A formula to evaluate type A webs and link polynomials

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Abstract. We give a closed formula to evaluate exterior webs (also called MOY webs) and the associated Reshetikhin–Turaev link polynomials.

Contents

1. Introduction

The Reshetikhin–Turaev invariants of links [RT91] provide one of the most important family of link invariants in quantum topology and its ramifications.

In this note we focus on the subfamily of these invariants given by coloring the strands of links with exterior powers of the vector representation of quantum $\mathfrak{sl}_n$ or quantum $\mathfrak{gl}_n$. We show that this family of polynomials can be computed by a closed formula that takes as input only combinatorial data associated to a fixed colored link diagram and root theoretic data associated to the type A Dynkin diagram. The formula works for all links, all exterior colorings, and all ranks n.

Let us stress that, by its very nature, the closed formula we give is a Weyl-character-type formula: On the one hand, it is general, completely algorithmic and might reveal abstract properties of the family of exterior colored link polynomials. But on the other hand, the formula does not necessarily give an efficient way to compute these invariants, at least not without massaging it a bit.

The diagrammatic incarnation of this family of link invariants is given by (exterior $\mathfrak{sl}_n$ or $\mathfrak{gl}_n$) webs as shown in many works, see e.g. [RTW32], [Kup96], [MOY98] or [CKM14]. In fact, the aforementioned formula is an application of a closed evaluation formula for webs that we will also state and prove. To use this formula one does not need to know any combinatorics of webs or tableaux; one, in fact, does not even need to know webs. Moreover, another application of this formula is an easy to check criterion for webs to represent dual canonical basis elements.

The exposition in this note, including the statement of our main results, is mostly self-contained and explicit. For example, we included Python based code, cf. Remark 1.1 below, that can do computations using, for example, the online calculator of SageMath. The main proofs, that are not necessary to understand the rest of the paper, however use techniques from categorification as we elaborate on now.

In breakthrough work Hu–Shi [HS21] found a closed formula for the dimension of the cyclotomic KLR algebra of any symmetrizable Kac–Moody type by using combinatorics of Fock spaces. Even more remarkable, their formula is a Weyl-character-type formula that can be computed without any prior knowledge of cyclotomic KLR algebras. A consequence of categorical skew Howe duality is that cyclotomic KLR algebras of type A and webs (web algebras to be precise – skew Howe duality itself it not enough to make the connection) are essentially the same object, see e.g. the pioneering works [BS11a], [BS10] and [BS11b] for the $\mathfrak{sl}_2$ version of this result. See also [MPT14] and [Tub14] for the $\mathfrak{sl}_3$ version, and [Mac14] and [Tub20] for the general version of this relationship.

In this paper we put both together and obtain the aforementioned closed formula for web evaluations and the computation of link polynomials.

Remark 1.1. The main formula in (4.3) is easy to compute via a machine, and so is its adjustment to the case of the link polynomials from (5.5). The reader may find Python code for SageMath that can do the calculations here [MT22]. (The second link in [MT22] has "klrdim-webevaluation.py" which is the version used for this paper. The two main examples Example 4.5 and Example 5.7 are included in that file. This code is also commented at the end of the source file.)

Remark 1.2. To not distract the reader's attention, we postpone all proofs to Section 6. This has the advantage that we can formulate the main formulas without any reference to categorical skew Howe duality or KLR algebras.

2. A reminder on webs

Fix $n \in \mathbb{Z}_{\geq 1}$. We will now recall the description of (exterior $\mathfrak{gl}_n$) webs from [CKM14] using the exposition from [LTV22] and [LT21]. Background on monoidal categories given by generators-relations can be found in e.g. [TV17], and background on monoidal categories related to our setting can be found in e.g. [EGNO15].

We start with our reading and other conventions:

Notation 2.1. All categories that we use are strict, and we read diagrams from bottom to top and left to right. The illustration

$$(1 \circ_h \mathrm{g}) \circ_v (\mathrm{f} \circ_h 1) = \circ_v \boxed{} \circ_v = \boxed{} = \circ_v \boxed{} \circ_v$$

$$= (\mathrm{f} \circ_h 1) \circ_v (1 \circ_h \mathrm{g}),$$

summarizes our reading conventions. The identity on an object X is denoted by 1_X.

The webs we will use below are certain labeled and oriented graphs. The labels and orientations arise in a predetermined way from few choices. Using this we tend to omit many of the labels and orientations.

Although the edges of webs are labeled with $a \in \mathbb{Z}_{\geq 0}$, we will allow labels $a \in \mathbb{Z}$ in formulas: by convention, the webs with negative labels are zero altogether.

We write $\mathbb{1}$ for the monoidal unit and $(\_)^*$ for the duality in a pivotal category. For webs $\mathbb{1}$ is the empty word and $(\_)^*$ is changing upward to downward orientations

for objects, extended to monoidal products using $(X \circ_h Y)^* = Y^* \circ_h X^*$, and turning pictures by 180 degrees for morphisms.

