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Hilbert schemes of points on smooth projective surfaces and generalized Kummer varieties with finite group actions

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Abstract. Göttsche and Soergel gave formulas for the Hodge numbers of Hilbert schemes of points on a smooth algebraic surface and the Hodge numbers of generalized Kummer varieties. When a smooth projective surface S admits an action by a finite group G, we describe the action of G on the Hodge pieces via point counting. Each element of G gives a trace on $\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(-1)^i H^i(S^{[n]},\mathbb{C})q^n$. In the case that S is a K3 surface or an abelian surface, the resulting generating functions give some interesting modular forms when G acts faithfully and symplectically on S.

1. Introduction

Let S be a smooth projective surface over $\mathbb{C}$. In [GS93], the Hodge numbers of the Hilbert scheme of points of S are computed via perverse sheaves/mixed Hodge modules:

$$\sum_{n=0}^{\infty} h(S^{[n]},u,v)t^n$$
$$=\prod_{m=1}^{\infty}\prod_{p,q}\left(\sum_{i=0}^{h^{pq}}(-1)^{i(p+q+1)}\binom{h^{pq}}{i}u^{i(p+m-1)}v^{i(q+m-1)}t^{mi}\right)^{(-1)^{p+q+1}},$$

where $S^{[n]}$ is the Hilbert scheme of n points of S, $h(S^{[n]},u,v)=\sum_{p,q}h^{pq}(S^{[n]})u^pv^q$ is the Hodge-Deligne polynomial, and h^{pq} are the dimensions of the Hodge pieces $H^{p,q}(S,\mathbb{C})$. The Hodge numbers of the higher order Kummer varieties (generalized

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Kummer varieties) of an abelian surface are also computed:

$$h(K_n(A),-u,-v)=\frac{1}{((1-u)(1-v))^2}$$
$$\times\sum_{\alpha\in P(n)} gcd(\alpha)^4(uv)^{n-|\alpha|}\left(\prod_{i=1}^{\infty}\sum_{\beta^i\in P(\alpha_i)}\prod_{j=1}^{\infty}\frac{1}{j^{\beta^i_j}\beta^i_j!}((1-u^j)(1-v^j))^{2\beta^i_j}\right),$$

where $\alpha=(1^{\alpha_1}2^{\alpha_2}...)$ is a partition of n, $|\alpha|$ is the number of parts, and $gcd(\alpha):=gcd\{i\in\mathbb{Z}|\alpha_i\neq 0\}$.

In this paper G will always be a finite group. We will consider a smooth projective K3 surface S over $\mathbb{C}$ with a G-action, and ask whether we can prove similar equalities for G-representations. We use an equivariant version of the idea in Göttsche [Göt90], which studies the cohomology groups by counting the number of rational points over finite fields. Then we lift the results to the Hodge level by p-adic Hodge theory.

We will consider the G-equivariant Hodge-Deligne polynomial for a smooth projective variety X

$$E(X;u,v)=\sum_{p,q}(-1)^{p+q}[H^{p,q}(X,\mathbb{C})]u^p v^q,$$

where the coefficients lie in the ring of virtual G-representations $R_{\mathbb{C}}(G)$, of which the elements are the formal differences of isomorphism classes of finite dimensional $\mathbb{C}$-representations of G. The addition is given by direct sum and the multiplication is given by tensor product.

Theorem 1.1. *Let S be a smooth projective surface over $\mathbb{C}$ with a G-action. Let $S^{[n]}$ be the Hilbert scheme of n points of S. Then we have the following equality as virtual G-representations.*

$$\sum_{n=0}^{\infty}E(S^{[n]})t^n=\prod_{m=1}^{\infty}\prod_{p,q}\left(\sum_{i=0}^{h_{p,q}}(-1)^i[\wedge^i H^{p,q}(S,\mathbb{C})]u^{i(p+m-1)}v^{i(q+m-1)}t^{mi}\right)^{(-1)^{p+q+1}},$$

where $h_{p,q}$ are the dimensions of the Hodge pieces $H^{p,q}(S,\mathbb{C})$.

Remark 1.2. Theorem 1.1 has been proved in [Zha21, Theorem 1.1], where the proof uses Nakajima operators. We give a new proof here using the Weil conjecture and p-adic Hodge theory.

For a complex K3/abelian surface S with an automorphism g of finite order n, $H^0(S,K_S)=\mathbb{C}\omega_S$ has dimension 1, and we say g acts symplectically on S if it acts trivially on ω_S, and g acts non-symplectically otherwise, namely, g sends ω_S to $\zeta_n^k\omega_S$, $0<k<n$, where ζ_n is a primitive n-th root of unity.

Denote by $[e(X)]$ the virtual graded G-representation $\sum_{i=0}^{\infty}(-1)^i[H^i(X,\mathbb{C})]$ for a smooth projective variety X over $\mathbb{C}$ with a G-action.

Theorem 1.3. *Let G be a finite group which acts faithfully and symplectically on a smooth projective $K3$ surface S over $\overline{\mathbb{F}}_q$. Suppose $p\nmid|G|$. Then*

$$\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n=\exp\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\varepsilon(\mathrm{ord}(g^k))t^{mk}}{k}\right)$$

for all $g\in G$, where $\varepsilon(n)=24\left(n\prod_{p|n}\left(1+\frac{1}{p}\right)\right)^{-1}$. In particular, if G is generated by a single element g of order N, then we deduce that

N	$\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n$
1	$t/\eta^{24}(t)$
2	$t/\eta^8(t)\eta^8(t^2)$
3	$t/\eta^6(t)\eta^6(t^3)$
4	$t/\eta^4(t)\eta^2(t^2)\eta^4(t^4)$
5	$t/\eta^4(t)\eta^4(t^5)$
6	$t/\eta^2(t)\eta^2(t^2)\eta^2(t^3)\eta^2(t^6)$
7	$t/\eta^3(t)\eta^3(t^7)$
8	$t/\eta^2(t)\eta(t^2)\eta(t^4)\eta^2(t^8)$

where $\eta(t)=t^{1/24}\prod_{n=1}^{\infty}(1-t^n)$.

Remark 1.4. If g acts symplectically on S, then g has order $N\leq 8$ by [DK09, Theorem 3.3] since the G-action is tame. These eta quotients coincide with the results in the characteristic zero case. See [BG19], [BO18, Lemma 3.1], or [Zha21].

Theorem 1.5. *Let g be a symplectic automorphism (fixing the origin) of order N on an abelian surface S over $\mathbb{C}$. Then*

N	$\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n$
1	1
2	$\eta^8(t^2)/\eta^{16}(t)$
3	$\eta^3(t^3)/\eta^9(t)$
4	$\eta^4(t^4)/\eta^4(t)\eta^6(t^2)$
6	$\eta^4(t^6)/\eta(t)\eta^4(t^2)\eta^5(t^3)$

Remark 1.6. If g is a symplectic automorphism on a complex abelian surface, then g has order $1,2,3,4$ or 6 by [Fuj88, Lemma 3.3]. These eta quotients coincide with the results of [Pie21, Theorem 1.1] when G is cyclic.

Define a multiplication $\odot$ on the ring of power series $R_{\mathbb{C}}(G)[[u,v,w]]$ by $u^{n_1}v^{m_1}w^{l_1}\odot u^{n_2}v^{m_2}w^{l_2}:=u^{n_1+n_2}v^{m_1+m_2}w^{gcd(l_1,l_2)}$.

