goal we recall the notions of set of relations $\mathcal{C}$, its associated graph, and Gelfand-Tsetlin tableaux. Associated with a set of relations $\mathcal{C}$, in Section 4, we introduce the concept of $\mathcal{C}$-pattern and three kind of polyhedra associated with $\mathcal{C}$, we exhibit the relation of these polyhedra with $\mathfrak{gl}_n$-modules. Also, we define a tiling and tiling matrix associated with a $\mathcal{C}$-pattern, and study some of their properties. In Section 5, we use the results of the previous section to obtain a combinatorial characterization for the dimension of a minimal face with respect to a point in the polyhedron.

2. Preliminaries on modules

Unless otherwise specified, the ground field will be $\mathbb{C}$, $\mathfrak{g}$ will denote a finite-dimensional Lie algebra with a Cartan subalgebra $\mathfrak{h}$, and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. By $U(\mathfrak{g})$ we denote the universal enveloping algebra of $\mathfrak{g}$ and by V^* the dual space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of a vector space V.

2.1. Highest weight modules

We begin this section recalling some definitions and basic results on weight modules. For more details we refer the reader to [Hum08].

Definition 2.1. Let M be an $U(\mathfrak{g})$-module. For each $\lambda \in \mathfrak{h}^*$, the λ-*weight space* is defined by $M_\lambda := \{v \in M \,|\, h \cdot v = \lambda(h)v,\ \text{for all } h \in \mathfrak{h}\}$. If $M_\lambda \neq 0$, we say that λ is a *weight of M*, and $\dim(M_\lambda)$ will be called *multiplicity* of λ. M is called *weight module* if it is equal to the direct sum of its weight spaces.

A $\mathfrak{g}$-module M is a *highest weight module* if there exist $\lambda \in \mathfrak{h}^*$, and a nonzero vector $v^+ \in M_\lambda$ such that $M = U(\mathfrak{g}) \cdot v^+$ and $U(\mathfrak{n}^+) \cdot v^+ = 0$. The vector v^+ is called a *highest weight vector* of M and λ a *highest weight* of M.

It is a well known result in representation theory that simple finite-dimensional $\mathfrak{gl}_n$-modules are highest weight modules, and there is a correspondence between simple finite-dimensional modules and integral dominant $\mathfrak{gl}_n$-weights (i.e. n-tuples $\lambda \in \mathbb{Z}^n$ satisfying $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$, for all $i = 1, 2, ..., n-1$).

2.2. Gelfand-Tsetlin modules

From now on we fix $n \geq 2$, $\mathfrak{g} := \mathfrak{gl}_n$ to be linear Lie algebra of $n \times n$ matrices over $\mathbb{C}$, the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, and the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+$ is the subalgebra of upper triangular matrices and $\mathfrak{n}^-$ the subalgebra of lower triangular matrices.

A large and important class of weight modules that have been extensively studied in the last 30 years is the so-called *Gelfand-Tsetlin modules*. This modules are characterized by a well-behaved action of a fixed maximal commutative subalgebra of $U(\mathfrak{gl}_n)$, known as *Gelfand-Tsetlin subalgebra*. In what follows, we give a formal definition of Gelfand-Tsetlin subalgebra and Gelfand-Tsetlin modules.

For $m \leq n$, $\mathfrak{gl}_m$ naturally identifies with the subalgebra of $\mathfrak{gl}_n$ spanned by $\{E_{ij} \,|\, i, j = 1, ..., m\}$, where E_{ij} is the (i, j)th elementary matrix. We have the chain $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset ... \subset \mathfrak{gl}_n$, which indices the chain $U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset ... \subset U(\mathfrak{gl}_n)$ for the universal enveloping algebras. The *Gelfand-Tsetlin subalgebra* Γ of $U(\mathfrak{gl}_n)$ ([DFO94]) is generated by $\{Z_m \,|\, m = 1, ..., n\}$, where Z_m stands for the center of $U(\mathfrak{gl}_m)$.

Definition 2.2. A finitely generated $U(\mathfrak{gl}_n)$-module M is called a *Gelfand-Tsetlin module (with respect to Γ)* if $M|_\Gamma = \bigoplus_{\chi \in \Gamma^*} M_\chi$, where

$$M_\chi = \left\{ v \in M \mid \text{for each } \gamma \in \Gamma, \ \exists k \in \mathbb{Z}_{\geq 0} \text{ such that } (\gamma - \chi(\gamma))^k v = 0 \right\}.$$

Remark 2.3. Any weight module with finite-dimensional weight spaces (in particular any module in category $\mathcal{O}$) is a Gelfand-Tsetlin module. Moreover, as $\mathfrak{h} \subseteq \Gamma$, any simple Gelfand-Tsetlin module is a weight module.

3. Relation modules

The class of *relation* Gelfand-Tsetlin modules was introduced in [FRZ19]. These modules generalize the construction of simple finite-dimensional modules [GT50] and generic Gelfand-Tsetlin modules [DFO94]. We recall the construction and main properties.

Set $\mathfrak{V} := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq j \leq i \leq n\}$, and $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subseteq \mathfrak{V} \times \mathfrak{V}$, where

$$\begin{aligned}
\mathcal{R}^+ &:= \{((i,j);(i-1,t)) \mid 2 \leq j \leq i \leq n, \ 1 \leq t \leq i-1\}, \\
\mathcal{R}^- &:= \{((i,j);(i+1,s)) \mid 1 \leq j \leq i \leq n-1, \ 1 \leq s \leq i+1\}, \\
\mathcal{R}^0 &:= \{((n,i);(n,j)) \mid 1 \leq i \neq j \leq n\}.
\end{aligned}$$

Any subset $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*. By $G(\mathcal{C})$ we denote the directed graph with set of vertices $\mathfrak{V}$, and an arrow from vertex (i,j) to (r,s) if and only if $((i,j);(r,s)) \in \mathcal{C}$. For convenience, we will picture the set of vertices $\mathfrak{V}$ as a triangular arrangement with n rows, where the kth row is $\{(k,1),...,(k,k)\}$.

Example 3.1. For any $1 \leq k \leq n$, we consider the relations $\mathcal{C}_k := \mathcal{C}_k^+ \cup \mathcal{C}_k^-$, where $\mathcal{C}_k^+ := \{((i+1,j);(i,j)) \mid k \leq j \leq i \leq n-1\}$, $\mathcal{C}_k^- := \{((i,j);(i+1,j+1)) \mid k \leq j \leq i \leq n-1\}$.

We will refer to $\mathcal{C}_1$ as the *standard set of relations*, and to $\mathcal{C}_n = \varnothing$ as the *generic set of relations*. For $n=4$, the graphs associated with $\mathcal{C}_1^+$, $\mathcal{C}_1$, $\mathcal{C}_2^-$, and $\mathcal{C}_2$ are:
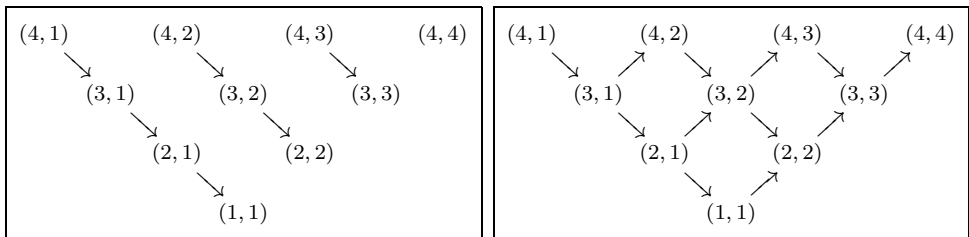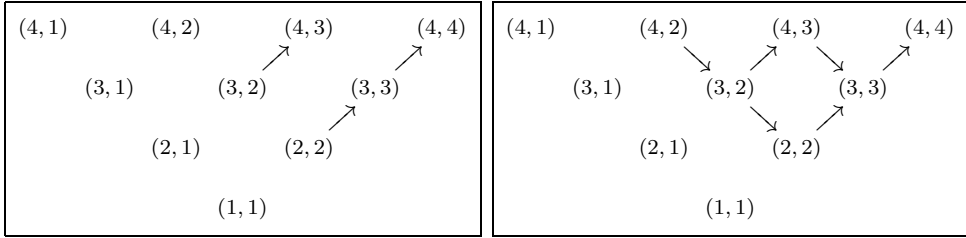
Definition 3.2. Let $\mathcal{C}$ be a set of relations.