Let q be a generic parameter. For $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$ we let $[0]_q = 0$, $[0]_q! = 1 = \begin{bmatrix} a \\ 0 \end{bmatrix}_q$, $[a]_q = -[-a]_q$ for $a < 0$ and otherwise
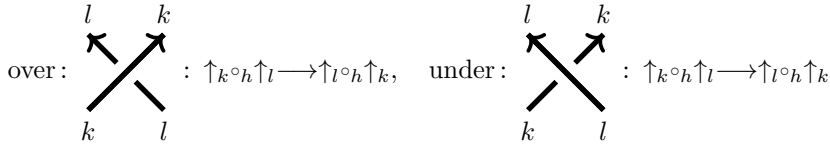
$$[a]_q = q^{a-1} + q^{a-3} + \ldots + q^{-a+3} + q^{-a+1}, \quad [b]_q! = [b]_q[b-1]_q \ldots [1]_q,$$
$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q[a-1]_q \ldots [a-b+1]_q}{[b]_q!}.$$
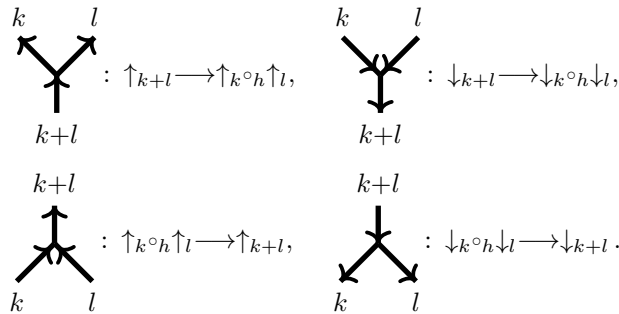
The web categories we study are:

Definition 2.2. The (exterior $\mathfrak{gl}_n$) web category $\mathbf{W}_q\mathfrak{gl}_n$ is the pivotal $\mathbb{Z}[q, q^{-1}]$-linear category with $\circ_h$-generating objects $\uparrow_k$ and $\downarrow_k$, for $k \in \mathbb{Z}_{\geq 0}$, of categorical dimension $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and $\downarrow_k = (\uparrow_k)^*$.

We further assume that $\mathbf{W}_q\mathfrak{gl}_n$ has a braid group action on upwards objects, meaning morphisms depicted

$$\text{over}: \quad : \uparrow_k \circ_h \uparrow_l \longrightarrow \uparrow_l \circ_h \uparrow_k, \quad \text{under}: \quad : \uparrow_k \circ_h \uparrow_l \longrightarrow \uparrow_l \circ_h \uparrow_k$$

for each simple braid group generator that satisfy the braid relations. We call these (k, l)-crossings (overcrossings and undercrossings). In these and similar pictures we tend to place the label under or over edges.

Finally, the $\circ_v$-$\circ_h$-generating morphisms are the ones coming from the pivotal structure and

$$: \uparrow_{k+l} \longrightarrow \uparrow_k \circ_h \uparrow_l, \qquad : \downarrow_{k+l} \longrightarrow \downarrow_k \circ_h \downarrow_l,$$

$$: \uparrow_k \circ_h \uparrow_l \longrightarrow \uparrow_{k+l}, \qquad : \downarrow_k \circ_h \downarrow_l \longrightarrow \downarrow_{k+l}.$$

The relations imposed on $\mathbf{W}_q\mathfrak{gl}_n$ are isotopies, the exterior relation, associativity, coassociativity, digon removal, and dumbbell-crossing relation together with

invertibility of the left mates of the (k, l)-overcrossings: we take the quotient by the $\circ_v$-$\circ_h$-ideal generated by isotopies, invertibility and

$$
\begin{array}{>n \atop >n} \Big| = 0, \qquad \bigwedge\limits_{k \ l \ m}^{k+l+m} = \bigwedge\limits_{k \ l \ m}^{k+l+m} ,
$$

$$
\bigvee\limits_{k+l+m}^{k \ l \ m} = \bigvee\limits_{k+l+m}^{k \ l \ m} , \qquad k \underset{k+l}{\overset{k+l}{\diamondsuit}} l = \begin{bmatrix} k+l \\ k \end{bmatrix}_q \cdot \underset{k+l}{\overset{k+l}{\uparrow}} ,
$$

$$
\bigtimes\limits_{k \ \ l}^{r \ \ s} = (-1)^{kl} \sum_{k-r=a-b} (-q)^{(k-a)(l-b)} \cdot \overset{r \ \ s}{\underset{k \ \ l}{\boxtimes}}
$$

$$
= (-1)^{kl} \sum_{k-r=a-b} (-q)^{-(k-a)(l-b)} \cdot \overset{r \ \ s}{\underset{k \ \ l}{\boxtimes}} ,
$$

together with their horizontally mirrored duals.

Remark 2.3. The final relation in Definition 2.2 is also known as the Schur relation. The non-quantum version comes from translating webs to the setting in Green's landmark book on the Schur algebra [Gr80], hence the name. In [LT21, Section 5] it is shown that this relation implies the more well-known crossing formula that we will use in (5.2) below. (Strictly speaking [LT21, Section 5] deals with symmetric webs, but the proof given therein works, mutatis mutandis, for exterior webs.)

Notation 2.4.

(a) We also write k for $\uparrow_k$ and $-k$ for $\downarrow_k$, where $k \in \mathbb{Z}_{\geq 0}$. In this notation a general object of $\mathbf{W}_q\mathfrak{gl}_n$ is of the form $\vec{k} = (k_1, ..., k_r) \in \mathbb{Z}^r$ for some $r \in \mathbb{Z}_{\geq 0}$.

(b) A web is a $\circ_v$-$\circ_h$-composition of the generating morphisms, i.e. not a $\mathbb{Z}[q, q^{-1}]$-linear combination.