Theorem 1.7. *Let A be an abelian surface over $\mathbb{C}$ with a G-action. Let $K_n(A)$ be the generalized Kummer variety. Then we have the following equality as virtual G-representations.*

$$\sum_{n=0}^{\infty} E(K_n(A);u,v)t^n = \frac{(w\frac{d}{dw})^4}{E(A)}$$

$$\bigodot_{m=1}^{\infty}\left(1+w^m\left(-1+\prod_{p,q}\left(\sum_{i=0}^{h_{p,q}}(-1)^i[\wedge^i H^{p,q}(S,\mathbb{C})]u^{i(p+m-1)}v^{i(q+m-1)}t^{mi}\right)^{(-1)^{p+q+1}}\right)\right)\Bigg|_{w=1}.$$

When we say S is a surface with a G-action over a field K, we mean that both S and the G-action can be defined over K.

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2. Preliminaries

Let X be a smooth projective variety over $\mathbb{C}$. Then we can choose a finitely generated $\mathbb{Z}$-subalgebra $\mathcal{R}\subset\mathbb{C}$ such that $X\cong\mathcal{X}\times_{\mathcal{S}}\mathrm{Spec}\mathbb{C}$ for a regular projective scheme $\mathcal{X}$ over $\mathcal{S}=\mathrm{Spec}\mathcal{R}$, and we can choose a maximal ideal $\mathfrak{q}$ of $\mathcal{R}$ such that $\mathcal{X}$ has good reduction modulo $\mathfrak{q}$. Since there are comparison theorems between étale cohomology and singular cohomology, we focus on characteristic p.

Now let X be a quasi-projective variety over $\overline{\mathbb{F}}_p$ with an automorphism σ of finite order. Suppose X and σ can be defined over some finite field $\mathbb{F}_q$. Let F_q be the corresponding geometric Frobenius. Then for $n\geq 1$, the composite $F_q^n\circ\sigma$ is the Frobenius map relative to some new way of lowering the field of definition of X from $\overline{\mathbb{F}}_p$ to $\mathbb{F}_{q^n}$ ([DL76, Proposition 3.3] and [Car85, Appendix(h)]). Then the Grothendieck trace formula implies that $\sum_{k=0}^{\infty}(-1)^k\mathrm{Tr}((F_q^n\sigma)^*,H_c^k(X,\mathbb{Q}_l))$ is the number of fixed points of $F_q^n\sigma$, where $H_c^k(X,\mathbb{Q}_l)$ are the compactly supported l-adic cohomology groups.

Lemma 2.1. *Let X and Y be two smooth projective varieties over $\overline{\mathbb{F}}_p$ with finite group G-actions. Suppose X,Y and the actions of G can be defined over $\mathbb{F}_q$, where q is a p power. If $|X(\overline{\mathbb{F}}_p)^{gF_{q^n}}|=|Y(\overline{\mathbb{F}}_p)^{gF_{q^n}}|$ for every $n\geq 1$ and $g\in G$, then $H^i(X,\mathbb{Q}_l)\cong H^i(Y,\mathbb{Q}_l)$ as G-representations for every $i\geq 0$.*

Proof. Fix $g\in G$. Denote by F_q the geometric Frobenius over $\mathbb{F}_q$. Since the finite group action is defined over $\mathbb{F}_q$, the action g commutes with F_q and the action of g on the cohomology group is semisimple. There exists a basis of the cohomology group such that the actions of g and F_q are in Jordan normal forms simultaneously. Let $\alpha_{i,j}, j=1,2,...,a_i$ (resp. $\beta_{i,j}, j=1,2,...,b_i$) denote the eigenvalues of F_q acting on $H^i(X,\mathbb{Q}_l)$ (resp. $H^i(Y,\mathbb{Q}_l)$) in such a basis, where a_i (resp. b_i) is the i-th betti number. Let $c_{i,j}, j=1,2,...,a_i$ (resp. $d_{i,j}, j=1,2,...,b_i$) denote the eigenvalues of g acting on the same basis of $H^i(X,\mathbb{Q}_l)$ (resp. $H^i(Y,\mathbb{Q}_l)$). Then the Grothendieck trace formula ([Car85, Appendix(h)] and [DL76, Proposition 3.3]) implies that

$$|X(\overline{\mathbb{F}}_p)^{gF_{q^n}}|=\sum_{i=0}^{\infty}(-1)^i\mathrm{Tr}((gF_{q^n})^*,H^i(X.\mathbb{Q}_l))$$

Since $|X(\overline{\mathbb{F}}_p)^{gF_{q^n}}|=|Y(\overline{\mathbb{F}}_p)^{gF_{q^n}}|$ for every $n\geq 1$, we have

$$\sum_{i=0}^{\infty}(-1)^i\sum_{j=1}^{a_i}c_{i,j}\alpha_{i,j}^n=\sum_{i=0}^{\infty}(-1)^i\sum_{j=1}^{b_i}d_{i,j}\beta_{i,j}^n$$

for every $n\geq 1$. By linear independence of the characters $\chi_\alpha:\mathbb{Z}^+\to\mathbb{C}, n\mapsto\alpha^n$ and the fact that $\alpha_{i,j},\beta_{i,j}, j=1,2,...$ all have absolute value $q^{i/2}$ by Weil's conjecture, we deduce that $a_i=b_i$ and $\sum_{j=1}^{a_i}c_{i,j}=\sum_{j=1}^{b_i}d_{i,j}$ for each i. But since g is arbitrary, this implies that the G-representations $H^i(X,\mathbb{Q}_l)$ and $H^i(Y,\mathbb{Q}_l)$ are the same. □

Proposition 2.2. *Let X be a smooth projective variety with a G-action over $\mathbb{F}_q$. Denote the dimension of X by N. Then*

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(-1)^i[H^i(X^{(k)}_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)]z^it^k=\prod_{j=0}^{2N}\left(\sum_{i=0}^{b_j}(-1)^i[\wedge^iH^j(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)]z^{ij}t^i\right)^{(-1)^{j+1}},$$

where the coefficients lie in $R_{\mathbb{Q}_l}(G)$.

Proof. By the Weil conjectures, we have

$$\exp\left(\sum_{r=1}^{\infty}|X(\mathbb{F}_{q^r})|\frac{t^r}{r}\right)=\sum_{k=0}^{\infty}|X^{(k)}(\mathbb{F}_q)|t^k$$
$$=\sum_{k=0}^{\infty}|X^{(k)}(\overline{\mathbb{F}}_p)^{F_q}|t^k=\prod_{j=0}^{2N}\left(\prod_{i=1}^{b_j}(1-\alpha_{j,i}t)\right)^{(-1)^{j+1}},$$

where $\alpha_{j,i}$ are the eigenvalues of F_q on $H^j(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)$.

By the discussion at the beginning of the section and the Grothendieck trace formula, we deduce that

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\sum_i h_{k,m,i}\beta_{k,m,i}^n t^k=\prod_{j=0}^{2N}\left(\prod_{i=1}^{b_j}(1-g_{j,i}\alpha_{j,i}^n t)\right)^{(-1)^{j+1}}$$
$$=\prod_{j=0}^{2N}\left(\sum_{i=0}^{b_j}(-1)^i\left(\sum_{1\le l_1<\dots<l_i\le b_j}g_{j,l_1}\alpha_{j,l_1}^n\cdots g_{j,l_i}\alpha_{j,l_i}^n\right)t^i\right)^{(-1)^{j+1}}$$

where $h_{k,m,i}$ (resp. $\beta_{k,m,i}$) are the eigenvalues of g (resp. F_q) on $H^m(X^{(k)}_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)$, and $g_{j,i}$ are the eigenvalues of g on $H^j(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)$. Hence we deduce that the trace of g on the left hand side equals the trace of g on the right hand side for each graded piece in the equality in Proposition 2.2 by the proof of Lemma 2.1. □

We obtain the information of Hodge pieces via p-adic Hodge theory by using an equivariant version of the method in [Ito03, §4].

