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A Whittaker category for the symplectic Lie algebra and differential operators

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Abstract. For any $n\in\mathbb{Z}_{\geq 2}$, let $\mathfrak{m}_n$ be the subalgebra of $\mathfrak{sp}_{2n}$ spanned by all long negative root vectors $X_{-2\varepsilon_i}$, $i=1,\dots,n$. Then $(\mathfrak{sp}_{2n},\mathfrak{m}_n)$ is a Whittaker pair in the sense of a definition given by Batra and Mazorchuk. In this paper, we use differential operators to study the category of $\mathfrak{sp}_{2n}$-modules that are locally finite over $\mathfrak{m}_n$. We show that when $\mathbf{a}\in(\mathbb{C}^*)^n$, each non-empty block $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ with the central character χ_μ is equivalent to the Whittaker category $\mathcal{W}^{\mathbf{a}}$ of the even Weyl algebra $\mathcal{D}_n^{ev}$ which is semi-simple. Any module in $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ has the minimal Gelfand-Kirillov dimension n. We also characterize all possible algebra homomorphisms from $U(\mathfrak{sp}_{2n})$ to the Weyl algebra $\mathcal{D}_n$ under a natural condition.

1. Introduction

Among the representation theory of Lie algebras, Whittaker modules are interesting non-weight modules which play an important role in the classification of irreducible modules for several Lie algebras. Whittaker modules were first introduced by Arnal and Pinzcon for $\mathfrak{sl}_2(\mathbb{C})$, see [AP]. The classification of the irreducible modules for $\mathfrak{sl}_2(\mathbb{C})$ in [B] illustrates the importance of Whittaker modules. It was shown that irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules can be divided into three families: weight modules, Whittaker modules, and modules obtained from irreducible elements in a noncommutative domain. Kostant studied Whittaker modules for any complex semisimple Lie algebra $\mathfrak{g}$ in [K]. Whittaker modules defined by Kostant are closely associated with the triangular decomposition $\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ of $\mathfrak{g}$. Every Whittaker module depends on a Lie algebra homomorphism $\psi:\mathfrak{n}_+\to\mathbb{C}$. The map ψ is called non-singular if $\psi(x_\alpha)\neq 0$ for any simple root vector x_α. Kostant gave a classification of all simple non-singular Whittaker modules. Some results on complex semisimple

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Lie algebras have been generalized to other algebras with triangular decompositions. For example, for Whittaker modules over algebras related to the Virasoro algebra, one can see [OW1], [OW2], [GLZ], [LPX], [LWZ] and [TWX]. Whittaker modules over quantum groups $U_h(\mathfrak{g})$, $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$ were studied in [S], [OM] and [XGL], respectively. Whittaker modules have also been studied for generalized Weyl algebras by Benkart and Ondrus, see [BO]. Whittaker modules for non-twisted affine Lie algebras and several similar algebras were studied in [ALZ], [C], [CF], [CJ] and [GZ]. In [BM], Batra and Mazorchuk have constructed a more general framework to describe the Whittaker modules. They considered Whittaker pairs $(\mathfrak{g},\mathfrak{n})$ of Lie algebras, where $\mathfrak{n}$ is a quasi-nilpotent Lie subalgebra of $\mathfrak{g}$ such that the adjoint action of $\mathfrak{n}$ on the quotient $\mathfrak{g}/\mathfrak{n}$ is locally nilpotent, and studied the category $\mathcal{WH}$ of $\mathfrak{g}$-modules such that $\mathfrak{n}$ acts locally finitely. Under this general Whittaker set-up in [BM], they also determined a block decomposition of the category $\mathcal{WH}$ according to the action of $\mathfrak{n}$. The characterizations of each block for most Lie algebras are still open.

Differential operators are important tools for studying representations of Lie algebras. To construct explicit representations of a Lie algebra $\mathfrak{g}$ by differential operators, it is actually to find algebra homomorphisms from $U(\mathfrak{g})$ to the Weyl algebra $\mathcal{D}_n$. Let $\mathfrak{m}_n$ be the subalgebra of $\mathfrak{sp}_{2n}$ spanned by root vectors $X_{-2\varepsilon_i}$, $i{\in}\{1,\dots,n\}$. Then $(\mathfrak{sp}_{2n},\mathfrak{m}_n)$ is a Whittaker pair. An $\mathfrak{sp}_{2n}$-module M is called a Whittaker module if the action of each element of $\mathfrak{m}_n$ on M is locally finite, see [BM]. Similar as Kostant's definition, a Lie algebra homomorphism $\phi{:}\mathfrak{m}_n{\to}\mathbb{C}$ is called non-singular if $\phi(X_{-2\varepsilon_i}){\neq}0$, for any $i{\in}\{1,\dots,n\}$. We will characterize non-singular Whittaker modules in this paper using differential operators.

The paper is organized as follows. In Section 2, we recall some basic definitions and important facts including the weighting functors and Nilsson's modules for $\mathfrak{sp}_{2n}$. In Section 3, we characterize the Whittaker category $\mathcal{WH}_{\mathbf{a}}$ when $\mathbf{a}{\in}(\mathbb{C}^*)^n$, where $\mathcal{WH}_{\mathbf{a}}$ consists of $\mathfrak{sp}_{2n}$-modules M such that $X_{-2\varepsilon_i}{+}a_i^2$ acts locally nilpotently for any $i{=}1,\dots,n$, and $\mathrm{wh}_{\mathbf{a}}(M){=}\{v{\in}M\,|\,X_{-2\varepsilon_i}v{=}{-}a_i^2v,\ i{=}1,\dots,n\}$ is finite dimensional. In Lemma 3, we show that any module in $\mathcal{WH}_{\mathbf{a}}$ has the minimal Gelfand-Kirillov dimension n. Let $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ be the full subcategory of $\mathcal{WH}_{\mathbf{a}}$ consisting of all $U(\mathfrak{sp}_{2n})$-modules M with the central character χ_μ given by the highest weight μ. Using the translation functor and weighting functor, we show that when $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ is non-empty, there is an equivalence between $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ and the category $\mathcal{W}^{\mathbf{a}}$ of finitely generated $\mathcal{D}_n^{ev}$-modules such that $\partial_i^2{-}a_i^2$ acts locally nilpotently for any i, where $\mathcal{D}_n^{ev}$ is the subalgebra of the Weyl algebra $\mathcal{D}_n$ generated by differential operators of even degree. In Section 4, we give a differential operators realization of $\mathfrak{sp}_{2n}$ from any $f{\in}A_n$, see Lemma 12. Thus we have constructed many simple modules P_n^f over $\mathfrak{sp}_{2n}$. Furthermore, we show that these operators realizations exhaust all algebra homomorphisms from $U(\mathfrak{sp}_{2n})$ to $\mathcal{D}_n$ which map each root vector $X_{\varepsilon_i+\varepsilon_j}$ to t_it_j.

In this paper, we denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{C}$ and $\mathbb{C}^*$ the sets of integers, positive integers, nonnegative integers, complex numbers, and nonzero complex numbers, respectively. All vector spaces and algebras are over $\mathbb{C}$. For a Lie algebra $\mathfrak{g}$ we denote by $U(\mathfrak{g})$ its universal enveloping algebra, $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We write $\otimes$ for $\otimes_{\mathbb{C}}$.

2. Preliminaries

In this section, we collect some preliminary definitions and related results that will be used throughout the paper. In particular, we introduce the notion of Whittaker modules in the sense of [BM] that are of main interest in this paper.

2.1. The symplectic algebra $\mathfrak{sp}_{2n}$

Throughout the whole text, we fix an integer n bigger than 1. Recall that $\mathfrak{sp}_{2n}$ is the Lie subalgebra of $\mathfrak{gl}_{2n}$ consisting of all $2n\times 2n$-matrices X satisfying $SX=-X^TS$ where

$$S=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

So $\mathfrak{sp}_{2n}$ consists of all $2n\times 2n$-matrices of the following form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

such that $B=B^T$, $C=C^T$, where $A,B,C\in\mathfrak{gl}_n$. Let e_{ij} denote the matrix unit whose (i,j)-entry is 1 and 0 elsewhere. Then

$$\mathfrak{h}_n=\operatorname{span}\{h_i:=e_{i,i}-e_{n+i,n+i}\,|\,1\leq i\leq n\}$$

is a Cartan subalgebra (a maximal abelian subalgebra whose adjoint action on $\mathfrak{sp}_{2n}$ is diagonalizable) of $\mathfrak{sp}_{2n}$. Let Λ^+ be the set of dominant integral weight, $\{\varepsilon_i|1\leq i\leq n\}\subset\mathfrak{h}_n^*$ be such that $\varepsilon_i(h_k)=\delta_{i,k}$. The root system of $\mathfrak{sp}_{2n}$ is precisely

$$\Delta=\{\pm\varepsilon_i\pm\varepsilon_j|1\leq i,j\leq n\}\setminus\{0\}.$$

The positive root system is

$$\Delta_+=\{\varepsilon_i-\varepsilon_j,\varepsilon_k+\varepsilon_l\,|\,1\leq i<j\leq n,1\leq k,l\leq n\}.$$

We list root vectors in $\mathfrak{sp}_{2n}$ as follows:

Root vector	Root
$X_{\varepsilon_i+\varepsilon_j} := e_{i,n+j}+e_{j,n+i}$	$\varepsilon_i+\varepsilon_j$
$X_{-\varepsilon_i-\varepsilon_j} := e_{n+i,j}+e_{n+j,i}$	$-\varepsilon_i-\varepsilon_j$
$X_{\varepsilon_i-\varepsilon_j} := e_{i,j}-e_{n+j,n+i}$	$\varepsilon_i-\varepsilon_j,$

where $i,j\in\{1,...,n\}$, with $i\neq j$ when we encounter $\varepsilon_i-\varepsilon_j$.

