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Fluctuations in depth and associated primes of powers of ideals

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Abstract. We count the numbers of associated primes of powers of ideals as defined in [2]. We generalize those ideals to monomial ideals $\mathrm{BHH}(m, r, s)$ for $r \geq 2$, $m, s \geq 1$; we establish partially the associated primes of powers of these ideals, and we establish completely the depth function of quotients by powers of these ideals: the depth function is periodic of period r repeated m times on the initial interval before settling to a constant value. The number of needed variables for these depth functions are lower than those from general constructions in [6].

This paper was motivated by results from Herzog and Hibi [7] and Bandari, Herzog and Hibi [2] that construct monomial ideals I with various properties of the depth function $n \mapsto \mathrm{depth}(R/I^n)$. In particular, Herzog and Hibi [7] constructed for any non-increasing eventually constant sequence $\{a_n\}$ a monomial ideal such that for all integers n, $\mathrm{depth}(R/I^n) = a_n$. In general, the depth function need not be monotone, as shown by an example in [7]. Bandari, Herzog and Hibi [2] constructed for each positive integer m a monomial ideal I for which the depth function takes on values $0, 1$, repeated m times, followed by 0 and then by constant 2. Thus this function has a global maximum, exactly m strict local maxima and exactly $m+1$ strict local minima. This was the first example of prescribed depth periodicity of period 2 on a segment of the domain. We point out that a later paper, [6], by Hà, Nguyen, Trung and Trung, establishes more generally for any eventually constant $\mathbb{N}_0$-valued sequence $\{a_n\}$ the existence of a monomial ideal Q in a polynomial ring S satisfying $\mathrm{depth}(S/Q^n) = a_n$ for all n. This completely determines all depth functions of powers of ideals.

Part of our long-term goal is to shed light similarly on the possible functions $n \mapsto \#\,\mathrm{Ass}(R/I^n)$ for ideals I in Noetherian rings R. Certainly these functions are all positive-integer valued and eventually constant by a result of Brodmann [4]. The second author and Weinstein proved in [11] that for every non-increasing sequence

$\{a_n\}$ of positive integers there exists a family of monomial ideals I such that for all n, the number of associated primes of I^n is a_n. For arbitrary (necessarily eventually constant) sequences of positive integers much less is known. If some a_n equals 1, then if we are to vary over monomial ideals it is necessary that all a_m for $m \geq n$ also be equal to 1. If we do not restrict to monomial ideals, then a big jump can occur from a_1 to a_2 even if we restrict to prime ideals; a result from [8] proves that a_2 is not bounded above by any polynomial function in the number of variables in the ring.

We present in Theorem 3.11 the function $n \mapsto \# \operatorname{Ass}(R/I^n)$ for ideals I introduced by Bandari, Herzog and Hibi in [2]. Once we completed the count of all associated primes and observed certain partial periodicity of period 2, we introduced a more general family of ideals, $\operatorname{BHH}(m, r, s)$ with $r \geq 2$, $m, s \geq 1$; in this notation, the original Bandari–Herzog–Hibi ideals are $\operatorname{BHH}(m, 2, 2)$, and in Theorem 3.11 we count more generally the associated primes of $\operatorname{BHH}(m, 2, s)^n$ as:

$$(3-\delta_{1=n})^m + \left(\sum_{\ell=0}^{m} \sum_{t=b(\ell)}^{m} \binom{m}{\ell} \binom{\ell}{\ell+t-m} \right) + \begin{cases} 0, & \text{if } n \leq 2m \text{ and } n \text{ is even;} \\ 1, & \text{otherwise,} \end{cases}$$

where $b(\ell) = \max\{n-1-\ell, m-\ell\}$ and δ_C equals 1 if the condition C is true and 0 otherwise. In particular, the number of associated primes of $\operatorname{BHH}(m, 2, s)$ is $2^m + 3^m + 1$, the number of associated primes of $\operatorname{BHH}(m, 2, s)^2$ is $2 \cdot 3^m$, and when $n \geq 2m+2$, the number of associated primes of $\operatorname{BHH}(m, 2, s)^n$ is $1+3^m$. In Theorem 3.13 we prove that the function $n \mapsto \# \operatorname{Ass}(R/\operatorname{BHH}(m, 2, s)^n)$ has exactly $\left\lceil \frac{m-1}{2} \right\rceil$ local maxima. The global maximum $2 \cdot 3^m + 1$ is achieved exactly at $n = 3, 5, \ldots, 2\left\lceil \frac{m-1}{2} \right\rceil + 1$. This function is periodic of period 2 when restricted to $[3, 2\left\lceil \frac{m-1}{2} \right\rceil + 1]$.

We present this count of associated primes in two different ways. As a result, Remark 3.12 proves an identity of binomial expressions that we have not found in the literature.

For $r > 2$ we completely describe and count all the associated primes that contain one of the special variables $c_1, \ldots, c_s$ (and hence all), and we give some properties and descriptions of the associated primes that do not contain these special variables. The latter associated primes satisfy persistence, namely that if a prime ideal not containing the special variable is associated to an nth power, then it is associated to all higher powers. This persistence is not in general satisfied by the associated primes containing the special variable.

Seidenberg proved in [10, Point 65] that there exists a primitive recursive function $B(n, d)$ such that any ideal I in a polynomial ring in n variables over a field with generators of degree at most d has at most $B(n, d)$ associated primes. Ananyan and Hochster [1] proved that there exists a primitive recursive function $E(g, d)$ such that any ideal I in a polynomial ring over a field with at most g generators of degrees d or less has at most $E(g,d)$ associated primes. The ideal $\mathrm{BHH}(m,2,1)$ is in a polynomial ring with $2m+3$ variables and has $2m+5$ generators of degrees up to 9. Its third power has $2\cdot 3^m+1$ associated primes, and by Lemma 1.2 it has at most $\binom{2m+1}{2}+\binom{2m+6}{3}$ generators of degrees up to 23. For large m, this number of generators is less than or equal to $2m^3$, showing that for large m and n,

$$B(2m+3,23)\geq 2\cdot 3^m+1,\ \text{ i.e., } B(n,23)\geq 2\cdot(\sqrt{3})^{n-3}+1,$$
$$E(2m^3,23)\geq 2\cdot 3^m+1,\ \text{ i.e., } E(n,23)\geq 2\cdot 3^{\sqrt[3]{n/2}}+1.$$

Asymptotically, the lower bound here for $B(n,d)$ is stronger than the bound $3^{n/3}$ in [8], but the lower bound for $E(n,d)$ here is weaker than the bound $3^{\sqrt{2n}-1}$ in [8].

In Theorem 4.2 we prove that the function $n\mapsto \operatorname{depth}(R/\mathrm{BHH}(m,r,s)^n)$ is periodic of period r when restricted to the interval $[1,...,rm+1]$, that it has exactly $m+1$ local minima, all on that interval and equal to 0, that all other values on that interval are 1, and that the only value outside of the interval is s. More generally, we show that an e-fold splitting of $\mathrm{BHH}(m,r,s)$ gives an ideal I in a ring A such that

$$\operatorname{depth}\left(\frac{A}{I^n}\right)=\begin{cases} e-1, & \text{if } n=ru+1 \text{ with } u=0,...,m;\\ e, & \text{if } n\leq rm+1 \text{ and } n\not\equiv 1 \mod r;\\ s+e-1, & \text{otherwise, i.e., if } n>mr+1.\end{cases}$$

We point out that the construction in [6] by Hà, Nguyen, Trung and Trung of the monomial ideal with the same depth function uses at least $e+4(rm-m)+3s$ variables, whereas our construction uses $rm+r+s+e-1$. The difference $3rm-4m-r+2s+1=2(r-2)m+(m-1)r+2s+1$ in the number of variables is always positive since $r\geq 2$ (for periodicity) and $m,s\geq 1$.

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1. Generalized Bandari–Herzog–Hibi ideals

Definition 1.1. Let m,r and s be positive integers with $r\geq 2$. Let $c_1,...,c_s,a_j$, $x_{i,j}$ be variables over a field k where $i\in[m]$ and $j\in[r]$. For the sake of notation, we identify $a_{r+1}=a_1$, $a_0=a_r$, $x_{i,r+1}=x_{i,1}$, $x_{i,0}=x_{i,r}$ for all i, and more generally, $x_{i,j}=x_{i,(j \mod r)}$. We define

$$\begin{aligned}
B_0(r) &= (a_j^6, a_j^5 a_{j+1} : j=1,\dots,r),\\
B_c(r,s) &= (c_1,\dots,c_s)a_1^4 a_2^4 \cdots a_r^4;\\
X(m,r) &= (a_j^4 x_{i,j} x_{i,j+1}^2 : i\in[m], j\in[r]),\\
\mathrm{BHH}(m,r,s) &= B_0(r)+B_c(r,s)+X(m,r).
\end{aligned}$$

We call $\mathrm{BHH}(m,r,s)$ the **Bandari-Herzog-Hibi** ideals. When $s=1$, we write $c=c_1$.

