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On local colorings of split graphs

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Abstract. A *semi-matching coloring* of a finite simple graph $G=(V,E)$ is a mapping φ from V to $\{1,\dots,k\}$ such that (i) every color class is an independent set, and (ii) the edge set of the graph induced by the union of any two consecutive color classes is a matching. A semi-matching coloring φ is a *local coloring* if, in addition, (iii) the union of any three consecutive color classes induces a triangle-free subgraph of G. In this paper, we give two counterexamples and one positive solution to three problems arisen in recent papers of You, Cao, Wang. In particular, we show that the local and semi-matching coloring problems are NP-complete on the class of split graphs.

The concept of local coloring, introduced in [1], has attracted some interest in recent publications because of its connections to other graph theoretical problems, which include *Kneser's conjecture* [6]. A subsequent paper [4] extends this notion by introducing the semi-matching coloring problem and demonstrates its relation to Kneser graphs. The papers [5], [7] contain a description of the algorithmic complexity of the problems under consideration, and both the local and semi-matching colorings turn out to be NP-complete even if the number of the colors is a fixed integer $k\geqslant 3$. The authors of [8], [9] undertake a further investigation of the complexity of the problem and pose several questions, which include the complexity status of the local colorings of *split graphs*. The conventional chromatic number is tractable on this class, but is the local coloring NP-complete for split graphs? Also, the papers [8], [9] contain an explicitly posed conjecture stating the NP-hardness of the same problem on *perfect graphs*, but, since this class includes the split graphs, our NP-completeness proof is valid for perfect graphs as well.

In Section 1, we present several relevant examples of the computation of the local and semi-matching chromatic numbers, and one of these examples refutes a statement in [8]. In Section 2, we construct a part of the reduction that we use

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in our NP-completeness proof and give a counterexample to a statement in [9]. In Section 3, we finalize the NP-completeness proof of the local and semi-matching colorings restricted to split graphs, and hence we prove a conjecture in [8], [9].

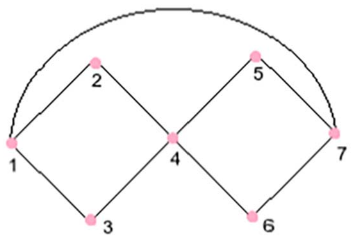
1. Examples

As in the previous research on the topic [1], [4], we define the *semi-matching chromatic number* $\chi_m(G)$ and *local chromatic number* $\chi_l(G)$ as the smallest possible maximal value of a color used in a semi-matching coloring and a local coloring of a graph G, respectively. Also, we recall a trivial inequality $\chi(G)\leqslant\chi_m(G)\leqslant\chi_l(G)$ involving the conventional chromatic number $\chi(G)$. For instance, the behavior of the complete graph K_n with respect to these notions is as follows.

Observation 1. (See [1].) $\chi_l(K_n)=\lfloor 1.5n-0.5\rfloor$ and $\chi_m(K_n)=\chi(K_n)=n$.

Proof. Every pair of vertices is adjacent, so there cannot be a smaller proper coloring than just to take the first n positive integers. Such a coloring does also possess a semi-matching property because the union of any two consecutive color classes is just an edge. In the local case, we are not allowed to use three consecutive numbers, so 1, 2, 4, 5, 7, 8, ... is the optimal labeling in this case, which corresponds to $\lfloor 1.5n-0.5\rfloor$ being the maximal number of a color used. $\square$

Example 2. The graph H defined as



satisfies $\chi(H)=3$, $\chi_m(H)=\chi_l(H)=4$.

Proof. The graph is not bipartite, so $\chi(H)\geqslant 3$. In fact, we have $\chi(H)=3$, since we can construct a proper coloring φ of the graph H as

$$\varphi(1)=\varphi(4)=3,\ \ \varphi(2)=\varphi(3)=\varphi(5)=\varphi(6)=1,\ \ \varphi(7)=2.$$

This mapping is neither a local coloring nor a semi-matching coloring because the union of the colors 1 and 2 contains the path (5, 7, 6), but the change of the value $\varphi(7)$ to 4 allows us to avoid this obstruction and get $\chi_m(H)\leqslant\chi_l(H)\leqslant 4$.