(i) By $\mathfrak{V}(\mathcal{C})$ we will denote the subset of $\mathfrak{V}$ of all vertices that are source or target of an arrow in $G(\mathcal{C})$.

(ii) Given $(i,j),(r,s)\in\mathfrak{V}$ we write $(i,j)\succeq_{\mathcal{C}}(r,s)$ if there exists a directed path in $G(\mathcal{C})$ from (i,j) to (r,s).

Entries of vectors $L\in\mathbb{C}^{n(n+1)/2}$ will be indexed by elements of $\mathfrak{V}$, ordered as $(l_{n1},...,l_{nn}|l_{n-1,1},...,l_{n-1,n-1}|...|l_{21},l_{22}|l_{11})$, and $T(L)$ will denote the triangular configuration of height n, with kth row $(l_{k1},...,l_{kk})$ for $k=1,...,n$. We will refer to $T(L)$ as *Gelfand-Tsetlin tableau*. For any $\mathbb{A}\subseteq\mathbb{C}^{n(n+1)/2}$, we denote by $\mathbf{T}(\mathbb{A})$ the set of all Gelfand-Tsetlin tableaux $T(L)$, with $L\in\mathbb{A}$. By $\mathbb{Z}_0^{n(n+1)/2}$ we will denote the set of vectors L in $\mathbb{Z}^{n(n+1)/2}$ such that $l_{ni}=0$ for $i=1,...,n$. Given a Gelfand-Tsetlin tableau $T(L)$, elements of the set $\mathbf{T}(L+\mathbb{Z}_0^{n(n+1)/2})$ will be called *L-integral tableaux*.

Whenever we refer to connected components of a directed graph G, we are referring to the connected components of the graph $\overline{G}$ obtained from G forgetting the orientation of the arrows. A *walk* in G is by definition a path in $\overline{G}$.

Definition 3.3. ([FRZ19, Definition 4.2]) Let $\mathcal{C}$ be a set of relations and $T(L)$ any Gelfand-Tsetlin tableau.

(i) We say that $T(L)$ *satisfies* $\mathcal{C}$, if $l_{ij}-l_{rs}\in\mathbb{Z}_{\geq 0}$ for any $((i,j);(r,s))\in\mathcal{C}$.
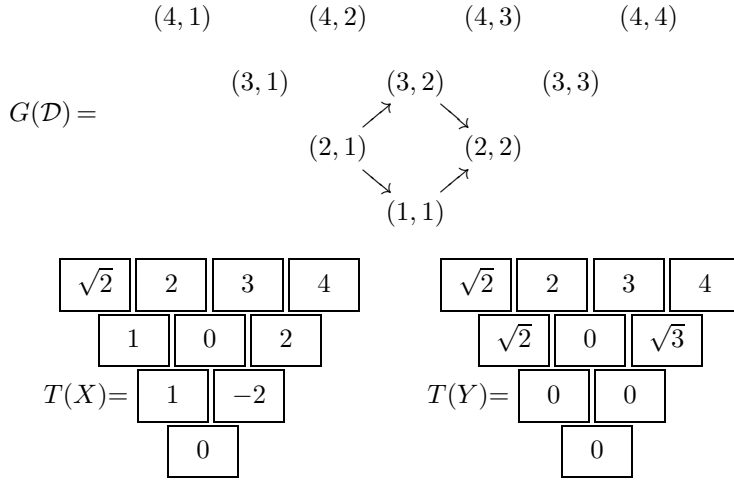
(ii) $T(L)$ is a $\mathcal{C}$-*realization*, if $T(L)$ satisfies $\mathcal{C}$ and for any $1\leq k\leq n-1$ we have, $l_{ki}-l_{kj}\in\mathbb{Z}$ if and only if (k,i) and (k,j) belong to the same connected component of $G(\mathcal{C})$.

(iii) We call $\mathcal{C}$ *noncritical* if for any $\mathcal{C}$-realization $T(L)$ one has $l_{ki}-l_{kj}+j-i\neq 0$, $1\leq k\leq n-1$, $i\neq j$, whenever (k,i) and (k,j) belong to the same connected component of $G(\mathcal{C})$.

(iv) Suppose that $T(L)$ satisfies $\mathcal{C}$. By $\mathcal{B}_{\mathcal{C}}(T(L))$ we denote the set of L-integral tableaux satisfying $\mathcal{C}$, and by $V_{\mathcal{C}}(T(L))$ the vector space over $\mathbb{C}$ with basis $\mathcal{B}_{\mathcal{C}}(T(L))$.

Remark 3.4. Whenever we use a Gelfand-Tsetlin tableau $T(M)$, we are considering it as an element of a vector space of the form $V_{\mathcal{C}}(T(L))$ for some set of relations $\mathcal{C}$. In particular, we should be careful when comparing M with $T(M)$. For instance, $T(M+N)\neq T(M)+T(N)$.

Example 3.5. Let $\mathcal{D}$ be the set of relations with associated graph $G(\mathcal{D})$, and Gelfand-Tsetlin tableaux $T(X)$ and $T(Y)$ defined as follows:

$$(4,1) \qquad\qquad (4,2) \qquad\qquad (4,3) \qquad\qquad (4,4)$$

$$G(\mathcal{D}) =$$

$$(3,1) \qquad (3,2) \qquad (3,3)$$
$$(2,1) \qquad\qquad (2,2)$$
$$(1,1)$$

$$T(X)=\quad \boxed{\sqrt{2}}\ \boxed{2}\ \boxed{3}\ \boxed{4}$$
$$\boxed{1}\ \boxed{0}\ \boxed{2}$$
$$\boxed{1}\ \boxed{-2}$$
$$\boxed{0}$$

$$T(Y)=\quad \boxed{\sqrt{2}}\ \boxed{2}\ \boxed{3}\ \boxed{4}$$
$$\boxed{\sqrt{2}}\ \boxed{0}\ \boxed{\sqrt{3}}$$
$$\boxed{0}\ \boxed{0}$$
$$\boxed{0}$$

In this case, $T(X)$ and $T(Y)$ satisfy $\mathcal{D}$, $T(Y)$ is a $\mathcal{D}$-realization, and $T(X)$ is not a $\mathcal{D}$-realization (for instance, $x_{31}-x_{32}\in\mathbb{Z}$). Moreover, as $(2,1)\succeq_{\mathcal{D}}(2,2)$, for any $\mathcal{D}$-realization $T(Z)$ we should have $z_{21}-z_{22}\in\mathbb{Z}_{\geq 0}$, in particular $\mathcal{D}$ is a noncritical set of relations.

Remark 3.6. Definitions 3.3(i), (iii) are slightly different from the original definitions in [FRZ19], however, they can be recovered with the shift $l_{kj}\mapsto l_{kj}+j-1$.

