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Decay of extremals of Morrey's inequality

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Abstract. We study the decay (at infinity) of extremals of Morrey's inequality in $\mathbb{R}^n$. These are functions satisfying

$$\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}}=C(p,n)\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where $p>n$ and $C(p,n)$ is the optimal constant in Morrey's inequality. We prove that if $n\geq 2$ then any extremal has a power decay of order β for any

$$\beta<-\frac{1}{3}+\frac{2}{3(p-1)}+\sqrt{\left(-\frac{1}{3}+\frac{2}{3(p-1)}\right)^2+\frac{1}{3}}.$$

1. Introduction

Morrey's classical inequality in $\mathbb{R}^n$ states that for $p>n$, there is a constant $C=C(p,n)$ such that

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)}=\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}}\leq C(p,n)\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{\frac{1}{p}}, \tag{1.1}$$

for all functions whose first order partial derivatives belong to $L^p(\mathbb{R}^n)$. In a series of papers (cf. [7]–[9]), Hynd and Seuffert study this inequality and prove that there is a smallest constant $C>0$ such that (1.1) holds and that there are extremals of this inequality. An extremal is a function for which equality is attained in (1.1). They also prove that up to translation, rotation, dilatation and multiplication by a constant, any extremal function u satisfies

1. $-\Delta_p u=c(\delta_{e_n}-\delta_{-e_n})$ in $\mathbb{R}^n$ for a constant $c>0$,

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2. $|u|\leq 1$, $u(e_n)=1$, $u(-e_n)=-1$,
3. u is antisymmetric with respect to the x_n-variable,
4. u is positive in $\mathbb{R}^n \cap \{x_n>0\}$.

See Theorem 2.4 and Propositions 3.1, 3.4 and 3.5 in [9]. Here $\Delta_p u := \nabla\cdot(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. In addition to this, they study the behavior at infinity of extremals in dimensions $n\geq 2$ and prove that there is $\beta>0$ and $C>0$ such that

$$\sup_{|x|\geq R} |u| \leq CR^{-\beta}, \quad \text{for all } R. \tag{1.2}$$

See Corollary 4.7 in [7]. However, no estimate of β is given.

The main objective of this paper is to provide an explicit exponent β. More precisely, we prove the following theorem.

Theorem 1.1. *Suppose $p>n\geq 2$, that u is an extremal of (1.1) satisfying properties (1)–(4) above and*

$$\beta < -\frac{1}{3}+\frac{2}{3(p-1)}+\sqrt{\left(-\frac{1}{3}+\frac{2}{3(p-1)}\right)^2+\frac{1}{3}}.$$

Then there is $C_1=C_1(\beta,p,n)$ such that

$$|u(x)|\leq C_1|x|^{-\beta},$$

for all $|x|\geq 1$.

As a corollary, we obtain the corresponding decay for the gradient.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there is $C_2=C_2(\beta,p,n)$ such that*

$$|\nabla u(x)|\leq C_2|x|^{-\beta-1},$$

for all $|x|\geq 2$.

Remark 1.3. A couple of remarks:

1. By (2) above the conclusion of Theorem 1.1 is valid also for $|x|\leq 1$. However, the same is not true for Corollary 1.2. Indeed, by [6, Proposition 2.8] $|\nabla u(x)|$ becomes unbounded as $x\to\pm e_n$.

2. In dimension one, the extremal satisfying (1)–(4) is explicitly given by

$$u(x)=\begin{cases} -1 & \text{for } x\leq -1,\\ x & \text{for } x\in(-1,1),\\ 1 & \text{for } x\geq 1.\end{cases}$$

Therefore, the assumption $n\geq 2$ is necessary in Theorem 1.1. However, the bound in Corollary 1.2 is trivially true when $n=1$.

Although it is of intrinsic interest to further understand the extremal functions of Morrey's inequality, our motivation for the results in this short note stem from a particular application. Namely, in [5] we address the existence of minimizers in a certain variational problem and an estimate for the decay of Morrey extremals and their gradients entered as a key technical ingredient.