When m,r and s are clear, we write these ideals as B_0, B_c, X, B, respectively. We will refer to elements $a_j^4 x_{i,j} x_{i,j+1}^2$ as $h_{i,j}$.

We name these ideals in honor of the Bandari-Herzog-Hibi paper [2] which originated the ideals $\mathrm{BHH}(m,2,2)$.

Understanding the associated primes of powers of these ideals is important for understanding the depth function of their quotients. For the primary decompositions part, we prove in Theorem 1.4 that it suffices to find the decompositions in case $s=1$. This reduction greatly simplifies the notation and speeds up any concrete calculations of the associated primes and thus the counting.

Lemma 1.2. *Let J_1, J_2, J_3 be ideals in a ring R such that $(J_1+J_2)^2\subseteq J_2^2$. Then for all positive integers n, $(J_1+J_2+J_3)^n=J_1J_3^{n-1}+(J_2+J_3)^n$.*

Thus with $B=\mathrm{BHH}(m,r,s)$, $B^n=B_cX^{n-1}+(B_0+X)^n=B_cB^{n-1}+(B_0+X)^n$.

Proof. The first display holds trivially for $n=1$. The equality $(J_1+J_2+J_3)^2= J_1^2+J_1J_2+J_1J_3+J_2^2+J_2J_3+J_3^2 =J_1J_3+J_2^2+J_2J_3+J_3^2 =J_1J_3+(J_2+J_3)^2$ proves the case $n=2$. Then by induction on $n\geq 2$,

$$\begin{aligned}
(J_1+J_2+J_3)^n &= (J_1+J_2+J_3)(J_1+J_2+J_3)^{n-1}\\
&= (J_1+J_2+J_3)\left(J_1J_3^{n-2}+(J_2+J_3)^{n-1}\right)\\
&= J_1^2J_3^{n-2}+J_1(J_2+J_3)^{n-1}+J_1J_2J_3^{n-2}+(J_2+J_3)^n+J_1J_3^{n-1}\\
&= J_1(J_2+J_3)^{n-1}+(J_2+J_3)^n\\
&= J_1\sum_{i=0}^{n-1} J_2^iJ_3^{n-1-i}+(J_2+J_3)^n\\
&= J_1J_3^{n-1}+\sum_{i=1}^{n-1} J_1J_2^iJ_3^{n-1-i}+(J_2+J_3)^n\\
&\subseteq J_1J_3^{n-1}+(J_2+J_3)^n.
\end{aligned}$$

Since also the last ideal is contained in the first in this display, the conclusion follows.

The second part follows with $J_1{=}B_c$, $J_2{=}B_0$ and $J_3{=}X$, since $J_1^2 \subseteq (a_1^6)(a_2^6) \subseteq J_2^2$ and $J_1 J_2 \subseteq (a_j^6, a_j^5 a_{j+1} : j \in [r])(a_1^4 \cdots a_r^4) \subseteq ((a_j^5 a_{j+1})^2, (a_j^6)(a_{j+1}^5 a_{j+2}) : j \in [r]) \subseteq J_2^2$. $\square$

Lemma 1.3. *Let I_1, I_2, I_3 be ideals in a Noetherian ring A such that $(I_1+I_2)^2 \subseteq I_2^2$ and let $c, c_1, ..., c_s$ be variables over A. Then the set of associated primes of $\frac{A[c_1,...,c_s]}{((c_1,...,c_s)I_1+I_2+I_3)^n}$ equals*

$$\{(P+(c_1,...,c_s))A[c_1,...,c_s] : P \subseteq A, P+(c) \in \operatorname{Ass}(A[c]/(cI_1+I_1+I_2)^n)\}$$
$$\cup \{PA[c_1,...,c_s] : P \in \operatorname{Ass}(A[c]/(cI_1+I_1+I_2)^n) \text{ and } c \notin P\}.$$

Proof. Let $\underline{c}$ stand either for the ideal (c) or for the ideal $(c_1, ..., c_s)$. By Lemma 1.2 and using $J_1{=}\underline{c}I_1$ and $J_2{=}I_2$, $J_3{=}I_3$, we get that for all positive integers n, $(\underline{c}I_1+I_1+I_2)^n = \underline{c}I_1 I_3^{n-1} + (I_2+I_3)^n$.

Define $\varphi\colon A[c_1, ..., c_s] \to A[c]$ to be the A-algebra homomorphism that takes all c_i to c and is the identity on A. We impose the $\mathbb{N}^s$-grading on $A[c_1, ..., c_s]$ with $\deg(c_i){=}e_i$ (the s-tuple with 1 in the ith position and 0 elsewhere) and we define the degrees of all other variables to be 0. Then φ is not a graded homomorphism, but it is a surjective spreading as defined in [8, Definition 2.1], and $((c_1, ..., c_s)I_1+I_2+I_3)^n = (c_1, ..., c_s)I_1 I_3^{n-1} + (I_2+I_3)^n$ is a spreading of $(cI_1+I_2+I_3)^n = cI_1 I_3^{n-1} + (I_2+I_3)^n$. By [8, Lemma 2.5], the spreading of an irredundant primary decomposition of $cI_1 I_3^{n-1} + (J_2+J_3)^n$ corresponds to an irredundant primary decomposition of $(c_1, ..., c_s)I_1 I_3^{n-1} + (J_2+J_3)^n$; specifically, any associated prime of the former ideal not containing c is associated to the latter ideal and does not contain any c_i, and furthermore any associated prime of the former ideal that contains c is spread to one unique associated prime of the latter ideal in which the generator c is replaced by the s generators $c_1, ..., c_s$. $\square$

The last two lemmas immediately prove that the number of associated primes of $(\mathrm{BHH}(m, r, s))^n$ is the same as the number of associated primes of $(\mathrm{BHH}(m, r, 1))^n$, via the following formalization:

Theorem 1.4. *Set $B{=}\mathrm{BHH}(m, r, 1)$ and $B(s){=}\mathrm{BHH}(m, r, s)$. For every positive integer n, the sets of associated primes of B^n and of $B(s)^n$ are in one-to-one correspondence*:

(1) *Associated primes of B^n not containing c have the same minimal generating sets as their corresponding primes associated to $B(s)^n$ that do not contain $c_1, ..., c_s$.*

(2) *Associated primes of B^n that contain c are of the form $P+(c)$ for some monomial prime P in variables $a_i, x_{i,j}$ and they correspond to associated primes of $B(s)^n$ of the form $P+(c_1, ..., c_s)$.* $\square$

2. Lemmas

Throughout this section, B stands for $\mathrm{BHH}(m,r,1)$ and n is a positive integer.

Lemma 2.1. *Let P be a prime ideal associated to B. Suppose that for each $j\in[r]$ there exists $i_j\in[m]$ such that $x_{i_j,j}$ and $x_{i_j,j+1}$ are both in P. Then $c\in P$. In particular, if $x_{i,1},\ldots,x_{i,r}\in P$ for some $i\in[m]$, then $c\in P$.*

Proof. By definition of associated primes there exists a monomial w such that $P=(B:w)$ and hence $w\in B:P\subseteq B:(x_{i_j,j},x_{i_j,j+1}: j\in[r])$. We have

$$\begin{aligned}
&B:(x_{i_j,j},x_{i_j,j+1})\\
&\quad=(B+(a_{j-1}^4x_{i_j,j-1}x_{i_j,j},a_j^4x_{i_j,j+1}^2))\cap(B+(a_j^4x_{i_j,j}x_{i_j,j+1},a_{j+1}^4x_{i_j,j+2}^2))\\
&\quad\subseteq B+(a_{j-1}^4)+(a_j^4x_{i_j,j}x_{i_j,j+1}^2,a_j^4a_{j+1}^4x_{i_j,j+1}^2x_{i_j,j+2}^2)\\
&\quad=B+(a_{j-1}^4),
\end{aligned}$$

so that

$$w\in\bigcap_{j=1}^r(B:(x_{i_j,j},x_{i_j,j+1}))\subseteq\bigcap_{j=1}^r(B+(a_{j-1}^4))=B+(a_1^4\cdots a_r^4).$$

In all cases, c multiplies this intersection into B, proving that $c\in P$. $\square$

Lemma 2.2. *Let P be a prime ideal associated to B^n containing some $x_{i,j}$. Write $P=B^n:w$ for some monomial w.*

(1) *Then $w\in a_{j-1}^4x_{i,j-1}x_{i,j}B^{n-1}\cup a_j^4x_{i,j+1}^2B^{n-1}$.*

(2) *If P also contains $x_{i,j+1}$, then*

$$\begin{aligned}
w\in{}&\left(a_{j-1}^4x_{i,j-1}x_{i,j}B^{n-1}\cup a_j^4x_{i,j+1}^2B^{n-1}\right)\cap\left(a_j^4x_{i,j}x_{i,j+1}B^{n-1}\cup a_{j+1}^4x_{i,j+2}^2B^{n-1}\right)\\
&\cap\left(a_{j-1}^4x_{i,j-1}x_{i,j}B^{n-1}\cup a_{j+1}^4x_{i,j+2}^2B^{n-1}\right).
\end{aligned}$$

Proof. (1) Since $B^n:w=P$, and by the form of the generators of B,

$$w\in B^n:P\subseteq B^n:x_{i,j}=B^n+(a_{j-1}^4x_{i,j-1}x_{i,j},a_j^4x_{i,j+1}^2)B^{n-1}.$$

But w cannot be in B^n (for otherwise $B^n:w=R$ is not a prime ideal), and since w is a monomial, (1) follows.