Proposition 2.3. ([Ser68, I. 2.3]) *Let K be a number field, $m,m'\geq 1$ be integers, and l be a prime number. Let*

$$\rho\colon\ \mathrm{Gal}(\overline{K}/K)\longrightarrow \mathrm{GL}(m,\mathbb{Q}_l),\qquad \rho'\colon\ \mathrm{Gal}(\overline{K}/K)\longrightarrow \mathrm{GL}(m',\mathbb{Q}_l)$$

be continuous l-adic $\mathrm{Gal}(\overline{K}/K)$-representations such that ρ and ρ' are unramified outside a finite set S of maximal ideals of $\mathcal{O}_K$. If

$$\mathrm{Tr}(\rho(\mathrm{Frob}_{\mathfrak{p}}))=\mathrm{Tr}(\rho'(\mathrm{Frob}_{\mathfrak{p}}))\quad \textit{for all maximal ideals } \mathfrak{p}\notin S,$$

then ρ and ρ' have the same semisimplifications as $\mathrm{Gal}(\overline{K}/K)$-representations. Here $\mathrm{Frob}_{\mathfrak{p}}$ is the geometric Frobenius at $\mathfrak{p}$.

Let p be a prime number and F be a finite extension of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be a p-adic completion of an algebraic closure $\overline{F}$ of F. Define $\mathbb{Q}_p(0)=\mathbb{Q}_p$, $\mathbb{Q}_p(1)=(\varprojlim \mu_{p^n})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, and for $n\geq 1$, $\mathbb{Q}_p(n)=\mathbb{Q}_p(1)^{\otimes n}$, $\mathbb{Q}_p(-n)=\mathrm{Hom}(\mathbb{Q}_p(n),\mathbb{Q}_p)$. Moreover, we define $\mathbb{C}_p(n)=\mathbb{C}_p\otimes_{\mathbb{Q}_p}\mathbb{Q}_P(n)$, on which $\mathrm{Gal}(\overline{F}/F)$ acts diagonally. It is known that $(\mathbb{C}_p)^{\mathrm{Gal}(\overline{F}/F)}=F$ and $(\mathbb{C}_p(n))^{\mathrm{Gal}(\overline{F}/F)}=0$ for $n\neq 0$.

Let $B_{HT}=\oplus_{n\in\mathbb{Z}}\mathbb{C}_p(n)$ be a graded $\mathbb{C}_p$-module with an action of $\mathrm{Gal}(\overline{F}/F)$. For a finite dimensional $\mathrm{Gal}(\overline{F}/F)$-representation V over $\mathbb{Q}_p$, we define a finite dimensional graded F-module $D_{HT}(V)$ by $D_{HT}(V)=(V\otimes_{\mathbb{Q}_p}B_{HT})^{\mathrm{Gal}(\overline{F}/F)}$. The graded module structure of $D_{HT}(V)$ is induced from that of B_{HT}. In general, it is known that

$$\dim_F D_{HT}(V)\leq \dim_{\mathbb{Q}_p} V.$$

If the equality holds, V is called a *Hodge-Tate representation.*

Theorem 2.4. ([Fal88], [Tsu99] (Hodge-Tate decomposition)) *Let X be a proper smooth variety over F and k be an integer. The p-adic étale cohomology $H^k_{ét}(X_{\overline{F}}, \mathbb{Q}_p)$ of $X_{\overline{F}} = X \otimes_F \overline{F}$ is a finite dimensional $\mathrm{Gal}(\overline{F}/F)$-representation over $\mathbb{Q}_p$. Then, $H^k_{ét}(X_{\overline{F}}, \mathbb{Q}_p)$ is a Hodge-Tate representation, Moreover, there exists a canonical and functorial isomorphism*

$$\bigoplus_{i+j=k} H^i(X, \Omega^j_X) \otimes_F \mathbb{C}_p(-j) \cong H^k_{ét}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

of $\mathrm{Gal}(\overline{F}/F)$-representations, where $\mathrm{Gal}(\overline{F}/F)$ acts on $H^i(X, \Omega^j_X)$ trivially and acts on $H^k_{ét}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ diagonally.

Now for a finite dimensional $\mathrm{Gal}(\overline{F}/F)$-representation V over $\mathbb{Q}_p$, suppose it is also a G-representation such that the G-action commutes with the $\mathrm{Gal}(\overline{F}/F)$-action. In this case, we call it a $\mathrm{Gal}(\overline{F}/F)$-$G$-representation and we define a G-representation over F:

$$[h^n(V)] := (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n))^{\mathrm{Gal}(\overline{F}/F)}.$$

Lemma 2.5. *Let W_2 be a Hodge-Tate $\mathrm{Gal}(\overline{F}/F)$-$G$-representation and*

$$0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W_3 \longrightarrow 0$$

be an exact sequence of finite dimensional $\mathrm{Gal}(\overline{F}/F)$-$G$-representations over $\mathbb{Q}_p$. Then W_1 and W_3 are Hodge-Tate representations and

$$[h^n(W_2)] = [h^n(W_1)] \oplus [h^n(W_3)] = [h^n(W_1 \oplus W_3)]$$

as G-representations for all n.

Proof. It follows from [Ito03, Lemma 4.4] that W_1 and W_3 are Hodge-Tate representations and we have the following short exact sequence of G-representations

$$0 \longrightarrow D_{HT}(W_1) \longrightarrow D_{HT}(W_2) \longrightarrow D_{HT}(W_3) \longrightarrow 0,$$

which implies that

$$[h^n(W_2)] = [h^n(W_1)] \oplus [h^n(W_3)] = [h^n(W_1 \oplus W_3)]. \quad \square$$

Corollary 2.6. *Let X be a proper smooth variety over F with a G-action. Then*

$$H^i(X, \Omega^j_X) = [h^j(H^{i+j}(X_{\overline{F}}, \mathbb{Q}_p)^{ss})] \text{ as } G\text{-representations for all } i, j,$$

where $H^{i+j}(X_{\overline{F}}, \mathbb{Q}_p)^{ss}$ denotes the semisimplification of $H^{i+j}(X_{\overline{F}}, \mathbb{Q}_p)$ as a $\mathrm{Gal}(\overline{F}/F)$-representation.