Then we can obtain a basis of $\mathfrak{sp}_{2n}$ as follows:

$$B:=\{X_\alpha|\alpha\in\Delta\}\cup\{h_i|1\leq i\leq n\}.$$

Set

$$\mathfrak{n}_\pm:=\bigoplus_{\alpha\in\Delta_\pm}\mathfrak{g}_\alpha,\ \text{where}\ \mathfrak{g}_\alpha:=\{x\in\mathfrak{sp}_{2n}\,|\,[h,x]=\alpha(h)x,\ \forall\ h\in\mathfrak{h}_n\}.$$

Then the decomposition

$$\mathfrak{sp}_{2n}=\mathfrak{n}_-\oplus\mathfrak{h}_n\oplus\mathfrak{n}_+$$

is a triangular decomposition of $\mathfrak{sp}_{2n}$, and the Lie subalgebra $\mathfrak{b}:=\mathfrak{h}_n\oplus\mathfrak{n}_+$ is a Borel subalgebra of $\mathfrak{sp}_{2n}$.

For the convenience of later calculations, we list some nontrivial Lie bracket of $\mathfrak{sp}_{2n}$ as follows:

$$\begin{aligned}
[X_{\varepsilon_i-\varepsilon_j},X_{\varepsilon_k-\varepsilon_l}]&=\delta_{jk}X_{\varepsilon_i-\varepsilon_l}-\delta_{li}X_{\varepsilon_k-\varepsilon_j},\\
[X_{\varepsilon_i+\varepsilon_j},X_{-\varepsilon_k-\varepsilon_l}]&=\delta_{jk}X_{\varepsilon_i-\varepsilon_l}+\delta_{il}X_{\varepsilon_j-\varepsilon_k}+\delta_{ik}X_{\varepsilon_j-\varepsilon_l}+\delta_{jl}X_{\varepsilon_i-\varepsilon_k},\\
[X_{\varepsilon_i-\varepsilon_j},X_{\varepsilon_k+\varepsilon_l}]&=\delta_{jk}X_{\varepsilon_i+\varepsilon_l}+\delta_{jl}X_{\varepsilon_i+\varepsilon_k},\\
[X_{\varepsilon_i-\varepsilon_j},X_{-\varepsilon_k-\varepsilon_l}]&=-\delta_{il}X_{-\varepsilon_k-\varepsilon_j}-\delta_{ki}X_{-\varepsilon_l-\varepsilon_j}.
\end{aligned}\tag{2.1}$$

In particular, $[X_{2\varepsilon_i},X_{-2\varepsilon_k}]=4[e_{i,n+i},e_{n+k,k}]=\delta_{ik}4h_i$, where $i,k\in\{1,...,n\}$.

2.2. Weight modules

An $\mathfrak{sp}_{2n}$-module V is called a weight module if $\mathfrak{h}_n$ acts diagonally on V, i.e.,

$$V=\oplus_{\lambda\in\mathfrak{h}_n^*}V_\lambda,$$

where $V_\lambda=\{v\in V\,|\,hv=\lambda(h)v,\forall\ h\in\mathfrak{h}_n\}$. For a weight module V, denote

$$\mathrm{supp}(V)=\{\lambda\in\mathfrak{h}_n^*\,|\,V_\lambda\neq 0\}.$$

For a weight module M, a nonzero vector $v\in M_\lambda$ is called a highest weight vector if $\mathfrak{n}_+v=0$. A module is called a highest weight module if it is generated by a highest weight vector. A weight module M is called a uniformly bounded module, if there is a $k\in\mathbb{N}$ such that $\dim M_\lambda\leq k$ for any weight $\lambda\in\mathrm{supp}(M)$. Let $\mathcal{B}$ be the category of uniformly bounded weight $\mathfrak{sp}_{2n}$-modules.

2.3. Whittaker modules

Let $\mathfrak{m}_n = \oplus_{1\le i\le n}\mathbb{C}X_{-2\varepsilon_i}$ which is a commutative subalgebra of $\mathfrak{sp}_{2n}$. Since the adjoint action of $\mathfrak{m}_n$ on the quotient $\mathfrak{sp}_{2n}/\mathfrak{m}_n$ is nilpotent, $(\mathfrak{sp}_{2n}, \mathfrak{m}_n)$ is a Whittaker pair in the sense of [BM]. An $\mathfrak{sp}_{2n}$-module M is called a Whittaker module if the action of $\mathfrak{m}_n$ on M is locally finite. For an $\mathbf{a}=(a_1, ..., a_n)\in(\mathbb{C})^n$, we can define a Lie algebra homomorphism $\phi_{\mathbf{a}}:\mathfrak{m}_n\to\mathbb{C}$ such that $\phi_{\mathbf{a}}(X_{-2\varepsilon_i})=-a_i^2$ for any $i\in\{1, ..., n\}$. A Whittaker module M is of type $\mathbf{a}$ if for any $v\in M$ there is a $k\in\mathbb{N}$ such that $(x-\phi_{\mathbf{a}}(x))^k v=0$ for all $x\in\mathfrak{m}_n$. We also define the subspace

$$\mathrm{wh}_{\mathbf{a}}(M)=\{v\in M\mid xv=\phi_{\mathbf{a}}(x)v,\ \forall\ x\in\mathfrak{m}_n\}$$

of M. An element in $\mathrm{wh}_{\mathbf{a}}(M)$ is called a Whittaker vector.

Such Whittaker modules are more complicated than the classical Whittaker modules defined by Kostant. For example, $\dim\mathrm{wh}_{\mathbf{a}}(M)$ is not necessarily 1 for a simple Whittaker module M. We consider Whittaker modules under some natural finite condition. Let $\mathcal{WH}_{\mathbf{a}}$ be the category of Whittaker $U(\mathfrak{sp}_{2n})$-modules M of type $\mathbf{a}$ such that $\mathrm{wh}_{\mathbf{a}}(M)$ is finite dimensional.

Remark 1. *The condition that* $\mathrm{wh}_{\mathbf{a}}(M)$ *is finite dimensional amounts to the condition that weight spaces are finite dimensional for a weight module, see the proof of Lemma 7.*

2.4. Central characters

Let $\mathfrak{X}=\mathrm{Hom}(Z(\mathfrak{sp}_{2n}),\mathbb{C})$ be the set of central characters of $\mathfrak{sp}_{2n}$. We have a map $\xi:\mathfrak{h}_n^*\to\mathfrak{X}$ which maps $\mu\in\mathfrak{h}_n^*$ to the central character χ_μ of the Verma module $M(\mu)$. By the Harish-Chandra's Theorem, the map ξ is surjective. Moreover for $\mu,\lambda\in\mathfrak{h}_n^*$, $\chi_\mu=\chi_\lambda$ if and only if $\mu\sim\lambda$, defined as there is an element σ in the Weyl group of $\mathfrak{sp}_{2n}$ such that $\sigma(\mu+\rho)=\lambda+\rho$, where ρ is the half sum of all positive roots of $\mathfrak{sp}_{2n}$. For each $\chi\in\mathfrak{X}$, denote by $\mathcal{WH}_{\mathbf{a}}^\chi$ the full subcategory of $\mathcal{WH}_{\mathbf{a}}$ of all $U(\mathfrak{sp}_{2n})$-modules M such that for any $v\in M$ there is a $k\in\mathbb{N}$ such that $(z-\chi(z))^k v=0$ for all $z\in Z(\mathfrak{sp}_{2n})$. Similarly, we have the full subcategory $\mathcal{B}^\chi$ of $\mathcal{B}$ for any $\chi\in\mathfrak{X}$. Moreover we have the block decompositions:

$$\mathcal{WH}_{\mathbf{a}}=\bigoplus_{\chi\in\mathfrak{X}}\mathcal{WH}_{\mathbf{a}}^\chi,\quad \mathcal{B}=\bigoplus_{\chi\in\mathfrak{X}}\mathcal{B}^\chi.$$

2.5. Weighting functor

We recall the weighting functor introduced in [N2]. For a point $\gamma\in\mathbb{C}^n$, let I_γ be the maximal ideal of $U(\mathfrak{h}_n)=\mathbb{C}[h_1, ..., h_n]$ generated by

$$h_1-\gamma_1, ..., h_n-\gamma_n.$$

For an $\mathfrak{sp}_{2n}$-module M and $\gamma\in\mathbb{C}^n$, set $M^\gamma:=M/I_\gamma M$. For a $\mu=(\mu_1, ..., \mu_n)\in\mathbb{C}^n$, let

$$\mathfrak{W}^\mu(M):=\bigoplus_{\gamma\in\mathbb{Z}^n} M^{\gamma+\mu}.$$

Here the module $\mathfrak{W}^\mu(M)$ is a submodule of the coherent family defined in [N2].

