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On the finiteness of certain factorization invariants

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Abstract. Let H be a monoid and π_H be the unique extension of the identity map on H to a monoid homomorphism $\mathscr{F}(H)\to H$, where we denote by $\mathscr{F}(X)$ the free monoid on a set X. Given $A\subseteq H$, an A-word $\mathfrak{z}$ (i.e., an element of $\mathscr{F}(A)$) is minimal if $\pi_H(\mathfrak{z})\neq\pi_H(\mathfrak{z}')$ for every permutation $\mathfrak{z}'$ of a proper subword of $\mathfrak{z}$. The minimal A-elasticity of H is then the supremum of all rational numbers m/n with $m,n\in\mathbb{N}^+$ such that there exist minimal A-words $\mathfrak{a}$ and $\mathfrak{b}$ of length m and n, resp., with $\pi_H(\mathfrak{a})=\pi_H(\mathfrak{b})$.

Among other things, we show that if H is commutative and A is finite, then the minimal A-elasticity of H is finite. This provides a non-trivial generalization of the finiteness part of a classical theorem of Anderson et al. from the case where H is cancellative, commutative, and finitely generated modulo units, and A is the set $\mathscr{A}(H)$ of atoms of H. We also demonstrate that commutativity is somewhat essential here, by proving the existence of an atomic, cancellative, finitely generated monoid with trivial group of units whose minimal $\mathscr{A}(H)$-elasticity is infinite.

1. Introduction

Let H be a (multiplicatively written) monoid, e.g., the multiplicative monoid of a (unital, associative) ring. An irreducible of H is an element $a\in H$ that is neither a divisor of the identity 1_H of H nor a product of two other elements $x,y\in H$ each of which is neither a divisor of 1_H nor a divisor of a. An atom of H is, on the other hand, a non-unit that cannot be factored as the product of two non-units. The existence itself of an atom implies that any divisor of 1_H is a unit [12, Lemma 2.2]. Hence, every atom is an irreducible, and the converse is true when the monoid is, e.g., commutative and cancellative (see Section 5).

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We define the (classical) elasticity of H as the supremum (with respect to the standard ordering of the non-negative real numbers) of the set of rational numbers of the form m/n with $m,n \in \mathbb{N}^+$ (the positive integers) such that $a_1 \ldots a_m = b_1 \ldots b_n$ for some irreducibles $a_1, \ldots, a_m, b_1, \ldots, b_n \in H$. Moreover, we say that a monoid is atomic if every non-unit has an atomic factorization, that is, the element factors as a (finite) product of atoms.

Since the late 1980s, elasticity has received wide attention in the literature; see Anderson's survey [3] for an overview of results prior to 1997, [17, Theorems 6.2 and 7.2] for some of the strongest finiteness criteria so far available in the cancellative commutative setting, and [5], [7], [14], [19], [24], and [25] for a non-exhaustive list of recent contributions. Introduced by Valenza [22] in his study of factorization in number rings, the notion was made popular by Zaks [23] who used it as a measure of the deviation of an atomic monoid from the condition of half-factoriality (a monoid is half-factorial if it is atomic and any two atomic factorizations of the same element have the same length, i.e., the same number of factors). Most notably, it is a classical result in the arithmetic theory of monoids (and rings) that the elasticity of a cancellative commutative monoid with finitely many non-associated atoms is a rational number and hence finite, where two elements $u, v \in H$ are associated if u divides v (namely, $v \in HuH$) and v divides u. This was first proved by Anderson et al. in [2, Theorem 7] and is herein referred to as *Anderson et al.'s theorem*. The result has been later extended to unit-cancellative commutative monoids with finitely many non-associated atoms by Fan et al. [11, Proposition 3.4(1)], where the monoid H is called unit-cancellative if $yx \neq x \neq xy$ for all $x, y \in H$ such that y is a non-unit (obviously, a cancellative monoid is unit-cancellative).

In this article, we adopt a new point of view set forth in [4], [10] and obtain a non-trivial generalization (Corollaries 3.3 and 3.4) of the *finiteness* part of Fan et al.'s result (and hence of Anderson et al.'s theorem) to any commutative monoid with finitely many non-associated irreducibles. The proof relies on *Dickson's lemma* (see, e.g., Theorem 9.18 in [8]), which is the one and only aspect in common with previous results in the same vein. A critical feature of our approach is the use of *minimal* factorizations to counter the blowup of factorization lengths and related invariants that is typical of a non-unit-cancellative or non-commutative setup (Examples 3.12 and 4.2). The conclusion then becomes an immediate consequence of an essentially combinatorial theorem (Theorem 2.3) that makes no reference to atoms, irreducibles, etc.

The price we pay is that we cannot say anything about the rationality of the invariants we introduce along the way to generalize the classical elasticity (Section 5). What we gain is that, in the very spirit of a series of recent papers by the same authors [9], [10], [20] and [21], many of our results are no longer phrased in the language of *monoids and irreducibles* (whose scope is, in a sense, too narrow) but rather in the more abstract language of *monoids and preorders* (see Section 3 for details). Loosely speaking, this allows us to use basically any sort of elements as "building blocks" in the factorization process (Proposition 3.10).

Other than that, we prove a couple or so of finiteness results on length sets and their unions in a (commutative or non-commutative) monoid with finitely many irreducibles (Definition 3.5, Propositions 3.6 and 3.7, and Corollary 3.9), and we show by way of example that our main theorem (Theorem 2.3) fails in the non-commutative setting (Section 4).

Notation

Through the paper, $\mathbb{N}$ is the set of non-negative integers, and for $a, b \in \mathbb{N} \cup \{\infty\}$ we let $[\![a, b]\!] := \{x \in \mathbb{N} \colon a \le x \le b\}$ be the discrete interval from a to b.

A preorder on a set S is a reflexive and transitive binary relation on S. Given $x, y \in S$ and a preorder $\preceq$ on S, we say that x is $\preceq$-equivalent to y if $x \preceq y \preceq x$, and we write $x \prec y$ to mean that $x \preceq y$ and $y \not\preceq x$. (This convention also applies to the symbols $\sqsubseteq$ and $\sqsubset$, which we will likewise employ for preorders.) We call $\preceq$ artinian if, for every $\preceq$-non-increasing sequence $(x_k)_{k \ge 0}$ in S, it holds $x_k \preceq x_{k+1}$ for all but finitely many $k \in \mathbb{N}$.

We refer to [16] for generalities on monoids. In particular, we denote by $\mathscr{F}(X)$ the free monoid on a set X and refer to the elements of $\mathscr{F}(X)$ as X-words. We use the symbols $*_X$ and ε_X, resp., for the operation and the identity of $\mathscr{F}(X)$. We take $\|\mathfrak{u}\|_X$ to be the (word) length of an X-word $\mathfrak{u}$, and for each $i \in [\![1, \|\mathfrak{u}\|_X]\!]$ we let $\mathfrak{u}[i]$ be the i^{th} letter of $\mathfrak{u}$. An X-word $\mathfrak{v}$ is then a (scattered) subword of $\mathfrak{u}$ if there is a strictly increasing function $\sigma \colon [\![1, \|\mathfrak{v}\|_X]\!] \to [\![1, \|\mathfrak{u}\|_X]\!]$ such that $\mathfrak{u}[\sigma(i)] = \mathfrak{v}[i]$ for each $i \in [\![1, \|\mathfrak{v}\|_X]\!]$. When there is no serious risk of ambiguity, we drop the subscript "X" from the above notation.

We use $H^\times$ for the group of units (or invertible elements) of a monoid H and $\langle X \rangle_H := \bigcup_{k \in \mathbb{N}} X^k$ for the submonoid of H generated by a set $X \subseteq H$, where X^k is the setwise product of k copies of X (in particular, $X^0 = \{1_H\}$). We call H reduced if its only unit is the identity 1_H and we write π_H for the unique extension of the identity map on H to a monoid homomorphism $\mathscr{F}(H) \to H$.

2. Elasticity

By the definition given in the introduction, the (classical) elasticity of a monoid H is the supremum of the set of all rational numbers of the form $\|\mathfrak{b}\|^{-1}\|\mathfrak{a}\|$ as $\mathfrak{a}$ and $\mathfrak{b}$ range over the non-empty $\mathscr{I}(H)$-words with $\pi_H(\mathfrak{a}) = \pi_H(\mathfrak{b})$, where $\mathscr{I}(H)$ is the set of irreducibles of H. We aim to generalize this idea.