Proof. By Theorem 2.4, if we take the $\mathrm{Gal}(\overline{F}/F)$-invariant of $H^{i+j}(X_{\overline{F}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\mathbb{C}_p(j)$, we have

$$H^i(X,\Omega_X^j)=[h^j(H^{i+j}(X_{\overline{F}},\mathbb{Q}_p))].$$

On the other hand, since $H^{i+j}(X_{\overline{F}},\mathbb{Q}_p)$ is a $\mathrm{Gal}(\overline{F}/F)$-$G$ Hodge-Tate representation,

$$[h^j(H^{i+j}(X_{\overline{F}},\mathbb{Q}_p))]=[h^j(H^{i+j}(X_{\overline{F}},\mathbb{Q}_p)^{ss})]$$

by Lemma 2.5. Hence we are done. □

Theorem 2.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be n-dimensional smooth projective varieties over a number field K with G-actions. Suppose for all but finitely many good reductions, we have*

$$|X(\overline{\mathbb{F}}_p)^{gF_{q^n}}|=|Y(\overline{\mathbb{F}}_p)^{gF_{q^n}}| \text{ for every } n\geq 1 \text{ and } g\in G,$$

where X,Y are the good reductions over $\mathbb{F}_q$. Then

$$H^{p,q}(\mathcal{X}_{\mathbb{C}})\cong H^{p,q}(\mathcal{Y}_{\mathbb{C}})$$

for all p,q as G-representations.

Proof. By the proof of Lemma 2.1 and Proposition 2.3, we deduce that $H^i(\mathcal{X}_{\overline{K}},\mathbb{Q}_l)$ and $H^i(\mathcal{Y}_{\overline{K}},\mathbb{Q}_l)$ have the same semisimplifications as $\mathrm{Gal}(\overline{K}/K)$-$G$-representations.

Now take a maximal ideal $\mathfrak{q}$ of $\mathcal{O}_K$ dividing l. Let F be the completion of K at $\mathfrak{q}$. Fix an embedding $\overline{K}\hookrightarrow\overline{F}$. Then we have an inclusion $\mathrm{Gal}(\overline{F}/F)\subset\mathrm{Gal}(\overline{K}/K)$. Therefore, $H^i(\mathcal{X}_{\overline{F}},\mathbb{Q}_l)$ and $H^i(\mathcal{Y}_{\overline{F}},\mathbb{Q}_l)$ have the same semisimplifications as $\mathrm{Gal}(\overline{F}/F)$-$G$-representations. By Corollary 2.6, we conclude that

$$H^q(\mathcal{X}_{\mathbb{C}},\Omega^p_{\mathcal{X}_{\mathbb{C}}})\cong H^q(\mathcal{Y}_{\mathbb{C}},\Omega^p_{\mathcal{Y}_{\mathbb{C}}})$$

for all p,q as G-representations. □

3. Hilbert scheme of points

We denote by $X^{[n]}$ the component of the Hilbert scheme of a projective scheme X parametrizing subschemes of length n of X. For properties of Hilbert scheme of points, see references [Göt94], [Iar77] and [Nak99].

Lemma 3.1. *Let S be a smooth projective surface with a G-action over $\mathbb{F}_q$. Suppose $g\in G$ and let F_q be the geometric Frobenius. Then*

$$\sum_{n=0}^{\infty}|S^{[n]}(\overline{\mathbb{F}}_q)^{gF_q}|t^n=\prod_{r=1}^{\infty}\left(\sum_{n=0}^{\infty}|\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})(\overline{\mathbb{F}}_q)^{g^rF_q^r}|t^{nr}\right)^{|P_r(S,gF_q)|},$$

where $\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})$ *is the punctual Hilbert scheme of* n *points at some* $g^rF_q^r$*-fixed point* $x\in S(\overline{\mathbb{F}}_q)$*, and* $P_r(S,gF_q)$ *is the set of primitive* 0*-cycles of degree* r *of* gF_q *on* S*, whose elements are of the form* $\sum_{i=0}^{r-1} g^iF_q^i(x)$ *with* $x\in S(\overline{\mathbb{F}}_q)^{g^rF_q^r}\setminus (\cup_{j<r}S(\overline{\mathbb{F}}_q)^{g^jF_q^j})$*.*

Proof. Let $Z\in S^{[n]}(\overline{\mathbb{F}}_q)^{gF_q}$. Suppose $(n_1,\dots,n_r)$ is a partition of n and $Z=(Z_1,\dots,Z_r)$ with Z_i being the closed subscheme of Z supported at a single point with length n_i. Then $\mathrm{Supp}Z$ decomposes into gF_q orbits. We can choose an ordering $\leq$ on $S(\overline{\mathbb{F}}_q)$. In each orbit, we can find the smallest $x_j\in S(\overline{\mathbb{F}}_q)$. Suppose Z_j with length l is supported on x_j and x_j has order k. Then the component of Z which is supported on the orbit of x_j is determined by Z_j, namely, it is $\cup_{i=0}^{k-1}g^iF_q^i(Z_j)$ with length kl. Also notice that Z_j is fixed by $g^kF_q^k$. Hence, to give an element of $S^{[n]}(\overline{\mathbb{F}}_q)^{gF_q}$ is the same as choosing some gF_q orbits and for each orbit choosing some element in $\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})(\overline{\mathbb{F}}_q)^{g^kF_q^k}$ for some $g^kF_q^k$-fixed point x in this orbit such that the final length altogether is n. Combining all of these into power series, we get the desired equality. □

The idea we used above is explained in detail in [Göt90, Lemma 2.7]. We implicitly used the fact that $\pi:(S^{[n]}_{(n)})_{\mathrm{red}}\to S$ is a locally trivial fiber bundle in the Zariski topology with fiber $\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])_{\mathrm{red}}$ [Göt94, Lemma 2.1.4], where $S^{[n]}_{(n)}$ parametrizes closed subschemes of length n that are supported on a single point.

We need the following key lemma.

Lemma 3.2. *Let* S *be a smooth projective surface with a* G*-action over* $\mathbb{F}_q$*. If* $x\in S(\overline{\mathbb{F}}_q)^{gF_q}$*, where* $g\in G$ *and* F_q *is the geometric Frobenius, then*

$$|\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})(\overline{\mathbb{F}}_q)^{gF_q}|=|\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])(\overline{\mathbb{F}}_q)^{F_q}|.$$

We will prove this lemma later in this section.

From Lemma 3.2, we observe that $|\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})(\overline{\mathbb{F}}_q)^{gF_q}|$ is a number independent of the choice of the gF_q-fixed point x.

We denote $\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])$ by V_n. Combining Lemma 3.1 and Lemma 3.2, we deduce that

$$\sum_{n=0}^{\infty}|S^{[n]}(\overline{\mathbb{F}}_q)^{gF_q}|t^n=\prod_{r=1}^{\infty}\left(\sum_{n=0}^{\infty}|V_n(\overline{\mathbb{F}}_q)^{F_q^r}|t^{nr}\right)^{|P_r(S,gF_q)|}.$$

Recall the following structure theorem for the punctual Hilbert scheme of points.

Proposition 3.3. ([ES87, Proposition 4.2]) *Let k be an algebraically closed field. Then* $\mathrm{Hilb}^n(k[[s,t]])$ *over k has a cell decomposition, and the number of d-cells is $P(d,n-d)$, where $P(x,y):=\#\{$partition of x into parts$\leq y\}$.*

Denote by $p(n,d)$ the number of partitions of n into d parts. Then $p(n,d)= P(n-d,d)$. Now we can proceed similarly as in the proof of [Göt90, Lemma 2.9].