1.1. Known results

The asymptotic behavior at infinity for solutions of PDEs has been studied before. See for instance [12] where it is proved that bounded p-harmonic functions in exterior domains has a limit at infinity. Related results can also be found in [10], [3] and [4].

1.2. Plan of the paper

In Section 2, we discuss notation, definitions and certain prerequisites for this paper. This is followed by Section 3, where Aronsson's p-harmonic functions obtained through separation of variables are discussed. In Section 4, we study the singularities of functions that are p-harmonic in punctured domains. Finally, we prove our main results in Section 5.

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2. Preliminaries

Throughout the paper we work in $\mathbb{R}^n$ with $p>n\geq 2$ and we will denote the exponent appearing in Theorem 1.1 by

$$\beta_p := -\frac{1}{3}+\frac{2}{3(p-1)}+\sqrt{\left(-\frac{1}{3}+\frac{2}{3(p-1)}\right)^2+\frac{1}{3}}.$$

We will need a few results regarding p-harmonic functions. The following assertion is contained in Theorem 1.1 and Remark 1.6 in [10].

Theorem 2.1. *Suppose that* $|u|\le 1$ *in* $B_1\setminus\{0\}$, $u\in W^{1,p}_{\mathrm{loc}}(B_1\setminus\{0\})$ *and that*

$$-\Delta_p u=0 \ \text{ in } B_1\setminus\{0\}.$$

Then $u\in W^{1,p}_{\mathrm{loc}}(B_1)$ *and there is* γ *such that*

$$-\Delta_p u=|\gamma|^{p-2}\gamma\delta_0 \quad \text{in } B_1.$$

The next result is Corollary 2.4 in [7].

Proposition 2.2. *Suppose* u *is bounded and satisfies*

$$-\Delta_p u=c\delta_{x_0}$$

in $\mathbb{R}^n$ *for some point* x_0 *and some constant* c*. Then* u *is necessarily constant and* $c=0$.

3. Solutions in the plane by separation of variables

In [1], Aronsson studies p-harmonic functions for $p>2$ in sectors of $\mathbb{R}^2$ which have the form $u(r,\phi)=r^{-\varkappa}f(\phi)$ for $\varkappa>0$([1]) and where (r,ϕ) are polar coordinates. In Lemma 1 case α) in [1], it is proved that u is p-harmonic in the cone $r>0$, $\phi\in I$ if and only if

$$g(\phi):=(f'(\phi))^2+\left(1+\frac{1}{a\varkappa}\right)\varkappa^2(f(\phi))^2>0, \quad a=\frac{p-1}{p-2} \tag{3.1}$$

and there is a constant $C>0$ such that

$$[(f'(\phi))^2+\varkappa^2(f(\phi))^2]^{-\varkappa}=C^2|g(\phi)|^{-\varkappa-1}. \tag{3.2}$$

Recall that $p>n=2$ so $a>0$. On p. 145 in [1], the following semi-explicit formula for a solution is given:

$$\phi=\theta-a(1+\varkappa)\int_0^\theta \frac{1}{\cos^2\theta'+a\varkappa}d\theta', \quad f=\left(1+\frac{\cos^2\theta}{a\varkappa}\right)^{\frac{-\varkappa-1}{2}}\cos\theta.$$

In order to see that this implies (3.1) and (3.2), it is sufficient to compute $f'(\phi)$ and find that

$$f'(\phi)=\varkappa\left(1+\frac{\cos^2\theta}{a\varkappa}\right)^{\frac{-\varkappa-1}{2}}\sin\theta$$

([1]) Note that $\varkappa$ here corresponds to $-k$ in Aronsson's notation and therefore the resulting equations differ accordingly. Aronsson considers k of arbitrary sign but here only singular solutions will be important.

so that

$$(f'(\phi))^2+\varkappa^2(f(\phi))^2=\varkappa^2\left(1+\frac{\cos^2\theta}{a\varkappa}\right)^{-\varkappa-1}.$$

It follows that

$$g(\phi)=\varkappa^2\left(1+\frac{\cos^2\theta}{a\varkappa}\right)^{-\varkappa}>0$$

and that (3.2) holds with $C=\varkappa^2$.