Definition 2.1. Given a set X, we denote by $\sqsubseteq_X$ the binary relation on the free monoid $\mathscr{F}(X)$ defined by $\mathfrak{a} \sqsubseteq_X \mathfrak{b}$, for some X-words $\mathfrak{a}$ and $\mathfrak{b}$, if and only if $\mathfrak{a}$ is a permuted subword of $\mathfrak{b}$, i.e., there is an injective function $\sigma\colon [\![1, \|\mathfrak{a}\|]\!] \to [\![1, \|\mathfrak{b}\|]\!]$ such that $\mathfrak{a}[i] = \mathfrak{b}[\sigma(i)]$.

Since the composition of two injections is still an injection, it is immediate that the relation $\sqsubseteq_X$ in Definition 2.1 is a preorder on $\mathscr{F}(X)$. More precisely, $\sqsubseteq_X$ is an artinian preorder, because $\mathfrak{a} \sqsubset_X \mathfrak{b}$ implies $\|\mathfrak{a}\| < \|\mathfrak{b}\|$. This makes it natural to talk about $\sqsubseteq_X$-minimality in $\mathscr{F}(X)$. Moreover, the pair $(\mathscr{F}(X), \sqsubseteq_X)$ is a *strongly positive monoid* in the sense of [10, Definition 2.3], meaning that (i) $\varepsilon \sqsubseteq_X \mathfrak{a}$ for every X-word $\mathfrak{a}$ and (ii) $\mathfrak{a} \sqsubseteq_X \mathfrak{b}$ yields $\mathfrak{u} * \mathfrak{a} * \mathfrak{v} \sqsubseteq_X \mathfrak{u} * \mathfrak{b} * \mathfrak{v}$ for all $\mathfrak{u}, \mathfrak{v} \in \mathscr{F}(X)$, with the latter inequality being strict if and only if the former is.

Definition 2.2. Given a monoid H and a set $A \subseteq H$, an A-word $\mathfrak{z}$ is minimal if $\pi_H(\mathfrak{z}') \neq \pi_H(\mathfrak{z})$ whenever $\mathfrak{z}' \sqsubset_H \mathfrak{z}$. The minimal A-elasticity $\varrho_A^{\mathsf{m}}(H)$ of H is then the supremum of the set of all rational numbers of the form m/n with $m, n \in \mathbb{N}^+$ such that there exist minimal A-words $\mathfrak{a}$ and $\mathfrak{b}$ with $\|\mathfrak{a}\| = m$, $\|\mathfrak{b}\| = n$, and $\pi_H(\mathfrak{a}) = \pi_H(\mathfrak{b})$, where it is understood that $\sup \varnothing := 0$.

With these preliminaries in place, we are just ready for the main theorem of the paper. In the proof, we will make use of Dickson's lemma [8, Theorem 9.18], stating that every non-empty subset of $\mathbb{N}^{\times n}$ $(n \in \mathbb{N}^+)$ has at least one minimal element with respect to the product order $\le_n$ induced on $\mathbb{N}^{\times n}$ by the standard order on $\mathbb{N}$, so that $\mathfrak{u} \le_n \mathfrak{v}$, for some $\mathfrak{u}, \mathfrak{v} \in \mathbb{N}^{\times n}$, if and only if $\mathfrak{u}[i] \le \mathfrak{v}[i]$ for each $i \in [\![1, n]\!]$ (here, we regard $\mathfrak{u}$ and $\mathfrak{v}$ as $\mathbb{N}$-words of length n).

Theorem 2.3. *The minimal A-elasticity of a commutative monoid H is finite for every finite $A \subseteq H$.*

Proof. Suppose A is a finite subset of H and put $s := |A| \in \mathbb{N}$. We may assume $s \neq 0$ and $A \neq \{1_H\}$, or else the only minimal A-word is the empty word ε and the conclusion is trivial. Accordingly, let $a_1, \dots, a_s$ be an enumeration of A with $a_1 \neq 1_H$, and for each $t \in [\![1, s]\!]$ denote by v_t the function $\mathscr{F}(A) \to \mathbb{N}$ that maps an A-word $\mathfrak{a}$ to its a_t-adic valuation, i.e., to the number of indices $i \in [\![1, \|\mathfrak{a}\|]\!]$ such that $a_t = \mathfrak{a}[i]$. It is clear that

$$\|\mathfrak{a}\| = \mathsf{v}_1(\mathfrak{a}) + \dots + \mathsf{v}_s(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in \mathscr{F}(A). \tag{1}$$

Let S be the set of all triples $(\mathfrak{a}, \mathfrak{b}, \mathfrak{z})$ of A-words with $(\mathfrak{a}, \mathfrak{b}) \neq (\varepsilon, \varepsilon)$ and $\pi_H(\mathfrak{a} * \mathfrak{z}) = \pi_H(\mathfrak{b} * \mathfrak{z})$ such that $\mathfrak{a} * \mathfrak{z}$ is a minimal A-word (the definition of S is intentionally "asymmetric", insomuch as we do not require that also $\mathfrak{b} * \mathfrak{z}$ is a minimal A-word).

If $(\mathfrak{a}, \mathfrak{b}, \mathfrak{z}) \in S$, then $\mathfrak{b} \neq \varepsilon$. Otherwise, $\mathfrak{a}$ would be a non-empty A-word; and since $(\mathscr{F}(A), \sqsubseteq_H)$ is a strongly positive monoid (see the comments after Definition 2.1), we would find that $\mathfrak{z}$ is a proper subword of $\mathfrak{a}*\mathfrak{z}$ with $\pi_H(\mathfrak{a}*\mathfrak{z})=\pi_H(\mathfrak{z})$, contradicting that $\mathfrak{a}*\mathfrak{z}$ is a minimal A-word. So, it makes sense to define

$$\varrho^* := \sup\{\|\mathfrak{b}\|^{-1}\|\mathfrak{a}\| \colon (\mathfrak{a},\mathfrak{b},\mathfrak{z}) \in S\}.$$

If $(\mathfrak{a},\mathfrak{b})$ is a pair of minimal A-words with $\pi_H(\mathfrak{a})=\pi_H(\mathfrak{b})$ and $\mathfrak{b}\neq\varepsilon$, then $(\mathfrak{a},\mathfrak{b},\varepsilon)\in S$; and there is at least one such pair, as we can take $\mathfrak{a}=\mathfrak{b}=a_1$. It follows that $\varrho_A^{\mathsf{m}}(H)\le\varrho^*$. Consequently, to prove that $\varrho_A^{\mathsf{m}}(H)$ is finite, it suffices to check that ϱ^* is.

For, let $\le_{3s}$ be the product order induced on $\mathbb{N}^{\times 3s}$ by the standard order on $\mathbb{N}$, and let f be the function

$$S \longrightarrow \mathbb{N}^{\times 3s} \colon (\mathfrak{a},\mathfrak{b},\mathfrak{z}) \longmapsto (\mathsf{v}_1(\mathfrak{a}),\dots,\mathsf{v}_s(\mathfrak{a}),\mathsf{v}_1(\mathfrak{b}),\dots,\mathsf{v}_s(\mathfrak{b}),\mathsf{v}_1(\mathfrak{z}),\dots,\mathsf{v}_s(\mathfrak{z})).$$

Since S is non-empty, $f(S)$ is a non-empty subset of $\mathbb{N}^{\times 3s}$. We thus gather from Dickson's lemma that the set T of $\le_{3s}$-minimal elements of $f(S)$ is finite and non-empty, with the result that

$$R := \left\{\|\mathfrak{b}\|^{-1}\|\mathfrak{a}\| \colon (\mathfrak{a},\mathfrak{b},\mathfrak{z}) \in f^{-1}(T)\right\}$$

is a non-empty finite subset of $\mathbb{Q}_{\ge 0}$. In particular, it is straightforward from Eq. (1) that

$$\begin{aligned} R &= \left\{\frac{\mathsf{v}_1(\mathfrak{a})+\dots+\mathsf{v}_s(\mathfrak{a})}{\mathsf{v}_1(\mathfrak{b})+\dots+\mathsf{v}_s(\mathfrak{b})} \colon (\mathfrak{a},\mathfrak{b},\mathfrak{z}) \in f^{-1}(T)\right\} \\ &= \left\{\frac{n_1+\dots+n_s}{d_1+\dots+d_s} \colon (n_1,\dots,n_s,d_1,\dots,d_s,v_1,\dots,v_s) \in T\right\}, \end{aligned}$$

so making it evident that $1\le|R|\le|T|<\infty$. Hence, R has a maximum element $r\in\mathbb{Q}_{\ge 0}$, and it is obvious that $r\le\rho^*$. We claim $r=\rho^*$ (of course, this will be enough to show that $\rho^*<\infty$).