Proof of Theorem 1.1. Since we have

$$\prod_{i=1}^{\infty}\left(\frac{1}{1-z^{i-1}t^i}\right)=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}p(n,n-i)t^n z^i,$$

by Proposition 3.3 we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\#\{\text{m-dim cells of } \mathrm{Hilb}^n(\overline{\mathbb{F}}_p[[s,t]])\}t^n z^m=\prod_{i=1}^{\infty}\frac{1}{1-z^{i-1}t^i}.$$

Fix $N\in\mathbb{N}$. Then by choosing sufficiently large q powers Q such that the cell decomposition of $V_{n,\overline{\mathbb{F}}_q}$ is defined over $\mathbb{F}_Q$ for $n\leq N$, we deduce that

$$\sum_{n=0}^{\infty}|V_{n,\overline{\mathbb{F}}_q}(\mathbb{F}_{Q^r})|t^{nr}\equiv\prod_{i=1}^{\infty}\frac{1}{1-Q^{r(i-1)}t^{ri}}\mod t^N.$$

Now consider a good reduction of S over $\mathbb{F}_q$.

$$\begin{aligned}\sum_{n=0}^{\infty}|S^{[n]}(\overline{\mathbb{F}}_q)^{gF_Q}|t^n&\equiv\prod_{r=1}^{\infty}\prod_{i=1}^{\infty}\left(\frac{1}{1-Q^{r(i-1)}t^{ri}}\right)^{|P_r(S,gF_Q)|}\mod t^N\\&=\exp\left(\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\sum_{h=1}^{\infty}|P_r(S,gF_Q)|Q^{hr(i-1)}t^{hri}/h\right)\\&=\exp\left(\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}(\sum_{r|m}r|P_r(S,gF_Q)|)Q^{m(i-1)}t^{mi}/m\right)\\&=\prod_{i=1}^{\infty}\exp\left(\sum_{m=1}^{\infty}|S(\overline{\mathbb{F}}_q)^{g^mF_Q^m}|Q^{m(i-1)}t^{mi}/m\right)\\&=\prod_{i=1}^{\infty}\sum_{n=0}^{\infty}|S^{(n)}(\overline{\mathbb{F}}_q)^{gF_Q}|Q^{n(i-1)}t^{ni}.\end{aligned}$$

By replacing Q by Q-powers and using the proof of Proposition 2.2 and Theorem 2.7, we obtain

$$\sum_{n=0}^{\infty}E(S^{[n]})t^n=\prod_{m=1}^{\infty}\prod_{p,q}\left(\sum_{i=0}^{h_{p,q}}(-1)^i[\wedge^i H^{p,q}(S,\mathbb{C})]u^{i(p+m-1)}v^{i(q+m-1)}t^{mi}\right)^{(-1)^{p+q+1}},$$

since we can reduce to the case where everything is defined over a number field K as in [Ito03, Proposition 5.1]. □

Corollary 3.4. *For a smooth projective surface S over $\overline{\mathbb{F}}_p$ or $\mathbb{C}$, we have*

$$\sum_{n=0}^{\infty}[e(S^{[n]})]t^n = \prod_{m=1}^{\infty}\prod_{j=0}^{4}\left(\sum_{i=0}^{b_j}(-1)^i[\wedge^i H^j(S,\mathbb{Q}_l)][-2i(m-1)]t^{mi}\right)^{(-1)^{j+1}},$$

where the coefficients lie in $R_{\mathbb{Q}_l}(G)$, and $[-2i(m-1)]$ indicates shift in degrees.

Remark 3.5. Notice that the generating series of the topological Euler characteristic of $S^{[n]}$ is $\sum_{n=0}^{\infty} e(S^{[n]})t^n = \prod_{m=1}^{\infty}(1-t^m)^{-e(S)}$. But this is not the case if we consider G-representations and regard $\prod_{m=1}^{\infty}(1-t^m)^{-[e(S)]}$ as

$$\exp(\sum_{m=1}^{\infty}[e(S)](-\log(1-t^m))) = \exp(\sum_{m=1}^{\infty}[e(S)](\sum_{k=1}^{\infty}t^{mk}/k))).$$

What we have is actually

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n &= \prod_{m=1}^{\infty}\left(\frac{(\prod_{i=1}^{b_1}(1-g_{1,i}t^m))(\prod_{i=1}^{b_3}(1-g_{3,i}t^m))}{(1-t^m)(\prod_{i=1}^{b_2}(1-g_{2,i}t^m))(1-t^m)}\right)\\ &= \exp\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{t^{mk}}{k}\left(1-\sum_{i=1}^{b_1}g_{1,i}^k+\sum_{i=1}^{b_2}g_{2,i}^k-\sum_{i=1}^{b_3}g_{3,i}^k+1\right)\right).\end{aligned}$$

We will use this expression to determine the G-representation $[e(S^{[n]})]$ later when S is a K3 surface or an abelian surface.

Now we start to prove Lemma 3.2.

Let S be a smooth projective surface over $\mathbb{F}_q$ with an automorphism g over $\mathbb{F}_q$ of finite order. If $x \in S(\overline{\mathbb{F}}_q)^{gF_q}$ where F_q is the geometric Frobenius, then x lies over a closed point $y \in S$. Denote the residue degree of y by N. Hence $x \in S(\mathbb{F}_{q^N})$ and there are N geometric points $x, F_q(x), \ldots, F_q^{N-1}(x)$ lying over y.

Let us study the relative Hilbert scheme of n points at a closed point.

$$\mathrm{Hilb}^n(\mathrm{Spec}(\widehat{\mathcal{O}_{S,y}})/\mathrm{Spec}\mathbb{F}_q) \cong \mathrm{Hilb}^n(\mathrm{Spec}(\mathbb{F}_{q^N}[[s,t]])/\mathrm{Spec}\mathbb{F}_q).$$

Since g and $\mathbb{F}_q$ fix y, they act on this Hilbert scheme. Over $\overline{\mathbb{F}}_q$, we have

$$\mathrm{Hilb}^n(\mathrm{Spec}(\widehat{\mathcal{O}_{S,y}})/\mathrm{Spec}\mathbb{F}_q)\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q \cong \mathrm{Hilb}^n(\mathrm{Spec}(\overline{\mathbb{F}}_q\otimes_{\mathbb{F}_q}\mathbb{F}_{q^N}[[s,t]])/\mathrm{Spec}\overline{\mathbb{F}}_q)$$

by the base change property of the Hilbert scheme. Denote by u a primitive element of the field extension $\mathbb{F}_{q^N}/\mathbb{F}_q$ and denote by $f(x)$ the irreducible polynomial of u over $\mathbb{F}_q$. Since we have an $\overline{\mathbb{F}}_q$-algebra isomorphism

$$\overline{\mathbb{F}}_q \otimes_{\mathbb{F}_q} \mathbb{F}_{q^N} \cong \overline{\mathbb{F}}_q \otimes_{\mathbb{F}_q} (\mathbb{F}_q[x]/(f(x))) \cong \overline{\mathbb{F}}_q[x]/(x-u) \times \ldots \times \overline{\mathbb{F}}_q[x]/(x-u^{q^{N-1}})$$

by the Chinese Remainder Theorem, we deduce that

$$\begin{aligned}\mathrm{Hilb}^n(\mathrm{Spec}(\widehat{\mathcal{O}_{S,y}})/\mathrm{Spec}\mathbb{F}_q) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q &\cong \mathrm{Hilb}^n(\mathrm{Spec}((\overline{\mathbb{F}}_q \times \ldots \times \overline{\mathbb{F}}_q)[[s,t]])/\mathrm{Spec}\overline{\mathbb{F}}_q) \\ &\cong \mathrm{Hilb}^n(\coprod \mathrm{Spec}\overline{\mathbb{F}}_q[[s,t]]/\mathrm{Spec}\overline{\mathbb{F}}_q).\end{aligned}$$

Hence the $\overline{\mathbb{F}}_q$-valued points of $\mathrm{Hilb}^n(\mathrm{Spec}(\widehat{\mathcal{O}_{S,y}})/\mathrm{Spec}\mathbb{F}_q)$ correspond to the closed subschemes of degree n of $\coprod \mathrm{Spec}\overline{\mathbb{F}}_q[[s,t]]$, i.e. the closed subschemes of degree n of S whose underlying space is a subset of the points $x, F_q(x), \ldots, F_q^{N-1}(x)$.