(c) There is no harm in thinking of webs as topological objects, meaning as labeled oriented graphs embedded in two-space. We will sometimes use this to simplify drawings.
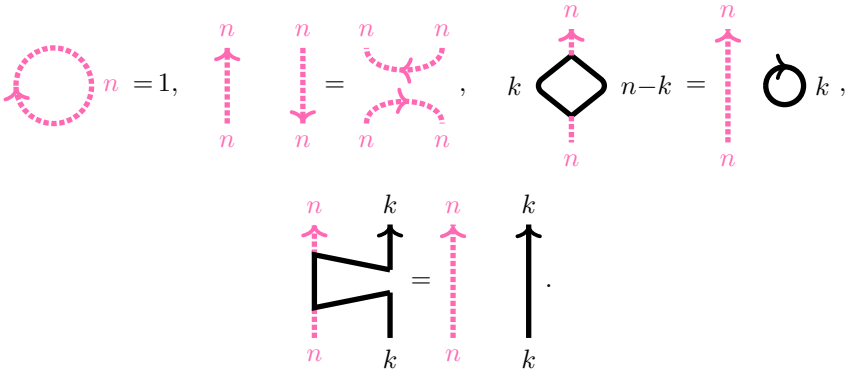
(d) The edges labeled n play a special role and we will illustrate them as

$$n\ \begin{smallmatrix}\uparrow\end{smallmatrix} = n\uparrow, \qquad n\ \begin{smallmatrix}\downarrow\end{smallmatrix} = n\downarrow.$$

We call them phantom edges. They should be thought of as nonexisting.

It follows from the defining relations that the phantom edge calculus is essentially trivial, i.e.:

Lemma 2.5. *We have*

There are more relations of a similar flavor which we omit to illustrate.

Motivated by Lemma 2.5, we call webs consisting of only phantom edges trivial.

Remark 2.6. We will stay with $\mathbf{W}_q\mathfrak{gl}_n$, which are webs for $\mathfrak{gl}_n$, in this paper. We however stress that all results are valid for the respective $\mathfrak{sl}_n$ version, thus, including the Temperley–Lieb calculus [RTW32], Kuperberg's $\mathfrak{sl}_3$ spiders [Kup96] and Cautis–Kamnitzer–Morrison's $\mathfrak{sl}_n$ webs [CKM14]. The (well-known) translation between these two pictures is a systematic identification of the form $\uparrow_k \cong \downarrow_{n-k}$. This, representation theoretically, corresponds to the fact that $\bigwedge^k \mathbb{C}^n$ is dual to $\bigwedge^{n-k} \mathbb{C}^n$ as an $\mathfrak{sl}_n$-module, but not as a $\mathfrak{gl}_n$-module.

It will turn out to be useful to only allow upwards (pointing) webs:

Definition 2.7. Let $\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n \subset \mathbf{W}_q\mathfrak{gl}_n$ be the full subcategory monoidally generated by $\{\uparrow_k | k \in \mathbb{Z}_{\geq 0}\}$.

We call webs in $\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n$ upwards webs. By definition, webs in $\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n$ have upwards pointing boundary only, but can, a priori, have downwards oriented edges away from the boundary. One can show that $\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n$ has morphism spaces spanned by webs with all edges pointing upwards, but we will not need this fact.

3. F-forms of webs

Let $\alpha_i = (0, ..., 0, 1, -1, 0, ..., 0)$, the ith simple root, with the one in the ith entry.

Definition 3.1. In this definition we work in $\mathbf{W}_q^\uparrow \mathfrak{gl}_n$ only. For all $a, i \in \mathbb{Z}_{\geq 0}$ we define the ath F-operator $F_i^{(a)}$ to be the operator that takes $1_{\vec{k}}$, with $\vec{k} \in \mathbb{Z}_{\geq 0}^l$, and returns $1_{\vec{k}-a\alpha_i}$, and we define the ath E-operator $E_i^{(a)}$ to be the operator that takes $1_{\vec{k}}$, with $\vec{k} \in \mathbb{Z}_{\geq 0}^l$, and returns $1_{\vec{k}+a\alpha_i}$ given by

$$1_{\vec{k}-a\alpha_i} F_i^{(a)} 1_{\vec{k}} = \quad , \quad 1_{\vec{k}+a\alpha_i} E_i^{(a)} 1_{\vec{k}} = \quad ,$$

and by the identity outside of these pictures. The associated webs are called ladder webs.

We will simplify notation using e.g. $1_{\vec{k}-a\alpha_i} F_i^{(a)} 1_{\vec{k}} = F_i^{(a)} 1_{\vec{k}} = 1_{\vec{k}-a\alpha_i} F_i^{(a)}$. Moreover, for all $k \in \{1, ..., n\}$ we use ladders to define

We call these webs and all their mates phantom crossings. The following lemma allows us to use phantom crossings essentially without cost. Note that these are not the (n, k)-crossings coming from the braid group action on $\mathbf{W}_q \mathfrak{gl}_n$, and in the picture above the strands "cross virtually". See (6.1) for the precise relation between the various crossings.

Lemma 3.2. *The phantom crossings satisfy all colored Reidemeister moves.*

*Definition* 3.3. We call all operations of the form

and all of its mates and mirrors forgetting phantom edges. These operations can be successively applied to webs and will relabel and reorient them as part of these operations.

Recall from Notation 2.4 that we think of webs as labeled oriented graphs. We say two such graphs are equivalent up to forgetting phantom edges if one can be obtained from the other by forgetting any finite number of phantom edges (including the relabeling and reorientation).

One can show that equivalence up to forgetting phantom edges defines an equivalence relation on webs. By definition, webs are compositions of the generators of $\mathbf{W}_q\mathfrak{gl}_n$, and we therefore can define:

Definition 3.4. An upwards-form of a web w is a web $U(w)$ in $\mathbf{W}_q^\uparrow\mathfrak{gl}_n$ that is equivalent to w up to forgetting phantom edges.