(2) Now suppose that P contains $x_{i,j}$ and $x_{i,j+1}$. Then by (1),

$$w\in\left(a_{j-1}^4x_{i,j-1}x_{i,j}B^{n-1}\cup a_j^4x_{i,j+1}^2B^{n-1}\right)\cap\left(a_j^4x_{i,j}x_{i,j+1}B^{n-1}\cup a_{j+1}^4x_{i,j+2}^2B^{n-1}\right).$$

Suppose for contradiction that

$$w\notin a_{j-1}^4x_{i,j-1}x_{i,j}B^{n-1}\cup a_{j+1}^4x_{i,j+2}^2B^{n-1}.$$

Then $w \in a_j^4 x_{i,j+1}^2 B^{n-1} \cap a_j^4 x_{i,j} x_{i,j+1} B^{n-1}$. So we have proved that with $u=0$, $w \in B^u(a_j^4 x_{i,j+1}^2 B^{n-1-u} \cap a_j^4 x_{i,j} x_{i,j+1} B^{n-1-u})$. We proceed from this:

$$\begin{aligned} w &\in B^u(a_j^4 x_{i,j+1}^2 B^{n-1-u} \cap a_j^4 x_{i,j} x_{i,j+1} B^{n-1-u}) \\ &= B^u a_j^4 x_{i,j+1} \left(x_{i,j+1} B^{n-1-u} \cap x_{i,j} B^{n-1-u}\right) \\ &= B^u a_j^4 x_{i,j} x_{i,j+1}^2 \left((B^{n-1-u} : x_{i,j}) \cap (B^{n-1-u} : x_{i,j+1})\right) \\ &\subseteq B^{u+1} \big((B^{n-1-u} + (a_{j-1}^4 x_{i,j-1} x_{i,j}, a_j^4 x_{i,j+1}^2) B^{n-2-u}) \\ &\qquad\qquad \cap (B^{n-1-u} + (a_j^4 x_{i,j} x_{i,j+1}, a_{j+1}^4 x_{i,j+2}^2) B^{n-2-u})\big)\,. \end{aligned}$$

But $w \notin B^n \cup a_{j-1}^4 x_{i,j-1} x_{i,j} B^{n-1} \cup a_{j+1}^4 x_{i,j+2}^2 B^{n-1}$, so necessarily

$$w \in B^{u+1}(a_j^4 x_{i,j+1}^2 B^{n-2-u} \cap a_j^4 x_{i,j} x_{i,j+1} B^{n-2-u}).$$

This is the induction step, so when $u=n-2$, we get that

$$w \in B^{n-1}((a_j^4 x_{i,j+1}^2) \cap (a_j^4 x_{i,j} x_{i,j+1})) = B^{n-1}(a_j^4 x_{i,j} x_{i,j+1}^2) \subseteq B^n,$$

which is a contradiction to $B^n : w$ being a prime ideal. This finishes the proof of (2). □

Lemma 2.3. *Let P be a prime ideal associated to B^n. Then the following properties hold:*

(1) $a_1, ..., a_r \in P$.

(2) *If P does not contain $x_{i,1} x_{i,2} \cdots x_{i,r}$ for some i, then $P = (a_1, ..., a_r)$.*

(3) *If P contains some $x_{i,j}$, then there exists $j_0 \in \{j-1, j\}$ such that $x_{e,j_0} x_{e,j_0+1} \in P$ for all $e \in [m]$.*

(4) *If P does not contain $x_{i,j}, x_{i,j+1}$ for some $i \in [m]$ and $j \in [r]$, then $c \notin P$.*

Proof. (1) Since $a_j^6 \in B$, it follows that a_j must be in every associated prime ideal of B^n.

(2) Suppose that $x_{i,1} x_{i,2} \cdots x_{i,r} \notin P$ for some i. Then P is associated to

$$B^n : (x_{i,1} \cdots x_{i,r})^\infty = \left(B_0 + (c(a_1 a_2 \cdots a_r)^4) + (a_j^4 : j \in [r])\right)^n = (a_j^4 : j \in [r])^n,$$

whose only associated prime is $(a_1, ..., a_r)$.

(3) follows from Lemma 2.2(1): there exists $j_0 \in \{j-1, j\}$ such that the witness w for P is in $a_{j_0}^4 B^{n-1}$. Hence for all e, $w x_{e,j_0} x_{e,j_0+1}^2 \in B^n$.

(4) By assumption, P is associated to

$$\begin{aligned} B^n : (x_{i,j} x_{i,j+1})^\infty &= \left(B + (a_j^4, a_{j-1}^4 x_{i,j-1}, a_{j+1}^4 x_{i,j+2}^2)\right)^n \\ &= \left(B_0 + X + (a_j^4, a_{j-1}^4 x_{i,j-1}, a_{j+1}^4 x_{i,j+2}^2)\right)^n, \end{aligned}$$

and c does not appear in any generator of the last ideal. Thus P cannot contain c. □

Lemma 2.4.
(1) *If* $r=2$, *then* $x_{i,j}a_1^4a_2^4h_{i,j}\in B^2$.
(2) *If* $r=2$, *then* $a_1^4a_2^4h_{i,j}^2\in B^3$.
(3) $x_{i,j+2}^2a_j^4a_{j+1}^4h_{i,j}\in B^2$.
(4) $a_{j-1}^4a_j^4x_{i,j-1}x_{i,j}h_{i,j}\in B^2$.
(5) $x_{i,j}a_{j-2}^4a_{j-1}^4a_j^4h_{i,j-2}h_{i,j}\in B^3$.
(6) $x_{i,j-1}a_{j-1}^4a_j^4h_{i,j}^2\in B^3$.
(7) $a_{j-2}^4a_{j-1}^4a_j^4h_{i,j-2}h_{i,j}^2\in B^4$.

Proof. For (1) and (2) we use that $x_{i,j-1}=x_{i,j+1}$ to rewrite

$$x_{i,j}a_1^4a_2^4h_{i,j}=x_{i,j}a_1^4a_2^4\left(a_j^4x_{i,j}x_{i,j+1}^2\right)\in(a_j^6)(a_{j-1}^4x_{i,j-1}x_{i,j}^2)\subseteq B^2,$$

and $a_1^4a_2^4h_{i,j}^2=a_1^4a_2^4(a_j^4x_{i,j}x_{i,j+1}^2)^2\in(a_j^6)^2(a_{j-1}^4x_{i,j-1}x_{i,j}^2)\subseteq B^3$. Part (3) follows from

$$x_{i,j+2}^2a_j^4a_{j+1}^4h_{i,j}=x_{i,j+2}^2a_j^4a_{j+1}^4(a_j^4x_{i,j}x_{i,j+1}^2)\in(a_j^6)(a_{j+1}^4x_{i,j+1}x_{i,j+2}^2)\subseteq B^2,$$

part (4) from $a_{j-1}^4a_j^4x_{i,j-1}x_{i,j}h_{i,j}=a_{j-1}^4a_j^8x_{i,j-1}x_{i,j}^2x_{i,j+1}^2\in(a_j^6)(a_{j-1}^4x_{i,j-1}x_{i,j}^2)\subseteq B^2$, part (5) from

$$\begin{aligned}x_{i,j}a_{j-2}^4a_{j-1}^4a_j^4h_{i,j-2}h_{i,j}&=x_{i,j}a_{j-2}^4a_{j-1}^4a_j^4\left(a_{j-2}^4x_{i,j-2}x_{i,j-1}^2\right)\left(a_j^4x_{i,j}x_{i,j+1}^2\right)\\&\in(a_{j-2}^6)(a_j^6)\left(a_{j-1}^4x_{i,j-1}x_{i,j}^2\right)\subseteq B^3,\end{aligned}$$

part (6) from $x_{i,j-1}a_{j-1}^4a_j^4h_{i,j}^2=x_{i,j-1}a_{j-1}^4a_j^{12}x_{i,j}^2x_{i,j+1}^4\in(a_j^6)^2(h_{i,j-1})\subseteq B^3$, and part (7) from

$$\begin{aligned}a_{j-2}^4a_{j-1}^4a_j^4h_{i,j-2}h_{i,j}^2&=a_{j-2}^8a_{j-1}^4a_j^{12}x_{i,j-2}x_{i,j-1}^2x_{i,j}^2x_{i,j+1}^4\\&\in(a_{j-2}^6)(a_j^6)^2(a_{j-1}^4x_{i,j-1}x_{i,j}^2)\subseteq B^4.\quad\square\end{aligned}$$

Corollary 2.5. *Let* $w=a_1^4\cdots a_r^4(\prod_{i,j}x_{i,j}^{v_{i,j}})(\prod_{i,j}h_{i,j}^{u_{i,j}})$ *with* $v_{i,j}$, $u_{i,j}$ *non-negative integers such that* $\sum_{i,j}u_{i,j}=n-1$.