Now we assume that some mapping ψ from the vertex set of H to $\{1,2,3\}$ gives a semi-matching coloring. If $\psi(1)=2$, then the vertices 2 and 3 have different colors in $\{1,3\}$, which forces the vertex 4 to be colored with 2 as well. Similarly, we get $\{\psi(5),\psi(6)\}=\{1,3\}$ and $\psi(7)=2$, which is impossible because of the edge $\{1,7\}$. A similar argument shows that $\psi(4)\neq 2$ and $\psi(7)\neq 2$, and, using the symmetry of our construction, we can assume without loss of generality that $\psi(1)\neq\psi(4)$. In this case, we have to take $\psi(2)=\psi(3)=2$ because ψ is a proper coloring, but this contradicts to the semi-matching assumption and implies $\chi_l(H)\geqslant\chi_m(H)>3$. □

The retracted Theorem 1.1 in [8] stated that the inequality $\chi_l(G)\leqslant 3$ holds if, and only if, the graph G is triangle-free and its vertices of degree three or more induce a bipartite graph. As we can see, the graph H in the above example is indeed triangle-free; all the vertices except $1,4,7$ are degree-two, but the subgraph induced by $1,4,7$ is bipartite. Since $\chi_l(H)=4$, we have a counterexample.

2. The reduction

We proceed with a consideration of *split graphs*, that is, graphs whose vertices can be partitioned into a clique and an independent set [3]. The chromatic number of such a graph, denoted S in what follows, is clearly equal to the order $\omega(S)$ of the largest clique, and the same applies to every induced subgraph of S, which means that S is a *perfect graph* [3]. Proposition 4 in [9] stated that

$$\chi_l(S)=\chi_l\left(K_{\omega(S)}\right) \quad \text{or} \quad \chi_l(S)=\chi_l\left(K_{\omega(S)}\right)+1, \tag{2.1}$$

but only a restricted version of this statement does actually hold.

Observation 3. *The inequalities*

$$\chi_l\left(K_{\omega(S)}\right)\leqslant\chi_l(S)\leqslant\chi_l\left(K_{\omega(S)}\right)+2,$$
$$\omega(S)\leqslant\chi_m(S)\leqslant\omega(S)+2$$

hold for any split graph S.

Proof. The complete graph $K_{\omega(S)}$ is a subgraph of S, so the left parts of these inequalities follow by Observation 1. On the other hand, one can construct a local coloring or a semi-matching coloring of S with the maximal color $c+2$ from the corresponding coloring of $K_{\omega(S)}$ that used the colors $1,\dots,c$ by picking the color $c+2$ to every vertex outside the largest clique. □

Let us construct a sequence (U_n) of split graphs satisfying $\omega(U_n)=n$,

$$\chi_m(U_n)=n+2 \quad \text{and} \quad \chi_l(U_n)=\chi_l(K_n)+2, \tag{2.2}$$

showing that the conditions (2.1) may fail and that the inequalities in Observation 3 are sharp. This construction is also used in the NP-completeness proof in Section 3.

Definition 4. Let $\mathcal{C}=\{C_1,\dots,C_t\}$ be a sequence of proper subsets of a finite set $V=\bigcup\mathcal{C}$ (that is, $C_i\subsetneq V$ for all i). We define the graph $\mathcal{S}=\mathcal{S}(\mathcal{C})$ as follows:

(i) the vertices are $W_1\cup\ldots\cup W_t\cup V$, where the sets $W_i=\{w_{i0},\dots,w_{i\,2|V|}\}$ are disjoint pairwise and disjoint with V;

(ii) V is a clique and $W_1\cup\ldots\cup W_t$ is an independent set in $\mathcal{S}$;

(iii) the vertices in every W_i are adjacent to those vertices $v\in V$ which belong to the corresponding set C_i and only to them.

Observation 5. *The graph $\mathcal{S}(\mathcal{C})$ is split, and its clique number equals $|V|$.*

Proof. It follows immediately from the item (ii) of Definition 4 that $\mathcal{S}$ is split and $\omega(\mathcal{S})\geqslant|V|$. Since $W_1\cup\ldots\cup W_t$ is an independent set, any clique larger than V should contain the whole of V and one other vertex, but such a set cannot actually be a clique because every C_i is a proper subset of V. □

Now we can construct an example satisfying the equalities (2.2).

Example 6. Let $V=\{v_1,\dots,v_n\}$, and let $\mathcal{C}=\{C_1,\dots,C_n\}$ be the set of the subsets $C_i=V\setminus\{v_i\}$. For $n\geqslant 6$, the equalities (2.2) hold with $U_n=\mathcal{S}(\mathcal{C})$.

Proof. First, assume that $\mathcal{S}(\mathcal{C})$ admits a semi-matching coloring φ with the colors $1,\dots,n+1$. By the pigeonhole principle, we can find two vertices in W_1 colored with the same color c. According to the semi-matching property, the colors $c-1,c,c+1$ are forbidden for the vertices in C_1, which means that C_1 is a clique of the size $n-1$ that is properly colored with $n-2$ colors; this is a contradiction.