For any $\lambda\in\mathfrak{h}_n^*$, we identify λ with the vector $(\lambda(h_1), ..., \lambda(h_n))$ in $\mathbb{C}^n$. Nilsson defined a weight module structure on $\mathfrak{W}^\mu(M)$; see Proposition 8 in [N2].

Proposition 2. *The vector space $\mathfrak{W}^\mu(M)$ becomes a weight $\mathfrak{sp}_{2n}$-module under the following action*:

$$X_\alpha\cdot(v+I_\gamma M):=X_\alpha v+I_{\gamma+\alpha}M, v\in M, \alpha\in\Delta, \gamma\in\mu+\mathbb{Z}^n. \tag{2.2}$$

We see that $h_i\cdot(v+I_\gamma M)=\gamma_i(v+I_\gamma M)$ for any i. So $\mathfrak{W}^\mu(M)$ is a weight module. In many cases, the $\mathfrak{sp}_{2n}$-module $\mathfrak{W}^\mu(M)$ is 0. For example, if M is a simple weight module with a weight not in $\mu+\mathbb{Z}^n$, one can easily see that $\mathfrak{W}^\mu(M)=0$. We also note that $\mathfrak{W}^\mu(M)=M$ if M is a simple weight $\mathfrak{sp}_{2n}$-module with $\mathrm{supp}(M)=\mu+\mathbb{Z}^n$. If M is a $U(\mathfrak{h}_n)$-torsion free module of finite rank when restricted to $U(\mathfrak{h}_n)$, then $\mathfrak{W}^\mu(M)$ is a uniformly bounded weight module with $\mathrm{supp}(\mathfrak{W}^\mu(M))=\mu+\mathbb{Z}^n$.

2.6. Nilsson's modules

Since $\mathfrak{h}_n$ is commutative, $U(\mathfrak{h}_n)=\mathbb{C}[h_1, ..., h_n]$ as an associative algebra. In [N2], Nilsson constructed an $\mathfrak{sp}_{2n}$-module structure on $U(\mathfrak{h}_n)$ as follows:

$$\begin{aligned}
h_i\cdot g&=h_i g,\\
X_{2\varepsilon_i}\cdot g&=(h_i-\frac{1}{2})(h_i-\frac{3}{2})\sigma_i^2(g),\\
X_{-2\varepsilon_i}\cdot g&=-\sigma_i^{-2}(g),\\
X_{\varepsilon_i+\varepsilon_j}\cdot g&=(h_i-\frac{1}{2})(h_j-\frac{1}{2})\sigma_i\sigma_j(g), i\neq j,\\
X_{-\varepsilon_i-\varepsilon_j}\cdot g&=-\sigma_i^{-1}\sigma_j^{-1}(g), i\neq j,\\
X_{\varepsilon_i-\varepsilon_j}\cdot g&=(h_i-\frac{1}{2})\sigma_i\sigma_j^{-1}(g), i\neq j,
\end{aligned}$$

where $g\in U(\mathfrak{h}_n)$ and $\sigma_i\in\mathrm{Aut}(U(\mathfrak{h}_n))$ such that $\sigma_i(h_k)=h_k-\delta_{ik}$. We denote by $N_{\mathbf{1}}$ this $\mathfrak{sp}_{2n}$-module. It is easy to see that $N_{\mathbf{1}}$ is a Whittaker module with respect to the pair $(\mathfrak{sp}_{2n},\mathfrak{m}_n)$ of type $\mathbf{1}=(1,...,1)$. In [N2], Nilsson has shown that any $\mathfrak{sp}_{2n}$-module structure on $U(\mathfrak{h}_n)$ can be twisted to be $N_{\mathbf{1}}$ by some automorphism of $\mathfrak{sp}_{2n}$. We will show that any simple module in the block $\mathcal{WH}_{\mathbf{1}}^{\chi_{-\frac{1}{2}\omega_n}}$ can be twisted to be $N_{\mathbf{1}}$ by some automorphism of $\mathfrak{sp}_{2n}$, where $\omega_n=(1,...,1)$ is the n-th fundamental weight of $\mathfrak{sp}_{2n}$.

3. Non-singular Whittaker modules

In this section, we will characterize the category $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ when $\mathbf{a}\in(\mathbb{C}^*)^n$ using the weighting functor. In this case, similar as Kostant's definition of Whittaker modules [K], a module in $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ is said to be non-singular. For convenience, set $t^{\mathbf{m}}=t_1^{m_1}...t_n^{m_n}$, $h^{\mathbf{m}}=h_1^{m_1}...h_n^{m_n}$, for any $\mathbf{m}=(m_1,...,m_n)\in\mathbb{Z}^n$.

3.1. The category $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$

We define the total order on $\mathbb{Z}_{\geq0}^n$ satisfying the condition: $\mathbf{r}<\mathbf{m}$ if $|\mathbf{r}|<|\mathbf{m}|$ or $|\mathbf{r}|=|\mathbf{m}|$ and there is an $l\in\{1,...,n\}$ such that $r_i=m_i$ when $1\leq i<l$ and $r_l<m_l$, where $|\mathbf{m}|=\sum_{i=1}^n m_i$. For each $\mathbf{m}$, the set $\{\mathbf{r}\in\mathbb{Z}_{\geq0}^n|\mathbf{r}<\mathbf{m}\}$ is finite. Hence as an ordered set $\mathbb{Z}_{\geq0}^n$ is isomorphic to $\mathbb{Z}_{\geq0}$. For a nonzero $\mathbf{m}\in\mathbb{Z}_{\geq0}^n$, denote by $\mathbf{m}'$ the predecessor of $\mathbf{m}$, i.e., $\mathbf{m}'$ is the maximal element in $\mathbb{Z}_{\geq0}^n$ such that $\mathbf{m}'<\mathbf{m}$.

The following lemma gives a rough characterization of modules in $\mathcal{WH}_{\mathbf{a}}$ which is important for the later discussions.

Lemma 3. *Any module M in $\mathcal{WH}_{\mathbf{a}}$ is a free $U(\mathfrak{h}_n)$-module of finite rank.*

Proof. First, we show that $M=U(\mathfrak{h}_n)\mathrm{wh}_{\mathbf{a}}(M)$.

Denote $Y^{\mathbf{m}}=(X_{-2\varepsilon_1}+a_1^2)^{m_1}...(X_{-2\varepsilon_n}+a_n^2)^{m_n}$, for any $\mathbf{m}\in\mathbb{Z}_{\geq0}^n$. Then the set $\{Y^{\mathbf{s}}|\mathbf{s}\in\mathbb{Z}_{\geq0}^n\}$ forms a basis of $U(\mathfrak{m}_n)$. Using the hypothesis that $a_i\neq0$ for any i and induction on $\mathbf{m}$, from $[h_i,X_{-2\varepsilon_i}]=-2X_{-2\varepsilon_i}$, we can show that for any $\mathbf{m},\mathbf{s}\in\mathbb{Z}_{\geq0}^n$ and nonzero $v\in\mathrm{wh}_{\mathbf{a}}(M)$, we have that $Y^{\mathbf{s}}h^{\mathbf{m}}v=0$ whenever $\mathbf{s}>\mathbf{m}$, $Y^{\mathbf{m}}h^{\mathbf{m}}v=k_{\mathbf{m}}v$ for some nonzero scalar $k_{\mathbf{m}}$.