Definition 3.7. ([FRZ19, Definition 4.4]) Let $\mathcal{C}$ be a set of relations. We call $\mathcal{C}$ *admissible* if for any $\mathcal{C}$-realization $T(L)$, $V_{\mathcal{C}}(T(L))$ has structure of a $\mathfrak{gl}_n$-module, endowed with the following action of the generators of $\mathfrak{gl}_n$ on any $T(M)\in\mathcal{B}_{\mathcal{C}}(T(L))$,

$$(1) \qquad E_{k,k+1}(T(M)) = -\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k+1}(m_{ki}-m_{k+1,j}+j-i)}{\prod_{j\neq i}^{k}(m_{ki}-m_{kj}+j-i)}\right)T(M+\delta^{ki}),$$

$$(2) \qquad E_{k+1,k}(T(M)) = \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(m_{ki}-m_{k-1,j}+j-i)}{\prod_{j\neq i}^{k}(m_{ki}-m_{kj}+j-i)}\right)T(M-\delta^{ki}),$$

$$(3) \qquad E_{kk}(T(M)) = \left(\sum_{i=1}^{k} m_{ki} - \sum_{i=1}^{k-1} m_{k-1,i} \right) T(M),$$

where δ^{ki} stands for the vector in $\mathbf{T}(\mathbb{Z}_0^{n(n+1)/2})$ such that $(\delta^{ki})_{rs} = \delta_{kr}\delta_{is}$. If the new tableau $T(M \pm \delta^{ki})$ does not belong to $\mathcal{B}_{\mathcal{C}}(T(L))$, then the corresponding summand of $E_{k,k+1}(T(M))$ or $E_{k+1,k}(T(M))$ is zero by definition. If $\mathcal{C}$ is admissible, and $T(L)$ a $\mathcal{C}$-realization, we will call $V_{\mathcal{C}}(T(L))$ a *relation module*.

Remark 3.8. By Equation (3), whenever $\mathcal{C}$ is an admissible set of relations, $\mathcal{B}_{\mathcal{C}}(T(L))$ is an eigenbasis for the action of the Cartan subalgebra $\{E_{11}, ..., E_{nn}\}$. Moreover, any relation module is a Gelfand-Tsetlin module, with diagonalizable action of the Gelfand-Tsetlin subalgebra Γ (see [FRZ19, Theorem 5.3]).

Definition 3.9. Let $\mathcal{C}$ be any set of relations.
(i) We call $G(\mathcal{C})$ *reduced*, if for any $(k,j) \in \mathfrak{V}$ the following is satisfied:
— There exist at most one arrow pointing up with target (k,j), and at most one arrow pointing down with target (k,j).
— There exist at most one arrow pointing up with source (k,j), and at most one arrow pointing down with source (k,j).
— There exist at most one horizontal arrow with target (k,j), and at most one arrow horizontal arrow with source (k,j).
(ii) Set $k \neq n$, we called $((k,i);(k,j)) \in \mathfrak{V} \times \mathfrak{V}$ an *adjoining pair* for $G(\mathcal{C})$, if:
— $i < j$,
— $(k,i) \succeq_{\mathcal{C}} (k,j)$,
— $(k,i) \succeq_{\mathcal{C}} (k,t) \succeq_{\mathcal{C}} (k,j)$ implies $t = i$ or $t = j$.

Theorem 3.10. ([*FRZ*19, *Theorem* 4.33]) *Suppose that* $\mathcal{C}$ *is a noncritical set of relations whose associated graph* $G = G(\mathcal{C})$ *satisfies the following conditions*:
(i) G *does not contain loops, and* $(k,i) \succeq_{\mathcal{C}} (k,j)$ *implies* $i \leq j$;
(ii) G *is reduced*;
(iii) *If* G *contains an arrow connecting* (k,i) *and* $(k+1,t)$, *then* $(k+1,s)$ *and* (k,j) *with* $i < j$, $s < t$ *are not connected in* G.
$\mathcal{C}$ *is an admissible set of relations if and only if for any connected component* $G(\mathcal{E})$ *of* $G(\mathcal{C})$ *and any adjoining pair* $((k,i);(k,j))$ *in* $G(\mathcal{E})$, *there exist* p, q *such that* $\mathcal{E}_1 \subseteq \mathcal{E}$ *or, there exist* $s < t$ *such that* $\mathcal{E}_2 \subseteq \mathcal{E}$, *where the graphs associated to* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are as follows*
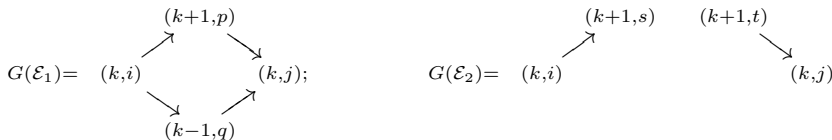
Remark 3.11. Follows from Theorem 3.10 that the sets of relations $\mathcal{C}_k$, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ from Example 3.1, are admissible set of relations for any $1 \leq k \leq n$.

Gelfand-Tsetlin construction [GT50], is one of the most remarkable results in representation theory and gives an explicit realization of any simple finite-dimensional module. The following theorem describes Gelfand-Tsetlin construction in terms of the standard set of relations.

Theorem 3.12. *If* $\lambda := (\lambda_1, ..., \lambda_n)$ *is an integral dominant* $\mathfrak{gl}_n$*-weight and* $T(\Lambda)$ *is the Gelfand-Tsetlin tableau of height n with entries* $\lambda_{ki} := \lambda_i$, *then* $V_{\mathcal{C}_1}(T(\Lambda))$ *is isomorphic to the simple finite-dimensional module* $L(\lambda)$. *Moreover,*

(i) $\mathcal{B}_{\mathcal{C}_1}(T(\Lambda))$ *is a basis of* $V_{\mathcal{C}_1}(T(\Lambda))$.

(ii) *For any* $\mu = (\mu_1, ..., \mu_n) \in \mathfrak{h}^*$, *the weight space* $L(\lambda)_\mu$ *has a basis*

$$\left\{ T(X) \in \mathcal{B}_{\mathcal{C}_1}(T(\Lambda)) \ \Big| \ \mu_k = \sum_{i=1}^{k} x_{ki} - \sum_{i=1}^{k-1} x_{k-1,i} \ for \ all \ k = 1, ..., n \right\}.$$

Example 3.13. Let $\mathcal{C}$ be one of the sets of relations defined in Example 3.1, $T(L)$ a $\mathcal{C}$-realization and $\lambda := (l_{n1}, ..., l_{nn})$. As we mention before, they are admissible sets of relations and:

(i) Set $\mathcal{C} = \mathcal{C}_1^+$, in this case $V_{\mathcal{C}}(T(L))$ is isomorphic to the *generic* Verma module $M(\lambda)$ (see [FRZ19, Example 5.10]).

(ii) Set $\mathcal{C} = \mathcal{C}_2$, and $\tilde{\lambda} := (l_{n2}, ..., l_{nn})$. In this case $V_{\mathcal{C}}(T(L))$ is an infinite-dimensional module with finite-dimensional weight spaces (see [Maz03, Section 3]), all of them of dimension $\dim(L(\tilde{\lambda}))$. In fact, by Theorem 3.12, $L(\tilde{\lambda})$ has a basis $\mathcal{B}$ parameterized by the set of standard tableaux in $\mathbf{T}(\mathbb{Z}^{n(n-1)/2})$ with top row $\tilde{\lambda}$. Given μ a weight of $V_{\mathcal{C}}(T(L))$, we consider the map $\psi_\mu : \mathcal{B} \longrightarrow \mathbf{T}(\mathbb{R}^{n(n+1)/2})$ defined by $\psi_\mu(T(\tilde{S})) = T(S)$, where:

$$s_{ij} = \begin{cases} \tilde{s}_{i-1,j-1}, & \text{if } i,j \geq 2; \\ \sum_{t=1}^{i} \mu_t - \sum_{r=2}^{i} \tilde{s}_{i-1,r-1}, & \text{if } j=1 \text{ and } 2 \leq i \leq n; \\ \mu_1, & \text{if } (i,j) = (1,1). \end{cases}$$

It is easy to check that ψ_μ is injective and $\psi(\mathcal{B})$ is a basis of $V_{\mathcal{C}}(T(L))_\mu$.

(iii) Set $\mathcal{C} = \mathcal{C}_k$ with $k \geq 3$. In this case, the module $V_{\mathcal{C}}(T(L))$ has infinite-dimensional weight spaces.