Upon integration, the relation between ϕ and θ simplifies to

$$\phi=\theta-\left(\frac{1}{\varkappa}+1\right)\mu\arctan\left(\mu\tan\theta\right), \quad \mu=\frac{\sqrt{a\varkappa}}{\sqrt{a\varkappa+1}},$$

for $\theta\in(-\pi/2,\pi/2)$. This implies that the range of possible ϕ is $I=(\phi(\pi/2),\phi(-\pi/2))$, which is an interval of length

$$\tilde{L}=\pi\left(\mu\left(1+\frac{1}{\varkappa}\right)-1\right).$$

We also note that f is positive when $\cos\theta$ is positive which is exactly on the interval I. Hence, this defines a positive solution of the p-Laplace equation in a cone with opening $\tilde{L}$, which is zero on the boundary rays of the cone.

Since we will be interested in solutions in cones with opening π or larger, we let $\tilde{L}=\pi L$ and obtain

$$L=\frac{\sqrt{a\varkappa}}{\sqrt{a\varkappa+1}}\left(1+\frac{1}{\varkappa}\right)-1, \tag{3.3}$$

which implies

$$(L+1)^2=\frac{(\varkappa+1)^2}{\varkappa^2+\frac{\varkappa}{a}}. \tag{3.4}$$

Upon recalling that $a=(p-1)/(p-2)$, it is clear that L is strictly decreasing in p. It is not hard to see that if $L=1$, (3.4) gives $\varkappa=\beta_p$. This corresponds to a half plane solution. Here we observe that β_p decreases to its limit $1/3$ as $p\to\infty$, hence $\beta_p>1/3$ for all $p>2$. In addition, differentiating (3.3) gives

$$\frac{dL}{d\varkappa}=\frac{\sqrt{a}(\varkappa(1-2a)-1)}{2\varkappa^{\frac{3}{2}}(a\varkappa+1)^{\frac{3}{2}}}<0.$$

Therefore, L is strictly decreasing in $\varkappa$. Hence, for $\delta>0$ there is a one to one correspondence between $L=1+\delta$ and the corresponding power $\varkappa(\delta)<\beta_p$. Moreover, $\varkappa(\delta)\to\beta_p$ as $\delta\to0$. Upon renaming δ, we may summarize our conclusions as:

For any power $\beta<\beta_p$, there is a $\delta>0$ and a p-harmonic function $u=r^{-\beta}f(\phi)$ in the cone $r>0$, $\phi\in(-\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta)$ which is positive for $\phi\in(-\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta)$ and satisfies $u(r,-\frac{\pi}{2}-\delta)=u(r,\frac{\pi}{2}+\delta)=0$.

4. Slow decay implies boundedness

In this section, we prove that certain global solutions of the p-Laplace equation cannot blow up at the origin if they blow up too slowly.

Proposition 4.1. *Suppose* $u \geq 0$, $\Delta_p u = 0$ *in* $\mathbb{R}^n \cap \{x_n > 0\}$, $u(x) = 0$ *for* $x_n = 0$ *except possibly at the origin,* $|u(x)| \leq 1$ *for* $|x| \geq 1$ *and that*

$$|u(x)| \leq |x|^{-\beta},$$

for $|x| \leq 1$ *where* $\beta < \beta_p$. *Then* $|u(x)| \leq 1$ *for all* $x \neq 0$.

Proof. Take $\tau > 0$ so that $\beta + \tau < \beta_p$. From the discussion in Section 3, it follows that there is a solution w of the form

$$w(r, \phi) = r^{-\beta-\tau} f(\phi)$$

valid in the cone $r > 0, \phi \in (-\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$ for some $\delta > 0$. Here the polar coordinates are chosen in the $x_{n-1}x_n$-plane so that $\phi = \pm\pi/2$ corresponds to $x_n = 0$ and $\phi = 0$ corresponds to the positive x_n-axis. Moreover, $f(\phi) > 0$ for $\phi \in (-\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$. This means in particular that that $r^{\beta+\tau} w(r, \phi) > c_f > 0$ in $B_1 \cap \{x_n \geq 0\} \setminus \{0\}$. If we are in dimension three or more, we extend this solution trivially to be a solution in $\mathbb{R}^n \cap \{x_n \geq 0\}$. Thus,

$$w(x_1, ..., x_n) = (x_{n-1}^2 + x_n^2)^{-\beta/2-\tau/2} f(\phi) \geq |x|^{-\beta-\tau} f(\phi).$$

The idea is to use w as a base for a barrier that will force u to be bounded.