Assume to the contrary that $r<\rho^*$. Considering that $(\mathfrak{a},\mathfrak{b},\mathfrak{z})\in f^{-1}(T)$ implies $\|\mathfrak{b}\|^{-1}\|\mathfrak{a}\|\le r$, the set $\overline{S}$ of all triples $(\mathfrak{a},\mathfrak{b},\mathfrak{z})\in S$ with $r<\|\mathfrak{b}\|^{-1}\|\mathfrak{a}\|$ is then a non-empty subset of $S\setminus f^{-1}(T)$. Let $\preceq_S$ be the binary relation on S defined by $(\mathfrak{a},\mathfrak{b},\mathfrak{z})\preceq_S (\mathfrak{a}',\mathfrak{b}',\mathfrak{z}')$, for some $(\mathfrak{a},\mathfrak{b},\mathfrak{z}),(\mathfrak{a}',\mathfrak{b}',\mathfrak{z}')\in S$, if and only if

$$\text{(i) } \|\mathfrak{a}\|+\|\mathfrak{b}\| < \|\mathfrak{a}'\|+\|\mathfrak{b}'\|, \quad \text{or} \quad \text{(ii) } \|\mathfrak{a}\|+\|\mathfrak{b}\| = \|\mathfrak{a}'\|+\|\mathfrak{b}'\| \text{ and } \|\mathfrak{z}\| \le \|\mathfrak{z}'\|.$$

It is routine to verify that $\preceq_S$ is an artinian preorder on S. Since $\overline{S}$ is a non-empty subset of S, we thus obtain from the well-foundedness of artinian preorders (see, e.g., Remark 3.9(3) in [20]) that $\overline{S}$ has a $\preceq_S$-minimal element $\mathfrak{p}=(\mathfrak{a},\mathfrak{b},\mathfrak{z})$. By construction, $\mathfrak{p} \in S$ and $\mathfrak{p} \notin f^{-1}(T)$. Consequently, there exists $\mathfrak{p}_1 = (\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{z}_1) \in S$ such that $f(\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{z}_1) <_{3s} f(\mathfrak{a}, \mathfrak{b}, \mathfrak{z})$, which means that

$$\text{(j) } \mathfrak{a}_1 \sqsubseteq_H \mathfrak{a}, \;\; \mathfrak{b}_1 \sqsubseteq_H \mathfrak{b}, \text{ and } \mathfrak{z}_1 \sqsubseteq_H \mathfrak{z}, \quad \text{and} \qquad (2)$$
$$\text{(jj) at least one of these inequalities is strict.}$$

Denote by $\approx_H$ the relation of $\sqsubseteq_H$-equivalence on $\mathscr{F}(H)$, and for all $i \in [\![1, s]\!]$ and $\mathfrak{u}, \mathfrak{v} \in \mathscr{F}(H)$ set $\Delta_i(\mathfrak{u}, \mathfrak{v}) := |\mathsf{v}_i(\mathfrak{u}) - \mathsf{v}_i(\mathfrak{v})|$. We let

$$\bar{\mathfrak{z}}_1 := a_1^{*\Delta_1(\mathfrak{z}, \mathfrak{z}_1)} * \ldots * a_s^{*\Delta_s(\mathfrak{z}, \mathfrak{z}_1)},$$

and we distinguish two cases depending on whether $\mathfrak{b}_1 \approx_H \mathfrak{b}$ or $\mathfrak{b}_1 \sqsubset_H \mathfrak{b}$ (by Eq. (2), there are no other possibilities). In each case, we will reach a contradiction, thus finishing the proof of the theorem.

Case 1: $\mathfrak{b}_1 \approx_H \mathfrak{b}$. If $\mathfrak{a}_1 \approx_H \mathfrak{a}$, then $r < \|\mathfrak{b}\|^{-1} \|\mathfrak{a}\| = \|\mathfrak{b}_1\|^{-1} \|\mathfrak{a}_1\|$ (as $\sqsubseteq_H$-equivalent H-words have the same length) and, by item (jj) of Eq. (2), $\mathfrak{z}_1 \sqsubset_H \mathfrak{z}$. It follows that if $\mathfrak{a}_1 \approx_H \mathfrak{a}$, then $\overline{S} \ni (\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{z}_1) \prec_S \mathfrak{p}$, in contradiction with the $\preceq_S$-minimality of $\mathfrak{p}$ in $\overline{S}$. Therefore, we conclude from item (j) of Eq. (2) that $\mathfrak{a}_1 \sqsubset_H \mathfrak{a}$.

Since H is commutative and the equivalence $\mathfrak{b}_1 \approx_H \mathfrak{b}$ translates to $\mathfrak{b}_1$ and $\mathfrak{b}$ being the same A-word up to a permutation of their letters, it is then seen that

$$\begin{aligned} \pi_H(\mathfrak{a}_1 * \mathfrak{z}) &= \pi_H(\mathfrak{a}_1 * \mathfrak{z}_1 * \bar{\mathfrak{z}}_1) = \pi_H(\mathfrak{a}_1 * \mathfrak{z}_1)\, \pi_H(\bar{\mathfrak{z}}_1) = \pi_H(\mathfrak{b}_1 * \mathfrak{z}_1)\, \pi_H(\bar{\mathfrak{z}}_1) \\ &= \pi_H(\mathfrak{b} * \mathfrak{z}_1)\, \pi_H(\bar{\mathfrak{z}}_1) = \pi_H(\mathfrak{b} * \mathfrak{z}) = \pi_H(\mathfrak{a} * \mathfrak{z}). \end{aligned}$$

Considering that $\mathfrak{a}_1 * \mathfrak{z} \sqsubset_H \mathfrak{a} * \mathfrak{z}$, this is however in contradiction with the fact that $\mathfrak{a} * \mathfrak{z}$ is a minimal A-word.

Case 2: $\mathfrak{b}_1 \sqsubset_H \mathfrak{b}$. Let $\mathfrak{a}_2$ and $\mathfrak{b}_2$ be, resp., the A-words $a_1^{*\Delta_1(\mathfrak{a}, \mathfrak{a}_1)} * \ldots * a_s^{*\Delta_s(\mathfrak{a}, \mathfrak{a}_1)}$ and $a_1^{*\Delta_1(\mathfrak{b}, \mathfrak{b}_1)} * \ldots * a_s^{*\Delta_s(\mathfrak{b}, \mathfrak{b}_1)}$, and set $\mathfrak{z}_2 := \mathfrak{a}_1 * \mathfrak{z}$. We have $\|\mathfrak{b}_1\| < \|\mathfrak{b}\|$ and hence $\mathfrak{b}_2 \neq \varepsilon$. Moreover, $\mathfrak{a}_2 * \mathfrak{z}_2 = \mathfrak{a}_2 * \mathfrak{a}_1 * \mathfrak{z} \approx_H \mathfrak{a} * \mathfrak{z}$ and $\pi_H(\mathfrak{a}_2 * \mathfrak{z}_2) = \pi_H(\mathfrak{a} * \mathfrak{z})$, since $\mathfrak{a}_2 * \mathfrak{a}_1$ and $\mathfrak{a}$ are the same word up to a permutation of their letters and H is commutative. This shows that $\mathfrak{a}_2 * \mathfrak{z}_2$ is a minimal A-word. In addition,

$$\begin{aligned} \pi_H(\mathfrak{b}_2 * \mathfrak{z}_2) &= \pi_H(\mathfrak{b}_2 * \mathfrak{a}_1 * \mathfrak{z}) = \pi_H(\mathfrak{b}_2 * \bar{\mathfrak{z}}_1)\, \pi_H(\mathfrak{a}_1 * \mathfrak{z}_1) \\ &= \pi_H(\mathfrak{b}_2 * \bar{\mathfrak{z}}_1)\, \pi_H(\mathfrak{b}_1 * \mathfrak{z}_1) = \pi_H(\mathfrak{b}_1 * \mathfrak{b}_2 * \mathfrak{z}_1 * \bar{\mathfrak{z}}_1) \\ &= \pi_H(\mathfrak{b} * \mathfrak{z}) = \pi_H(\mathfrak{a} * \mathfrak{z}) = \pi_H(\mathfrak{a}_2 * \mathfrak{z}_2). \end{aligned}$$

It then follows that $(\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{z}_1)$ and $(\mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{z}_2)$ are both in S, and

$$\|\mathfrak{a}_i\| + \|\mathfrak{b}_i\| \le \|\mathfrak{a}\| + \|\mathfrak{b}_i\| < \|\mathfrak{a}\| + \|\mathfrak{b}\| \quad (i = 1, 2).$$

So, we get from the $\preceq_S$-minimality of $\mathfrak{p}$ in $\overline{S}$ that $\|\mathfrak{b}_i\|^{-1} \|\mathfrak{a}_i\| \le r$, and hence

$$r < \frac{\|\mathfrak{a}\|}{\|\mathfrak{b}\|} = \frac{\|\mathfrak{a}_1\| + \|\mathfrak{a}_2\|}{\|\mathfrak{b}_1\| + \|\mathfrak{b}_2\|} \le \max\left\{ \frac{\|\mathfrak{a}_1\|}{\|\mathfrak{b}_1\|}, \frac{\|\mathfrak{a}_2\|}{\|\mathfrak{b}_2\|} \right\} \le r,$$

which is however absurd. (For the second inequality in the last display, see, e.g., [6, Lemma 1.41].) □

It remains an open question whether, given a commutative monoid H and a finite set $A \subseteq H$, the A-elasticity of H is not only finite (as guaranteed by Theorem 2.3) but also rational (cf. Section 5).