Example 3.5. Let $n=3$. The web w

$$w = \quad = \quad \rightsquigarrow U(w) = \quad ,$$
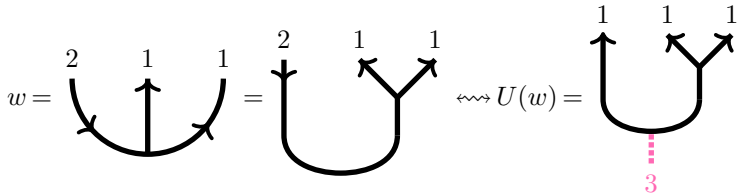
has the illustrated upwards-form. $\Diamond$

Lemma 3.6. *Every web has at least one upwards-form.*

Definition 3.7. An F-form $F(w)=F_{i_k}^{(a_r)}...F_{i_1}^{(a_1)}1_{\vec{k}}$ of an upwards-pointing web w is a string of F-operators such that the graphs of w and $F(w)$ are the same as labeled oriented graphs.

In general, an F-form of a web w is an F-form for any upwards-form $U(w)$.

Definition 3.7 is best understood by example:

Example 3.8. For $n=3$, an F-form for w as below is:
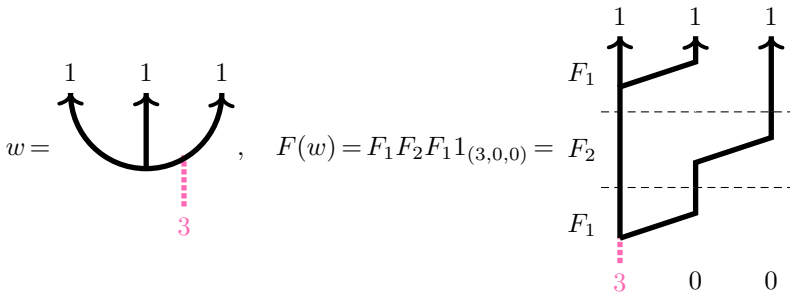
$$w = \quad , \quad F(w) = F_1 F_2 F_1 1_{(3,0,0)} = \quad .$$

Here and throughout, the horizontal slices are a visual aid only. Note that F-forms are not unique and $F_3 F_1 F_2 F_1 1_{(3,0,0,0)}$ would be another F-form of w. $\Diamond$

We call objects of $\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n$ of the form $(n, ..., n, 0, ..., 0) = (n^{\ell}, 0, ..., 0)$ a level.

Lemma 3.9. *Every web has at least one F-form in* $\mathrm{Hom}_{\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n}(\Lambda, \vec{k})$ *for some level* Λ.

Definition 3.10. Let $I = \{i^{(a)} | i \in \mathbb{Z}_{\geq 1}, a \in \mathbb{Z}_{\geq 0}\}$. Fix an F-form $F(w) = F_{i_r}^{(a_r)} ... F_{i_1}^{(a_1)} 1_{\Lambda}$ of w. The residue sequence for a web w and its F-form is the tuple $r_w = (i_1^{(a_1)}, ..., i_r^{(a_r)}) \in I^r$.

Note the reversed reading conventions when going from $F(w)$ to its residue sequence.

4. Evaluation of webs

Let $\star$ be the operation on webs that flips them upside down and reverses orientations. We now define a pairing on $\mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$, which we call the evaluation pairing:

Definition 4.1. Given two webs $u, w \in \mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$ we let $(u, w) \in \mathrm{End}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1})$ be the element given by $(u, w) = w^{\star} \circ_v u$, and we then extend this $\mathbb{Z}[q, q^{-1}]$-linearly to all of $\mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$.

Our main goal is to give a closed formula for (u, w). To this end, we need some preparation.

Definition 4.2. We define the following.
(a) Assume that we have an F-form $F(u) = F_{i_r}^{(a_r)} ... F_{i_1}^{(a_1)} 1_{\Lambda}$ of level $\Lambda = (n^{\ell}, 0, ..., 0)$ that ends at $\vec{k} = (k_1, ..., k_m)$. For $r_u = (i_1^{(a_1)}, ..., i_r^{(a_r)})$ we use the exploded sequence

$$\overline{r}_u = (\underbrace{i_1, ..., i_1}_{a_1 \text{ times}}, ..., \underbrace{i_r, ..., i_r}_{a_r \text{ times}}) \text{ of length } \overline{r} = a_1 + ... + a_r.$$

We also let $[r_u]_q! = [a_1]_q! ... [a_r]_q!$. We will use $\frac{q^d}{[r_u]_q! [r_w]_q!}$ which we call scaling, where $d = d(\vec{k}) = -\frac{1}{2}\left(n(n-1)\ell - \sum_{i=1}^{m} k_i(k_i-1)\right)$ is a shift.

(b) We let $S_{\overline{r}} = \mathrm{Aut}(\{1, ..., \overline{r}\})$ denote the symmetric group whose unit we denote by e. For $\overline{r}_u, \overline{r}_w \in I^{\overline{r}}$ we let $S_{\overline{r}_u}^{\overline{r}_w} = \{\sigma \in S_{\overline{r}} | \sigma.\overline{r}_u = \overline{r}_w\}$ be the set of possible crossings where $\sigma.\overline{r}_u$ is the permutation of the entries of $\overline{r}_u$ determined by σ.