4. Polyhedra and its faces

We begin the section with some generalities about polyhedra and faces of poly-hedra. Then we introduce the main objects to be studied in this paper, namely, polyhedra associated with sets of relations. We finish the section with some techni-cal lemmas that will help us to characterize the dimension of a face associated with a given point.

Let V be a finite-dimensional $\mathbb{R}$-vector space. Given $v \in V$ and W a vector subspace of V, the set $v + W$ is called *affine subspace* of V. The dimension of $v + W$ is $\dim(W)$. For a subset X of V, we will denote by $\mathrm{aff}(X)$ the smallest affine subspace of V containing X.

A subset P of an $\mathbb{R}$-vector space V is called a *polyhedron* if it is the intersection of finitely many closed halfspaces. The *dimension* of P is given by $\dim(\mathrm{aff}(P))$. A *polytope* is a bounded polyhedron.

A hyperplane H is called a *support hyperplane* of the polyhedron P, if $H \cap P \neq \varnothing$, and P is contained in one of the two closed halfspaces bounded by H. The intersection $F = H \cap P$ is a *face* of P, and H is called a *support hyperplane associated with F*. Faces of dimension 0 are called *vertices*, and an *edge* is a face of dimension 1. In general, a face F of dimension k is called a *k-face*.

Proposition 4.1. ([BG09, Theorem 1.10(d)]) *Let P be a polyhedron in $\mathbb{R}^d$, and $x \in P$. There exists a unique face F such that $x \in \mathrm{int}(F)$, where $\mathrm{int}(F)$ denotes the relative interior of the affine subspace $\mathrm{aff}(F)$ with respect to the standard topology of $\mathbb{R}^d$.*

The face from Proposition 4.1 is the unique minimal element in the set of faces of P containing x and is called *minimal face* for x.

Lemma 4.2. *Let P be a polyhedron of an $\mathbb{R}$-vector space V and let F be a face containing $x \in P$ with $\mathrm{aff}(F) = x + H$. If there exists v such that $x \pm v \in P$, then $v \in H$.*

Proof. Suppose that $x - v, x + v \in P$. Let us consider a support hyperplane $H_\alpha = \{z \in V \mid \alpha(z) = 0\}$ of P such that $F = H_\alpha \cap P$ and $\alpha : V \longrightarrow \mathbb{R}$ its affine form, that is,

$$\alpha(z) = \beta(z) + a_0, \quad a_0 = \alpha(0)$$

for some linear map $\beta : V \longrightarrow \mathbb{R}$. As $x \in F = H_\alpha \cap P$, we can easily check that $\alpha(x \pm v) = \pm\beta(v)$. This implies that, $x \pm v \in H_\alpha^-$ if and only if $x \mp v \in H_\alpha^+$. Since H_α is a support hyperplane of P, either

$$x \pm v \in P \subseteq H_\alpha^+ := \{z \in V \mid \alpha(z) \geq 0\}, \quad \text{or} \quad x \pm v \in P \subseteq H_\alpha^- := \{z \in V \mid \alpha(z) \leq 0\}.$$

Hence $x \pm v \in H_\alpha$. Furthermore $x \pm v \in F \subseteq \mathrm{aff}(F) = x + H$, which implies $v \in H$. $\square$

Given a polyhedron $P \subseteq \mathbb{R}^{n(n+1)/2}$ and $X \in P$, by $D_X(P) \subseteq \mathbb{R}^{n(n+1)/2}$ we will denote the set $\{Y \in \mathbb{R}^{n(n+1)/2} \mid X+Y, X-Y \in P\} = (X-P) \cap (X+P)$.

Remark 4.3. Let P be a polyhedron and $X \in P$. If $\{Y_1, ..., Y_k\}$ is a maximal set of linearly independent vectors in $D_X(P)$, then the dimension of the minimal face in P containing X is k.

4.1. Polyhedra associated with sets of relations

In this section, associated with any set of relations $\mathcal{C}$ we will define several polyhedra in $\mathbb{R}^{n(n+1)/2}$, which will be the main objects of study until the end of this paper.

Definition 4.4. Let $\mathcal{C}$ be any set of relations, $X \in \mathbb{R}^{n(n+1)/2}$ is called a $\mathcal{C}$-*pattern*, if $x_{ij} \geq x_{rs}$ for any $((i,j);(r,s)) \in \mathcal{C}$.

Remark 4.5. If $\mathcal{C}$ is a set of relations and $T(L) \in \mathbf{T}(\mathbb{R}^{n(n+1)/2})$ satisfies $\mathcal{C}$ (i.e. $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ if $((i,j),(r,s)) \in \mathcal{C}$), then L is a $\mathcal{C}$-pattern. The converse is not necessarily true; in fact, let $\mathcal{D}$ be the set of relations of Example 3.5 and $T(X)$ the tableau
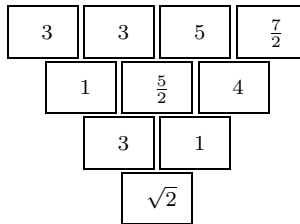
in this case X is a $\mathcal{D}$-pattern, and $T(X)$ does not satisfies $\mathcal{D}$.

For $1 \leq k \leq n$ we define $R_k : \mathbb{C}^{n(n+1)/2} \longrightarrow \mathbb{C}$ given by $R_k(X) := \sum_{i=1}^{k} x_{ki}$. The *kth weight linear map*, $w_k : \mathbb{C}^{n(n+1)/2} \longrightarrow \mathbb{C}$ is defined by

$$w_k(X) := \begin{cases} R_k(X) - R_{k-1}(X), & \text{if } 2 \leq k \leq n; \\ x_{11}, & \text{if } k=1. \end{cases}$$

Definition 4.6. Fix $\lambda, \mu \in \mathbb{R}^n$ and $\mathcal{C}$ a set of relations. We consider the following polyhedra in $\mathbb{R}^{n(n+1)/2}$ associated with $\mathcal{C}$

$$P_{\mathcal{C}} := \left\{ X \in \mathbb{R}^{n(n+1)/2} \mid X \text{ is a } \mathcal{C}\text{-pattern} \right\},$$
$$P_{\mathcal{C}}(\lambda) := \left\{ X \in P_{\mathcal{C}} \mid x_{nj} = \lambda_j \text{ for all } 1 \leq j \leq n \right\},$$
$$P_{\mathcal{C}}(\lambda, \mu) := \left\{ X \in P_{\mathcal{C}}(\lambda) \mid w_i(X) = \mu_i \text{ for all } 1 \leq i \leq n \right\}.$$

Given $y \in \mathbb{R}$, the point Y with entries $y_{ij} = y$ is a $\mathcal{C}$-pattern for any $\mathcal{C}$, in particular $P_{\mathcal{C}}$ is always unbounded. However, $P_{\mathcal{C}}(\lambda)$ is a polytope if and only if the maximal and minimal points with respect to $\succeq_{\mathcal{C}}$, belong to the set $\{(n, 1), ..., (n, n)\}$.

Remark 4.7. If $\lambda \in \mathbb{Z}^n$ is an integral dominant $\mathfrak{gl}_n$-weight with $\lambda_n \geq 0$ and $\mu \in \mathbb{Z}^n$ a weight of $L(\lambda)$, then $P_{\mathcal{C}_1}(\lambda, \mu)$ is a polytope called the *Gelfand-Tsetlin polytope* associated with λ and μ.