3. Factorizations and [unions of] length sets

Let $\mathcal{H}=(H,\preceq)$ be a premon (or premonoid), i.e., a monoid H paired with a preorder $\preceq$ on its underlying set (note that, in general, we require no compatibility between the operation in H and the preorder $\preceq$). In particular, we write $|_H$ for the divisibility preorder on H (that is, $u|_H v$ if and only if $v \in H$ and $u \in HvH$) and $H^{\rm div}$ for the divisibility premon $(H,|_H)$ of H.

An element $u \in H$ is a $\preceq$-unit (of H) if it is $\preceq$-equivalent to the identity 1_H (viz., $u \preceq 1_H \preceq u$); otherwise, u is a $\preceq$-non-unit. A $\preceq$-non-unit $a \in H$ is then a $\preceq$-irreducible if $a \neq xy$ for all $\preceq$-non-units $x, y \in H$ with $x \prec a$ and $y \prec a$. We use $\mathcal{H}^\times$ for the set of $\preceq$-units, and $\mathscr{I}(\mathcal{H})$ for the set of $\preceq$-irreducibles of the monoid H. We may also refer to the elements of $\mathscr{I}(\mathcal{H})$ as the irreducibles of the premon $\mathcal{H}$. These notions were first considered in [20, Definition 3.6] and further studied in [9], [10], [21].

Following [10, Section 3], we denote by $\sqsubseteq_{\mathcal{H}}$ the shuffling preorder induced by $\preceq$, that is, the preorder on $\mathscr{F}(H)$ defined by $\mathfrak{a} \sqsubseteq_{\mathcal{H}} \mathfrak{b}$, for some H-words $\mathfrak{a}$ and $\mathfrak{b}$, if and only if there is an injective function $\sigma\colon [\![1, \|\mathfrak{a}\|]\!] \to [\![1, \|\mathfrak{b}\|]\!]$ such that $\mathfrak{a}[i] \preceq \mathfrak{b}[\sigma(i)] \preceq \mathfrak{a}[i]$ for every $i \in [\![1, \|\mathfrak{a}\|]\!]$. It turns out that, similar to the case of the preorder $\sqsubseteq_H$ introduced in Definition 2.1, $\sqsubseteq_{\mathcal{H}}$ is artinian.

Accordingly, we let a $\preceq$-factorization of an element $x \in H$ be any $\mathscr{I}(\mathcal{H})$-word $\mathfrak{a} \in \pi_H^{-1}(x)$, and we set $\mathcal{Z}_{\mathcal{H}}(x) := \pi_H^{-1}(x) \cap \mathscr{F}(\mathscr{I}(\mathcal{H}))$. A minimal $\preceq$-factorization of x is then a $\sqsubseteq_{\mathcal{H}}$-minimal word in $\mathcal{Z}_{\mathcal{H}}(x)$, namely, an $\mathscr{I}(\mathcal{H})$-word $\mathfrak{a} \in \pi_H^{-1}(x)$ with the property that there exists no $\mathscr{I}(\mathcal{H})$-word $\mathfrak{b} \in \pi_H^{-1}(x)$ with $\mathfrak{b} \sqsubset_{\mathcal{H}} \mathfrak{a}$. We denote the set of minimal $\preceq$-factorizations of x by $\mathcal{Z}_{\mathcal{H}}^{\sf m}(x)$. It follows from the artinianity of $\sqsubseteq_{\mathcal{H}}$ that x has a $\preceq$-factorization if and only if it has a minimal $\preceq$-factorization (see Remark 3.3(1) in [10]).

Definition 3.1. (1) Given a premon $\mathcal{H}=(H,\preceq)$, the $\preceq$-elasticity $\varrho_{\mathcal{H}}(x)$ (resp., the minimal $\preceq$-elasticity $\varrho_{\mathcal{H}}^{\sf m}(x)$) of an element $x \in H$ is the supremum of $\|\mathfrak{b}\|^{-1}\|\mathfrak{a}\|$ as $\mathfrak{a}$ and $\mathfrak{b}$ range over the non-empty $\preceq$-factorizations (resp., the non-empty minimal $\preceq$-factorizations) of x. (It is understood that $\sup \emptyset := 0$.)

(2) The elasticity $\varrho(\mathcal{H})$ (resp., the minimal elasticity $\varrho^{\sf m}(\mathcal{H})$) of the premon $\mathcal{H}$ is then the supremum of $\varrho_{\mathcal{H}}(x)$ (resp., of $\varrho_{\mathcal{H}}^{\sf m}(x)$) as x ranges over the $\preceq$-non-units of H. In particular, we let the elasticity (resp., the minimal elasticity) of the monoid H be the elasticity (resp., the minimal elasticity) of the divisibility premon $H^{\rm div}$ of H.

In the notation of Definition 3.1, we have $\varrho_{\mathcal{H}}(x)=\varrho^{\mathsf{m}}_{\mathcal{H}}(x)=0$ for every $x\in H$ whose set of $\preceq$-factorizations is empty. It is also worth noticing that $\varrho^{\mathsf{m}}_{\mathcal{H}}(1_H)=0$, since the only minimal $\preceq$-factorization of the identity 1_H is the empty word. Lastly, observe that the elasticity of the monoid H is nothing different from what we called the *classical elasticity* of H in Section 1.

Theorem 3.2. *The minimal elasticity of a commutative premon with finitely many irreducibles is finite.*

Proof. Let $\mathcal{H}=(H,\preceq)$ be a commutative premon with finitely many irreducibles, and denote by $\varrho^{\mathsf{m}}_{\mathrm{irr}}(H)$ the minimal $\mathscr{I}(\mathcal{H})$-elasticity of H. Since $\mathscr{I}(\mathcal{H})$ is a finite set (by hypothesis), we have from Theorem 2.3 that $\varrho^{\mathsf{m}}_{\mathrm{irr}}(H)<\infty$. This suffices to finish the proof, as it is immediate that a minimal $\preceq$-factorization of a $\preceq$-non-unit is a minimal $\mathscr{I}(\mathcal{H})$-word in the sense of Definition 2.2 and hence $\varrho^{\mathsf{m}}(\mathcal{H})\le\varrho^{\mathsf{m}}_{\mathrm{irr}}(H)$. □

As mentioned in Section 1, it was proved by Fan et al. in [11, Proposition 3.4(1)] that, if H is a commutative unit-cancellative monoid H such that the quotient $H/H^\times$ is finitely generated (i.e., H is finitely generated modulo units), then the classical elasticity of H is rational (and hence finite). In the next corollaries, we show how Theorem 3.2 can be used to recover the *finiteness* part of Fan et al.'s result.

Corollary 3.3. *If a commutative monoid has finitely many irreducibles modulo units, then its minimal elasticity is finite.*

Proof. Let H be a commutative monoid. An element $u\in H$ divides the identity 1_H if and only if u is a unit. Thus, an irreducible of H is a non-unit $a\in H$ that does not factor as a product of two non-units each of which is not divisible by a; and the irreducibles of the quotient $H/H^\times$ (where two elements of H belong to the same class if and only if they differ by a unit) are the cosets $bH^\times$ of the irreducibles $b\in H$. It follows that every minimal factorization of a non-unit $x\in H$ maps (through the canonical projection $H\to H/H^\times$) to a minimal factorization of $xH^\times$ in $H/H^\times$. So, the minimal elasticity of H is bounded above by the minimal elasticity of $H/H^\times$. Since $H/H^\times$ has finitely many irreducibles (by hypothesis), it is then clear from Theorem 3.2 that the minimal elasticity of H is finite. □

Corollary 3.4. *If a commutative unit-cancellative monoid is finitely generated modulo units, then its* (*classical*) *elasticity is finite.*

Proof. Let H be a commutative unit-cancellative monoid. Every irreducible of H (i.e., every $|_H$-irreducible) is then an atom [20, Corollary 4.4], and every minimal $|_H$-factorization is a $|_H$-factorization [4, Proposition 4.7(v)]. If, on the other hand, H is finitely generated up to units, then [10, Example 2.5(1) and Remark 4.9(2)] ensures that there is a finite set $A\subseteq H$ of irreducibles (and hence atoms) such that any atom (and hence any irreducible) belongs to $H^\times A H^\times$. So, putting it all together, the desired conclusion follows from Corollary 3.3. □

The following definitions generalize sets of lengths and related invariants from the classical theory of factorization, see [4, Sections 2.3 and 4.1] and [10, Definition 3.2].