(c) Recall the simple roots α_i and let $\langle \alpha_i, \alpha_j \rangle = a_{ij}$ with be the usual Cartan pairing (so $a_{ii} = 2$, $a_{ij} = -1$ for $|i-j| = 1$ and $a_{ij} = 0$ else) which we will use as indicated. Let $J_{\sigma}^{<t} = \{1 \leq j < t | \sigma(j) < \sigma(t)\}$. We define the following number of that we

think as counting weights from previous steps:

$$N^\Lambda(\sigma, \overline{r}_w, t) = \Big\langle \Lambda - \sum_{j \in J_\sigma^{\leq t}} \alpha_{(\overline{r}_w)_j}, \alpha_{(\overline{r}_w)_t} \Big\rangle.$$

We also write $N(\sigma, t) = N^\Lambda(\sigma, \overline{r}_w, t)$ for short.

(d) We write $X_\sigma = \prod_{t=1}^{\overline{r}} [N^\Lambda(\sigma, \overline{r}_u, t)]_q q^{N^\Lambda(e, \overline{r}_u, t) - 1}$. We let

(4.3) $$\chi(u, w) = \frac{q^d}{[r_u]_q! [r_w]_q!} \cdot \sum_{\sigma \in S_{\overline{r}_u}^{\overline{r}_w}} X_\sigma.$$

We encourage the reader to compare (4.3) to [HS21, Theorem 1.1]. The setting of [HS21] looks quite different from ours, but we will see in Section 6 why this comparison makes sense.

Theorem 4.4. *For webs $u, w \in \mathrm{Hom}_{\mathbf{W}_q \mathfrak{gl}_n}(\mathbb{1}, \vec{k})$ we have that $\chi(u, w)$ is independent of all choices involved, and*
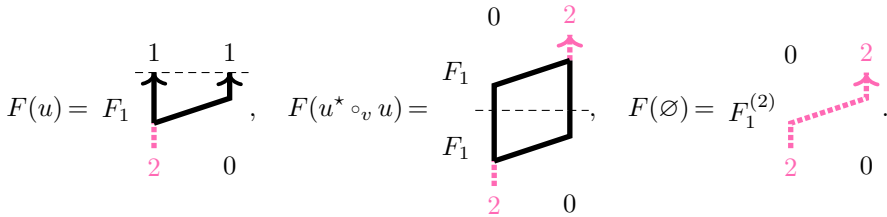
$$(u, w) = \chi(u, w) \in \mathbb{Z}_{\geq 0}[q, q^{-1}].$$

Example 4.5. Readers are encouraged to refer to the SageMath code provided in Remark 1.1 for further insight.

(a) Let $n = 2$. Let $u = w \in \mathrm{Hom}_{\mathbf{W}_q \mathfrak{gl}_n}(\mathbb{1}, (-1, 1))$ be, so that $(u, u) = [2]_q$:

$$u = \underset{ \quad 1}{\overset{1 \qquad 1}{\smile}}, \quad (u, u) = u^\star \circ_v u = \bigcirc = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = [2]_q.$$

As illustrated below, an F-form of u is $F(u) = F_1 1_{(2,0)}$, and an F-form of $u^\star \circ_v u$ is $F(u^\star \circ_v u) = F_1 F_1 1_{(2,0)}$. We will also use the trivial web with F-form $F(\varnothing) = F_1^{(2)} 1_{(2,0)}$.

The residue sequences from left to right are (1), $(1, 1)$ and $(1^{(2)})$, and in all cases $\Lambda = (2, 0)$, and $\ell = 1$, while $\vec{k} = (1, 1)$ for $F(u)$ and $\vec{k} = (0, 2)$ for the other two cases.

The main formula can be applied to either the pair (u, u) or to $(\varnothing, u^\star \circ_v u)$, giving the same result. That is, for (u, u) we have $\bar{r} = 1$, $S_{\bar{r}_u}^{\bar{r}_w} = \{e\}$ and

$$\sigma = e\colon X_\sigma = \underbrace{[2]_q q}_{[N(e,1)]_q q^{N(e,1)-1}} = 1 + q^2.$$
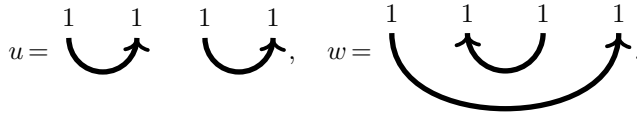
Moreover, $d = -1$, no explosion was needed and scaling by q^{-1} gives $\chi(u, u) = [2]_q$.

For $(\varnothing, u^\star \circ_v u)$ we have $\bar{r} = 2$, $S_{\bar{r}_u}^{\bar{r}_w} = \{e, (1, 2)\}$ and

$$\sigma = e\colon X_\sigma = 0, \quad \sigma = (1, 2)\colon X_\sigma = \underbrace{[2]_q q}_{[N((1,2),1)]_q q^{N(e,1)-1}} \overbrace{[2]_q q^{-1}}^{[N((1,2),2)]_q q^{N(e,2)-1}} = [2]_q^2.$$

This time we need to scale by $[2]_q^{-1}$ since we exploded $(1^{(2)})$ to $(1, 1)$ during this computation and $d = 0$. The result is the same.
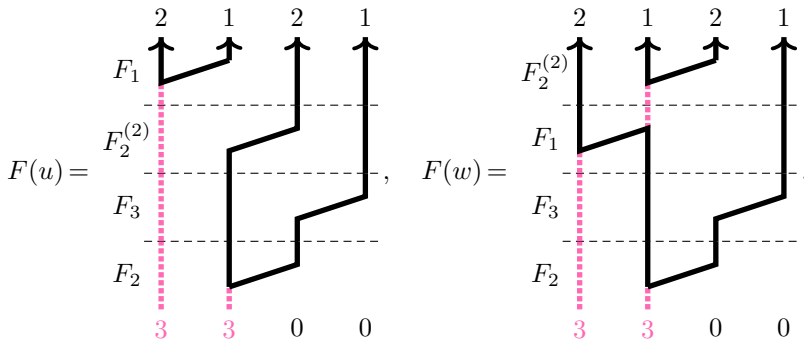
(b) Let $n = 3$. We consider the following two webs $u, w \in \mathrm{Hom}_{\mathbf{W}_q \mathfrak{gl}_n}\big(\mathbb{1}, (-1, 1, -1, 1)\big)$:

We, of course, immediately get

We get the same result from the main formula as follows. First, F-forms of u and w are

Here $\Lambda = (3^2, 0, 0)$ and $\ell = 2$. Using the associated residue sequences $(2, 3, 2^{(2)}, 1)$ and $(2, 3, 1, 2^{(2)})$ we get the same result: Firstly, the exploded sequences are $(2, 3, 2, 2, 1)$

and $(2, 3, 1, 2, 2)$ so we remember that we have to multiply in the end by $q^{-4}[2]_q^{-2}$. We compute

$$S_{\overline{r}_u}^{\overline{r}_w} = \{(3, 4, 5), (1, 4)(3, 5), (1, 5, 3, 4), (3, 5), (1, 4, 5, 3), (1, 5, 3)\}$$

where we use the usual notation for permutations. The six relevant summands are then

$$\sigma \neq (1, 4, 5, 3)\colon X_\sigma = 0, \quad \sigma = (1, 4, 5, 3)\colon X_\sigma = q^4[2]_q^2[3]_q = (1+q^2)^2(1+q^2+q^4).$$

Thus, scaling by $q^{-4}[2]_q^{-2}$ gives the desired result. Here we explode twice, so we get $[2]_q^{-2}$ and $d = -4$.

As in (a), we could also use an F-form for $w^\star \circ_v u$, e.g. $(2, 3, 2^{(2)}, 1, 2, 1^{(2)}, 3^{(2)}, 2^{(2)})$, which we pair with the trivial web $(2^{(3)}, 3^{(3)}, 1^{(3)}, 2^{(3)})$. Applying the formula for the exploded residue sequences $(2, 3, 2, 2, 1, 2, 1, 1, 3, 3, 2, 2)$ and $(2, 2, 2, 3, 3, 3, 1, 1, 1, 2, 2, 2)$ gives the same result.

For completeness, let $(n^\ell, 0, ..., 0)$ have ℓ symbols n and $m - \ell$ symbols 0. In general a residue sequence of the trivial web see as a web in $\mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}\big((n^\ell, 0, ..., 0), (0, ..., 0, n^\ell)\big)$ is given by the residue sequence $(\ell^{(n)}, ..., (m-1)^{(n)}, (\ell-1)^{(n)}, ..., (m-2)^{(n)}, ...)$. $\diamond$

Due to their relation to invariant tensors, the space $\mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$ has an important basis known as Lusztig–Kashiwara's dual canonical basis. (For details see e.g. [KK99] or [Tub20, Section 4.1.5].) We can thus ask whether a fixed $w \in \mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$ corresponds to a dual canonical basis element, and Theorem 4.4 gives a complete answer that can be checked using the main formula:

Proposition 4.6. *A* $w \in \mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$ *is dual canonical if and only if* $q^{-d}\chi(w, w) \in 1 + q\mathbb{Z}_{\geq 0}[q]$.

5. Evaluation of link polynomials

In order to compute link polynomials using F-forms we first need to explain how to interpret crossings. To this end, note that the category $\mathbf{W}_q\mathfrak{gl}_n$ has no generating sideways or downwards crossings as extra generators. However, these crossings can be obtained as compositions of upwards-pointing crossings and the generators of $\mathbf{W}_q\mathfrak{gl}_n$. For example:
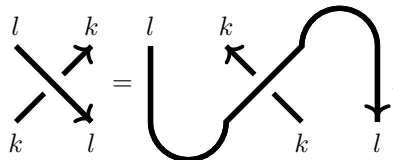
We consider framed oriented colored links $L_{\vec{c}}$ in $\mathbb{R}^3$, whose components are colored by $\vec{c} \in \mathbb{Z}_{\geq 0}^r$ where $r \in \mathbb{Z}_{\geq 0}$ is the number of components. We associate F-forms to their diagrams $D_{\vec{c}}$:

Definition 5.1. Given an oriented link diagram $D_{\vec{c}}$. An F-form of $D_{\vec{c}}$ is the linear combinations of webs obtained by:

(a) Choosing a Morse positioning (in terms of the $\circ_v$-$\circ_h$ generators, including upwards-pointing crossings) of $D_{\vec{c}}$;

(b) Replace all (k, l)-crossings by

(5.2)

and obtain a linear combinations of webs $a_1 w_1 + ... + a_s w_s$ where $a_i \in \mathbb{Z}[q, q^{-1}]$;

(c) Replace all w_i by a choice of F-form $F(w_i)$;

(d) The associated F-form of $D_{\vec{c}}$ is $F(D_{\vec{c}}) = a_1 F(w_1) + ... + a_s F(w_s)$.

Lemma 5.3. *Every colored link diagram has at least one F-form.*

Definition 5.4. Let $L_{\vec{c}}$ be a framed oriented colored link and let $F(D_{\vec{c}}) = a_1 F(w_1) + ... + a_s F(w_s)$ be any F-form of any of its diagrams. We define:

(5.5)
$$\chi(L_{\vec{c}}) = \sum_{i=1}^{s} a_i \chi(w_i) \in \mathbb{Z}[q, q^{-1}].$$

Remark 5.6. For the reader who wants to work with links and not framed links: One can easily verify that a k-colored Reidemeister I move gives the scalars $q^{\pm(k(-k+n+1))}$ (plus for the overcrossing and minus for the undercrossing). This in turn determines how the invariant in Definition 5.4 changes under framing changes.