To see this, let $v = 1 + \varepsilon w$ for $\varepsilon > 0$. We now wish to compare u with v in $B_1 \setminus B_\rho \cap \{x_n \geq 0\}$. Take $\rho = \rho(\varepsilon) \in (0, 1)$ such that $\rho^{-\beta} \leq \varepsilon c_f \rho^{-\beta-\tau}$. On $\partial B_1 \cap \{x_n > 0\}$ we have $v \geq 1 \geq u$, on $\partial B_\rho \cap \{x_n \geq 0\}$ we have

$$|u(x)| \leq |\rho|^{-\beta} \leq \varepsilon c_f \rho^{-\beta-\tau} \leq \varepsilon w \leq v$$

and on $B_1 \setminus B_\rho \cap \{x_n = 0\}$ we have $v \geq 0 = u$. The comparison principle implies $u \leq v$ in $B_1 \setminus B_\rho$. Moreover, since $\rho^{-\beta} \leq \varepsilon c_f \rho^{-\beta-\tau}$, we trivially have $u \leq |x|^{-\beta} \leq \varepsilon c_f |x|^{-\beta-\tau} \leq v$ in $B_\rho \setminus \{0\}$. We conclude that $u \leq v$ in $B_1 \setminus \{0\}$. This inequality does not depend on ε so by letting $\varepsilon \to 0$, we obtain $u \leq 1$ in $B_1 \setminus \{0\}$ and the proof is complete. □

5. Proof of the main theorem

Proposition 5.1. *Assume* $\Delta_p u = 0$ *in* $\mathbb{R}^n \setminus \overline{B}_1$, $|u(x)| \leq 1$ *for* $|x| \geq 1$, $u \geq 0$ *for* $x_n \geq 0$ *and that* u *is antisymmetric with respect to the* x_n*-variable. Then for each* $\beta < \beta_p$, *there is a constant* $C = C(n, p, \beta)$ *such that*

$$\sup_{|x| \geq r} |u(x)| \leq C r^{-\beta}, \quad r \geq 1.$$

We prove the proposition by proving the lemma below.

Lemma 5.2. *Assume the hypotheses of Proposition 5.1. Then for each* $\beta<\beta_p$, *there is a constant* $C=C(n,p,\beta)>0$ *such that for all* $r\geq 1$ *at least one of the following properties hold*:

1. $S_r:=\sup_{|x|\geq r}|u(x)|\leq Cr^{-\beta}$,
2. *There is a* $k\geq 1$ *such that* $2^{-k}r\geq 1$ *and* $S_r\leq 2^{-k\beta}S_{2^{-k}r}$.

We first explain how Proposition 5.1 follows from this lemma.

Proof of Proposition 5.1. If alternative (1) of Lemma 5.2 holds for all $r\geq 1$, then we are done. If not, we pick an r for which (1) fails so that, by alternative (2),

$$S_r\leq 2^{-k_1\beta}S_{2^{-k_1}r},$$

for some integer k_1 with $2^{-k_1}r\geq 1$. If (1) holds for $2^{-k_1}r$, then

$$S_r\leq 2^{-k_1\beta}S_{2^{-k_1}r}\leq 2^{-k_1\beta}C(2^{-k_1}r)^{-\beta}=Cr^{-\beta}$$

and again we are done. If not, we continue with

$$S_{2^{-k_1}r}\leq 2^{-k_2\beta}S_{2^{-k_2}2^{-k_1}r},$$

where $2^{-k_2}2^{-k_1}r\geq 1$. Iterating this as long as alternative (1) fails, we obtain