(1) *Suppose that it is possible to rewrite* w *in the same format but with different* $v_{i,j},u_{i,j}$. *Then* $w\in B^n$.

(2) *Suppose that* $n>mr+1$ *and that* w *multiplies* $(x_{i,j}: i\in[m], j\in[r])$ *into* B^n. *Then* $w\in B^n$.

Proof. We set $A=a_1^4\cdots a_r^4$, $w_0=\prod_{i,j}x_{i,j}^{v_{i,j}}$, and $w_1=\prod_{i,j}h_{i,j}^{u_{i,j}}$.

(1) By assumption, there exists a positive $v_{i,j}$ such that $x_{i,j}$ gets incorporated in the rewriting of w either into a new $h_{i,j-1}$ or into a new $h_{i,j}$.

Suppose that $x_{i,j}$ is incorporated into a new $h_{i,j}$. Then $x_{i,j+1}^2$ needs to be a factor of w. This factor can come either from $u_{i,j}>0$ or from $v_{i,j+1}+u_{i,j+1}\geq 2$. If we use $x_{i,j+1}^2$ from $h_{i,j}^{u_{i,j}}$, then our $x_{i,j}$ is not making a new $h_{i,j}$, so necessarily

$v_{i,j+1}+u_{i,j+1}\geq 2$. By definition, Lemma 2.4(4) and (6), $A\cdot x_{i,j}x_{i,j+1}^2\in(h_{i,j})\subseteq B$, $A\cdot x_{i,j}x_{i,j+1}h_{i,j+1}\in B^2$, $A\cdot x_{i,j}h_{i,j+1}^2\in B^3$. This proves that $w\in B^n$.

Now suppose that $x_{i,j}$ is incorporated into a new $h_{i,j-1}$. Then $x_{i,j-1}x_{i,j}$ needs to be a factor of $w/x_{i,j}$. The $x_{i,j-1}$ factor can come from $v_{i,j-1}>0$, $u_{i,j-2}>0$ or from $u_{i,j-1}>0$. We can eliminate the option $u_{i,j-1}>0$ as it does not generate a new $h_{i,j-1}$. Similarly, the additional factor $x_{i,j}$ can only be taken from $v_{i,j}>1$ or $u_{i,j}>0$. If $v_{i,j-1}>0$ and $v_{i,j}>1$, then $A\cdot w_0\in(h_{i,j-1})\subseteq B$ and $w\in B^n$. If $v_{i,j-1}>0$ and $u_{i,j}>0$, then $w\in(A\cdot x_{i,j-1}x_{i,j}h_{i,j}\cdot\frac{w_1}{h_{i,j}})\in B^n$ by Lemma 2.4(4). If $u_{i,j-2}>0$ and $v_{i,j}>1$, then $w\in(A\cdot x_{i,j}^2h_{i,j-2}\cdot\frac{w_1}{h_{i,j-2}})\in B^n$ by Lemma 2.4(3). Finally, if $u_{i,j-2}>0$ and $u_{i,j}>0$, then $w\in(A\cdot x_{i,j}h_{i,j}h_{i,j-2}\cdot\frac{w_1}{h_{i,j}h_{i,j-2}})\in B^n$ by Lemma 2.4(5).

(2) Since $n-1>mr$, there exists (i,j) such that $u_{i,j}\geq 2$. By assumption, $x_{i,j}w$ and $x_{i,j+1}w$ are both in B^n. Thus for both variables, the rewriting needs to happen as in (1). As in the proof of (1), one of the following conditions holds for $x_{i,j}w$:

a) $v_{i,j+1}+u_{i,j+1}\geq 2$,

b) ($v_{i,j-1}>0$ or $u_{j-2}>0$) and ($v_{i,j}>0$ or $u_{i,j}>0$);

and one of the following conditions holds for $x_{i,j+1}w$:

a') $v_{i,j+2}+u_{i,j+2}\geq 2$,

b') ($v_{i,j}>0$ or $u_{i,j-1}>0$) and ($v_{i,j+1}>0$ or $u_{i,j+1}>0$).

If b) holds, then $x_{i,j-1}|w_0$ or $h_{i,j-2}|w_1$. It follows that $w\in B^n$ since $u_{i,j}\geq 2$ and hence $w\in(A\cdot x_{i,j-1}h_{i,j}^2\cdot\frac{w_1}{h_{i,j}^2})\in B^n$ due to Lemma 2.4(6) or $w\in(A\cdot h_{i,j-2}h_{i,j}^2\cdot\frac{w_1}{h_{i,j-2}h_{i,j}^2})\in B^n$ due to Lemma 2.4(7). Similarly, if a') holds, then $x_{i,j+2}^2|w_0$ or ($x_{i,j+2}|w_0$ and $h_{i,j+2}|w_1$) or $h_{i,j+2}^2|w_1$. Since $h_{i,j}|w_1$, it follows that $w\in(A\cdot x_{i,j+2}^2h_{i,j}\cdot\frac{w_1}{h_{i,j}})\in B^n$ by Lemma 2.4(3) or $w\in(A\cdot x_{i,j+2}h_{i,j+2}h_{i,j}\cdot\frac{w_1}{h_{i,j}h_{i,j+2}})\in B^n$ by Lemma 2.4(5) or $w\in(A\cdot h_{i,j+2}^2h_{i,j}\cdot\frac{w_1}{h_{i,j}h_{i,j+2}^2})\in B^n$ by Lemma 2.4(7).

So we may assume that we have conditions a) and b'). If $x_{i,j+1}^2|w_0$ and ($x_{i,j}|w_0$ or $h_{i,j-1}|w_1$), then either $w\in(A\cdot x_{i,j}x_{i,j+1}^2\cdot w_1)\in B^n$ or $w\in(A\cdot h_{i,j-1}x_{i,j+1}^2\cdot\frac{w_1}{h_{i,j-1}})\in B^n$ by Lemma 2.4(3). If $x_{i,j+1}|w_0$ and $h_{i,j+1}|w_1$ and ($x_{i,j}|w_0$ or $h_{i,j-1}|w_1$), then $w\in(A\cdot x_{i,j}x_{i,j+1}h_{i,j+1}\cdot\frac{w_1}{h_{i,j+1}})\in B^n$ by Lemma 2.4(4) or $w\in(A\cdot x_{i,j+1}h_{i,j-1}h_{i,j+1}\cdot\frac{w_1}{h_{i,j-1}h_{i,j+1}})\in B^n$ by Lemma 2.4(5). Finally, if $h_{i,j+1}^2|w_1$ and ($x_{i,j}|w_0$ or $h_{i,j-1}|w_1$), then $w\in(A\cdot x_{i,j}h_{i,j+1}^2\cdot\frac{w_1}{h_{i,j+1}^2})\in B^n$ by Lemma 2.4(6) or $w\in(A\cdot h_{i,j-1}h_{i,j+1}^2\cdot\frac{w_1}{h_{i,j-1}h_{i,j+1}^2})\in B^n$ by Lemma 2.4(7). $\square$

Lemma 2.6. *Let P be a prime ideal that contains c and is associated to B^n. We know that $P=B^n:w$ for some monomial w. Then*

(1) *$w=a_1^4\cdots a_r^4w_0$ for some $w_0\in X^{n-1}$.*

(2) *If $x_{i,j}$ and $x_{i,j+1}$ are in P and $x_{i,j}$ is not a factor of w, then $r\geq 3$ and $x_{i,j+3}\in P$.*

(3) *If $x_{i,j}$, $x_{i,j+1}$, $x_{i,j+2}\in P$ and $r\geq 3$, then $x_{i,j}$ is a factor of w.*

(4) *Suppose that* $x_{i,j}x_{i,j+1}^2$ *divides* w_0 *and that* $x_{i,j+1}\in P$. *Then* $x_{i,j}^2$ *divides* w.