For each $\mathbf{m}\in\mathbb{Z}_{\geq0}^n$, let $\mathbb{I}_{\mathbf{m}}$ be the ideal of $U(\mathfrak{m}_n)$ spanned by $Y^{\mathbf{s}}$ with $\mathbf{s}>\mathbf{m}$, and $M_{\mathbf{m}}=\{w\in M|\mathbb{I}_{\mathbf{m}}w=0\}$. Clearly $\mathrm{wh}_{\mathbf{a}}(M)=M_{\mathbf{0}}$. For any nonzero $w\in M$, by the definition of M, there is an $\mathbf{m}\in\mathbb{Z}_{\geq0}^n$ such that $w\in M_{\mathbf{m}}\setminus M_{\mathbf{m}'}$, i.e., $Y^{\mathbf{m}}w\neq0$ and $Y^{\mathbf{s}}w=0$ for any $\mathbf{s}>\mathbf{m}$. So $Y^{\mathbf{m}}w\in\mathrm{wh}_{\mathbf{a}}(M)$. We call $\mathbf{m}$ the degree of w. By the above discussion, $Y^{\mathbf{m}}h^{\mathbf{m}}Y^{\mathbf{m}}w=k_{\mathbf{m}}Y^{\mathbf{m}}w$. We use induction on the degree $\mathbf{m}$ of w

to show that $w \in U(\mathfrak{h}_n)\mathrm{wh}_{\mathbf{a}}(M)$. Let $w' = w - \frac{1}{k_{\mathbf{m}}} h^{\mathbf{m}} Y^{\mathbf{m}} w$. Then

$$Y^{\mathbf{m}} w' = Y^{\mathbf{m}} w - \frac{1}{k_{\mathbf{m}}} Y^{\mathbf{m}} h^{\mathbf{m}} Y^{\mathbf{m}} w = 0.$$

This implies that the degree of w' is smaller than $\mathbf{m}$. By the induction hypothesis, $w' \in U(\mathfrak{h}_n)\mathrm{wh}_{\mathbf{a}}(M)$. Consequently $w \in U(\mathfrak{h}_n)\mathrm{wh}_{\mathbf{a}}(M)$.

Next we show that $M \cong U(\mathfrak{h}_n) \otimes \mathrm{wh}_{\mathbf{a}}(M)$. Suppose that $\{v_i | i=1,...,k\}$ is a basis of the vector space $\mathrm{wh}_{\mathbf{a}}(M)$. We need to show that $\{h^{\mathbf{m}} v_i | \mathbf{m} \in \mathbb{Z}^n_{\geq 0}, i=1,...,k\}$ is linearly independent. Suppose that $w := \sum_{\mathbf{r} \leq \mathbf{m}} \sum_{i=1}^{k} c_{\mathbf{r},i} h^{\mathbf{r}} v_i = 0$. Then from $Y^{\mathbf{m}} w = 0$, we see that $c_{\mathbf{m},i} = 0$. Consequently, by induction on $\mathbf{m}$, $c_{\mathbf{r},i} = 0$ for any $\mathbf{r} < \mathbf{m}$ and i. Thus $\{h^{\mathbf{m}} v_i | \mathbf{m} \in \mathbb{Z}^n_{\geq 0}, i=1,...,k\}$ is linearly independent. The proof is complete. $\square$

By Lemma 3, the Gelfand-Kirillov dimension of any module in $\mathcal{WH}_{\mathbf{a}}$ is n which is minimal among all infinite dimensional $\mathfrak{sp}_{2n}$-modules. So any module in $\mathcal{WH}_{\mathbf{a}}$ is a minimal representation. With the characterizations of modules in Lemma 3, we can use the weighting functor and the category $\mathcal{B}$ of uniformly bounded weight modules to study $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$.

Proposition 4. *We have the following statements.*

(a) *For any* $\mathbf{a} \in (\mathbb{C}^*)^n$ *and* $\mu \notin \Lambda^+$, *if the block* $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ *is non-empty, then* $\mu(h_i - h_{i+1}) \in \mathbb{Z}_{\geq 0}$, *for any* $i \neq n$, $\mu(h_n) \in \frac{1}{2} + \mathbb{Z}$ *and* $\mu(h_{n-1} + h_n) \in \mathbb{Z}_{\geq -2}$.

(b) *For any* $\mu \in \mathfrak{h}_n^*$ *and* $\mathbf{a} \in (\mathbb{C}^*)^n$, *if* $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ *is non-empty, then* $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ *is equivalent to* $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$.

Proof. (a) Let $M \in \mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$. By Lemma 3, M is a free $U(\mathfrak{h}_n)$-module of finite rank. Then the module $\mathfrak{W}^\mu(M)$ is a bounded weight $\mathfrak{sp}_{2n}$-module, i.e, $\mathfrak{W}^\mu(M) \in \mathcal{B}^{\chi_\mu}$. By Lemmas 9.1 and 9.2 in [M], one can prove (a). We should note that the symbols h_i in [M] represent the simple coroots which are different from our h_i.

(b) For a $\lambda \in \Lambda^+$, let $L(\lambda)$ be the simple $\mathfrak{sp}_{2n}$-module of highest weight λ. The condition $\lambda \in \Lambda^+$ implies that $L(\lambda)$ is finite dimensional. Recall that the translation functor $T^{\mu}_{-\frac{1}{2}\omega_n}$ is defined by

$$T^{\mu}_{-\frac{1}{2}\omega_n}(M) = \{v \in L(\lambda) \otimes M \mid (z - \chi_\mu(z))^k v = 0, \text{for some } \ k \in \mathbb{Z}_+, \forall\ z \in Z(\mathfrak{sp}_{2n})\},$$

for any $M \in \mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$. If $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$ is nonempty, then by the proof of Lemma 9.2 in [M], we can choose $\lambda \in \Lambda^+$ such that the functor $T^{\mu}_{-\frac{1}{2}\omega_n}$ gives an equivalence between $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$ and $\mathcal{WH}_{\mathbf{a}}^{\chi_\mu}$, see also [BG]. $\square$

In order to study the category $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$, we use the Weyl algebra $\mathcal{D}_n$. Let $A_n=\mathbb{C}[t_1,...,t_n]$ be the polynomial algebra in n variables. Then the subalgebra of $\mathrm{End}_{\mathbb{C}}(A_n)$ generated by

$$\{t_i, \partial_i := \frac{\partial}{\partial t_i} \mid 1 \leq i \leq n\}$$

is called the Weyl algebra $\mathcal{D}_n$ over A_n. Namely, $\mathcal{D}_n$ is the unital associative algebra over $\mathbb{C}$ generated by $t_1,...,t_n$, $\partial_1,...,\partial_n$ subject to the following relations

$$[\partial_i, \partial_j] = [t_i, t_j] = 0, \quad [\partial_i, t_j] = \delta_{i,j},\ 1 \leq i, j \leq n.$$

Let $\mathcal{D}_n^{ev}$ be the subalgebra of $\mathcal{D}_n$ spanned by

$$\{t^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{Z}_+^n, |\alpha|+|\beta| \in 2\mathbb{Z}_+\},$$

where $\partial^\beta = \partial_1^{\beta_1}...\partial_n^{\beta_n}$. We call $\mathcal{D}_n^{ev}$ the even Weyl algebra of rank n. In the following lemma, we recall a differential operator realization of $\mathfrak{sp}_{2n}$, see [BL].

Lemma 5. *The map*

$$\begin{aligned} \theta_0:\ U(\mathfrak{sp}_{2n}) &\longrightarrow \mathcal{D}_n^{ev}, \\ X_{\varepsilon_i+\varepsilon_j} &\longmapsto t_i t_j, \\ X_{\varepsilon_i-\varepsilon_j} &\longmapsto t_i \partial_j, \quad i \neq j, \\ h_i &\longmapsto t_i \partial_i + \frac{1}{2}, \\ X_{-\varepsilon_i-\varepsilon_j} &\longmapsto -\partial_i \partial_j, 1 \leq i, j \leq n, \end{aligned} \tag{3.1}$$

defines a surjective algebra homomorphism.

Let P_n be the unital subalgebra of A_n generated by $t_i t_j$, $i,j \in \{1,...,n\}$. By Lemma 5, P_n can be made to be an $\mathfrak{sp}_{2n}$-module called the Weil representation, see [M]. It is easy to see that P_n is isomorphic to the simple highest weight module $L(-\frac{1}{2}\omega_n)$ of the highest weight $-\frac{1}{2}\omega_n$ up to an involution of $\mathfrak{sp}_{2n}$, where $\omega_n = \sum_{i=1}^n \varepsilon_i$ is the n-th fundamental weight of $\mathfrak{sp}_{2n}$.

By Theorem 5.2 in [GS1], we obtain the following description of $\mathcal{B}^{\chi_{-\frac{1}{2}\omega_n}}$.

Lemma 6. *If M is a module in $\mathcal{B}^{\chi_{-\frac{1}{2}\omega_n}}$, then* $\ker \theta_0 M = 0$.

Using Lemma 6 and the weighting functor, we show that any module in $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$ is actually a $\mathcal{D}_n^{ev}$-module.