Example 5.7. This example can also be found in the SageMath code as in Remark 1.1.

Let $n=2$ and let $H_{(1,1)}$ denote the $(1,1)$-colored Hopf link coming from the braid word with two positive crossings. The reader may convince themselves that an F-form of the standard diagram $D_{(1,1)}$ of $H_{(1,1)}$ is

$$F(D_{(1,1)}) = F_4F_5F_3F_4(q^2F_2F_3F_1F_2 - qF_2F_3F_2F_1 - qF_3F_2F_1F_2$$
$$+ F_3F_2F_2F_1)F_4F_3F_5F_4F_2F_3F_1F_2 1_{(2,2,0,0,0,0)},$$

which we compare to the trivial web with residue sequence $(2^{(2)}, 3^{(2)}, 4^{(2)}, 5^{(2)}, 1^{(2)}, 2^{(2)}, 3^{(2)}, 4^{(2)})$. Using the main formula from Theorem 4.4 we get

$$(\varnothing, H_{(1,1)}) = q^2(q+q^{-1})^2 - 2q(q+q^{-1}) + (q+q^{-1})^2 = q^{-1}[4]_q,$$

which is the expected result up to a power of q. (The precise power depends on the conventions one wants to compare the above to.) $\diamond$

Theorem 5.8. *The Laurent polynomial $\chi(L_{\vec{c}})$ is well-defined, i.e. it is an invariant of $L_{\vec{c}}$ and independent of all choices involved. Moreover, up to potentially adjusting conventions, this invariant agrees with the Reshetikhin–Turaev exterior colored $\mathfrak{gl}_n$ link invariant.*

Remark 5.9. The reader who wants to work with $\mathfrak{sl}_n$ instead of $\mathfrak{gl}_n$ needs to shift the Laurent polynomial $\chi(L_{\vec{c}})$ in Theorem 5.8 by $q^{1/n}$ to match e.g. the Reshetikhin–Turaev exterior colored $\mathfrak{sl}_n$ link invariant in [CKM14, Corollary 6.2.3].

6. The proofs

We now give all the proofs, sometimes collected into one proof.

Proof of "The phantom calculus is trivial, i.e. Lemma 2.5 and Lemma 3.2".
(Part a.) All of the relations displayed in Lemma 2.5 follow directly from the definitions except the right-hand equation for which we use [LT21, Lemma 2A.14].
*(Part b.)* It follows from (the exterior version of) [LT21, Section 5] that

(6.1)

and similarly for the other phantom crossings. (In the above picture note the difference between the honest (n,k)-crossings and the phantom crossings.) As a consequence of e.g. (the exterior version of) [LT21, Section 2], the honest (k,l)-crossings satisfy the Reidemeister relations, with Reidemeister I only up to the

scalar $q^{\pm(k(-k+n+1))}$ (here $k=l$), and the other defining relations of the colored tangle category (see e.g. for an uncolored list of these relations). Thus, the statement of Lemma 3.2 follows from (6.1). $\square$

Proof of "F-forms exist, i.e. Lemma 3.6, Lemma 3.9 and Lemma 5.3". (Part a.) That every web, more precisely any expression in the $\circ_v$-$\circ_h$ generators, has an upwards-form can be seen inductively: Let $h \in \mathbb{Z}_{\geq 0}$ denote the number of Morse points. If $h=0$ and the web we start with is already upwards-pointing, then there is nothing to show. If the starting web is downwards pointing, then we can just reverse all orientations. The analog for $h=1$ is also easily verified. So assume $h>1$. Pick any Morse point and perform either of

(6.2)
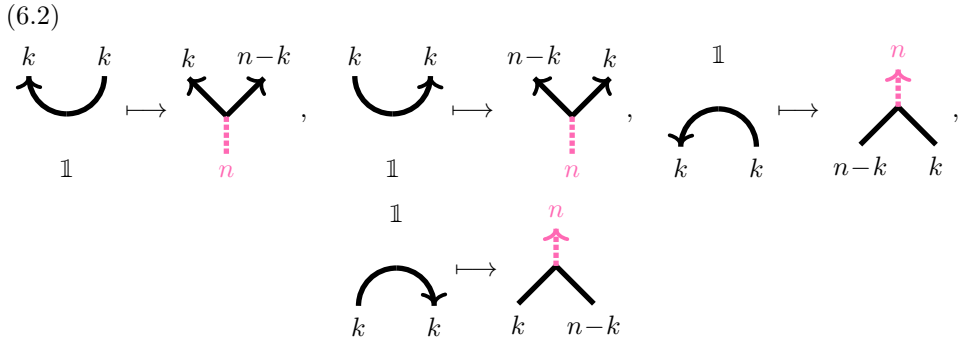
once, if h is odd, or twice (at two different Morse points), if h is even. Using phantom crossings we connected the dangling phantom strands in those pictures to anywhere point at the bottom and top, respectively. Relabeling and reorienting the result gives a legal web with smaller h by the combinatorics of oriented plane trivalent graphs (we can ignore the precise positions of the phantom edges by Lemma 3.2), so the claim of Lemma 3.6 follows.

(Part b.) This can be proven by using Lemma 3.6 and [Tub20, Lemma 4.9].