(5) *Suppose that* $x_{i,1}^2x_{i,2}^2\cdots x_{i,r}^2$ *divides* w. *Then* $n\geq r+1$ *and* w_0 *is an element of* $h_{i,1}h_{i,2}\cdots h_{i,r}X^{n-1-r}$.

Proof. Since $c\in P$, we know by Lemma 1.2 that $w\in B^n:c=a_1^4\cdots a_r^4X^{n-1}+(B_0+X)^n$. Since w is a monomial not in B^n, necessarily $w\in a_1^4\cdots a_r^4X^{n-1}$. This proves (1).

To simplify notation we assume in the rest of the proof that $j=1$.

(2) Suppose that $x_{i,1}$ does not divide w_0 (or w). Then by Lemma 2.2(2), $w\in a_1^4x_{i,2}^2B^{n-2}\cap a_2^4x_{i,3}^2B^{n-2}$ and so necessarily $r\geq 3$. This means that w_0 is a multiple of $x_{i,2}^2x_{i,3}^2$. Write $w_0=h_{i,2}^e h_{i,3}^{e'}w'$ for some non-negative integers e,e' and some $w'\in X^{n-1-e-e'}$. We may take e to be maximal possible, and for the maximal e we choose maximal possible e', so that in particular $h_{i,2}$ and $h_{i,3}$ are not factors of w'. By assumption also no $h_{i,1},h_{i,r}$ appear in w'. First suppose that $e=0$. Then by the $(x_{i,2},x_{i,3})$-degree count, $w_0\in x_{i,2}x_{i,3}^2X^{n-1}+x_{i,2}x_{i,3}h_{i,3}X^{n-2}+x_{i,2}h_{i,3}^2X^{n-3}$, whence by Lemma 2.4, $w\in B^n$. So necessarily $e\geq 1$. Hence by Lemma 2.4(3), $x_{i,4}^2w\in B^n$, so that $x_{i,4}\in P$. This proves (2).

We continue with the proof of (3). Recall that we assume that $x_{1,i}$ does not divide w. Since $x_{i,2}w\in B^n$, necessarily in the rewriting of $x_{i,2}w$ as an element of B^n, $x_{i,2}$ must combine with $x_{i,3}^2$ into a new $h_{i,2}$, i.e., $w/h_{i,2}^e\in a_2^4x_{i,3}^2B^{n-1-e}$. Thus, $x_{i,2}$ is not a factor of $w_0/h_{i,2}^e$ for otherwise $w\in B^n$. In addition, $x_{i,3}w\in B^n$ which implies that $x_{i,3}$ needs to recombine with w into a new element of B which necessarily is $h_{i,3}$. Thus, w_0 must have a factor of $x_{i,4}^2$, which comes either as $x_{i,4}^2$, $x_{i,4}h_{i,4}$, or $h_{i,4}^2$. But since the exponent on $h_{i,2}$ in w is at least 1, then by Lemma 2.4(3), (5), and (7), $w\in B^n$, which is a contradiction, and thus proves (3).

(4) Suppose that $x_{i,1}^2$ does not divide w_0. By assumption $x_{i,1}x_{i,2}^2$ is a factor of w_0, by the $(x_{i,1},x_{i,2})$-degree count, $w_0\in h_{i,1}X^{n-2}+x_{i,1}x_{i,2}^2X^{n-1}+x_{i,1}x_{i,2}h_{i,2}X^{n-2}+x_{i,1}h_{i,2}^2X^{n-3}$. If w_0 is in one of the last three summands, then $w\in B^n$ by Definition or Lemma 2.4(4) and (6). So we may assume that $w_0\in h_{i,1}X^{n-2}$. Since $x_{i,2}w\in B^n$, this $x_{i,2}$ must recombine with w into a new $h_{i,1}$ or $h_{i,2}$, but since there are no spare $x_{i,1}$ in w, necessarily $x_{i,3}^2$ is a factor of w_0. Thus, by the $x_{i,3}$-degree count, $w_0\in x_{i,3}^2h_{i,1}X^{n-2}+x_{i,3}h_{i,1}h_{i,3}X^{n-3}+h_{i,1}h_{i,3}^2X^{n-4}+h_{i,1}h_{i,2}X^{n-3}$. If w_0 is in the last summand, then no new $h_{i,2}$ would be formed. We can, therefore, assume that w_0 is in the first three summands. But then $w\in B^n$ by Lemma 2.4(3), (5), and (7). Thus $x_{i,1}^2$ must be a factor of w.

(5) Let E be the largest subset of $[r]$ such that $w_0\in(\prod_{j\in E}h_{i,j})X^{n-1-|E|}$. If E is empty, then $n=1$ and $w\in a_j^4x_{i,j}x_{i,j+1}^2\subseteq B=B^n$, which is a contradiction. Thus E is not empty. By symmetry we may assume that $h_{i,1}$ is a factor and for contradiction we assume that $h_{i,2}$ is not a factor. By the $(x_{i,1},x_{i,2})$-degree count, $w_0\in x_{i,1}h_{i,1}X^{n-2}+h_{i,1}^2X^{n-3}+h_{i,r}h_{i,1}X^{n-3}$. If $r=2$, the last summand is not possi-

ble by assumption and the first two summands make w be in B^n by Lemma 2.4(1) and (2), which proves that $r \geq 3$.

If $h_{i,3}$ is also not a factor of w_0, then by the $x_{i,3}$-degree count, $w_0 \in h_{i,1}x_{i,3}^2 X^{n-2}$, which means that $w \in B^n$ by Lemma 2.4(3). This proves that $h_{i,3}$ must be a factor of w_0, and consequently that E contains at least every other $h_{i,j}$ as j varies. Now say that $h_{i,1}, h_{i,3}$ are factors but $h_{i,2}$ is not. By the $x_{i,3}$-degree count again, $w_0 \in h_{i,1}x_{i,3}h_{i,3}X^{n-3} + h_{i,1}h_{i,3}^2 X^{n-3}$, so that $w \in B^n$ by Lemma 2.4(5) and (7). This proves (5). □

3. G-good primes

The set-up is as in Section 2 with $B = \mathrm{BHH}(m, r, 1)$ and n a positive integer. In this section we characterize all associated primes of powers of B that are g-good. We prove that all associated primes that contain c are g-good, which characterizes and counts all associated primes of powers of B that contain c. Theorem 3.11 counts associated primes of any power of $\mathrm{BHH}(m, 2, s)$ and Theorem 3.13 determines the maxima of the numbers of these associated primes. In Proposition 3.10 we prove the persistence property of associated primes of powers not containing c.

We think of the mr variables $x_{i,j}$ as appearing in an $m \times r$ matrix. If a monomial prime ideal does not contain all $x_{i,j}$ in some row i, then we talk about **gaps**, and if a prime ideal omits some k consecutive $x_{i,j}$ in a row i, we refer to that as a gap of **length** k. Keep in mind that we identify $x_{i,r}$ with $x_{i,0}$, et cetera, so that the gaps are counted in the round.

Definition 3.1. Let P be a monomial prime ideal containing $(a_1, \dots, a_r)$. We say that P is **g-good** if it has no gaps of length 2 or larger in any of the rows.

We characterize in this section all associated primes of B^n that contain c in terms of g-good primes. The characterization enables a count, see Theorem 3.6. G-good primes also play a role for primes that do not contain c; see Theorems 3.8 and 3.9.

Proposition 3.2. *Let P be a g-good prime ideal containing c such that for each $i \in [m]$, the set $P \cap \{x_{i,1}, \dots, x_{i,r}\}$ has either r or exactly $r/2$ elements.* (*The latter happens only if r is even.*) *Let $U = \{i : x_{i,1}, \dots, x_{i,r} \in P\}$ and $V = \{(i,j) : x_{i,j+1} \notin P\}$.*

(1) *Suppose that $n = ur + v + 1$, where u and v are any non-negative integers such that $u \leq |U|$ and $v \leq |V|$. Then P is associated to B^n.*

(2) *Suppose that n cannot be written as in* (1). *Then P is not associated to B^n.*

Proof. Observe that $V = \{(i,j) : x_{i,j} \in P \text{ and } i \notin U\}$.