Lemma 4.8. *Let X be a $\mathcal{C}$-pattern in $P_{\mathcal{C}}(\lambda, \mu)$, and Y a $\mathcal{C}$-pattern.*
(i) $X + Y \in P_{\mathcal{C}}(\lambda)$ *if and only if $y_{ni} = 0$ for any $1 \leq i \leq n$.*
(ii) $X + Y \in P_{\mathcal{C}}(\lambda, \mu)$ *if and only if $X + Y \in P_{\mathcal{C}}(\lambda)$, and $R_k(Y) = 0$ for $1 \leq k \leq n$.*

Proof. As the tableau associated with the sum of $\mathcal{C}$-patterns is a $\mathcal{C}$-pattern, in order to prove (i) we just note that $x_{ni} = \lambda_i = x_{ni} + y_{ni}$ if and only if $y_{ni} = 0$. To prove (ii) we note that $X + Y \in P_{\mathcal{C}}(\lambda)$ belongs to $P_{\mathcal{C}}(\lambda, \mu)$ if and only if $\sum_{i=1}^{k} \mu_i = R_k(X + Y)$, and $R_k(X + Y) = R_k(X) + R_k(Y) = \sum_{i=1}^{k} \mu_i + R_k(Y)$ for any $1 \leq k \leq n$. $\square$

The following results establish a direct connection between relation modules and the polyhedra $P_{\mathcal{C}}$, $P_{\mathcal{C}}(\lambda)$, and $P_{\mathcal{C}}(\lambda, \mu)$ from Definition 4.6. Recall that tableaux in $\mathbf{T}(L + \mathbb{Z}_0^{n(n+1)/2})$ are called L-integral, and X in $L + \mathbb{Z}_0^{n(n+1)/2}$ is called L-*integral point*.

Theorem 4.9. *Let $\mathcal{C}$ be any admissible set of relations, $T(L)$ a $\mathcal{C}$-realization, and $V = V_{\mathcal{C}}(T(L))$ the corresponding relation $\mathfrak{gl}_n$-module. Set $\lambda = (l_{n1}, ..., l_{nn})$, and $\mu = (w_1(L), w_2(L), ..., w_n(L)) \in \mathfrak{h}^*$.*
(i) *The polyhedra $P_{\mathcal{C}}$ and $P_{\mathcal{C}}(\lambda)$ have the same number of L-integral points, and this number is equal to $\dim(V)$.*
(ii) *The number of L-integral points in $P_{\mathcal{C}}(\lambda, \mu)$ is equal to $\dim(V_{\mu})$.*

Proof. First, we note that by definition the sets of L-integral points of $P_{\mathcal{C}}$ and $P_{\mathcal{C}}(\lambda)$ coincide. By construction, $\mathbf{T}(P_{\mathcal{C}}(\lambda)) \cap \mathbf{T}(L + \mathbb{Z}_0^{n(n+1)/2})$ is a basis of the module $V_{\mathcal{C}}(T(L))$, which implies (i). Analogously, $\mathbf{T}(P_{\mathcal{C}}(\lambda, \mu)) \cap \mathbf{T}(L + \mathbb{Z}_0^{n(n+1)/2})$ is a basis of the weight space V_{μ}, implying (ii). $\square$

As a consequence of Theorem 4.9, we have the following:

Corollary 4.10. *Let $\mathcal{C}$ be an admissible set of relations, $T(L)$ a $\mathcal{C}$-realization, $\lambda = (l_{n1}, ..., l_{nn})$, and μ a weight of $V_{\mathcal{C}}(T(L))$.*

(i) *Set $\mathcal{C}=\mathcal{C}_1$. In this case, the module $V_{\mathcal{C}}(T(L))$ is isomorphic to the simple finite-dimensional module $L(\lambda)$.*

     – *The number of L-integral points in $P_{\mathcal{C}}(\lambda)$ is finite and equal to the dimension of $L(\lambda)$.*

     – *The number of L-integral points in $P_{\mathcal{C}}(\lambda, \mu)$ is finite and equal to the dimension of the weight space $L(\lambda)_\mu$ (cf. [GZ85, Proposition 1]).*

(ii) *Set $\mathcal{C}=\mathcal{C}_1^+$. In this case, the module $V_{\mathcal{C}}(T(L))$ is isomorphic to the generic Verma module $M(\lambda)$ (see [FRZ19, Example 5.10]), and*

     – *$P_{\mathcal{C}}(\lambda)$ contains infinitely many L-integral points.*

     – *The number of L-integral points in $P_{\mathcal{C}}(\lambda, \mu)$ is $\dim(M(\lambda)_\mu) < \infty$.*

(iii) *Set $\mathcal{C}=\mathcal{C}_2$. In this case, $\tilde{\lambda}=(l_{n2}, ..., l_{nn})$ is a dominant $\mathfrak{gl}_{n-1}$-weight, and $V_{\mathcal{C}}(T(L))$ is an infinite-dimensional module with finite weight spaces of dimension $\dim(L(\tilde{\lambda}))$ (see Example 3.13).*

   – *$P_{\mathcal{C}}(\lambda)$ contains infinitely many L-integral points.*

     – *If μ is a weight of $V_{\mathcal{C}}(T(L))$, the number of L-integral points in $P_{\mathcal{C}}(\lambda, \mu)$ is equal to $\dim(L(\tilde{\lambda}))$.*

Definition 4.11. Let $\mathcal{C}$ be a set of relations, and $X \in \mathbb{R}^{n(n+1)/2}$ be a $\mathcal{C}$-pattern. Associated with $\mathcal{C}$ and $T(X)$ we have an equivalence relation in $\mathfrak{V}$, where (i, j) is related with (r, s) if and only if there exists a walk in $G(\mathcal{C})$ connecting (i, j) and (r, s) with the entries of X associated with the vertices in the walk being equal. The partition of $\mathfrak{V}$ induced by the relation is called *tiling*, and is denoted by $\mathcal{M}_{\mathcal{C}}(X)$. The equivalence classes will be called *tiles*. In particular, if $(i, j) \notin \mathfrak{V}(\mathcal{C})$, the set $\{(i, j)\}$ is a tile.

Set $\Lambda \subseteq \{1, ..., n\}$. A tile $\mathcal{M}$ will be called Λ-*free* if $\mathcal{M} \cap \{(k, j) \mid 1 \leq j \leq k\} = \varnothing$ for any $k \in \Lambda$. Moreover, if (i, j) belongs to a Λ-free tile, it will be called Λ-*free*.

Remark 4.12. In this paper, we will be interested in the particular cases $\Lambda = \{n\}$, and $\Lambda = \{1, n\}$. It is worth to mention that in [LM04] the authors call free tiles what we will call here $\{1, n\}$-free tiles.
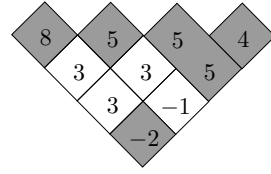
Example 4.13. Let $\mathcal{D}$ be the set of relations from Example 3.5, and $\mathcal{C}_1^+$, $\mathcal{C}_2^+$ as described in Example 3.1. The following are the tilings of a $\mathcal{D}$-pattern, and a tableau seeing as a $\mathcal{C}_1^+$-pattern and as a $\mathcal{C}_2^+$-pattern. For the picture to the left, the tiles without colouring are the $\{n\}$-free tiles, and for the other two pictures, the tiles without colouring are the $\{1, n\}$-free tiles.
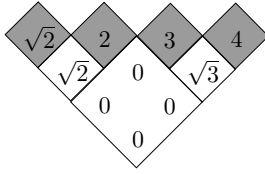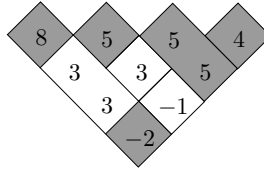
Tiling 1   Tiling 2   Tiling 3

Lemma 4.14. *Let $\mathcal{C}$ be a set of relations, X a $\mathcal{C}$-pattern, $\lambda=(x_{n1},...,x_{nn})$, and $\mu=(w_1(X), w_2(X),...,w_n(X))$.*

(i) *If $Y \in D_X(P_\mathcal{C})$, then $y_{ij}=y_{rs}$, whenever (i,j) and (r,s) belong to the same tile of $\mathcal{M}_\mathcal{C}(X)$.*

(ii) *If $Y \in D_X(P_\mathcal{C}(\lambda))$, then $y_{ij}=0$, whenever (i,j) is not $\{n\}$-free.*

(iii) *If $Y \in D_X(P_\mathcal{C}(\lambda,\mu))$, then $y_{ij}=0$, whenever (i,j) is not $\{1,n\}$-free.*

Proof. If $(i,j)=(r,s)$ there is nothing to prove, so we can assume (i,j) and (r,s) are in the same tile and $(i,j) \neq (r,s)$. Consider a walk in $G(\mathcal{C})$ connecting (i,j) and (r,s) with the entries associated with the vertices in the walk being equal. Arguing by induction on the length of the walk, it is enough to prove the lemma for walks of length 1. Without loss of generality we can assume that such a walk is given by the relation $((i,j);(r,s)) \in \mathcal{C}$. In this case, the entries of any $\mathcal{C}$-pattern Z should satisfy $z_{ij} \geq z_{rs}$. Now, by the hypothesis

$$x_{ij}+y_{ij} \geq x_{rs}+y_{rs}, \quad x_{ij}-y_{ij} \geq x_{rs}-y_{rs}, \quad \text{and} \quad x_{ij}=x_{rs}$$

which implies (i).