(Part c.) Lemma 5.3 follows directly from Lemma 3.9 via the Skein relations (5.2). $\square$

Proof of "The formulas work, i.e. Theorem 4.4 and Theorem 5.8". (Part a.) In the proof below we will use various statements about $\mathfrak{gl}_n$ web algebras, all of which, more or less explicitly, can be found in [Mac14] or [Tub20], using a matrix factorization description.

For $u, w \in \mathrm{Hom}_{\mathbf{W}_q\mathfrak{gl}_n}(\mathbb{1}, \vec{k})$ let $W(u,w)$ denote the associated free $\mathbb{Z}[q, q^{-1}]$-module summand of the $\mathfrak{gl}_n$ web algebra, see [Tub20, Section 3.3.4] for details. More precisely, $W(u,w)$ is the idempotent truncation of the $\mathfrak{gl}_n$ web algebra obtained by using the two webs u and w. By the construction of the $\mathfrak{gl}_n$ web algebra

via $\mathfrak{gl}_n$ foams or $\mathfrak{gl}_n$ matrix factorizations, respectively, and the universal construction, we have that $(u,w){=}q^d\mathrm{rank}_{\mathbb{Z}[q,q^{-1}]}\big(W(u,w)\big)$ (the graded rank) and we will use this throughout. (This graded rank formula can be obtained in many ways. The paper [RoWa20] gives a self-contained summary how the universal construction applied to $\mathfrak{gl}_n$ foams gives a categorification of the MOY calculus and thus the formula $(u,w){=}q^d\mathrm{rank}_{\mathbb{Z}[q,q^{-1}]}\big(W(u,w)\big)$ follows.) Here the shift by q^d comes simply from the desire to have the unit of the $\mathfrak{gl}_n$ web algebra sitting in degree zero, which corresponds to using Gaussians (e.g. $1{+}q^2$) instead of quantum numbers.

Let us first assume that we have $u,w{\in}\mathrm{Hom}_{\mathbf{W}_q^{\uparrow}\mathfrak{gl}_n}(\Lambda,\vec{k})$ for some level Λ with fixed F-forms. The result in [Tub20, Theorem 5.16] identifies $W(u,w)$ as an idempotent truncation of a thick version of the cyclotomic KLR algebra of type $A_{\mathbb{Z}}$ and level Λ. The idempotent truncation is exactly given by the images of the F-forms of u,w under the categorified skew Howe duality in terms of the $\mathfrak{gl}_n$ web algebra, see e.g. [Tub20, Section 3.3]. The thickening, as explained in [Tub20, Section 3.3] is just the KLR diagram version of the usual thick calculus from [KLMS12], and it is easy to see that explosion in this case corresponds to isomorphisms on both sides of categorified skew Howe duality. As explained in [Tub20, Section 5] these various isomorphism patched together provide the factors $[r_u]_q!$ and $[r_w]_q!$ in the formula from (4.3). Otherwise, (4.3) is the type $A_{\mathbb{Z}}$ version of [HS21, Theorem A], so we can use [Tub20, Theorem 5.16] to push it over to the $\mathfrak{gl}_n$ web algebra.

It remains to argue that choosing different F-forms does not change the result. If $F(u)$ and $F(u)'$ are two F-forms for u that are equal as topological webs and similarly $F(w)$ and $F(w)'$ are two F-forms for u that are equal as topological webs, then $(F(u),F(w)){=}(F(u)',F(w)')$ using the topological invariance of the $\mathfrak{gl}_n$ web algebra. By the previous point we further know that $(F(u),F(w)){=}\chi(u,w)$ and $(F(u)',F(w)'){=}\chi(u,w)'$, with the prime indicating that we use the ingredients coming from $F(w)$ and $F(w)'$ in $\chi(u,w)'$. Thus, $\chi(u,w){=}\chi(u,w)'$ which is what we wanted to show.

We next claim that (u,w) stays the same under the moves in (6.2) and the consequent reorientation. Indeed, there is a bijection of the bases on either side since the counting of flows (another way to index $\mathrm{rank}_{\mathbb{Z}[q,q^{-1}]}\big(W(u,w)\big)$) comes out to be the same. To see this, recall that a flow is a labeling of the edges of thickness k with a k element subset of $\{1,...,n\}$ such that locally e.g.

$$\underset{A\cup B}{\overset{A \quad B}{\bigvee}} \,, A,B\subset\{1,...,n\},$$
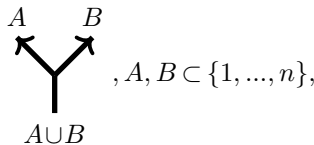
holds. Then we first note that changing the orientation does not change the number of flows. The operations in (6.2) also do not change the flow, so the claim follows.

Finally, we need to argue that the rank of the $\mathfrak{gl}_n$ web algebra does not change when using phantom edges and crossings freely. To this end, note that Lemma 3.2 implies that webs seen as objects in three space do not change under the usage of phantom crossings. In particular, the evaluation formula that gives (u, w) does not actually see them at all.

(Part b.) By Theorem 4.4 we have that the formula in (4.3) matches the evaluation on webs. The latter, by [MOY98, Theorem 5.1] (strictly speaking we use different conventions but the arguments of [MOY98, Theorem 5.1] still apply in our conventions), is known to give a link invariant. That this invariant is, up to conventions again, the Reshetikhin–Turaev polynomial of interest follows then from [CKM14, Corollary 6.2.3]. □

Proof of "The dual canonical webs, i.e. Proposition 4.6". Directly from Theorem 4.4 and [Tub20, Theorem 4.19]. □

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