Definition 3.5. (1) Given a premon $\mathcal{H}=(H,\preceq)$ and an element $x \in H$, we let

$$\mathsf{L}_{\mathcal{H}}(x) := \{\|\mathfrak{a}\|_H : \mathfrak{a} \in \mathcal{Z}_{\mathcal{H}}(x)\} \subseteq \mathbb{N} \quad \text{and} \quad \mathsf{L}_{\mathcal{H}}^{\mathsf{m}}(x) := \{\|\mathfrak{a}\|_H : \mathfrak{a} \in \mathcal{Z}_{\mathcal{H}}^{\mathsf{m}}(x)\} \subseteq \mathbb{N}$$

be, resp., the **length set** and the **minimal length set** of x (relative to $\mathcal{H}$). Accordingly, we refer to

$$\mathscr{L}(\mathcal{H}) := \{\mathsf{L}_{\mathcal{H}}(x) : x \in H \setminus \mathcal{H}^\times\} \quad \text{and} \quad \mathscr{L}^{\mathsf{m}}(\mathcal{H}) := \{\mathsf{L}_{\mathcal{H}}^{\mathsf{m}}(x) : x \in H \setminus \mathcal{H}^\times\},$$

resp., as the **system of length sets** and the **system of minimal length sets** of $\mathcal{H}$; and given $k \in \mathbb{N}$, we call

$$\mathscr{U}_k(\mathcal{H}) := \bigcup \{L \in \mathscr{L}(\mathcal{H}) : k \in L\} \quad \text{and} \quad \mathscr{U}_k^{\mathsf{m}}(\mathcal{H}) := \bigcup \{L \in \mathscr{L}^{\mathsf{m}}(\mathcal{H}) : k \in L\}$$

resp., the **union of length sets** and the **union of minimal length sets** containing k.

(2) In particular, we take the system of length sets (resp., the system of minimal length sets) of the monoid H to be the system of length sets (resp., of minimal length sets) of the divisibility premon H^{div} of H, and we write $\mathscr{L}(H)$ for $\mathscr{L}(H^{\text{div}})$ and $\mathscr{L}^{\mathsf{m}}(H)$ for $\mathscr{L}^{\mathsf{m}}(H^{\text{div}})$. The same goes with [minimal] length sets and unions of [minimal] length sets.

Note that, in the notation of Definition 3.5, the sets $\mathscr{U}_0(\mathcal{H})$ and $\mathscr{U}_0^{\mathsf{m}}(\mathcal{H})$ are both empty, because there is no $\preceq$-non-unit whose length set contains 0 (the only H-word of length 0 is the empty word, and the empty word is a $\preceq$-factorization of the identity 1_H).

Proposition 3.6. *The following hold for a premon* $\mathcal{H}=(H,\preceq)$:

(i) $\varrho(\mathcal{H}) \le 1$ *(resp.,* $\varrho^{\mathsf{m}}(\mathcal{H}) \le 1$*) if and only if* $|\mathsf{L}_{\mathcal{H}}(x)| \le 1$ *(resp.,* $|\mathsf{L}_{\mathcal{H}}^{\mathsf{m}}(x)| \le 1$*) for each* $x \in H \setminus \mathcal{H}^\times$.

(ii) *If* $\varrho(\mathcal{H})$ *(resp.,* $\varrho^{\mathsf{m}}(\mathcal{H})$*) is finite, then* $\mathscr{U}_k(\mathcal{H})$ *(resp.,* $\mathscr{U}_k^{\mathsf{m}}(\mathcal{H})$*) is finite for every* $k \in \mathbb{N}$.

Proof. We focus on length sets and leave the corresponding statements on *minimal* length sets to the reader (the proofs are essentially the same).

(i) If $|\mathsf{L}_{\mathcal{H}}(x)| \le 1$ for a $\preceq$-non-unit $x \in H$, then either $\varrho_{\mathcal{H}}(x)=0$ or $\varrho_{\mathcal{H}}(x)=1$. So, if $|\mathsf{L}_{\mathcal{H}}(x)| \le 1$ for every $\preceq$-non-unit $x \in H$, then $\varrho(\mathcal{H}) \le 1$.

As for the converse, assume $\varrho(\mathcal{H}) \le 1$ and suppose by way of contradiction that $|\mathsf{L}_\mathcal{H}(x_0)| \ge 2$ for some $\preceq$-non-unit $x_0 \in H$. There then exist $k_1, k_2 \in \mathsf{L}_\mathcal{H}(x_0)$ with $1 \le k_1 < k_2$. This implies $\varrho(\mathcal{H}) \ge \varrho_\mathcal{H}(x_0) \ge k_2/k_1 > 1$, which is absurd.

(ii) Let $\varrho(\mathcal{H})$ be finite and suppose for a contradiction that $|\mathscr{U}_k(\mathcal{H})| = \infty$ for some $k \in \mathbb{N}$. Since $\mathscr{U}_0(\mathcal{H})$ is empty, k is then a *positive* integer and there is a sequence $(\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2), \ldots$ of pairs of $\mathscr{I}(\mathcal{H})$-words such that, for every $i \in \mathbb{N}^+$, $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are $\preceq$-factorizations of the same $\preceq$-non-unit $x_i \in H$, with the additional property that $k = \|\mathfrak{a}_i\| \le \|\mathfrak{b}_i\| < \|\mathfrak{b}_{i+1}\|$. This, however, contradicts the finiteness of $\varrho(\mathcal{H})$. $\square$

Proposition 3.7. *Let $\mathcal{H} = (H, \preceq)$ be a premon. Then either there is a bound $M \in \mathbb{N}$ such that $\|\mathfrak{a}\| \le M$ for every minimal $\preceq$-factorization $\mathfrak{a}$, or for each $k \in \mathbb{N}$ there is a $\preceq$-minimal factorization of length k.*

Proof. Suppose that there is an integer $k \ge 1$ such that the set of minimal $\preceq$-factorizations of length k is empty. We claim that there is no minimal $\preceq$-factorization of length $n \ge k$, and we proceed to prove it by induction on n.

If $n = k$, the assertion is obvious. Otherwise, let $\mathfrak{a} = a_1 * \ldots * a_n$ be an $\mathscr{I}(\mathcal{H})$-word of length $n \ge k+1$ and assume inductively that there is no minimal $\preceq$-factorization of length $n-1$. In particular, this means that $\mathfrak{a}' := a_1 * \ldots * a_{n-1}$ is not a minimal $\preceq$-factorization, viz., there exists an $\mathscr{I}(\mathcal{H})$-word $\mathfrak{b}$ with $\mathfrak{b} \sqsubset_\mathcal{H} \mathfrak{a}'$ and $\pi_H(\mathfrak{b}) = \pi_H(\mathfrak{a}')$. It follows that $\mathfrak{b} * a_n \sqsubset_\mathcal{H} \mathfrak{a}$; and since $\pi_H(\mathfrak{b} * a_n) = \pi_H(\mathfrak{a})$, we conclude that $\mathfrak{a}$ is not a minimal $\preceq$-factorization either. $\square$

In [13, Proposition 2 and Corollary 1], Geroldinger and Lettl proved that the unions of length sets (of the divisibility premon) of a cancellative, commutative, finitely generated monoid H are all finite. This result is generalized to a non-commutative, non-cancellative context by Corollary 3.9 below.