(1) Let U_0 be a subset of U of cardinality u and V_0 a subset of V of cardinality v. Let M be a large integer and set

$$(*)\qquad \begin{aligned} w_0 &= a_1^4 \cdots a_r^4 \Big(\prod_{i\in U\setminus U_0} \prod_{j=1}^{r} x_{i,j} \Big) \Big(\prod_{x_{i,j}\notin P} x_{i,j}^M \Big), \\ w &= w_0 \Big(\prod_{i\in U_0} \prod_{j=1}^{r} h_{i,j} \Big) \Big(\prod_{(i,j)\in V_0} h_{i,j} \Big). \end{aligned}$$

Then $w\in B^{ur+v}=B^{n-1}$. We will prove that $P=B^n:w$.

Since $a_j(a_1^4\cdots a_r^4)\in(a_j^5 a_{j+1})\in B_0\subseteq B$ and $c(a_1^4\cdots a_r^4)\in B_c\subseteq B$ it follows that $(a_1,\dots,a_r,c)\subseteq B^n:w$. Suppose that $i\in U_0$. Then for all $j\in[r]$, $x_{i,j}\in B^n:w$ by Lemma 2.4(1) in case $r=2$ and by Lemma 2.4(5) in case $r>2$. If $i\in U\setminus U_0$, then $x_{i,j}w\in x_{i,j}(a_{j-1}^4 x_{i,j-1}x_{i,j})X^{n-1}\subseteq X^n\subseteq B^n$. Thus $x_{i,j}\in B^n:w$ for all $i\in U$ and all $j\in[r]$. If $(i,j)\in V$, then $x_{i,j+1}\notin P$, so that $x_{i,j}w_0\in(a_j^4 x_{i,j}x_{i,j+1}^2)\subseteq X$ and thus $x_{i,j}\in B^n:w$. This proves that $P\subseteq B^n:w$.

To prove that $P=B^n:w$ it remains to show that every $x_{i,j}\notin P$ is a non-zero-divisor modulo B^n. By possibly taking M even larger it suffices to prove that $w\notin B^n$. In the given form w is an element of B^{n-1}. Any rewriting of w to make it an element of B^n has to involve the variables $x_{i,j}$ whose exponents are at least two. The only such $x_{i,j}$ are those not in P and those with $i\in U_0$. If $x_{i,j}\notin P$, then $x_{i,j+1}^2$ is not a factor of w and either $x_{i,j-1}$ is not a factor of w or else $x_{i,j-1}$ is a factor of w but tied up in $h_{i,j-1}$. Thus there is no possible way of using the rewriting with $x_{i,j}$ not in P. If $i\in U_0$, then this i contributes to w the factor $a_1^4\cdots a_r^4 x_{i,1}^3\cdots x_{i,r}^3\in B^r$, and there is no possible way of rewriting this part to put w into B^n. Thus $P=B^n:w$. It follows that P is associated to B^n. This finishes the proof of (1).

(2) Let P be associated to B^n and suppose for contradiction that n cannot be written as in (1). Then in particular $n>1$. By Lemma 2.6(1), $P=B^n:w$, where w is a monomial of the form $a_1^4\cdots a_r^4 w_0$ for some $w_0\in X^{n-1}$. The product of all $x_{i,j}$ with $i\in U$ divides w_0 by Lemma 2.6(2) in case $r=2$ and by Lemma 2.6(3) in case $r>2$. Let U_0 be the set of all $i\in U$ such that $h_{i,j}$ is a factor of w_0 for some $j\in[r]$. Let $i\in U_0$. Then by Lemma 2.6(4), $x_{i,j}^2$ divides w_0. Since $x_{i,j-1}$ also divides w_0, then again by Lemma 2.6(4), $x_{i,j-1}^2$ divides w_0. By continuing in this way we get that $\prod_{j=1}^r x_{i,j}^2$ divides w_0. Hence by Lemma 2.6(5), $\prod_{j=1}^r h_{i,j}$ divides w_0. By Lemma 2.4(2) and (7), $h_{i,j}^2$ is not a factor of w for all such i,j. Thus $n-1\ge|U_0|r$. Let $v=n-1-|U_0|r$. We just proved that w_0 is a product of the $|U_0|r$ factors $h_{i,j}$ with $(i,j)\in U_0\times[r]$ and v factors $h_{i,j}$ with $i\notin U$. By Lemma 2.4(3), necessarily for any such latter factor we have $x_{i,j+2}^2\in B^n:w=P$. Since P is g-good and the ith row has $r/2$ elements, necessarily $x_{i,j}\in P$ and $x_{i,j-1}$ is not in P. Then by Lemma 2.4(6), the squares of

these $h_{i,j}$ do not divide w. Thus these v factors are all distinct, which means that n must be written as in (1). $\square$

Theorem 3.3. *We consider the set S of all g-good prime ideals P containing c for which in each row of the matrix $[x_{i,j}]$, P contains either r or $r/2$ elements.*

(1) *If r is odd, then the maximal ideal is the only such prime ideal, and it is associated to B^n if and only if $n{=}ur{+}1$ for some $u{\le}m$.*

(2) *If r is even, then S contains 3^m prime ideals.*

(a) *For each $i{\in}\{0,...,m\}$ there exist $2^i\binom{m}{i}$ prime ideals in S of height $(m{+}1)r{-}i\frac{r}{2}{+}1$, and these are associated to B^n exactly when n equals $ur{+}v{+}1$ with $u{\in}\{0,...,m{-}i\}$ and $v{\in}\{0,...,i\frac{r}{2}\}$.*

(b) *The number $h(m,r,n)$ of elements of S that are associated to B^n equals*

$$\sum_{i=0}^{m} 2^i \binom{m}{i} \delta_{(n-1)/r-i/2\le\min\{q,m-i\}},$$

where $q{=}\lfloor\frac{n-1}{r}\rfloor$. For all $n{>}1{+}rm$, $h(m,r,n){=}0$.

Proof. (1) is an immediate corollary of Proposition 3.2.

To prove (2), observe that for $i{\in}[m]$, one of three things happen for $P{\in}S$: P contains the full ith row of $[x_{i,j}]$, P contains $x_{i,j}$ with j odd, and P contains $x_{i,j}$ with j even. Thus the count of elements of S is 3^m. For each $i{\in}\{0,1,...,m\}$, there are $\binom{m}{i}$ possibilities where exactly $m{-}i$ of the rows are fully in P, and the remaining i rows have two options. All these prime ideals contain also $a_1,...,a_r,c$, so that their height is $r{+}1{+}(m{-}i)r{+}i\frac{r}{2}={(m{+}1)r{-}i\frac{r}{2}{+}1}$.

According to Proposition 3.2, P is associated to B^n if and only if there exist integers $u{\in}\{0,...,m{-}i\}$ and $v{\in}\{0,...,i\frac{r}{2}\}$ such that $n{-}1{=}ur{+}v$. The rest of (2)(a) is an immediate corollary of Proposition 3.2.

For (2)(b) we need to account which n are possible. Note that $n{-}1{=}ur{+}v{\le}(m{-}i)r{+}ir/2{\le}mr$. Thus $h(m,r,n){=}0$ if $n{-}1{>}mr$. The possible u are $0,1,...,m{-}i$, if simultaneously $0{\le}v{=}n{-}1{-}ur{\le}ir/2$. Another way of recording this is with $\max\{0,(n{-}1)/r{-}i/2\}{\le}u{\le}\min\{q,m{-}i\}$. The assertion in (2)(b) follows because $\min\{q,m{-}i\}{\ge}0$. $\square$

Theorem 3.4. *Let P be a g-good monomial prime ideal containing c. Suppose that there exists $(i_0,j_0){\in}[m]{\times}[r]$ such that $x_{i_0,j_0-1},x_{i_0,j_0}{\in}P$ and $x_{i_0,j_0+1}{\notin}P$. Then P is associated to B^n for all $n{\ge}1$.*

Proof. The assumption on i_0,j_0 forces $r{\ge}3$. Let T_1 be the set of all $x_{i,j}$ not in P and T_2 the set of all $x_{i,j}{\in}P$ such that $x_{i,j+1}{\in}P$. In case $n{\ge}3$ we correct T_2 to not include x_{i_0,j_0-1}. For any large integer M we set

$$w_0=a_1^4\cdots a_r^4\left(\prod_{t\in T_1}t^M\right)\left(\prod_{t\in T_2}t\right),\quad w=w_0h_{i_0,j_0}^{n-1}.$$

Since $(a_1,...,a_r,c)a_1^4\cdots a_r^4\in B$, it follows that $(a_1,...,a_r,c)\subseteq B^n{:}w$. If $x_{i,j}\in P$ and $x_{i,j+1}\notin P$, then $x_{i,j}w\in B^n$ since $a_j^4x_{i,j+1}^2$ is a factor of w_0 and $h_{i,j}=a_j^4x_{i,j}x_{i,j+1}^2\in B$. If $n\geq 3$, then $x_{i_0,j_0-1}w\in(h_{i_0,j_0-1})(a_{j_0}^{12})(h_{i_0,j_0}^{n-3})\subseteq B^{1+2+n-3}=B^n$. In all other cases, if $x_{i,j},x_{i,j+1}\in P$, then $x_{i,j}w\in B^n$ since $a_{j-1}^4x_{i,j-1}x_{i,j}$ is a factor of w_0 and $h_{i,j-1}=a_{j-1}^4x_{i,j-1}x_{i,j}^2\in B$. This proves that $P\subseteq B^n{:}w$.