Since F_q acts on $\mathbb{F}_{q^N}[[s,t]]$ by sending s to s^q, t to t^q and $c \in \mathbb{F}_{q^N}$ to c^q, we deduce from the above discussion that F_q acts on $(\overline{\mathbb{F}}_q \times \ldots \times \overline{\mathbb{F}}_q)[[s,t]]$ by sending s to s^q, t to t^q and $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \overline{\mathbb{F}}_q \times \ldots \times \overline{\mathbb{F}}_q$ to $(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_0)$. This is actually an algebraic assertion, which can also be seen geometrically. For example, F_q is a $\overline{\mathbb{F}}_q$-morphism from $\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q}, F_q(x)}} \cong (\{0\} \times \overline{\mathbb{F}}_q \times \ldots \times \{0\})[[s,t]]$ to $\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q}, x}} \cong (\overline{\mathbb{F}}_q \times \{0\} \times \ldots \times \{0\})[[s,t]]$.

Let σ be an element of $\mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q)$. Recall that for an $\mathbb{F}_{q^N}$-vector space V, a σ-linear map $f: V \to V$ is an additive map on V such that $f(\alpha v) = \sigma(\alpha) f(v)$ for all $\alpha \in \mathbb{F}_{q^N}$ and $v \in V$.

Lemma 3.6. *Let $H = \langle g \rangle$. Suppose $p \nmid |H|$. Then we can choose s and t such that g acts on $\mathbb{F}_{q^N}[[s,t]]$ σ-linearly, where σ is the inverse of the Frobenius automorphism of* $\mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q)$.

Proof. The automorphism g acts as an $\mathbb{F}_q$-automorphism on $\mathbb{F}_{q^N}[[s,t]]$ fixing the ideal (s,t) and sending $\mathbb{F}_{q^N}$ to $\mathbb{F}_{q^N}$. Since we know F_q sends $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \overline{\mathbb{F}}_q \times \ldots \times \overline{\mathbb{F}}_q$ to $(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_0)$ and gF_q fixes the geometric points x, $F_q(x), \ldots, F_q^{N-1}(x)$, we deduce that g sends $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \overline{\mathbb{F}}_q \times \ldots \times \overline{\mathbb{F}}_q$ to $(\alpha_{N-1}, \alpha_0, \ldots, \alpha_{N-3}, \alpha_{N-2})$. Hence $g(\alpha) = \sigma(\alpha)$ for all $\alpha \in \mathbb{F}_{q^N}$ where σ is the inverse of the Frobenius automorphism.

For any element $h \in H$, we write $h(s) = as + bt + \ldots$ and $h(t) = cs + dt + \ldots$ where $a, b, c, d \in \mathbb{F}_q$ since h commutes with F_q. Define an automorphism $\rho(h)$ of $\mathbb{F}_{q^N}[[s,t]]$ by $\rho(h)(s) = as + bt$, $\rho(h)(t) = cs + dt$ and the action of $\rho(h)$ on $\mathbb{F}_{q^N}$ is the same as the action of h. Then we denote the $\mathbb{F}_{q^N}$-automorphism $\frac{1}{|H|}\sum_{h \in H} h\rho(h)^{-1}$ by θ. Notice that θ is an automorphism because the linear term of θ is an invertible matrix, and here is the only place we use the assumption that $p \nmid |G|$. We deduce that $g\theta = \theta\rho(g)$, which implies $\theta^{-1} g \theta = \rho(g)$. Hence we are done. $\square$

The above discussion implies that the g-action on $(\overline{\mathbb{F}}_q\times\ldots\times\overline{\mathbb{F}}_q)[[s,t]]$ is given by sending s to $(a,\ldots,a)s+(b,\ldots,b)t$, t to $(c,\ldots,c)s+(d,\ldots,d)t$ and $(\alpha_0,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1})\in\overline{\mathbb{F}}_q\times\ldots\times\overline{\mathbb{F}}_q$ to $(\alpha_{N-1},\alpha_0,\ldots,\alpha_{N-3},\alpha_{N-2})$.

Hence the action of gF_q on $(\overline{\mathbb{F}}_q\times\ldots\times\overline{\mathbb{F}}_q)[[s,t]]$ is given by sending s to $(a,\ldots,a)s^q+(b,\ldots,b)t^q$, t to $(c,\ldots,c)s^q+(d,\ldots,d)t^q$ and $(\alpha_0,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1})\in\overline{\mathbb{F}}_q\times\ldots\times\overline{\mathbb{F}}_q$ to itself. This implies that gF_q acts on each complete local ring, which is what we expected since gF_q fixes each geometric point over y. In particular, it acts on $\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}}\cong(\overline{\mathbb{F}}_q\times\{0\}\times\ldots\times\{0\})[[s,t]]\cong\overline{\mathbb{F}}_q[[s,t]]$.

Recall that $\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})(\overline{\mathbb{F}}_q)$ parametrizes closed subschemes of degree n of $S_{\overline{\mathbb{F}}_q}$ supported on x.

Proof of Lemma 3.2. First we define an $\mathbb{F}_q$-automorphism $\tilde{g}$ on $\mathbb{F}_q[[s,t]]$ by

$$\tilde{g}(s)=as+bt \quad \text{and} \quad \tilde{g}(t)=cs+dt$$

Recall that the action of F_q on $\mathbb{F}_q[[s,t]]$ is an $\mathbb{F}_q$-endomorphism sending s to s^q and t to t^q. By the above discussion, we observe that the action of gF_q on $\overline{\mathbb{F}}_q[[s,t]]$ on the left is the same as the action of $\tilde{g}F_q$ on $\overline{\mathbb{F}}_q[[s,t]]$ on the right. Hence we have

$$|\mathrm{Hilb}^n(\widehat{\mathcal{O}_{S_{\overline{\mathbb{F}}_q},x}})(\overline{\mathbb{F}}_q)^{gF_q}|=|\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])(\overline{\mathbb{F}}_q)^{\tilde{g}F_q}|.$$

Now for the right hand side, $\tilde{g}$ is an automorphism of finite order and F_q is the geometric Frobenius. Then by the Grothendieck trace formula, we have

$$|\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])(\overline{\mathbb{F}}_q)^{\tilde{g}F_q}|=\sum_{k=0}^{\infty}(-1)^k\mathrm{Tr}((\tilde{g}F_q)^*,H_c^k(\mathrm{Hilb}^n(\overline{\mathbb{F}}_q[[s,t]]),\mathbb{Q}_l)).$$