Lemma 7. *If M is a module in $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$, then* $\ker \theta_0 M = 0$, *i.e. M is a $\mathcal{D}_n^{ev}$-module.*

Proof. By lemma 3, M is a free $U(\mathfrak{h}_n)$-module. The module $\mathfrak{W}^{-\frac{1}{2}\omega_n}(M)$ is a uniformly bounded weight $\mathfrak{sp}_{2n}$-module, i.e. $\mathfrak{W}^{-\frac{1}{2}\omega_n}(M)\in\mathcal{B}^{\chi_{-\frac{1}{2}\omega_n}}$. By Lemma 6, $\ker\theta_0(\mathfrak{W}^{-\frac{1}{2}\omega_n}(M))=0$. So $\ker\theta_0 M\subset I_{\alpha-\frac{1}{2}\omega_n}M$ for any $\alpha\in\mathbb{Z}^n$. Since M is a free $U(\mathfrak{h}_n)$-module of finite rank, we have that $\cap_{\alpha\in\mathbb{Z}^n}(I_{\alpha-\frac{1}{2}\omega_n}M)=0$. So $\ker\theta_0 M=0$. $\square$

Let $\mathcal{W}^{\mathbf{a}}$ be the category of $\mathcal{D}_n^{ev}$-modules V such that $\partial_i^2-a_i^2$ acts locally nilpotently on V for any $i\in\{1,...,n\}$, and $\mathrm{wh}'_{\mathbf{a}}(V):=\{v\in V\,|\,\partial_i^2 v=a_i^2 v,\ \forall\ i=1,...,n\}$ is finite dimensional. Then by Lemma 7, we have the following equivalence.

Theorem 8. *The category $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$ is equivalent to the category $\mathcal{W}^{\mathbf{a}}$ of $\mathcal{D}_n^{ev}$-modules.*

3.2. Modules over the even Weyl algebra

By Theorem 8, we need to study the category $\mathcal{W}^{\mathbf{a}}$ for $\mathcal{D}_n^{ev}$. For a $\mathbf{b}=(b_1,...,b_n)\in(C^*)^n$ such that $b_i^2=a_i^2$ for all i, we define a $\mathcal{D}_n^{ev}$-module $M_{\mathbf{b}}:=\mathbb{C}[x_1,...,x_n]$ as follows:

$$\begin{aligned}
\partial_i\partial_j x^{\mathbf{m}} &= b_ib_j\tau_i^{-1}\tau_j^{-1}(x^{\mathbf{m}}),\\
t_it_j x^{\mathbf{m}} &= b_i^{-1}b_j^{-1}x_ix_j\tau_i\tau_j(x^{\mathbf{m}}),\quad i\neq j,\\
t_i^2 x^{\mathbf{m}} &= b_i^{-2}x_i^2\tau_i^2(x^{\mathbf{m}})-b_i^{-2}x_i\tau_i^2(x^{\mathbf{m}}),\\
t_i\partial_j x^{\mathbf{m}} &= b_i^{-1}b_jx_i\tau_i\tau_j^{-1}(x^{\mathbf{m}}),
\end{aligned}$$

where $x^{\mathbf{m}}=x_1^{m_1}...x_n^{m_n}$, $\tau_i\in\mathrm{Aut}(\mathbb{C}[x_1,...,x_n])$ such that $\tau_i(x_k)=x_k-\delta_{ik}$.

Proposition 9. (a) *Any simple module M in $\mathcal{W}^{\mathbf{a}}$ is isomorphic to $M_{\mathbf{b}}$, where $\mathbf{b}\in(\mathbb{C}^*)^n$ such that $b_i^2=a_i^2$ for all i.*

(b) *The category $\mathcal{W}^{\mathbf{a}}$ of $\mathcal{D}_n^{ev}$-modules is semi-simple.*

Proof. (a) Suppose that N is a nonzero submodule of $M_{\mathbf{b}}$. Since $\partial_i^2-a_i^2$ decreases the degrees of x_i, we must have that $x^0:=1\in N$. Note that x^0 generates $M_{\mathbf{b}}$. So $N=M_{\mathbf{b}}$, $M_{\mathbf{b}}$ is simple.

Suppose that M is a simple module in $\mathcal{W}^{\mathbf{a}}$. Since $\mathrm{wh}'_{\mathbf{a}}(M)$ is finite dimensional and $[\partial_i,\partial_j]=0$, there are a nonzero $v\in M$ and $\mathbf{b}\in(\mathbb{C}^*)^n$ such that $\partial_i\partial_j v=b_ib_jv$ and $b_i^2=a_i^2$ for all i,j. We can define a $\mathcal{D}_n^{ev}$-module isomorphism τ from M to $M_{\mathbf{b}}$ such that $\tau((t_1\partial_1)^{m_1}...(t_n\partial_n)^{m_n}v)=x^{\mathbf{m}}$, for all $\mathbf{m}\in\mathbb{Z}_+^n$. So $M\cong M_{\mathbf{b}}$.

(b) It suffices to show that $\mathrm{Ext}^1_{\mathcal{D}_n^{ev}}(M_{\mathbf{b}},M_{\mathbf{b}'})=0$. If there are i,j such that $b_ib_j\neq b'_ib'_j$, then from that the eigenvalues of $\partial_i\partial_j$ on $M_{\mathbf{b}}$ and $M_{\mathbf{b}'}$ are different, $\mathrm{Ext}^1_{\mathcal{D}_n^{ev}}(M_{\mathbf{b}},M_{\mathbf{b}'})=0$. So it suffices to consider that $\mathbf{b}=\mathbf{b}'$. We will show that the short exact sequence

$$0\longrightarrow M_{\mathbf{b}}\xrightarrow{\alpha}V\xrightarrow{\beta}M_{\mathbf{b}}\longrightarrow 0\tag{3.2}$$

of $\mathcal{D}_n^{ev}$-modules is split. By the similar proof in Lemma 3, we can show that $V=\mathbb{C}[t_1\partial_1,...,t_n\partial_n]\otimes \mathrm{wh}'_{\mathbf{b}}(V)$, and $\dim \mathrm{wh}'_{\mathbf{b}}(V)=2$, since $\dim \mathrm{wh}'_{\mathbf{b}}(M_{\mathbf{b}})=1$. Explicitly we can replace h_i and $X_{-2\varepsilon_i}$ by $t_i\partial_i$ and ∂_i^2 respectively in the proof of Lemma 3. Choose $v\in \mathrm{wh}'_{\mathbf{b}}(V)\setminus\alpha(M_{\mathbf{b}})$. By the proof of (a), the submodule $\mathcal{D}_n^{ev}v\cong M_{\mathbf{b}}$. Note that $M_{\mathbf{b}}$ is a free $\mathbb{C}[t_1\partial_1,...,t_n\partial_n]$-module of rank one. So the sequence (3.2) is split. The proof is complete. □

Combining Theorem 8 and Proposition 9, we obtain the following characterization of $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$.

Theorem 10. *The category $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$ is semi-simple. Moreover, any simple module in $\mathcal{WH}_{\mathbf{a}}^{\chi_{-\frac{1}{2}\omega_n}}$ is isomorphic to $M_{\mathbf{b}}$, where $\mathbf{b}=(b_1,...,b_n)\in(\mathbb{C}^*)^n$ such that $b_i^2=a_i^2$ for all i, and $M_{\mathbf{b}}$ is an $\mathfrak{sp}_{2n}$-module under the map (3.1).*

Note that the $\mathfrak{sp}_{2n}$-module $M_{\mathbf{b}}$ is a free $U(\mathfrak{h}_n)$-module of rank one. By the result in [N2], any $\mathfrak{sp}_{2n}$-module that is a free $U(\mathfrak{h}_n)$-module of rank one can be twisted to be $N_{\mathbf{1}}$ defined in subsection 2.6 by some automorphism of $\mathfrak{sp}_{2n}$. Then by Theorem 10, we have the following result.

Corollary 11. *Any simple module in $\mathcal{WH}_{\mathbf{1}}^{\chi_{-\frac{1}{2}\omega_n}}$ can be twisted to be $N_{\mathbf{1}}$ by some automorphism of $\mathfrak{sp}_{2n}$.*

4. General Weil representations

In Section 3, we see that the algebra homomorphism θ_0 from $U(\mathfrak{sp}_{2n})$ to the Weyl algebra $\mathcal{D}_n$ is useful for the study of representations of $\mathfrak{sp}_{2n}$. In this section, we will find more algebra homomorphisms from $U(\mathfrak{sp}_{2n})$ to the Weyl algebra $\mathcal{D}_n$.

4.1. General Weil representations

In the following lemma, we give a differential operators realization of $\mathfrak{sp}_{2n}$ from any $f\in A_n$.