We next prove that $B^n{:}w\subseteq P$, i.e., that no power of a variable in T_1 is in $B^n{:}w$. By possibly taking M larger it suffices to prove that $w\notin B^n$. In the given form w is an element of B^{n-1}. Any rewriting of w to make it an element of B^n has to involve the variables $x_{i,j}$ whose exponents are at least two. The only such variables are those in T_1 and additionally x_{i_0,j_0} if $n-1\geq 2$. By the g-goodness assumption, the variables in T_1 do not have consecutive second indices and T_2 does not contain suitable "predecessors" to form a new $h_{i,j-1}$ with a variable $x_{i,j}\in T_1$. So necessarily $n\geq 3$, but then x_{i_0,j_0-1} is not a factor of w so it is not possible to recombine x_{i_0,j_0}^2 with that missing factor and no other rewriting is possible. Thus $w\notin B^n$.

Thus $P=B^n{:}w$ so that P is associated to B^n. □

Lemma 3.5. *The number of g-good primes* (*either all containing* c *or none containing* c) *equals the Lucas number* L_r^m (*with* $L_1=1$, $L_2=3$, $L_{r+2}=L_{r+1}+L_r$).

Proof. Note that the number of g-good primes of either type is equal to L_r^m where L_r is their number for the case $m=1$. We will ignore containments of $a_1,...,a_r$ in this proof.

In case $r=1$, the only g-good prime contains $x_{1,1}$, so L_1 is 1. The g-good options in case $r=2$ are $(x_{1,1})$, $(x_{1,2})$, and $(x_{1,1},x_{1,2})$, so $L_2=3$. Let U_r be the number of g-good primes that contain $x_{1,1}$ and $x_{1,r}$, and for $r>1$ let V_r be the number of g-good primes that contain $x_{1,1}$ and not $x_{1,r}$, and let $\overline{V}_r$ be the number of g-good primes that contain $x_{1,r}$ and not $x_{1,1}$. Clearly $\overline{V}_r=V_r$, $V_r=U_{r-1}$, and $L_r=U_r+V_r+\overline{V}_r=U_r+2V_r$. But $U_{r+1}=U_r+U_{r-1}$ (depending on whether $r-1$ is or is not in the subset), $U_1=1$, $U_2=1$, and $U_3=2$, which says that $U_1,U_2,U_3,...$ are the usual Fibonacci numbers, and so $V_2,V_3,V_4,...$ are also the usual Fibonacci numbers. Then

$$\begin{aligned}L_{r+1}+L_r&=U_{r+1}+2U_r+U_r+2U_{r-1}=(U_{r+1}+U_r)+2(U_r+U_{r-1})=U_{r+2}+2U_{r+1}\\&=L_{r+2},\end{aligned}$$

and so these numbers are the Lucas numbers. □

Theorem 3.6. *The number of prime ideals associated to* $\mathrm{BHH}(m,r,1)^n$ *that contain* c *is equal to*

$$\begin{cases} L_r^m - 3^m + h(m,r,n), & \text{if } r \text{ even;} \\ L_r^m, & \text{if } r \text{ odd, } n \equiv 1 \mod r \text{ and } n \leq rm+1; \\ L_r^m - 1, & \text{otherwise,} \end{cases}$$

where L_r *is the* r*th Lucas number with* $L_1=1$ *and* $L_2=3$ *and* $h(m,n,r)$ *refers to the number in Theorem* 3.3(b).

Proof. By Lemma 2.3(4), every prime ideal associated to B^n that contains c must be g-good.

Assume first that r is odd. In this case, according to Theorem 3.3(1), the maximal ideal is associated if and only if $n=ur+1$ for some integer $u \leq m$. Any other one of the L_r^m possible prime ideals satisfies condition of Theorem 3.4 and is thus associated to B^n for all n. This proves the theorem for odd r by Lemma 3.5.

Now, assume that r is even. Of the L_r^m possible prime ideals as accounted for by Lemma 3.5, those for which some row in the matrix $[x_{i,j}]$ is neither half-full nor full are covered by Theorem 3.4 and are thus associated to all powers of B. It remains to count those prime ideals associated to B^n for which each row in $[x_{i,j}]$ is either half- full or full. According to Theorem 3.3(2), there are 3^m prime ideals with only full and half-full levels, of which $h(m,r,n)$ are associated to B^n. The theorem follows. □

Example 3.7. The following tables of numbers of associated primes of $\mathrm{BHH}(m,r,1)^n$ that contain c are taken from Theorem 3.6 and agree with the calculations[1] by Macaulay2 [5] and Magma [3] of associated primes for low values of n.

$r=2$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	3	2	1	0	0	0								
2	9	8	9	4	1	0								
3	27	26	27	26	19	6	1	0						
4	81	80	81	80	81	64	33	8	1	0				
5	243	242	243	242	243	242	211	130	51	10	1	0		
6	729	728	729	728	729	728	729	664	473	232	73	12	1	0

[1] The program code associated with this paper is available as ancillary file on the arXiv page of this paper (arXiv:2309.15083).

$r=4$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	7	6	6	4	5	4	4	4	4	4	4	4
2	49	48	48	44	49	44	44	40	41	40	40	40
3	343	342	342	336	343	342	342	328	335	322	322	316
4	2401	2400	2400	2392	2401	2400	2400	2392	2401	2384	2384	2344
5	16807	16806	16806	16796	16807	16806	16806	16796	16807	16806	16806	16764

We have finished a characterization of all associated primes of B^n that contain c.

In contrast, we do not have a complete characterization of the prime ideals associated to B^n that do not contain c. Of these, we understand the g-good ones well: by Theorem 3.8, the number of such is L_r^n if $n\geq 2$, but the count is smaller for $n=1$ by Theorem 3.9.

Theorem 3.8. *Let P be a g-good monomial prime ideal that does not contain c and let $n\geq 2$. Then P is associated to B^n. The number of such primes is L_r^n.*

Proof. Set

$$e_n=\begin{cases}5n-5, & \text{if } n=2,3,4;\\ 6n-9, & \text{if } n\geq 4.\end{cases}$$

Note that

$$a_1^{e_n+3}=\begin{cases}a_1^{5n-2}\in(a_1^6)^{n-1}\subseteq B^{n-1}, & \text{if } n=2,3,4;\\ a_1^{6n-6}\in(a_1^6)^{n-1}\subseteq B^{n-1}, & \text{if } n\geq 4,\end{cases}$$

$$a_1^{e_n}a_2^4=\begin{cases}a_1^{5(n-1)}a_2^4\in(a_1^5a_2)^{n-1}\subseteq B^{n-1}, & \text{if } n=2,3,4;\\ a_1^{6n-9}a_2^4\in(a_1^5a_2)^4(a_1^6)^{n-5}\subseteq B^{n-1}, & \text{if } n\geq 5,\end{cases}$$

$$a_1^{e_n+5}a_2^4=\begin{cases}a_1^{5n}a_2^4\in(a_1^5a_2)^{n}\subseteq B^{n}, & \text{if } n=2,3,4;\\ a_1^{6n-4}a_2^4\in(a_1^5a_2)^4(a_1^6)^{n-4}\subseteq B^{n}, & \text{if } n\geq 4.\end{cases}$$

With $w_0=a_1^{e_n}a_1^4\cdots a_r^4$, we have that

$$\begin{aligned}a_1w_0&\in(a_1^{e_n+5}a_2^4)\subseteq B^n,\\ a_jw_0&\in(a_j^5a_{j+1})a_1^{e_n+3}\subseteq B^n \quad \text{if } j\in\{2,\ldots,r\},\end{aligned}$$

so that $(a_1,\ldots,a_r)w_0\subseteq B^n$.