By Lemma 4.8(i) we have $y_{ni}=0$ for $1 \leq i \leq n$, therefore (ii) is a consequence of item (i) and $P_\mathcal{C}(\lambda) \subseteq P_\mathcal{C}$. Moreover, under the assumptions of item (iii), Lemma 4.8(ii) implies also $y_{11}=0$, now (iii) follows from the definition of $\{1,n\}$-free tile and item (i). $\square$

Lemma 4.15. *Let $\mathcal{C}$ be a set of relations, X a $\mathcal{C}$-pattern, $\lambda=(x_{n1},...,x_{nn})$, and $\mu=(w_1(X), w_2(X),...,w_n(X))$.*

(i) *If $\mathcal{M}_\mathcal{C}(X)$ does not have $\{n\}$-free tiles, then X is a vertex of $P_\mathcal{C}(\lambda)$.*

(ii) *If $\mathcal{M}_\mathcal{C}(X)$ does not have $\{1,n\}$-free tiles, then X is a vertex of $P_\mathcal{C}(\lambda,\mu)$.*

Proof. To show that X is a vertex of a polyhedron P it is enough to prove that $D_X(P)=\{0\}$. With this in mind, (i) follows from Lemma 4.14(ii), and (ii) follows from Lemma 4.14(iii). $\square$

Definition 4.16. Given a $\mathcal{C}$-pattern X, with tiling $\mathcal{M}_{\mathcal{C}}(X)$ and set of $\{n\}$-free tiles $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_{\mathfrak{f}}$, we define a *tiling matrix* $A_{\mathcal{M}_{\mathcal{C}}(X)}$ to be the matrix

$$(a_{ik})_{\substack{1 \leq i \leq n-1 \\ 1 \leq k \leq \mathfrak{f}}} \quad \text{where} \quad a_{ik} = |\{j \,|\, (i, j) \in \mathcal{M}_k\}| \,.$$

For $\mathcal{C}$-patterns without $\{n\}$-free tiles we consider $A_{\mathcal{M}_{\mathcal{C}}(X)}$ to be the identity matrix of order $n-1$.

Remark 4.17. Note that the matrix $A_{\mathcal{M}_{\mathcal{C}}(X)}$ depends of the chosen order of the $\{n\}$-free tiles, but the dimension of the kernel of $A_{\mathcal{M}_{\mathcal{C}}(X)}$ does not. In fact, given two different orders of the $\{n\}$-free tiles, the corresponding matrices are related by a permutation of the columns.

Example 4.18. Let us consider the tilings $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ from Example 4.13, and enumerate the $\{n\}$-free tales from left to right and from bottom to top. Under this conditions the corresponding tiling matrices are:

$$A_{\mathcal{T}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A_{\mathcal{T}_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad A_{\mathcal{T}_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

5. Main results

From now on and until the end of this paper we will fix a set of relations $\mathcal{C}$, a $\mathcal{C}$-pattern X, $\lambda = (x_{n1}, ..., x_{nn})$, and $\mu = (w_1(X), w_2(X), ..., w_n(X))$. By $\mathfrak{t}$ we will denote the number of tiles, by $\mathfrak{f}$ the number of $\{n\}$-free tiles, and $\mathfrak{k}$ the dimension of $\ker\left(A_{\mathcal{M}_{\mathcal{C}}(X)}\right) \subseteq \mathbb{R}^{\mathfrak{f}}$, so $\mathfrak{k} \leq \mathfrak{f} \leq \mathfrak{t}$. By Remark 4.17, we can assume that $\mathcal{M}_{\mathcal{C}}(X) = \{\mathcal{M}_1, ..., \mathcal{M}_{\mathfrak{t}}\}$, where $\{\mathcal{M}_1, ..., \mathcal{M}_{\mathfrak{f}}\}$ is the set of all $\{n\}$-free tiles.

Notation 5.1. *Let us fix a basis $\left\{\bar{\mathbf{v}}^{(1)}, ..., \bar{\mathbf{v}}^{(\mathfrak{f})}\right\}$ of $\mathbb{R}^{\mathfrak{f}}$ such that $\left\{\bar{\mathbf{v}}^{(1)}, ..., \bar{\mathbf{v}}^{(\mathfrak{k})}\right\}$ is a basis of $\ker\left(A_{\mathcal{M}_{\mathcal{C}}(X)}\right) \subseteq \mathbb{R}^{\mathfrak{f}}$. Let $\iota : \mathbb{R}^{\mathfrak{f}} \longrightarrow \mathbb{R}^{\mathfrak{t}}$ the canonical inclusion on the first $\mathfrak{f}$ entries, and $\left\{\mathbf{v}^{(1)}, ..., \mathbf{v}^{(\mathfrak{t})}\right\}$ the basis of $\mathbb{R}^{\mathfrak{t}}$ given by*

$$\mathbf{v}^{(m)} = \begin{cases} \iota(\bar{\mathbf{v}}^{(m)}), & \text{if } m \leq \mathfrak{f}; \\ ith \text{ canonical vector of } \mathbb{R}^{\mathfrak{t}}, & \text{if } m > \mathfrak{f}. \end{cases}$$

Now, consider the linear map $\psi : \mathbb{R}^{\mathfrak{t}} \longrightarrow \mathbb{R}^{n(n+1)/2}$ defined by $\psi(\mathbf{v}^{(m)}) = Y^{(m)}$, where $y_{ij}^{(m)} = \mathbf{v}_k^{(m)}$ whenever $(i, j) \in \mathcal{M}_k$. Finally, denote by $\mathcal{Y}_r$ the set $\left\{Y^{(1)}, ..., Y^{(r)}\right\}$, for $r = 1, 2, ..., \mathfrak{t}$. By convention, $\mathcal{Y}_0 = \{0\}$.