Lemma 12. *For any $f\in A_n$, the map*

$$\begin{aligned}\theta_f:\ U(\mathfrak{sp}_{2n}) &\longrightarrow \mathcal{D}_n,\\ X_{\varepsilon_i+\varepsilon_j} &\longmapsto t_it_j,\\ X_{\varepsilon_i-\varepsilon_j} &\longmapsto t_i\partial_j(f)+t_i\partial_j, \quad i\neq j,\\ h_i &\longmapsto t_i\partial_i(f)+t_i\partial_i+\frac{1}{2},\\ X_{-\varepsilon_i-\varepsilon_j} &\longmapsto -(\partial_i(f)+\partial_i)(\partial_j(f)+\partial_j), 1\leq i,j\leq n,\end{aligned}\tag{4.1}$$

defines an algebra homomorphism.

Proof. By Lemma 5, θ_0 is an algebra homomorphism. For general f, the map θ_f is the composition $\sigma_f \circ \theta_0$, where σ_f is the algebra isomorphism of $\mathcal{D}_n$ defined by

$$t_i \longmapsto t_i, \partial_i \longmapsto \partial_i + \partial_i(f).$$

Therefore θ_f is an algebra homomorphism. $\square$

Next we will show that θ_f in Lemma 12 exhausts all algebra homomorphisms θ from $U(\mathfrak{sp}_{2n})$ to $\mathcal{D}_n$ such that $\theta(X_{\varepsilon_i+\varepsilon_j}) = t_i t_j$ for any i, j.

Firstly we give some formulas in $U(\mathfrak{sp}_{2n})$ that will be used in the subsequent text.

Lemma 13. *For any* $k \in \mathbb{N}$, *these formulas hold as follows*:
(1) $[X_{\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}^k] = \delta_{jl} 2k X_{2\varepsilon_l}^{k-1} X_{\varepsilon_i+\varepsilon_j}, 1 \le i \ne j \le n$;
(2) $[X_{-\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}^k] = -2k X_{2\varepsilon_l}^{k-1}\big(\delta_{jl} X_{\varepsilon_j-\varepsilon_i} + \delta_{il} X_{\varepsilon_i-\varepsilon_j}\big), 1 \le i \ne j \le n$;
(3) $[X_{-2\varepsilon_i}, X_{2\varepsilon_l}^k] = -\delta_{il} 4k X_{2\varepsilon_l}^{k-1}\big(h_i + k - 1\big), 1 \le i \le n.$

Proof. (1) According to $[X_{\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}] = 2\delta_{jl} X_{\varepsilon_i+\varepsilon_j}$ and $[X_{\varepsilon_i+\varepsilon_l}, X_{2\varepsilon_l}] = 0$, we can compute that

$$\begin{aligned}[X_{\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}^k] &= \sum_{t=1}^{k} X_{2\varepsilon_l}^{k-t} [X_{\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}] X_{2\varepsilon_l}^{t-1} \\ &= \sum_{t=1}^{k} 2\delta_{jl} X_{2\varepsilon_l}^{k-1} X_{\varepsilon_i+\varepsilon_j} = \delta_{jl} 2k X_{2\varepsilon_l}^{k-1} X_{\varepsilon_i+\varepsilon_j}.\end{aligned}$$

(2) From $[X_{-\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}] = -2\big(\delta_{jl} X_{\varepsilon_j-\varepsilon_i} + \delta_{il} X_{\varepsilon_i-\varepsilon_j}\big)$ and

$$[X_{\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}^{t-1}] = \delta_{jl} 2(t-1) X_{2\varepsilon_l}^{t-2} X_{\varepsilon_i+\varepsilon_j},$$

we have that

$$\begin{aligned}&[X_{-\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}^k] \\ =& \sum_{t=1}^{k} X_{2\varepsilon_l}^{k-t} [X_{-\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}] X_{2\varepsilon_l}^{t-1} \\ =& \sum_{t=1}^{k} -2 X_{2\varepsilon_l}^{k-t} \big(\delta_{jl} X_{\varepsilon_j-\varepsilon_i} X_{2\varepsilon_l}^{t-1} + \delta_{il} X_{\varepsilon_i-\varepsilon_j} X_{2\varepsilon_l}^{t-1}\big) \\ =& \sum_{t=1}^{k} -2 X_{2\varepsilon_l}^{k-t} \big(\delta_{il} X_{2\varepsilon_l}^{t-1} X_{\varepsilon_i-\varepsilon_j} + \delta_{jl} X_{2\varepsilon_l}^{t-1} X_{\varepsilon_j-\varepsilon_i} \\ &\quad + 2(t-1) X_{2\varepsilon_l}^{t-2} \delta_{il}\delta_{jl} (X_{\varepsilon_i+\varepsilon_l} + X_{\varepsilon_j+\varepsilon_l})\big) \\ =& -2k X_{2\varepsilon_l}^{k-1} \big(\delta_{il} X_{\varepsilon_i-\varepsilon_j} + \delta_{jl} X_{\varepsilon_j-\varepsilon_i}\big) - 2k(k-1)\delta_{il}\delta_{jl} X_{2\varepsilon_l}^{k-2} (X_{\varepsilon_i+\varepsilon_l} + X_{\varepsilon_j+\varepsilon_l}).\end{aligned}$$

Thus, we can see that

$$[X_{-\varepsilon_i-\varepsilon_j},X_{2\varepsilon_l}^k]=-2kX_{2\varepsilon_l}^{k-1}\big(\delta_{jl}X_{\varepsilon_j-\varepsilon_i}+\delta_{il}X_{\varepsilon_i-\varepsilon_j}\big),\quad i\neq j.$$

(3) By the similar computation in (2), we can obtain that

$$\begin{aligned}[X_{-2\varepsilon_i},X_{2\varepsilon_l}^k]&=-2kX_{2\varepsilon_l}^{k-1}2\delta_{il}h_i-2\delta_{il}k(k-1)X_{2\varepsilon_l}^{k-2}2X_{2\varepsilon_l}\\&=-\delta_{il}4kX_{2\varepsilon_l}^{k-1}\big(h_i+k-1\big).\end{aligned}$$

The proof is complete. □

In the following lemma, we give a preliminary description of algebra homomorphisms from $U(\mathfrak{sp}_{2n})$ to $\mathcal{D}_n$.

Lemma 14. *If θ is an algebra homomorphism from $U(\mathfrak{sp}_{2n})$ to $\mathcal{D}_n$ such that*

$$\theta(X_{\varepsilon_i+\varepsilon_j})=t_it_j,$$

for any i,j, then there exist $p_{ij},q_{ij}\in A_n$ such that
(1) $\theta(X_{\varepsilon_i-\varepsilon_j})=p_{ij}+t_i\partial_j, 1\le i\neq j\le n$;
(2) $\theta(h_i)=p_{ii}+t_i\partial_i, 1\le i\le n$;
(3) $\theta(X_{-2\varepsilon_i})=q_{ii}+(1-2p_{ii})t_i^{-1}\partial_i-\partial_i^2$, $1\le i\le n$;
(4) $\theta(X_{-\varepsilon_i-\varepsilon_j})=q_{ij}-p_{ij}t_i^{-1}\partial_i-p_{ji}t_j^{-1}\partial_j-\partial_i\partial_j, 1\le i\neq j\le n$.

Proof. We consider the action of $\theta(X_\alpha)$ on $R_n:=\mathbb{C}[t_1^{\pm},...,t_n^{\pm1}]$. For the convenience, we denote $X_{2\varepsilon_1}^{m_1}...X_{2\varepsilon_n}^{m_n}$ by $X^{\mathbf{m}}$, and $\theta(X_\alpha)(g(t))$ by $X_\alpha\cdot g(t)$ for any $\alpha\in\Delta$ and $g(t)\in R_n$.

By definition, $X_{\varepsilon_i+\varepsilon_j}\cdot t^{\mathbf{m}}=t_it_jt^{\mathbf{m}}$, for any $\mathbf{m}\in\mathbb{Z}^n$.