$$S_r\leq 2^{-k_n\beta}...2^{-k_1\beta}S_{2^{-k_n}...2^{-k_1}r}=2^{-(k_1+...+k_n)\beta}S_{2^{-k_1-...-k_n}r},$$

where $2^{-k_1-...-k_n}r\geq 1$. Since every $k_j\geq 1$, the procedure must stop after a finite number of steps (depending r), say after n steps. Then alternative (1) holds for the radius $2^{-k_1-...-k_n}r$ and so, finally,

$$S_r\leq 2^{-(k_1+...+k_n)\beta}S_{2^{-k_1-...-k_n}r}\leq 2^{-(k_1+...+k_n)\beta}C(2^{-k_1-...-k_n}r)^{-\beta}\leq Cr^{-\beta}.$$

This proves the claim. □

Proof of Lemma 5.2. We assume towards a contradiction that the statement is false. Then, for each $j=1,2,3,...$, we may find $r_j\geq 1$ such that

1. $S_{r_j}\geq jr_j^{-\beta}$,
2. $S_{r_j}\geq 2^{-k\beta}S_{2^{-k}r_j}$, for all $k\geq 1$ such that $2^{-k}r_j\geq 1$.

Note that the point 1) above forces $r_j\to\infty$, since u is bounded. Define

$$v_j(x)=\frac{u(r_jx)}{S_{r_j}}.$$

Setting $S_r(v_j):=\sup_{|x|\geq r}|v_j(x)|$, it follows that v_j satisfies

(a) $S_1(v_j)=1$,

(b) $S_{2^{-k}}(v_j)\leq 2^{k\beta}$, for all k such that $2^{-k}r_j\geq 1$,
(c) $\Delta_p v_j=0$ in $\mathbb{R}^n\setminus\overline{B}_{\frac{1}{r_j}}$.
Using local estimates for the p-Laplace equation, we may therefore extract a subsequence converging locally uniformly in $\mathbb{R}^n\setminus\{0\}$ to a function v. We also note that by Corollary 4.2 in [7], we know that

$$S_1(v_j)=\sup_{|x|\geq 1} v_j=\sup_{|x|=1} v_j.$$

Therefore, the local uniform convergence assures that v satisfies
(a') $\sup_{|x|=1} v=1$,
(b') $S_{2^{-k}}(v)\leq 2^{k\beta}$, for all $k\geq 1$,
(c') $\Delta_p v=0$ in $\mathbb{R}^n\setminus\{0\}$.
We also note that since each v_j is antisymmetric with respect to the x_n-variable and non-negative in $\{x_n\geq 0\}$, so is the limit v. By Proposition 4.1, $|v(x)|\leq 1$ for all $x\neq 0$. We can then apply Theorem 2.1 combined with Proposition 2.2 and conclude that v has to be identically zero. This contradicts (a') above. □

The proof of Theorem 1.1 is now immediate.

Proof of Theorem 1.1. As mentioned in the introduction, up to translation, rotation and dilatation, any extremal function u is antisymmetric with respect to the x_n-variable, positive in $\mathbb{R}^n\cap\{x_n>0\}$, p-harmonic outside $\overline{B}_1$ and satisfies $|u|\leq 1$. Therefore, Proposition 5.1 applies and the proof is complete. □

From this and interior estimates for the p-Laplace equation, Corollary 1.2 follows.

Proof of Corollary 1.2. Take x such that $|x|=R\geq 2$ and $\beta<\beta_p$. Then Theorem 1.1 implies

$$\sup_{B_{R/4}(x)} |u|\leq CR^{-\beta}.$$

Since $R\geq 2$, $B_{R/4}(x)\cap B_1=\varnothing$ so that u is p-harmonic in $B_{R/4}(x)$. By interior gradient estimates (cf. [2], [11] or [13])

$$\sup_{B_{R/8}(x)} |\nabla u|\leq CR^{-1}\sup_{B_{R/4}(x)} |u(x)|\leq CR^{-\beta-1}$$

and in particular

$$|\nabla u(x)|\leq CR^{-\beta-1},$$

which completes the proof of Corollary 1.2. □

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