Theorem 3.8. *Let $\mathcal{H} = (H, \preceq)$ be a premon and suppose there is a finite set $A \subseteq \mathscr{I}(\mathcal{H})$ such that every $\preceq$-irreducible is $\preceq$-equivalent to an element of A. Then the minimal length sets of $\mathcal{H}$ are all finite.*

Proof. Denote by $\sim_\mathcal{H}$ the relation of $\preceq$-equivalence on H and by $\approx_\mathcal{H}$ the relation of $\sqsubseteq_\mathcal{H}$-equivalence on $\mathscr{F}(H)$, and assume without loss of generality that the elements of A are pairwise $\preceq$-inequivalent. Next, let $\{q_1, \ldots, q_s\}$ be an enumeration of the elements of A, where $s := |A| \in \mathbb{N}^+$; and for each $i \in [\![1, s]\!]$, let $\mathsf{v}_i : \mathscr{F}(\mathscr{I}(\mathcal{H})) \to \mathbb{N}$ be the function that maps the empty word to 0 and a non-empty $\mathscr{I}(\mathcal{H})$-word $a_1 * \ldots * a_n$ of length n to the number of indices $j \in [\![1, n]\!]$ such that $a_j \sim_\mathcal{H} q_i$.

By assumption, we are given that, for every $a \in \mathscr{I}(\mathcal{H})$, there is a unique $i \in [\![1, s]\!]$ such that $a \sim_\mathcal{H} q_i$. Moreover, $q_i \sim_\mathcal{H} q_j$, for some $i, j \in [\![1, s]\!]$, if and only if $i = j$. It is then readily seen that

$$\|\mathfrak{a}\| = \mathsf{v}_1(\mathfrak{a}) + \ldots + \mathsf{v}_s(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in \mathscr{F}(\mathscr{I}(\mathcal{H})). \tag{3}$$

Now, fix $x \in H$. We need to show that the minimal length set $\mathsf{L}_{\mathcal{H}}^{\mathsf{m}}(x)$ of x is finite. If $\mathsf{L}_{\mathcal{H}}^{\mathsf{m}}(x)$ is empty, the conclusion is obvious. So, suppose that x has at least one minimal $\preceq$-factorization, and let $\le_s$ be the product order induced on $\mathbb{N}^{\times s}$ by the standard order on $\mathbb{N}$ and f be the function

$$\mathsf{Z}_{\mathcal{H}}^{\mathsf{m}}(x) \longrightarrow \mathbb{N}^{\times s} \colon \mathfrak{a} \longmapsto (\mathsf{v}_1(\mathfrak{a}), \dots, \mathsf{v}_s(\mathfrak{a})).$$

Since $f(\mathsf{Z}_{\mathcal{H}}^{\mathsf{m}}(x))$ is a non-empty subset of $\mathbb{N}^{\times s}$, we get from Dickson's lemma that the set $\mathcal{M}$ of $\le_s$-minimal elements of $f(\mathsf{Z}_{\mathcal{H}}^{\mathsf{m}}(x))$ is finite and non-empty.

We claim that $f(\mathfrak{a}) \in \mathcal{M}$ for every $\mathfrak{a} \in \mathsf{Z}_{\mathcal{H}}^{\mathsf{m}}(x)$; note that this will finish the proof, as it implies by Eq. (3) that

$$\begin{aligned} |\mathsf{L}_{\mathcal{H}}^{\mathsf{m}}(x)| &= |\{\mathsf{v}_1(\mathfrak{a}) + \dots + \mathsf{v}_s(\mathfrak{a}) \colon \mathfrak{a} \in \mathsf{Z}_{\mathcal{H}}^{\mathsf{m}}(x)\}| \\ &= |\{k_1 + \dots + k_s \colon (k_1, \dots, k_s) \in \mathcal{M}\}| \le |\mathcal{M}| < \infty. \end{aligned}$$

For the claim, suppose to the contrary that there exists $\mathfrak{a} \in \mathsf{Z}_{\mathcal{H}}^{\mathsf{m}}(x)$ with $f(\mathfrak{a}) \notin \mathcal{M}$. Then $f(\mathfrak{b}) <_s f(\mathfrak{a})$ for some $\mathfrak{b} \in f^{-1}(\mathcal{M})$, meaning that $\mathsf{v}_i(\mathfrak{b}) \le \mathsf{v}_i(\mathfrak{a})$ for each $i \in [\![1, s]\!]$ and at least one of these inequalities is strict. By the definition of $\sqsubseteq_{\mathcal{H}}$, it then follows that

$$\mathfrak{b} \approx_{\mathcal{H}} q_1^{\mathsf{v}_1(\mathfrak{b})} * \dots * q_s^{\mathsf{v}_s(\mathfrak{b})} \sqsubseteq_{\mathcal{H}} q_1^{\mathsf{v}_1(\mathfrak{a})} * \dots * q_s^{\mathsf{v}_s(\mathfrak{a})} \approx_{\mathcal{H}} \mathfrak{a},$$

which yields $\mathfrak{b} \sqsubset_{\mathcal{H}} \mathfrak{a}$, because two $\mathscr{I}(\mathcal{H})$-words are $\sqsubseteq_{\mathcal{H}}$-equivalent only if they have the same length and, by Eq. (3) and the above, $\|\mathfrak{b}\| < \|\mathfrak{a}\|$. We have thus reached a contradiction, because $\mathfrak{a}$ and $\mathfrak{b}$ are both minimal $\preceq$-factorizations of x. □

Corollary 3.9. *In a premon $\mathcal{H} = (H, \preceq)$ with finitely many $\preceq$-irreducibles, unions of minimal length sets are all finite.*

Proof. Assume for a contradiction that $|\mathscr{U}_k^{\mathsf{m}}(\mathcal{H})| = \infty$ for some $k \in \mathbb{N}$. There is then a sequence $(\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2), \dots$ of pairs of $\mathscr{I}(\mathcal{H})$-words such that, for each $i \in \mathbb{N}^+$, $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are minimal $\preceq$-factorizations of the same $\preceq$-non-unit $x_i \in H$, with the further property that $k = \|\mathfrak{a}_i\| \le \|\mathfrak{b}_i\| < \|\mathfrak{b}_{i+1}\|$. However, the set of $\mathscr{I}(\mathcal{H})$-words of length k is finite, since the basis $\mathscr{I}(\mathcal{H})$ is finite. So, we can find a (strictly) increasing sequence $i_1, i_2, \dots$ of positive integers with $\mathfrak{a}_{i_1} = \mathfrak{a}_{i_j}$ for all $j \in \mathbb{N}^+$, implying that the minimal length set of x_{i_1} is unbounded (as it contains the increasing sequence $\|\mathfrak{b}_{i_1}\|, \|\mathfrak{b}_{i_2}\|, \dots$) and hence contradicting Theorem 3.8. □

The next proposition shows that, under mild conditions on the premon $\mathcal{H}$ (see Remark 3.11) and up to a suitable modification of Definitions 3.1 and 3.5, the results of the present section can be extended to the case when the factors used in the factorization process are taken from an arbitrary set $A \subseteq \mathscr{I}(\mathcal{H})$.

Proposition 3.10. *Let $\mathcal{H}=(H,\preceq)$ be a premon in which the product of any two $\preceq$-non-units is a $\preceq$-non-unit, and let A be a set of $\preceq$-irreducibles. There then exists a preorder $\preceq_A$ on H such that each of the following conditions is satisfied*:

(i) *$u\in H$ is a $\preceq_A$-non-unit if and only if $u\in\langle A\rangle_H\setminus\mathcal{H}^\times$.*

(ii) *$a\in H$ is a $\preceq_A$-irreducible if and only if $a\in A$.*

(iii) *$a\preceq b\preceq a$, for some $a,b\in A$, if and only if $a\preceq_A b\preceq_A a$.*

Proof. Set $S:=\langle A\rangle_H\setminus\mathcal{H}^\times$ and let ϕ be the function $H\to\mathbb{N}$ that maps an element $x\in S$ to the smallest integer $n\geq 1$ such that $x\in A^n$ and an element in $H\setminus S$ to 0. We define a binary relation $\preceq_A$ on H by taking $x\preceq_A y$ if and only if one of the following conditions is satisfied:

$$(1)\ x,y\in H\setminus S,\quad (2)\ x,y\in S \text{ and } \phi(x)<\phi(y),\quad \text{or}$$
$$(3)\ x,y\in S,\ \phi(x)=\phi(y),\ \text{and } x\preceq y.$$

It is routine to check that $\preceq_A$ is a preorder, so we focus below on proving that $\preceq_A$ satisfies (i)–(iii).

(i) It is clear that $u\in H$ is a $\preceq_A$-unit if and only if $u\notin S$ (by the fact that $1_H\notin S$). Therefore, an element of H is a $\preceq_A$-non-unit if and only if it belongs to S.