(1) Define $p_{ij}=X_{\varepsilon_i-\varepsilon_j}\cdot 1$. From $[X_{\varepsilon_i-\varepsilon_j},X_{2\varepsilon_l}^k]=\delta_{jl}2kX_{2\varepsilon_l}^{k-1}X_{\varepsilon_i+\varepsilon_j}$, we obtain that

$$\begin{aligned}X_{\varepsilon_i-\varepsilon_j}\cdot t^{2\mathbf{m}}&=X_{\varepsilon_i-\varepsilon_j}\cdot X^{\mathbf{m}}\cdot 1\\&=[X_{\varepsilon_i-\varepsilon_j},X^{\mathbf{m}}]\cdot 1+X^{\mathbf{m}}\cdot X_{\varepsilon_i-\varepsilon_j}\cdot 1\\&=2m_jX^{\mathbf{m}-e_j}X_{\varepsilon_i+\varepsilon_j}\cdot 1+X^{\mathbf{m}}\cdot X_{\varepsilon_i-\varepsilon_j}\cdot 1\\&=2m_jt^{2\mathbf{m}+e_i-e_j}+p_{ij}t^{2\mathbf{m}}\\&=(p_{ij}+t_i\partial_j)(t^{2\mathbf{m}}).\end{aligned}$$

(2) It is similar as (1)

(3) For $i=j$, by $[X_{-2\varepsilon_i},X_{2\varepsilon_l}^k]=-\delta_{il}4kX_{2\varepsilon_l}^{k-1}\big(h_i+k-1\big)$, we can calculate that

$$[X_{-2\varepsilon_i},X^{\mathbf{m}}]=[X_{-2\varepsilon_i},\prod_{l=1}^nX_{2\varepsilon_l}^{m_l}]$$

$$
\begin{aligned}
&=\prod_{j=1}^{i-1} X_{2\varepsilon_j}^{m_j}[X_{-2\varepsilon_i}, X_{2\varepsilon_i}^{m_i}] \prod_{s=i+1}^{n} X_{2\varepsilon_s}^{m_s}\\
&=\prod_{j=1}^{i-1} X_{2\varepsilon_j}^{m_j}\Big(-4m_i X_{2\varepsilon_i}^{m_i-1}(h_i+m_i-1)\Big) \prod_{s=i+1}^{n} X_{2\varepsilon_s}^{m_s}\\
&=-4m_i X^{\mathbf{m}-e_i}(h_i+m_i-1).
\end{aligned}
$$

Hence, set $q_{ii}=X_{-2\varepsilon_i}\cdot 1$, we can determine the action of $X_{-2\varepsilon_i}$ as follows:

$$
\begin{aligned}
X_{-2\varepsilon_i}\cdot t^{2\mathbf{m}} &= X_{-2\varepsilon_i}\cdot X^{\mathbf{m}}\cdot 1\\
&=[X_{-2\varepsilon_i}, X^{\mathbf{m}}]\cdot 1+X^{\mathbf{m}}\cdot X_{-2\varepsilon_i}\cdot 1\\
&=-4m_i X^{\mathbf{m}-e_i}(h_i+m_i-1)\cdot 1+X^{\mathbf{m}}\cdot X_{-2\varepsilon_i}\cdot 1\\
&=(1-2p_{ii})t_i^{-1}\partial_i t^{2\mathbf{m}}-\partial_i^2 t^{2\mathbf{m}}+q_{ii}t^{2\mathbf{m}}\\
&=\big(q_{ii}+(1-2p_{ii})t_i^{-1}\partial_i-\partial_i^2\big)t^{2\mathbf{m}}.
\end{aligned}
$$

(4) When $i\neq j$, let's suppose $i<j$, and the case for $i>j$ is similar. According to $[X_{-\varepsilon_i-\varepsilon_j}, X_{2\varepsilon_l}^k]=-2kX_{2\varepsilon_l}^{k-1}\big(\delta_{jl}X_{\varepsilon_j-\varepsilon_i}+\delta_{il}X_{\varepsilon_i-\varepsilon_j}\big)$, we obtain that

$$
\begin{aligned}
&[X_{-\varepsilon_i-\varepsilon_j}, X^{\mathbf{m}}]=[X_{-\varepsilon_i-\varepsilon_j}, \prod_{l=1}^{n} X_{2\varepsilon_l}^{m_l}]\\
&=-2m_i X^{\mathbf{m}-e_i}X_{\varepsilon_i-\varepsilon_j}-4m_i m_j X^{\mathbf{m}-e_i-e_j}X_{\varepsilon_i+\varepsilon_j}-2m_j X^{\mathbf{m}-e_j}X_{\varepsilon_j-\varepsilon_i}.
\end{aligned}
$$

Denote $q_{ij}=X_{-\varepsilon_i-\varepsilon_j}\cdot 1$. Then, we have

$$
\begin{aligned}
X_{-\varepsilon_i-\varepsilon_j}\cdot t^{2\mathbf{m}} &= X_{-\varepsilon_i-\varepsilon_j}\cdot X^{\mathbf{m}}\cdot 1\\
&=[X_{-\varepsilon_i-\varepsilon_j}, X^{\mathbf{m}}]\cdot 1+X^{\mathbf{m}}\cdot X_{-\varepsilon_i-\varepsilon_j}\cdot 1\\
&=\big(q_{ij}-p_{ij}t_i^{-1}\partial_i-p_{ji}t_j^{-1}\partial_j-\partial_i\partial_j\big)(t^{2\mathbf{m}}).
\end{aligned}
$$

Consequently, the proof is completed. □

The following easy lemma will be used in the proof of Theorem 16

Lemma 15. *For any $i\in\{1,\dots,n\}$ and $p\in A_n$, the differential equation*

$$(t_i\partial_i+1)(q)=p$$

has at most one solution q in A_n.

Proof. The proof follows from the fact that the map $t_i\partial_i+1\colon A_n\to A_n$ is injective. □

Combining the above preparatory arguments, we can give all possible algebra homomorphisms from $U(\mathfrak{sp}_{2n})$ to the Weyl algebra $\mathcal{D}_n$ which map each root vector $X_{\varepsilon_i+\varepsilon_j}$ to t_it_j.

Theorem 16. *If θ is an algebra homomorphism from $U(\mathfrak{sp}_{2n})$ to $\mathcal{D}_n$ such that*

$$\theta(X_{\varepsilon_i+\varepsilon_j}) = t_it_j,$$

for any i,j, then there is an $f\in A_n$ such that $\theta=\theta_f$, that is
(1) $\theta(X_{\varepsilon_i+\varepsilon_j})=t_it_j$,
(2) $\theta(X_{\varepsilon_i-\varepsilon_j})=t_i\partial_j(f)+t_i\partial_j, i\neq j$,
(3) $\theta(h_i)=t_i\partial_i(f)+t_i\partial_i+\frac{1}{2}$,
(4) $\theta(X_{-\varepsilon_i-\varepsilon_j})=-(\partial_i(f)+\partial_i)(\partial_j(f)+\partial_j)$,
where $1\leq i,j\leq n$.

Proof. It is sufficient to determine p_{ij}, q_{ij} in Lemma 14. For any $f\in A_n$, we set $f_{(i)}=\partial_i(f)$.

Specifically, according to $[h_i,h_j]=0$, which means $[p_{ii}+t_i\partial_i, p_{jj}+t_j\partial_j]=0$, we can calculate

$$t_i\partial_i(p_{jj}) = t_j\partial_j(p_{ii}), 1\leq i,j\leq n.$$

Then we can obtain that $p_{ii}=t_if_{(i)}+c_i$ for some $f\in A_n$ and $\mathbf{c}=(c_1,...,c_n)\in\mathbb{C}^n$.

By $[X_{\varepsilon_i-\varepsilon_j}, X_{\varepsilon_j-\varepsilon_i}]=h_i-h_j$, we deduce that

$$p_{ij}t_j\partial_i+t_i\partial_jp_{ji}-p_{ji}t_i\partial_j-t_j\partial_ip_{ij}=p_{ii}-p_{jj}.$$

Consequently, we have $c_i=c_j$ for any $1\leq i,j\leq n$. Set $b=c_i$ for any $1\leq i\leq n$, then $p_{ii}=t_if_{(i)}+b$.

For $i\neq j$, $[h_i, X_{\varepsilon_i-\varepsilon_j}]=X_{\varepsilon_i-\varepsilon_j}$ implies that $(t_i\partial_i-1)(p_{ij})=t_i\partial_j(p_{ii})$. Solving this differential equation, we can check that $p_{ij}=t_if_{(j)}$ is the unique solution of it.

For $i\neq j$, from $[h_i, X_{-\varepsilon_i-\varepsilon_j}]=-X_{-\varepsilon_i-\varepsilon_j}$, we get

$$t_i\partial_i(q_{ij})+q_{ij} = -p_{ij}t_i^{-1}\partial_i(p_{ii})-p_{ji}t_j^{-1}\partial_j(p_{ii})-\partial_i\partial_j(p_{ii}). \tag{4.2}$$

We can calculate it more explicitly,

$$\begin{aligned}
&t_i\partial_i(q_{ij})+q_{ij}\\
&=-p_{ij}t_i^{-1}\partial_i(p_{ii})-p_{ji}t_j^{-1}\partial_j(p_{ii})-\partial_i\partial_j(p_{ii})\\
&=-\Big(t_if_{(j)}t_i^{-1}\partial_i\big(t_if_{(i)}+b\big)+t_jf_{(i)}t_j^{-1}\partial_j\big(t_if_{(i)}+b\big)+\partial_i\partial_j\big(t_if_{(i)}+b\big)\Big)\\
&=-\Big(f_{(i)}f_{(j)}+t_if_{(j)}\partial_i(f_{(i)})+f_{(i)}t_i\partial_j(f_{(i)})+\partial_j(f_{(i)})+t_i\partial_i\partial_j(f_{(i)})\Big).
\end{aligned}$$

We can check that $q_{ij}=-f_{(i)}f_{(j)}-\partial_j(f_{(i)})$ is the unique solution of the equation (4.2).