Lemma 5.2. *Under the previous notation.*

(i) *If $X \in P_{\mathcal{C}}$, then $\mathcal{Y}_{\mathfrak{t}} \subseteq D_X(P_{\mathcal{C}})$.*

(ii) *If $X \in P_{\mathcal{C}}(\lambda)$, then $\mathcal{Y}_{\mathfrak{f}} \subseteq D_X(P_{\mathcal{C}}(\lambda))$.*

(iii) *If $X \in P_{\mathcal{C}}(\lambda, \mu)$, then $\mathcal{Y}_{\mathfrak{k}} \subseteq D_X(P_{\mathcal{C}}(\lambda, \mu))$.*

Proof. To prove (i) we show that $X \pm Y^{(m)} \in P_{\mathcal{C}}$ for $1 \le m \le \mathfrak{t}$. As $X \pm Y \in P_{\varnothing}$ for any $Y \in \mathbb{R}^{n(n+1)/2}$, we can assume without lose of generality $\mathcal{C} \ne \varnothing$. Let us consider a relation $((i,j); (r,s)) \in \mathcal{C}$. Suppose first that $x_{ij} = x_{rs}$, in this case (i,j) and (r,s) belong to the same tile $\mathcal{M}_k$, and therefore $y_{ij}^{(m)} = \mathbf{v}_k^{(m)} = y_{rs}^{(m)}$, which implies $\left(x_{ij} \pm y_{ij}^{(m)}\right) - \left(x_{rs} \pm y_{rs}^{(m)}\right) = 0$. Suppose now that $x_{ij} > x_{rs}$ and assume

$$\left|\mathbf{v}_k^{(m)}\right| < \min\left\{\frac{|x_{rs} - x_{pq}|}{2} : x_{rs} \ne x_{pq}\right\}, \quad \text{for } 1 \le m, k \le \mathfrak{t}.$$

Under these conditions,

$$\pm\left(y_{ij}^{(m)} - y_{rs}^{(m)}\right) \le \left|y_{ij}^{(m)} - y_{rs}^{(m)}\right| \le \left|y_{ij}^{(m)}\right| + \left|y_{rs}^{(m)}\right|$$
$$\le 2\max\left\{\left|\mathbf{v}_k^{(m)}\right| : 1 \le k \le \mathfrak{t}\right\}$$
$$< \min\left\{|x_{rs} - x_{pq}| : x_{rs} \ne x_{pq}\right\}$$
$$\le |x_{ij} - x_{rs}| = x_{ij} - x_{rs}.$$

This implies $x_{ij} - x_{rs} \pm \left(y_{ij}^{(m)} - y_{rs}^{(m)}\right) \ge 0$.

To prove (ii) we show that $X \pm Y^{(m)} \in P_{\mathcal{C}}(\lambda)$ for all $1 \le m \le \mathfrak{f}$, by item (i) we have $Y^{(m)} \in D_X(P_{\mathcal{C}})$ for all $m = 1, 2, ..., \mathfrak{f}$, and by Lemma 4.8(i), it is enough to prove that $y_{nj}^{(m)} = 0$ for $1 \le j \le n$, which follows from the fact that (n, j) does not belong to any $\{n\}$-free tile.

Let us show that $X \pm Y^{(m)} \in P_{\mathcal{C}}(\lambda, \mu)$ for all $1 \le m \le \mathfrak{k}$. For any $1 \le i \le n - 1$ we have

$$R_i\left(Y^{(m)}\right) = \sum_{j=1}^{i} y_{ij}^{(m)} = \sum_{k=1}^{\mathfrak{f}} a_{ik}\mathbf{v}_k^{(m)} = \sum_{k=1}^{\mathfrak{f}} a_{ik}\bar{\mathbf{v}}_k^{(m)},$$

where $a_{ik} = |\{j \,|\, (i,j) \in \mathcal{M}_k\}|$. The right hand side of the previous equality is the dot product between $\bar{\mathbf{v}}^{(m)} \in \ker\left(A_{\mathcal{M}_{\mathcal{C}}(X)}\right)$ and the ith row of $A_{\mathcal{M}_{\mathcal{C}}(X)}$, therefore equal to zero. It follows from item (ii) and Lemma 4.8(ii) that $X \pm Y^{(m)} \in P_{\mathcal{C}}(\lambda, \mu)$, which completes the proof of (iii). $\square$

Definition 5.3. Let $\mathcal{C}$ be a set of relations, and $P \subseteq P_{\mathcal{C}}$ be any polyhedron. Associated with P and $\mathcal{C}$ we will consider the following polyhedron:

$$P^+ := \{X \in P \,|\, x_{ij} \ge 0 \text{ whenever } (i,j) \in \mathfrak{V}(\mathcal{C})\}.$$

$\mathcal{C}$ will be called *top-connected*, if for each $(i,j) \in \mathfrak{V}(\mathcal{C})$ with $i \ne n$, there exists r such that $(i,j) \succeq_{\mathcal{C}} (n, r)$.

Remark 5.4. In the classical definition of Gelfand-Tsetlin pattern it is required for the entries of the tableau to be non-negative (cf. [BZ89], [KB95], [LM04], [ABS11], [LMD19]). More concretely, the term Gelfand-Tsetlin pattern is used for elements in $P_{\mathcal{C}_1}^+$. In particular, Gelfand-Tsetlin polytopes are defined as $P_{\mathcal{C}_1}^+(\lambda, \mu)$. It became clear the necessity of discussing which properties of a polyhedra P are also satisfied by P^+.

If λ is an integral dominant weight with non-negative entries we have $P_{\mathcal{C}_1}(\lambda, \mu) = P_{\mathcal{C}_1}^+(\lambda, \mu)$. Note also that $\mathcal{C}_k$ and $\mathcal{C}_k^-$ from Example 3.1 are top-connected for any $1 \le k \le n$, however $\mathcal{C}_k^+$ is top-connected only if $k = n$.

Lemma 5.5. *Let $\mathcal{C}$ be a top-connected set of relations.*
(i) *If $X \in P_{\mathcal{C}}^+(\lambda)$, then $\mathcal{Y}_{\mathfrak{f}} \subseteq D_X\left(P_{\mathcal{C}}^+(\lambda)\right)$.*
(ii) *If $X \in P_{\mathcal{C}}^+(\lambda, \mu)$, then $\mathcal{Y}_{\mathfrak{t}} \subseteq D_X\left(P_{\mathcal{C}}^+(\lambda, \mu)\right)$.*

Proof. We only prove (i), the proof of item (ii) is analogue using Lemma 4.14(iii). By Lemma 5.2(ii) we have $\mathcal{Y}_{\mathfrak{f}} \subseteq D_X\left(P_{\mathcal{C}}(\lambda)\right)$, so it is enough to prove that $x_{ij} \pm y_{ij}^{(m)} \ge 0$ whenever $(i, j) \in \mathfrak{V}(\mathcal{C})$ and $1 \le m \le \mathfrak{f}$. If (i, j) is not $\{n\}$-free, then by Lemma 4.14(ii) we have $y_{ij}^{(m)} = 0$ and therefore $x_{ij} \pm y_{ij}^{(m)} = x_{ij} \ge 0$. On the other hand, suppose (i, j) belongs to some $\{n\}$-free tile $\mathcal{M}_k$, that implies $i \ne n$. As $\mathcal{C}$ is top-connected, there exists $1 \le r \le n$ such that $(i, j) \succeq_{\mathcal{C}} (n, r)$, so $(n, r) \in \mathfrak{V}(\mathcal{C})$, and $x_{ij} \ge x_{nr} \ge 0$. Moreover $x_{ij} \ne x_{nr}$ because (i, j) and (n, r) do not belong to the same tile. Rescaling $Y^{(m)}$ if necessary, we can assume

$$\left| \mathbf{v}_k^{(m)} \right| < \min \left\{ \frac{|x_{rs} - x_{pq}|}{2} : x_{rs} \ne x_{pq} \right\}$$

for $1 \le m, k \le \mathfrak{t}$, so we have

$$\pm y_{ij}^{(m)} = \pm \mathbf{v}_k^{(m)} \le \left| \mathbf{v}_k^{(m)} \right| < \frac{1}{2} \min \left\{ |x_{rs} - x_{pq}| : x_{rs} \ne x_{pq} \right\}$$
$$\le |x_{ij} - x_{nr}| = x_{ij} - x_{nr} \le x_{ij}.$$

Hence $x_{ij} \pm y_{ij}^{(m)} \ge 0$ for all $(i, j) \in \mathfrak{V}(\mathcal{C})$. $\square$

Remark 5.6. Note that in general it is not true that $\mathcal{Y}_{\mathfrak{t}} \subseteq D_X\left(P_{\mathcal{C}}^+\right)$. For instance, consider X to be the tableau with all entries being 0, and $\mathcal{C}$ any set of relations with $\mathfrak{V}(\mathcal{C}) = \mathfrak{V}$. It is trivial to show that $D_X\left(P_{\mathcal{C}}^+\right) = \{0\}$.