Further, assume that $\mathcal{S}(\mathcal{C})$ admits a local coloring ψ with the colors $1,\dots,\lambda+1$, where $\lambda=\lfloor 1.5n-0.5\rfloor$ is the local chromatic number of K_n. As in the previous paragraph, we can find two vertices in every W_i colored with the same color c_i, and the colors c_i-1,c_i,c_i+1 are forbidden for the vertices in C_i.

Case 1. If we have $c_i=c_j$ for two different indexes i and j, then the colors c_i-1, c_i, c_i+1 are forbidden for the vertices in the whole V. Since $\chi_l(K_n)=\lambda$, we can have neither $c_i=1$ nor $c_i=\lambda+1$. According to Observation 1, the colors $1,\dots,c-2$ can color a clique of the size at most $(2c-2)/3$, and the colors $c+2,\dots,\lambda+1$ can color a clique containing at most $(2\lambda-2c+2)/3$ vertices; the total number of the vertices in V cannot exceed $(2c-2)/3+(2\lambda-2c+2)/3=2\lambda/3<n$, which is a contradiction.

Case 2. Now we assume that all the colors (c_i) are pairwise different. Then we can find two pairs of indexes (i,j) for which c_i and c_j are consecutive colors, because otherwise we would need a total of at least $2n-2>\lambda+1$ colors. In other words, we have $c_i+1=c_j\leqslant c_p=c_q-1$, for some indexes i,j,p,q, and hence the colors c_i, c_i+1, c_p, c_p+1 are forbidden for the whole clique V. As we can check, the mapping

$$\psi_-(v_t)=\begin{cases}\psi(v_t), & \text{if } \psi(v_t)<c_i,\\ \psi(v_t)-1, & \text{if } c_i<\psi(v_t)<c_p,\\ \psi(v_t)-2, & \text{if } \psi(v_t)>c_p,\end{cases}$$

is a local coloring of the clique V with the colors $1,\ldots,\lambda-1$, and, since the cases 1 and 2 cover all the possibilities, the proof is complete. □

3. The proof

In this section, we prove that the local and semi-matching coloring problems are NP-complete in the class of split graphs. We record the formal definitions of these questions for the ease of further reference.

Problem 7. Given: A split graph G and an integer k.
Question 1: Is $\chi_l(G)\leqslant k$?
Question 2: Is $\chi_m(G)\leqslant k$?

It is easy to see that both questions in Problem 7 belong to NP, and we are going to prove their NP-hardness by constructing polynomial reductions directly from the *Boolean satisfiability problem* [2]. We proceed with a lemma describing the left extremal cases of Observation 3.

Lemma 8. *Let $\mathcal{S}$, $\mathcal{C}$, V be as in Definition 4; assume $|V|=n$. Then*
(i) $\chi_l(\mathcal{S})=1.5n-1$ if, and only if, n is even and there is a permutation $(v_1,\ldots,v_n)$ of V such that, for every C_i, there are two consecutive elements v_j,v_{j+1} that do not belong to C_i;
(ii) $\chi_m(\mathcal{S})=n$ if, and only if, there is a permutation $(v_1,\ldots,v_n)$ of V such that, for every C_i, either $v_1,v_2\notin C_i$ or $v_{n-1},v_n\notin C_i$, or else there are three consecutive elements v_j,v_{j+1},v_{j+2} that do not belong to C_i.

Proof. According to Observation 5, the clique number of $\mathcal{S}$ is n, so we can use Observation 1 and get $\chi_l(\mathcal{S})\geqslant 1.5n-1$ and $\chi_m(\mathcal{S})\geqslant n$. If these inequalities hold with the equalities, then the vertices of V should be colored as

$$\varphi(v_1)=1,\varphi(v_2)=2,\varphi(v_3)=4,\varphi(v_4)=5,\ldots,\varphi(v_{n-1})=3q-2,\varphi(v_n)=3q-1$$

with $q=n/2\in\mathbb{Z}$ in the local case and $\psi(v_1)=1,...,\psi(v_n)=n$ in the semi-matching case. As said in the proof of Example 6, every W_i should contain a pair of vertices both colored with a color c_i, and the colors c_i-1,c_i,c_i+1 are forbidden for the elements of C_i. In the local case, the colors c_i-1,c_i,c_i+1 do always cover exactly two consecutive vertices in $v_1,...,v_n$, hence the condition from the item (i). In the semi-matching case, the colors c_i-1,c_i,c_i+1 cover three consecutive vertices, except the possibilities $c_i=1$ or $c_i=n$ corresponding to the vertices v_1,v_2 or vertices v_{n-1},v_n being forbidden, respectively. This proves the 'only if' parts of our statements, and we can get the 'if' part by reversing the current argument. □

In our reductions, we use the following standard NP-complete problem.