Similarly, for $i=j$, we can get $q_{ii}=-f_{(i)}^2-\partial_i(f_{(i)})-(2b-1)t_i^{-1}f_{(i)}$.

Furthermore, for $i\neq j$, by $[X_{\varepsilon_i-\varepsilon_j},X_{-2\varepsilon_i}]=-2X_{-\varepsilon_i-\varepsilon_j}$, we deduce that

$$t_i\partial_j(q_{ii})-(1-2p_{ii})t_i^{-1}\partial_i(p_{ij})+\partial_i^2(p_{ij})=-2q_{ij}.$$

So we have $(1-2b)t_i^{-1}f_{(j)}=0$, that is, $b=\frac{1}{2}$ and $q_{ij}=-f_{(i)}f_{(j)}-\partial_j(f_{(i)})$ for $1\leq i,j\leq n$. Therefore, from Lemma 14, we can complete the proof. $\square$

4.2. New $\mathfrak{sp}_{2n}$-modules

Recall that P_n is the unital subalgebra of A_n generated by t_it_j, for $i,j\in\{1,...,n\}$. For any $f\in P_n$, via the homomorphism θ_f in (3.1), P_n becomes an $\mathfrak{sp}_{2n}$-module P_n^f. In Theorem 16, we actually have classified all $\mathfrak{sp}_{2n}$-module structures on P_n satisfying $X_{\varepsilon_i+\varepsilon_j}\cdot g(t)=t_it_jg(t)$ for any $i,j\in\{1,...,n\}, g(t)\in P_n$.

Corollary 17. *If there is an $\mathfrak{sp}_{2n}$-module structure on P_n such that $X_{\varepsilon_i+\varepsilon_j}\cdot g(t)=t_it_jg(t)$ for any $i,j\in\{1,...,n\}, g(t)\in P_n$, then this $\mathfrak{sp}_{2n}$-module structure is isomorphic to P_n^f for some $f\in P_n$.*

The following proposition gives a description of P_n^f.

Proposition 18. *Let $f,f'\in P_n$.*
(1) The $\mathfrak{sp}_{2n}$-module P_n^f is simple.
(2) As $\mathfrak{sp}_{2n}$-modules, $P_n^f\simeq P_n^{f'}$ if and only if $f-f'\in\mathbb{C}$.

Proof. (1) One can check directly that P_n is a simple $\mathcal{D}_n^{ev}$-module. Note that the image of θ_f is $\mathcal{D}_n^{ev}$. So P_n^f is a simple $\mathfrak{sp}_{2n}$-module.

(2) The sufficiency is obvious, it is enough to consider the necessity. Let φ: $P_n^f\to P_n^{f'}$ be an $\mathfrak{sp}_{2n}$-module isomorphism. Following $\varphi(X_{\varepsilon_i+\varepsilon_j}\cdot t^0)=X_{\varepsilon_i+\varepsilon_j}\cdot\varphi(t^0)$, we know that $\varphi(t^0)\neq 0$. From $\theta_f(X_{\varepsilon_i+\varepsilon_j})=t_it_j$ and $\varphi(X_{\varepsilon_i+\varepsilon_j}\cdot t^0)=X_{\varepsilon_i+\varepsilon_j}\cdot\varphi(t^0)$, we see that $\varphi(g(t))=g(t)\varphi(t^0)$ for any $g(t)\in P_n$. Then for any $i\neq j$, we have that

$$\begin{aligned}0&=\varphi(X_{\varepsilon_i-\varepsilon_j}\cdot t^0)-X_{\varepsilon_i-\varepsilon_j}\cdot\varphi(t^0)\\&=\varphi(t_if_{(j)})-(t_if'_{(j)}+t_i\partial_j)\cdot\varphi(t^0)\\&=-t_i\big(\partial_j+f'_{(j)}-f_{(j)}\big)\varphi(t^0).\end{aligned}$$

Consequently, we see that $\partial_j+f'_{(j)}-f_{(j)}$ is not injective on P_n. So we have $f_{(j)}=f'_{(j)}$ and $\varphi(t^0)\in\mathbb{C}^*t^0$ for any $j\in\{1,...,n\}$. That is $f-f'\in\mathbb{C}$, which completes the proof. $\square$

Therefore, we have constructed several simple modules P_n^f over $\mathfrak{sp}_{2n}$ generalizing the Weil representation.

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References

[ALZ] Adamović, D., Lü, R. and Zhao, K., Whittaker modules for the affine Lie algebra $A_1^{(1)}$, *Adv. Math.* **289** (2016), 438–479. MR3439693

[AP] Arnal, D. and Pinczon, G., On algebraically irreducible representations of the Lie algebra sl(2), *J. Math. Phys.* **15** (1974), 350–359. MR0357527

[B] Block, R., The irreducible representations of the Lie algebra $\mathfrak{sl}_2$ and of the Weyl algebra, *Adv. Math.* **39** (1981), 69–110. MR0605353

[BG] Bernstein, J. and Gelfand, S., Tensor products of finite and infinite-dimensional representations of semisimple Lie algebras, *Compos. Math.* **41** (1980), 245–285. MR0581584

[BL] Britten, D. J. and Lemire, F. W., A classification of simple Lie modules having a 1-dimensional weight space, *Trans. Am. Math. Soc.* **299** (1987), 683–697. MR0869228

[BM] Batra, P. and Mazorchuk, V., Blocks and modules for Whittaker pairs, *J. Pure Appl. Algebra* **215** (2011), 1552–1568. MR2771629

[BO] Benkart, G. and Ondrus, M., Whittaker modules for generalized Weyl algebras, *Represent. Theory* **13** (2009), 141–164. MR2497458

[C] Christodoupoulou, K., Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras, *J. Algebra* **320** (2008), 2871–2890. MR2442000

[CF] Cardoso, M. C. and Futorny, V., Affine Lie algebras representations induced from Whittaker modules, 2022. `arXiv:2203.13033v1`. MR4531637

[CJ] Chen, X. and Jiang, C., Whittaker modules for the twisted affine Nappi-Witten Lie algebra $H[\tau]$, *J. Algebra* **546** (2020), 37–61. MR4032277

[GLZ] Guo, X., Lu, R. and Zhao, K., Irreducible modules over the Virasoro algebra, *Doc. Math.* **16** (2011), 709–721. MR2861395

[GS1] Grantcharov, D. and Serganova, V., Category of $\mathfrak{sp}(2n)$-modules with bounded weight multiplicities, *Mosc. Math. J.* **6** (2006), 119–134. MR2265951

[GZ] Guo, X. and Zhao, K., Irreducible representations of untwisted affine Kac-Moody algebras, 2013. `arXiv:1305.4059v2`.

[K] Kostant, B., On Whittaker vectors and representation theory, *Invent. Math.* **48** (1978), 101–184. MR0507800

[LPX] Liu, D., Pei, Y. and Xia, L., Whittaker modules for the super-Virasoro algebras, *J. Algebra Appl.* **18**, 1950211 (2019). (13 pp). MR3994431

[LWZ] Liu, D., Wu, Y. and Zhu, L., Whittaker modules for the twisted Heisenberg-Virasoro algebra, *J. Math. Phys.* **51**, 023524 (2010). MR2605075

[M] Mathieu, O., Classification of irreducible weight modules, *Ann. Inst. Fourier* **50** (2000), 537–592. MR1775361

[N2] Nilsson, J., $U(\mathfrak{h})$-free modules and coherent families, *J. Pure Appl. Algebra* **220** (2016), 1475–1488. MR3423459

[OM] Ondrus, M., Whittaker modules for $U_q(\mathfrak{sl}_2)$, *J. Algebra* **289** (2005), 192–213. MR2139098

[OW1] Ondrus, M. and Wiesner, E., Whittaker modules for the Virasoro algebra, *J. Algebra Appl.* **8** (2009), 363–377. MR2535995

[OW2] Ondrus, M. and Wiesner, E., Whittaker categories for the Virasoro algebra, *Commun. Algebra* **41** (2013), 3910–3930. MR3169498

[S] Sevostyanov, A., Quantum deformation of Whittaker modules and the Toda lattice, *Duke Math. J.* **105** (2000), 211–238. MR1793611

[TWX] Tan, S., Wang, Q. and Xu, C., On whittaker modules for a Lie algebra arising from the 2-dimensional torus, *Pac. J. Math.* **273** (2015), 147–167. MR3290448

[XGL] Xia, L., Guo, X. and Zhang, J., Classification on irreducible Whittaker modules over quantum group $U_q(\mathfrak{sl}_3, \Lambda)$, *Front. Math. China* **16** (2021), 1089–1097. MR4307366

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