The following theorem generalizes [LM04, Theorem 1.5].

Theorem 5.7. *Let $\mathcal{C}$ be any set of relations, X a $\mathcal{C}$-pattern, and $\mathcal{M}_{\mathcal{C}}(X)$ its associated tiling. Set $\lambda = (x_{n1}, ..., x_{nn})$, and $\mu = (w_1(X), w_2(X), ..., w_n(X))$. Then*

(i) *The dimension of the minimal face of $P_\mathcal{C}$ containing X is equal to the number of tiles in $\mathcal{M}_\mathcal{C}(X)$.*

(ii) *The dimension of the minimal face of $P_\mathcal{C}(\lambda)$ containing X is equal to the number of $\{n\}$-free tiles in $\mathcal{M}_\mathcal{C}(X)$.*

(iii) *The dimension of the minimal face of $P_\mathcal{C}(\lambda, \mu)$ containing X is equal to the dimension of the kernel of $A_{\mathcal{M}_\mathcal{C}(X)}$.*

Proof. Let $H_{\lambda\mu}, H_\lambda$ and H be the $\mathbb{R}$-vector subspaces of $\mathbb{R}^{n(n+1)/2}$ such that $H_{\lambda\mu}+X$, $H_\lambda+X$, and $H+X$ are the affine span of the minimal face of $P_\mathcal{C}(\lambda, \mu)$, $P_\mathcal{C}(\lambda)$ and $P_\mathcal{C}$ containing X, respectively. Let us consider the bases and linear maps as mentioned in Notation 5.1.

Since $\{\mathbf{v}^{(1)}, ..., \mathbf{v}^{(\mathfrak{t})}\}$ is linearly independent, we conclude that $Y^{(1)}, ..., Y^{(\mathfrak{t})}$ are linearly independent. Due to Lemma 4.2 and Lemma 5.2 we also have $\mathcal{Y}_\mathfrak{t} \subseteq H_{\lambda\mu}$, $\mathcal{Y}_\mathfrak{f} \subseteq H_\lambda$ and $\mathcal{Y}_\mathfrak{t} \subseteq H$. We finish the proof showing that $H_{\lambda\mu}$ is spanned by $\mathcal{Y}_\mathfrak{t}$, H_λ is spanned by $\mathcal{Y}_\mathfrak{f}$, and H is spanned by $\mathcal{Y}_\mathfrak{t}$.

Set $Y \in H$ such that $X \pm Y \in P_\mathcal{C}$, and consider $\mathbf{w} := (\mathbf{w}_1, ..., \mathbf{w}_\mathfrak{t}) \in \mathbb{R}^\mathfrak{t}$, where $\mathbf{w}_k := y_{ij}$ whenever $(i, j) \in \mathcal{M}_k$ (note that $\mathbf{w}_k$ is well-defined by Lemma 4.14(i)). By construction of ψ, we have $\psi(\mathbf{w}) = Y$, and therefore $Y \in \text{span } \mathcal{Y}_\mathfrak{t}$, because $\{\mathbf{v}^{(1)}, ..., \mathbf{v}^{(\mathfrak{t})}\}$ is a basis of $\mathbb{R}^\mathfrak{t}$.

If $\mathcal{M}_\mathcal{C}(X)$ does not have $\{n\}$-free tiles, items (ii) and (iii) are consequence of Lemma 4.15. Suppose that $\mathcal{M}_\mathcal{C}(X)$ has at least one $\{n\}$-free tile.

If $Y \in H_\lambda$ is such that $X \pm Y \in P_\mathcal{C}(\lambda)$, Lemma 4.14(ii) implies that $y_{ij} = 0$ if (i, j) is not $\{n\}$-free. As $H_\lambda \subseteq H$ and $P_\mathcal{C}(\lambda) \subseteq P_\mathcal{C}$ we have $\mathbf{w}_k = 0$ for $k > \mathfrak{f}$. Finally, as $\psi(\mathbf{w}) = Y$, we have $Y \in \text{span } \mathcal{Y}_\mathfrak{f}$, by the construction of ψ and the basis $\{\bar{\mathbf{v}}^{(1)}, ..., \bar{\mathbf{v}}^{(\mathfrak{f})}\}$ of $\mathbb{R}^\mathfrak{f}$.

Consider now $Y \in H_{\lambda\mu}$ such that $X \pm Y \in P_\mathcal{C}(\lambda, \mu)$. Lemma 4.14(iii) implies that $y_{ij} = 0$ if (i, j) is not $\{1, n\}$-free, hence $\mathbf{w}_i = 0$ for $i \in \{1, \mathfrak{f}+1, \mathfrak{f}+2, ..., \mathfrak{t}\}$. Consider $\overline{\mathbf{w}} := (\mathbf{w}_1, ..., \mathbf{w}_\mathfrak{f})$. By the conditions on Y, we get $\overline{\mathbf{w}} \in \ker(A_{\mathcal{M}_\mathcal{C}(X)})$, in fact, from Lemma 4.8(ii) we have $\sum_{k=1}^{\mathfrak{f}} a_{ik}\mathbf{w}_k = \sum_{j=1}^{i} y_{ij} = 0$, where $a_{ik} = |\{j \,|\, (i, j) \in \mathcal{M}_k\}|$. Similarly, since $\psi(\mathbf{w}) = Y$ and $\{\bar{\mathbf{v}}^{(1)}, ..., \bar{\mathbf{v}}^{(\bar{\mathfrak{t}})}\}$ is a basis of $\ker(A_{\mathcal{M}_\mathcal{C}(X)})$. We conclude that $Y \in \text{span } \mathcal{Y}_\mathfrak{t}$, by the construction of ψ and the basis $\{\mathbf{v}^{(1)}, ..., \mathbf{v}^{(\mathfrak{t})}\}$ of $\mathbb{R}^\mathfrak{t}$. $\quad \square$

Example 5.8. Follows from Theorem 5.7 and Example 4.18 that the dimension of minimal faces containing the tableaux from Example 4.13 are given by

	λ	μ	$P_\mathcal{C}$	$P_\mathcal{C}(\lambda)$	$P_\mathcal{C}(\lambda, \mu)$
Tiling 1	$(\sqrt{2}, 2, 3, 4)$	$(0, 0, \sqrt{2}+\sqrt{3}, 9-\sqrt{3})$	7	3	1
Tiling 2	$(8, 5, 5, 4)$	$(-2, 4, 9, 11)$	8	4	1
Tiling 3	$(8, 5, 5, 4)$	$(-2, 4, 9, 11)$	9	5	2

Corollary 5.9. *Let $\mathcal{C}$ be a top-connected set of relations, X a $\mathcal{C}$-pattern, and $\mathcal{M}_{\mathcal{C}}(X)$ its associated tiling. Set $\lambda = (x_{n1}, ..., x_{nn})$, and $\mu = (w_1(X), ..., w_n(X))$. Suppose that $\mathcal{M}_{\mathcal{C}}(X)$ has at least one $\{n\}$-free tile. Then*

(i) *The dimension of the minimal face of $P_{\mathcal{C}}^+(\lambda)$ containing X is equal to the number of $\{n\}$-free tiles in $\mathcal{M}_{\mathcal{C}}(X)$.*

(ii) *The dimension of the minimal face of $P_{\mathcal{C}}^+(\lambda, \mu)$ containing X is equal to the dimension of the kernel of $A_{\mathcal{M}_{\mathcal{C}}(X)}$.*

Proof. Follows directly from Theorem 5.7 and Lemma 5.5. $\square$

Remark 5.10. In general, it is not true that the dimension of the minimal face of $P_{\mathcal{C}}^+$ containing X is equal to the number of tiles in $\mathcal{M}_{\mathcal{C}}(X)$. In fact, under the conditions of Remark 5.6, X is a vertex for $P_{\mathcal{C}}^+$, but $\mathcal{M}_{\mathcal{C}}(X)$ has one tile.

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