Problem 9. (CNF-SAT.) Given: A family of variables $\zeta=(\zeta_1,...,\zeta_\tau)$ and a family c of clauses of the form

$$\lambda_1\vee...\vee\lambda_k \tag{3.1}$$

in which every λ_i is either a variable in ζ or its negation. Question: Does there exist an assignment $\zeta\to\{0,1\}^n$ so that all clauses in c are satisfied?

The following two lemmas describe the complexity of the combinatorial problems arisen in the items (i) and (ii) of Lemma 8.

Lemma 10. *For a given family F of non-empty subsets $F_1,...,F_t$ of a finite set V of even cardinality, it is NP-hard to determine if V admits a permutation $(v_1,...,v_n)$ such that, for every F_i, there are two consecutive elements v_j,v_{j+1} in F_i.*

Proof. For an even integer τ, we define the set $V=A\cup B\cup R\cup X\cup Y$, where $A=\{a_1,...,a_{\tau+1}\}$, $B=\{b_1,...,b_{\tau+1}\}$, $R=\{r_1,...,r_\tau\}$, $X=\{x_1,...,x_\tau\}$, $Y=\{y_1,...,y_\tau\}$ are pairwise disjoint sets. We define F as the family containing the set $\{a_{\tau+1},b_{\tau+1}\}$, all the sets

$$\{a_i,b_i\},\ \{b_i,x_i,y_i\},\ \{x_i,r_i\},\ \{y_i,r_i\},\ \{x_i,y_i,a_{i+1}\}$$

for $i\in\{1,...,\tau\}$ (we call them *main*) and also several sets of the form

$$\{\xi_{i_1},b_{i_1},...,\xi_{i_k},b_{i_k}\} \tag{3.2}$$

with every ξ_j being either x_j or y_j; we call the sets of the latter type *optional.*

The conditions imposed by the main sets say that a desired permutation should look, up to reading it from the right to the left, like

$$a_1,b_1,z_1,r_1,\overline{z_1},a_2,b_2,z_2,r_2,\overline{z_2},...,a_{\tau+1},b_{\tau+1},$$

where $\{z_j, \overline{z_j}\} = \{x_j, y_j\}$. Now we take an instance of CNF-SAT as in Problem 9 and proceed with the reduction as follows. For any optional set (3.2), we take the clause of the form (3.1) such that

$$\lambda_q = \begin{cases} \zeta_{i_q} & \text{if } \xi_{i_q} = x_{i_q}, \\ \overline{\zeta_{i_q}} & \text{if } \xi_{i_q} = y_{i_q}, \end{cases}$$

and we identify an assignment $\zeta_j = 1$ to the choice $z_j = x_j$, and also $\zeta_j = 0$ to $z_j = y_j$, which certifies a desired reduction from CNF-SAT. □

Lemma 11. *For a given sequence $F_1, ..., F_t$ of non-empty subsets of a finite set V, it is NP-hard to determine if V admits a permutation $(v_1, ..., v_n)$ such that, for every F_i, either $v_1, v_2 \in F_i$ or $v_{n-1}, v_n \in F_i$, or else there are three consecutive elements v_j, v_{j+1}, v_{j+2} in F_i.*

Proof. The proof is similar to Lemma 10, except that we add a new symbol α to V, and also we define the main sets as $\{\alpha, a_1\}$, $\{a_{\tau+1}, b_{\tau+1}\}$, and

$$\{a_i, b_i, r_i\}, \quad \{r_i, x_i, y_i\}, \quad \{x_i, y_i, a_{i+1}\}$$

for all $i \in \{1, ..., \tau\}$. These sets restrict our attention to the permutations

$$\alpha, a_1, b_1, r_1, z_1, \overline{z_1}, ..., a_\tau, b_\tau, r_\tau, z_\tau, \overline{z_\tau}, a_{\tau+1}, b_{\tau+1},$$

and then the optional sets of the form

$$\{b_{i_1}, r_{i_1}, \xi_{i_1}, ..., b_{i_k}, r_{i_k}, \xi_{i_k}\}$$

correspond to the reduction from CNF-SAT given in Lemma 10. □

The description of the property

$$\chi_l(\mathcal{S}) = 1.5n - 1$$

obtained in Lemma 8(i) is equivalent to the problem as in Lemma 10 by taking F_i to be the complement of C_i. We have a similar situation with Lemma 8(ii) and Lemma 11, so we arrive at the main result of this section.

Theorem 12. *Questions 1 and 2 in Problem 7 are NP-complete.*

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