But the action of $\tilde{g}$ factors through $\mathrm{GL}_2(\overline{\mathbb{F}}_q)$. Now we use the fact that if G is a connected algebraic group acting on a separated and finite type scheme X, then the action of $g\in G$ on $H_c^*(X,\mathbb{Q}_l)$ is trivial [DL76, Corollary 6.5]. Hence we have

$$\begin{aligned}|\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])(\overline{\mathbb{F}}_q)^{\tilde{g}F_q}|&=\sum_{k=0}^{\infty}(-1)^k\mathrm{Tr}((F_q)^*,H_c^k(\mathrm{Hilb}^n(\overline{\mathbb{F}}_q[[s,t]]),\mathbb{Q}_l))\\&=|\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])(\overline{\mathbb{F}}_q)^{F_q}|. \quad \square\end{aligned}$$

Suppose S is a smooth projective K3 surface over $\overline{\mathbb{F}}_p$ with a G-action. Recall that a *Mathieu representation* of a finite group G is a 24-dimensional representation on a vector space V over a field of characteristic zero with character

$$\chi(g)=\varepsilon(\mathrm{ord}(g)),$$

where

$$\varepsilon(n)=24(n\prod_{p|n}(1+\frac{1}{p}))^{-1}.$$

Proposition 3.7. ([DK09, Proposition 4.1]) *Let G be a finite group of symplectic automorphisms of a K3 surface X defined in characteristic $p>0$. Assume that $p\nmid G$. Then for any prime $l\neq p$, the natural representation of G on the l-adic cohomology groups $H^*(X,\mathbb{Q}_l)\cong\mathbb{Q}_l^{24}$ is Mathieu.*

Proof of Theorem 1.3. By Remark 3.5, we deduce that

$$\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n=\exp\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mathrm{Tr}(g^k,[e(S)])t^{mk}}{k}\right).$$

Then by Proposition 3.7, we obtain the equality we want.

When G is a cyclic group of order N, we know that $N\leq 8$ by [DK09, Theorem 3.3]. Then the proof is the same as the proof in [Zha21] in the characteristic zero case. □

Proof of Theorem 1.5. If g is a symplectic automorphism (fixing the origin) on a complex abelian surface, then g has order $1,2,3,4$ or 6 by [Fuj88, Lemma 3.3]. We will do the case when the order $N=4$, and the calculation for other cases are similar. By [Fuj88, Page 33], we know the explicit action of g on the torus $S=\mathbb{C}^2/\bigwedge$ in each case. If $N=4$, then the action on $H^1(S,\mathbb{C})$ is given by

$$\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}.$$

Hence we deduce that $\mathrm{Tr}(g,[e(S)])=\mathrm{Tr}(g^3,[e(S)])=4$, and $\mathrm{Tr}(g^2,[e(S)])=16$. Now

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n&=\exp\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mathrm{Tr}(g^k,[e(S)])t^{mk}}{k}\right)\\&=\exp\left(\sum_{m=1}^{\infty}\sum_{k\equiv 1,3}\frac{4t^{mk}}{k}+\sum_{m=1}^{\infty}\sum_{k\equiv 2}\frac{16t^{mk}}{k}\right)\\&=\exp\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{4t^{mk}}{k}-\sum_{k=1}^{\infty}\frac{4t^{2mk}}{2k}\right)\right.\\&\qquad\left.+\sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{16t^{2mk}}{2k}-\sum_{k=1}^{\infty}\frac{16t^{4mk}}{4k}\right)\right)\\&=\frac{\prod_{m=1}^{\infty}(1-t^m)^{-4}}{\prod_{m=1}^{\infty}(1-t^{2m})^{-2}}\frac{\prod_{m=1}^{\infty}(1-t^{2m})^{-8}}{\prod_{m=1}^{\infty}(1-t^{4m})^{-4}}\\&=\frac{\eta^4(t^4)}{\eta^4(t)\eta^6(t^2)}.\end{aligned}$$

□

Remark 3.8. Fix a smooth projective surface S and an automorphism g of finite order. From the proof of Theorem 1.3 or Theorem 1.5, we notice that if $\mathrm{Tr}(g^k,[e(S)])$ only depends on the order of g^k in the cyclic group $\langle g\rangle$ for $k\geq 0$, then the generating function $\sum_{n=0}^{\infty}\mathrm{Tr}(g,[e(S^{[n]})])t^n$ is an eta quotient by the inclusion-exclusion principle.

4. Generalized Kummer varieties

Let A be an abelian surface over $\mathbb{C}$. Let $\omega_n:A^{[n]}\to A^{(n)}$ be the Hilbert-Chow morphism and let $g_n:A^{(n)}\to A$ be the addition map. The generalized Kummer variety of A is defined to be

$$K_n(A):=\omega_n^{-1}(g_n^{-1}(0)).$$

This is a smooth projective holomorphic symplectic variety. We follow the strategy in [Göt94].

Now suppose A is an abelian surface with a G-action over $\mathbb{F}_q$. Define the map γ_n by

$$\gamma_n:\coprod_{\alpha\in P(n)}\left(\left(\prod_{i=1}^{\infty}S^{(\alpha_i)}(\overline{\mathbb{F}}_q)^{gF_q}\right)\times\mathbb{A}^{n-|\alpha|}(\mathbb{F}_q)\right)\longrightarrow S^{(n)}(\overline{\mathbb{F}}_q)^{gF_q},$$
$$((\zeta_i)_i,v)\longmapsto\sum i\cdot\zeta_i.$$

Lemma 4.1. *For any* $\zeta\in S^{(n)}(\overline{\mathbb{F}}_q)^{gF_q}$*, we have* $|\gamma_n^{-1}(\zeta)=|\omega_n^{-1}(\zeta)|$.

Proof. Let $\zeta=\sum_{i=1}^{r}n_i\zeta_i\in S^{(n)}(\overline{\mathbb{F}}_q)^{gF_q}$, where ζ_i are distinct primitive cycles of degree d_i. Then

$$\begin{aligned}|\omega_n^{-1}(\zeta)|&=\prod_{i=1}^{r}|V_{n_i}(\overline{\mathbb{F}}_q)^{(gF_q)^{d_i}}|\\&=\prod_{i=1}^{r}|V_{n_i}(\overline{\mathbb{F}}_q)^{(F_q)^{d_i}}|\\&=\prod_{i=1}^{r}\sum_{\beta_j^i\in P(n_i)}q^{d_i(n_i-|\beta_j^i|)},\end{aligned}$$

where $V_n=\mathrm{Hilb}^n(\mathbb{F}_q[[s,t]])$. Here we use the key Lemma 3.2.

For $i=1,\dots,r$, let $\beta^i=(1^{\beta_1^i},2^{\beta_2^i},\dots)$ be a partition of n_i, and let $\alpha=(1^{\alpha_1},2^{\alpha_2},\dots)$ be the union of d_i copies of each β^i, where $\alpha_j=\sum_i d_i\beta_j^i$. Let

$$\eta_j=\sum_{i=1}^{r}\beta_j^i\zeta_i.$$

Let η be the sequence $(\eta_1, \eta_2, \eta_3, \ldots)$. Then for all $w \in \mathbb{A}^{n-|\alpha|}$ we have

$$\gamma_n((\eta, w)) = \zeta,$$

and in this way we get all the elements of $\gamma_n^{-1}(\zeta)$. Hence

$$|\gamma_n^{-1}(\zeta)| = \sum_{\beta^1 \in P(n_1)} \sum_{\beta^2 \in P(n_2)} \ldots \sum_{\beta^r \in P(n_r)} q^{n - \sum d_i |\beta^i|} = |\omega_n^{-1}(\zeta)|. \quad \square$$

Lemma 4.2. *Denote by $h_n : A^{(n)}(\overline{\mathbb{F}}_q)^{gF_q} \to A(\overline{\mathbb{F}}_q)^{gF_q}$ the restriction of g_n. Then h_n is onto and $|h_n^{-1}(x)|$ is independent of $x \in A(\overline{\mathbb{F}}_q)^{gF_q}$.*

Proof. Since gF_q is the Frobenius map of some twist of A, we can replace gF_q by F_q in the statement, and this is true by [Göt94, Lemma 2.4.8]. ☐

For each $l \in \mathbb{N}$, let $A(\overline{\mathbb{F}}_q)_l^{gF_q}$ be the image of the multiplication $(l) : A(\overline{\mathbb{F}}_q)^{gF_q} \to A(\overline{\mathbb{F}}_q)^{gF_q}$.