(ii) To start with, fix $a\in A$ and assume for a contradiction that a is not a $\preceq_A$-irreducible. Then, by item (i), $a=bc$ for some $b,c\in S$ with $b\prec_A a$ and $c\prec_A a$. Since $\phi(a)=1$, it follows that $b,c\notin\mathcal{H}^\times$ and $\phi(b)=\phi(c)=1$. So, $b\prec_A a$ and $c\prec_A a$ translate to $b\prec a$ and $c\prec a$, which contradicts the $\preceq$-irreducibility of a and ultimately proves that every $a\in A$ is a $\preceq_A$-irreducible.

As for the converse, let $a\in H$ be a $\preceq_A$-irreducible and assume for a contradiction that $a\notin A$. Then, since a is a $\preceq_A$-non-unit, we infer from item (i) that there exist an integer $n\geq 2$ and elements $a_1,\ldots,a_n\in A$ such that $a=a_1\ldots a_n$. Thus $a=bc$, with $b:=a_1$ and $c:=a_2\ldots a_n$. But this contradicts that a is a $\preceq_A$-irreducible, since b and c are both in S by the hypothesis that a non-empty product of $\preceq$-non-units is still a $\preceq$-non-unit.

(iii) Let $a,b\in A$. It is clear that $\phi(a)=\phi(b)=1$. Since A is a set of $\preceq$-irreducibles (by hypothesis) and a $\preceq$-irreducible is a $\preceq$-non-unit, a and b are both elements of S. Therefore, the definition itself of the preorder $\preceq_A$ implies that $a\preceq_A b\preceq_A a$ if and only if $a\preceq b\preceq a$. □

Remark 3.11. (1) Let H be a monoid. If $x,y\in H$ and $xy\mid_H 1_H$ (i.e., xy is a $\mid_H$-unit), then also x and y divide 1_H. That is, the product of any two $\mid_H$-non-units is again a $\mid_H$-non-unit, and hence Proposition 3.10 applies to the divisibility premon $(H,\mid_H)$ of H.

(2) Following [10, Definition 2.3], let a weakly positive monoid be a premon $\mathcal{H}=(H,\preceq)$ such that $1_H\preceq x$ and $uxv\preceq x\preceq yxz$ for all $x,y,z\in H$ and $u,v\in\mathcal{H}^\times$. By [10, Remark 2.4(3)] , the product of two $\preceq$-non-units is then a $\preceq$-non-unit. That is, Proposition 3.10 applies also to weakly positive monoids.

We conclude with a couple of examples that show how the paradigm of *minimal* factorizations (as opposite to "ordinary factorizations") can mitigate the effects of blowup phenomena that would otherwise affect the invariants considered through this section, making them lose most of their significance.

Example 3.12. (1) Let H be the multiplicative monoid of the integers modulo p^n, where $p\in\mathbb{N}^+$ is a prime and n is an integer ≥ 2. By [10, Example 3.4], H is an atomic monoid and the atoms (resp., the units) of H are precisely the $|_H$-irreducibles (resp., the $|_H$-units). In addition, every *non-zero* non-unit of H has an essentially unique atomic factorization, with "essentially unique" meaning that any two atomic factorizations of the same element are equivalent with respect to the shuffling preorder induced by the divisibility preorder $|_H$. On the other hand, the residue class of 0 modulo p^n has an essentially unique minimal $|_H$-factorization (of length n), but atomic factorizations of any length $\geq n$. It follows that the (classical) elasticity of H is ∞; the minimal elasticity is 1; and for every $k\in\mathbb{N}^+$, we have

$$\mathscr{U}_k(H)=\begin{cases}\{k\} & \text{if } 1\leq k<n,\\ [\![k,\infty]\!] & \text{if } k\geq n\end{cases}\quad\text{and}\quad \mathscr{U}_k^{\mathsf{m}}(H)=\begin{cases}\{k\} & \text{if } 1\leq k\leq n,\\ \varnothing & \text{if } k>n.\end{cases}$$

(2) Fix an integer $n\geq 2$ and let $\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$ be the reduced power monoid of the additive group $\mathbb{Z}_n$ of integers modulo n, i.e., the (additively written) monoid obtained by endowing the subsets of $\mathbb{Z}_n$ containing the zero element $[0]_n\in\mathbb{Z}_n$ with the (binary) operation of setwise addition $(X,Y)\mapsto\{x+y\colon x\in X,\ y\in Y\}$. Here, we use $[k]_n$ for the residue class modulo n of an integer k.

By [20, Proposition 4.11(ii)], every $X\in\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$ factors as a product of irreducibles. On the other hand, it is obvious that $\{[0]_n,x\}$ is an irreducible of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$ for every non-zero $x\in\mathbb{Z}_n$. Consequently, we get from the *proof* of [4, Proposition 4.12(i) and Lemma 5.5] (reworked in terms of irreducibles) that every minimal factorization in $\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$ has length smaller than n and the interval $[\![2,n-1]\!]$ is a minimal length set of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$. Since $\mathbb{Z}_n$ is a proper idempotent of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$, it follows that, for every integer $k\geq 2$,

$$\mathscr{U}_k(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n))=\mathbb{N}_{\geq 2}\quad\text{and}\quad \mathscr{U}_k^{\mathsf{m}}(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n))=\begin{cases}[\![2,n-1]\!] & \text{if } 2\leq k<n,\\ \varnothing & \text{if } k\geq n.\end{cases}$$

Note also that $\mathscr{U}_1(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n))=\{1\}$ when n is odd, since every irreducible of H is then an atom [20, Theorem 4.12]; and $\mathscr{U}_1(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n))=\mathbb{N}^+$ when n is even, because in this latter case the set $\{[0]_n,[n/2]_n\}$ is an idempotent irreducible of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z}_n)$.

4. An example

Given a set X and a binary relation R on the free monoid $\mathscr{F}(X)$, we define $R^\sharp$ as the smallest monoid congruence on $\mathscr{F}(X)$ containing R. This means that $\mathfrak{u} \equiv \mathfrak{v} \bmod R^\sharp$ if and only if there are $\mathfrak{z}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_n \in \mathscr{F}(X)$ with $\mathfrak{z}_0 = \mathfrak{u}$ and $\mathfrak{z}_n = \mathfrak{v}$ such that, for each $i \in [\![0, n-1]\!]$, there exist X-words $\mathfrak{p}_i$, $\mathfrak{q}_i$, $\mathfrak{q}_i'$, and $\mathfrak{r}_i$ with the following properties:

(i) either $\mathfrak{q}_i = \mathfrak{q}_i'$, or $\mathfrak{q}_i R \mathfrak{q}_i'$, or $\mathfrak{q}_i' R \mathfrak{q}_i$; (ii) $\mathfrak{z}_i = \mathfrak{p}_i * \mathfrak{q}_i * \mathfrak{r}_i$ and $\mathfrak{z}_{i+1} = \mathfrak{p}_i * \mathfrak{q}_i' * \mathfrak{r}_i$.

We denote by $\mathrm{Mon}\langle X \mid R \rangle$ the monoid obtained by taking the quotient of $\mathscr{F}(X)$ by the congruence $R^\sharp$; we write $\mathrm{Mon}\langle X \mid R \rangle$ multiplicatively and call it a (monoid) presentation. We refer to the elements of X as the generators of the presentation, to each pair $(\mathfrak{q}, \mathfrak{q}') \in R$ as a defining relation, and to each X-word in a defining relation as a defining word. If there is no danger of confusion, we systematically identify an X-word $\mathfrak{z}$ with its equivalence class in $\mathrm{Mon}\langle X \mid R \rangle$.

The left graph of a presentation $\mathrm{Mon}\langle X \mid R \rangle$ is the undirected multigraph with vertex set X and an edge from y to z for each pair $(y * \mathfrak{u}, z * \mathfrak{v}) \in R$ with $y, z \in X$ and $\mathfrak{u}, \mathfrak{v} \in \mathscr{F}(X)$; this results in a loop when $y = z$, and in multiple (or parallel) edges between y and z if there are two or more defining relations of the form $(y * \mathfrak{u}, z * \mathfrak{v})$. The right graph is defined analogously, using the right-most (instead of left-most) letters of each word from a defining relation. The left and the right graphs of a presentation were first considered by Adian [1], whence we refer to them as the Adian graphs of $\mathrm{Mon}\langle X \mid R \rangle$.