Lemma 4.3. *Let $\mu = (n_1, \ldots, n_t)$ be a partition of a number $n \in \mathbb{N}$. Then*

$$\sigma_\mu : (A(\overline{\mathbb{F}}_q)^{gF_q})^t \longrightarrow A(\overline{\mathbb{F}}_q)_{gcd(\mu)}^{gF_q}$$
$$(x_1, \ldots, x_t) \longmapsto \sum_{i=1}^{t} n_i x_i$$

is onto and $|\sigma_\mu^{-1}(x)|$ is independent of $x \in A(\overline{\mathbb{F}}_q)_{gcd(\mu)}^{gF_q}$.

Proof. As the above lemma, we can replace gF_q by F_q, and this is true by [Göt94, Lemma 2.4.9]. ☐

We denote $\left(\left(\prod_{i=1}^{\infty} S^{(\alpha_i)}(\overline{\mathbb{F}}_q)^{gF_q}\right) \times \mathbb{A}^{n-|\alpha|}(\mathbb{F}_q)\right.$ by $A[\alpha]$. Denote the restriction map of γ_n on $A[\alpha]$ by $\gamma_{n,\alpha} : A[\alpha] \to S^{(n)}(\overline{\mathbb{F}}_q)^{gF_q}$.

Lemma 4.4.

$$|K_n(A)(\overline{\mathbb{F}}_q)^{gF_q}| = \frac{1}{|A(\overline{\mathbb{F}}_q)^{gF_q}|} \sum_{\alpha \in P(n)} \left(gcd(\alpha)^4 q^{n-|\alpha|} \prod_{i=1}^{\infty} |A^{(\alpha_i)}(\overline{\mathbb{F}}_q)^{gF_q}| \right).$$

Proof. By Lemma 4.1, we have

$$|K_n(A)(\overline{\mathbb{F}}_q)^{gF_q}| = |\gamma_n^{-1}(h_n^{-1})| = \sum_{\alpha \in P(n)} |\gamma_{n,\alpha}^{-1}(h_n^{-1}(0))|.$$

Suppose $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$. Let

$$\mu = (m_1, \ldots, m_t) : = (1^{\mu_1}, 2^{\mu_2}, \ldots),$$

where $\mu_i{=}\min(1,\alpha_i)$ for all i. Let

$$\begin{aligned} f_\alpha : S[\alpha] &\longrightarrow (A(\overline{\mathbb{F}}_q)^{gF_q})^t \\ ((\zeta_1,\dots,\zeta_t),w) &\longmapsto (g_{\alpha_{m_1}}(\zeta_1),\dots,g_{\alpha_{m_t}}(\zeta_t)). \end{aligned}$$

Then the following diagram commutes:

$$\begin{array}{ccc} A[\alpha] & \xrightarrow{\gamma_{n,\alpha}} & A^{(n)}(\overline{\mathbb{F}}_q)^{gF_q} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle h_n} \\ (A(\overline{\mathbb{F}}_q)^{gF_q})^t & \xrightarrow{\sigma_\mu} & A(\overline{\mathbb{F}}_q)^{gF_q}. \end{array}$$

By Lemma 4.2 and Lemma 4.3, $\sigma_\mu\circ f_\alpha$ maps $S[\alpha]$ onto $A(\overline{\mathbb{F}}_q)^{gF_q}_{gcd(\alpha)}{=}A(\overline{\mathbb{F}}_q)^{gF_q}_{gcd(\mu)}$, and $|f_\alpha^{-1}(\sigma_\mu^{-1}(x))|$ is independent of $x{\in}A(\overline{\mathbb{F}}_q)^{gF_q}_{gcd(\alpha)}$. Since the multiplication with $gcd(\alpha)$ is an étale morphism of degree $(gcd(\alpha))^4$, we have

$$\begin{aligned} |K_n(A)(\overline{\mathbb{F}}_q)^{gF_q}| &= \sum_{\alpha\in P(n)} |f_\alpha^{-1}(\sigma_\mu^{-1}(x))| \\ &= \sum_{\alpha\in P(n)} \frac{|A[\alpha]|}{|A(\overline{\mathbb{F}}_q)^{gF_q}_{gcd(\alpha)}|} \\ &= \frac{1}{|A(\overline{\mathbb{F}}_q)^{gF_q}|}\sum_{\alpha\in P(n)}\left(gcd(\alpha)^4 q^{n-|\alpha|}\prod_{i=1}^{\infty}|A^{(\alpha_i)}(\overline{\mathbb{F}}_q)^{gF_q}|\right). \quad \square \end{aligned}$$

Proof of Theorem 1.7. By Lemma 4.4, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}|K_n(A)(\overline{\mathbb{F}}_q)^{gF_q}|t^n \\ &= \sum_{n=0}^{\infty}\frac{1}{|A(\overline{\mathbb{F}}_q)^{gF_q}|}\sum_{\alpha\in P(n)}\left(gcd(\alpha)^4 q^{n-|\alpha|}\prod_{i=1}^{\infty}|A^{(\alpha_i)}(\overline{\mathbb{F}}_q)^{gF_q}|\right)t^n \\ &= \frac{(w\frac{d}{dw})^4}{|A(\overline{\mathbb{F}}_q)^{gF_q}|}\sum_{n=0}^{\infty}\sum_{\alpha\in P(n)} w^{gcd(\alpha)}\prod_{i=1}^{\infty}\left(|A^{(\alpha_i)}(\overline{\mathbb{F}}_q)^{gF_q}|q^{(i-1)\alpha_i}t^{i\alpha_i}\right)\Bigg|_{w=1} \\ &= \frac{(w\frac{d}{dw})^4}{|A(\overline{\mathbb{F}}_q)^{gF_q}|}\bigodot_{m=1}^{\infty}\left(1{+}w^m(-1{+}\sum_{n=0}^{\infty}|A^{(n)}(\overline{\mathbb{F}}_q)^{gF_q}|q^{(m-1)n}t^{mn})\right)\Bigg|_{w=1}. \end{aligned}$$

Then by the proof of Proposition 2.2 and Theorem 2.7, the theorem follows. $\square$

Remark 4.5. It is calculated in [Göt94, Corollary 2.4.13] that

$$\sum_{n=1}^{\infty} e(K_n(A))q^n = \frac{(q\frac{d}{dq})^3}{24} E_2,$$

where $E_2 := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n$ is a quasi-modular form. As in the case of Hilbert schemes of points, we can calculate $\sum_{n=0}^{\infty} \mathrm{Tr}(g, [e(K_n(A))])t^n$, where g is a symplectic automorphism of finite order on the abelian surface A. But it is not obvious to the author whether or not the sum can be expressed by quasi-modular forms.

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