On the other hand, a piece of a presentation $\mathrm{Mon}\langle X \mid R \rangle$ is a non-empty X-word $\mathfrak{u}$ for which there exist $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{q}' \in \mathscr{F}(X)$ with $\mathfrak{p} \neq \mathfrak{p}'$ or $\mathfrak{q} \neq \mathfrak{q}'$ such that $\mathfrak{v} := \mathfrak{p} * \mathfrak{u} * \mathfrak{q}$ and $\mathfrak{v}' := \mathfrak{p}' * \mathfrak{u} * \mathfrak{q}'$ are defining words; in particular, it is not required that $\mathfrak{v} \neq \mathfrak{v}'$ or $(\mathfrak{v}, \mathfrak{v}') \in R$. The notion was first conceived in the study of *group* presentations and later extended by Kashintsev to semigroups (see [18, Section 1]).

Following [18], we say that a monoid H is of class K_p^q, for some $p, q \in \mathbb{N}^+$, if H is isomorphic to a monoid presentation $\mathrm{Mon}\langle X \mid R \rangle$ with finitely many generators such that (i) no defining word can be expressed as the concatenation in $\mathscr{F}(X)$ of less than p pieces and (ii) the Adian graphs of the presentation have both girth $\geq q$ (we recall that the girth of an undirected multigraph G is the shortest length of a cycle in G, with the understanding that the girth of a cycle-free multigraph is ∞). Our interest in these definitions is linked to the following result, first proved by Guba in [15, Theorem 1] and hence referred to as *Guba's* (*embedding*) *theorem* (note that, in [15], there is a typo in the very definition of a piece, as discussed on MathOverflow at https://mathoverflow.net/questions/353340/).

Theorem 4.1. *Every monoid of class K_3^2 embeds into a group* (*and hence is cancellative*).

Guba's theorem will come in handy in the next example, which shows that Theorem 2.3 does not carry over to the non-commutative setting in any obvious way.

Example 4.2. Let H be the presentation $\mathrm{Mon}\langle A \mid R\rangle$, where A is the 3-element set $\{a, b, c\}$ and R is the set $\{(\mathfrak{s}_n, \mathfrak{t}_n) : n = 2, 3, \ldots\} \subseteq \mathscr{F}(A) \times \mathscr{F}(A)$, with

$$\mathfrak{s}_n := c * a^{*n} * b^{*2^n} * a^{*n} * c \quad \text{and} \quad \mathfrak{t}_n := a * c^{*n} * b^{*n} * c^{*n} * a.$$

It is routine to check that H is a 3-generated monoid with trivial group of units whose $|_H$-irreducibles are $\underline{a}$, $\underline{b}$, and $\underline{c}$, where we write $\underline{\mathfrak{u}}$ for the $R^\sharp$-congruence class (in H) of an A-word $\mathfrak{u}$. Each of $\underline{a}$, $\underline{b}$, and $\underline{c}$ is on the other hand an atom, because the defining words in R have all length ≥ 2. Then, H is an atomic monoid and every $|_H$-factorization is an atomic factorization and vice versa. In addition, it is immediate that, for each $k \in \mathbb{N}^+$, none of the A-words

$$(4) \qquad c * a^{*k} * b, \quad b * a^{*k} * c, \quad a * c^{*k} * b, \quad \text{or} \quad b * c^{*k} * a$$

is a piece of $\mathrm{Mon}\langle A \mid R\rangle$; if, e.g., a defining word factors as $\mathfrak{p} * (a * c^{*k} * b) * \mathfrak{q}$ for some A-words $\mathfrak{p}$ and $\mathfrak{q}$, then necessarily $\mathfrak{p} = \varepsilon_A$ and $\mathfrak{q} = b^{*(k-1)} * c^{*k} * a$ (and the other cases are similar). Since a non-empty A-word $\mathfrak{s} := x_1 * \ldots * x_l$ is a piece of $\mathrm{Mon}\langle A \mid R\rangle$ only if so is any subword of the form $x_i * \ldots * x_j$ with $1 \leq i \leq j \leq l$, it follows that the support $\{x_1, \ldots, x_l\}$ of $\mathfrak{s}$ is a *proper* subset of A. As a matter of fact, a piece is in the first place a subword of a defining word, so that the shortest pieces with support A (if there were any) would be those listed in Eq. (4).

It follows that no defining word of $\mathrm{Mon}\langle A \mid R\rangle$ is the concatenation in $\mathscr{F}(A)$ of less than three pieces (note that each defining word is palindromic and its support is A). Consequently, H is a monoid of class K_3^2, which implies, by Guba's theorem, that H is cancellative. So, it remains to see that, for each $n \in \mathbb{N}^+$, there are minimal atomic factorizations $\mathfrak{a}_n$ and $\mathfrak{b}_n$ of an element $x_n \in H$ such that $\|\mathfrak{a}_n\| \geq n\, \|\mathfrak{b}_n\|$.

For, fix an integer $n \geq 2$. The H-words $\mathfrak{a}_n := \underline{c} * \underline{a}^{*n} * \underline{b}^{*2^n} * \underline{a}^{*n} * \underline{c}$ and $\mathfrak{b}_n := \underline{a} * \underline{c}^{*n} * \underline{b}^{*n} * \underline{c}^{*n} * \underline{a}$ are non-empty atomic factorizations of the same element, with the further property that $\|\mathfrak{b}_n\|^{-1} \|\mathfrak{a}_n\| \to \infty$ as $n \to \infty$. Therefore, it suffices to check that $\mathfrak{a}_n$ and $\mathfrak{b}_n$ are minimal atomic factorizations.

Assume to the contrary that $\mathfrak{a}_n$ is not a minimal atomic factorization (the minimality of $\mathfrak{b}_n$ can be proved in a similar fashion). Then $\mathfrak{s}_n \equiv \mathfrak{u} \bmod R^\sharp$ for some permutation $\mathfrak{u}$ of a *proper* subword of $\mathfrak{s}_n$; that is, there exist a smallest $k \in \mathbb{N}^+$ and A-words $\mathfrak{z}_0, \mathfrak{z}_1, \ldots, \mathfrak{z}_k$ with $\mathfrak{z}_0 = \mathfrak{s}_n$ and $\mathfrak{z}_k = \mathfrak{u}$ such that, for each $i \in [\![0, k-1]\!]$, there are an integer $m_i \geq 2$ and $\mathfrak{p}_i, \mathfrak{r}_i \in \mathscr{F}(A)$ with

(i) $\mathfrak{z}_i = \mathfrak{p}_i * \mathfrak{s}_{m_i} * \mathfrak{r}_i$ and $\mathfrak{z}_{i+1} = \mathfrak{p}_i * \mathfrak{t}_{m_i} * \mathfrak{r}_i$, or

(ii) $\mathfrak{z}_i = \mathfrak{p}_i * \mathfrak{t}_{m_i} * \mathfrak{r}_i$ and $\mathfrak{z}_{i+1} = \mathfrak{p}_i * \mathfrak{s}_{m_i} * \mathfrak{r}_i$.

However, it is readily seen (by induction on i) that this can only happen if $\mathfrak{z}_i \in \{\mathfrak{s}_n, \mathfrak{t}_n\}$ for every $i \in [\![0, k]\!]$, because, for an integer $m \ge 2$ with $m \ne n$, neither $\mathfrak{s}_m$ nor $\mathfrak{t}_m$ is a divisor of $\mathfrak{s}_n$ (resp., of $\mathfrak{t}_n$) in $\mathscr{F}(A)$. It follows that $\mathfrak{u} = \mathfrak{t}_n$, which is impossible since neither $\mathfrak{s}_n$ nor $\mathfrak{t}_n$ is a proper subword of $\mathfrak{s}_n$.

5. Prospects for future research

We know from Corollary 3.3 that, if a commutative monoid H has finitely many irreducibles modulo units, then its minimal elasticity $\rho^{\mathrm{m}}(H)$ is finite. In view of [11, Proposition 3.4(1)], it is therefore natural to ask whether $\rho^{\mathrm{m}}(H)$ is, in fact, a rational number (cf. the last lines of Section 2).

On the other hand, we gather from Section 4 that the minimal elasticity of an atomic, cancellative, finitely generated, reduced monoid need not be finite. Is this still the case with an acyclic, finitely generated, and reduced monoid? Here, we say a monoid H is acyclic if $uxv \ne x$ for all $u, v, x \in H$ such that u or v is a non-unit. Acyclic monoids were introduced in [20, Definition 4.2] and further studied in [10]; from an arithmetical point of view, they provide an interesting alternative to cancellativity in the non-commutative setting. For instance, we get from [20, Corollary 4.4] that, in an acyclic monoid, irreducibles and atoms are the same thing, which generalizes an observation made in the first lines of Section 1. It is also worth noting that every acyclic monoid is unit-cancellative; the two notions (of acyclicity and unit-cancellativity) coincide in the commutative setting, but not in general [20, Example 4.8].

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