Let X_P be the product of powers of the $x_{i,j}$, where

$$\text{the exponent of } x_{i,j} \text{ in } X_P=\begin{cases}2, & \text{if } x_{i,j}\notin P;\\ 1, & \text{if } x_{i,j}\in P \text{ and } x_{i,j+1}\in P;\\ 0, & \text{if } x_{i,j}\in P \text{ and } x_{i,j+1}\notin P.\end{cases}$$

Set $w = cw_0 X_P$. We will prove that $P = B^n : w$. We have established that $(a_1, \ldots, a_r)w \subseteq B^n$. Now let $x_{i,j} \in P$. If $x_{i,j+1} \notin P$, then $x_{i,j+1}^2$ divides X_P, and so

$$x_{i,j} w \in (a_j^4 x_{i,j} x_{i,j+1}^2) \cdot \begin{cases} a_1^{e_n} a_2^4, & \text{if } j = 1; \\ a_1^{e_n + 3}, & \text{if } j \neq 1, \end{cases}$$

which is in B^n. If instead $x_{i,j+1} \in P$, then $x_{i,j-1} x_{i,j}$ divides X_P, and so

$$x_{i,j} w \in (a_{j-1}^4 x_{i,j-1} x_{i,j}^2) \cdot \begin{cases} a_1^{e_n} a_2^4, & \text{if } j = 2; \\ a_1^{e_n + 3}, & \text{if } j \neq 2, \end{cases}$$

which is again in B^n. This proves that $Pw \subseteq B^n$.

To prove that $B^n : w \subseteq P$, we need to prove that $zw \notin B^n$, where z is a high power of a product of c and all the $x_{i,j}$ that are not in P. Any rewriting of zw as an n-fold product of elements in B cannot use any generator of $\underline{a}$-degree 4 because X_P contains no factors of the form $x_{i,j} x_{i,j+1}^2$. Thus the n factors in this rewriting are taken from the following list: $a_1^6, a_1^5 a_2, c a_1^4 \cdots a_r^4$. If the latter factor appears, then zw would have to be a multiple of $(a_1^6)^{n-1} c a_1^4 \cdots a_r^4$, but the a_1-degree is then too high. Thus the only possible factors are a_1^6 and $a_1^5 a_2$. It is easy to see that this is not possible if $n = 2, 3$, and for $n \geq 4$, the total degree $e_n + 8 = 6n - 1$ of a_1 and a_2 in zw would have to be at least $6n$, which is a contradiction. This finishes the proof that $B^n : w = P$, so that P is associated to B^n.

The number of such primes was determined in Lemma 3.5. □

Theorem 3.9. *Let P be a g-good monomial prime ideal that does not contain c. Then P is associated to B if and only if there exists $j_0 \in [r]$ such that for all $i \in [m]$, either $x_{i,j_0} \notin P$ or $x_{i,j_0+1} \notin P$.*

When $r = 2$, the number of such P is exactly 2^m.

Proof. By Lemma 2.1, if P is associated then such a j_0 must exist. Now suppose that j_0 exists. By possibly replacing j_0 with $j_0 + 1$ we may assume that there exists $i \in [m]$ such that $x_{i,j_0} \notin P$. By re-indexing we may assume that $j_0 = 1$.

Let X_P be the product of various powers of the $x_{i,j}$, where

$$\text{the exponent of } x_{i,j} \text{ in } X_P = \begin{cases} 2, & \text{if } x_{i,j} \notin P; \\ 1, & \text{if } x_{i,j} \in P \text{ and } j = r; \\ 1, & \text{if } x_{i,j} \in P,\ j \neq r \text{ and } x_{i,j+1} \in P; \\ 0, & \text{if } x_{i,j} \in P,\ j \neq r \text{ and } x_{i,j+1} \notin P, \end{cases}$$

and set $w = c a_r^3 \cdot (\prod_{j=1}^{r-1} a_j^4) X_P$. We will prove that $P = B : w$.

We have that $a_r w = (c a_1^4 a_2^4 \cdots a_r^4) \subseteq B$, and for $j \in \{1, ..., r-1\}$, we have that $a_j w \in (a_j^5 a_{j+1}) \subseteq B$. This proves that $(a_1, ..., a_r) w \subseteq B$.

Now let $x_{i,j} \in P$. We need to prove that $x_{i,j} w \in B$. If $j=r$, then $x_{i,r-1} x_{i,r}$ divides X_P and a_{r-1}^4 divides w. Hence, $x_{i,r} w \in (a_{r-1}^4 x_{i,r-1} x_{i,r}^2) \subseteq B$. If $j=1$, by the definition of j_0, $x_{i,2}$ is not in P. Thus $a_1^4 x_{i,2}^2$ is a factor of w, so that $x_{i,1} w \in (a_1^4 x_{i,1} x_{i,2}^2) \subseteq B$. Now let $j \in \{2, ..., r-1\}$. We need to prove that $x_{i,j} w \in B$. So necessarily $r>2$. Then X_P is a multiple of $x_{i,j+1}^2$ if $x_{i,j+1} \notin P$, or else it is a multiple of $x_{i,j-1} x_{i,j}$, so that

$$x_{i,j} w \in \begin{cases} (a_j^4 x_{i,j} x_{i,j+1}^2) \subseteq B, & \text{if } x_{i,j+1} \notin P; \\ (a_{j-1}^4 x_{i,j-1} x_{i,j}^2) \subseteq B, & \text{if } x_{i,j+1} \in P. \end{cases}$$

This finishes the proof that $P \subseteq B{:}w$.

It remains to prove that $B{:}w \subseteq P$. It suffices to prove that $zw \notin B$, where z is a high power of a product of c with all $x_{i,j}$ that are not in P. Since the a_r-degree of w is 3, the rewriting of zw as an element of B would not use the one generator of B that involves c. So c plays no role in this rewriting. If $x_{i,j} \notin P$, then both $x_{i,j-1}$ and $x_{i,j+1}$ must be in P. Thus $x_{i,j+1}^2$ is not a factor of zw and $x_{i,j-1}$ is a factor of zw exactly if $j-1=r$. In that case, a_{j-1}^4 is not a factor of zw, which means that no rewriting of zw as an element of B can use $a_{j-1}^4 x_{i,j-1} x_{i,j}^2$ or $a_j^4 x_{i,j} x_{i,j+1}^2$. Also, no factor of this form already appears in w. But then by the consideration of exponents of the a_j in w, $zw \notin B$. □

We have handled all the g-good prime ideals associated to powers of B. There are further associated primes that do not contain c. In the rest of this section we prove their persistence property and we completely describe and enumerate them in case $r=2$. Persistence definitely fails on associated primes that do contain c.

Proposition 3.10. (Persistence of associated primes) *Let P be a prime ideal associated to B^n that does not contain c. Then P is associated to B^{n+1}.*

Proof. If P is g-good, then P is associated to all B^{n+1} by Theorem 3.8.

So we may assume that there exists $i \in [m]$ and $j \in [r]$ such that $x_{i,j}, x_{i,j+1} \notin P$. Then P is associated to B^n if and only if it is associated to B^n after inverting $x_{i,j} x_{i,j+1}$. But then a_j^4 is a minimal generator of B, and the only other minimal generator of B (after this inversion) in which a_j appears is $a_{j-1}^5 a_j$. Write $P = B^n{:}w$ for some monomial w. Then

$$\begin{aligned} B^{n+1} : a_j^4 w &= (B^{n+1} : a_j^4) : w \\ &= (B^n + a_{j-1}^5 B^n + a_{j-1}^{10} B^{n-1} + a_{j-1}^{15} B^{n-2} + a_{j-1}^{20} B^{n-3}) : w \\ &= B^n : w = P, \end{aligned}$$

so that P is associated to B^{n+1} as well. □

Theorem 3.11. *Let m, $s \geq 1$. The set of associated primes of* $\mathrm{BHH}(m,2,s)^n$ *is the union* $\{(a_1,a_2)\} \cup \bigcup_{i=1}^m Q_c^{(i)} \cup Q_1 \cup Q_2$ *where*

$$\begin{aligned} Q_c^{(i)} &= \{P \mid n = 2u+v+1 \text{ with } 0 \leq u \leq m-i, 0 \leq v \leq i \text{ and} \\ &\qquad P \text{ has } i \text{ half-full and } m-i \text{ full rows}\}, \\ Q_1 &= \{P \text{ g-good} \mid c \notin P, n=1, \exists j_0 \in [r]\, \forall i \in [m] : x_{i,j_0} \notin P \text{ or } x_{i,j_0+1} \notin P\}, \\ Q_2 &= \{P \text{ g-good} \mid c \notin P, n \geq 2\}. \end{aligned}$$

The number of associated primes of $\mathrm{BHH}(m,2,s)^n$ *is equal to*

$$(3-\delta_{1=n})^m + 1 + \sum_{i=0}^m 2^i \binom{m}{i} \delta_{(n-1-i)/2 \leq \min\{q, m-i\}},$$

where $q = \lfloor \frac{n-1}{2} \rfloor$.

Proof. By Theorem 1.4, the set of associated primes of $\mathrm{BHH}(m,2,s)^n$ is equal to the set of associated primes of $\mathrm{BHH}(m,2,1)^n$. By Theorems 3.6 and 3.3, the set of associated primes of $\mathrm{BHH}(m,2,1)^n$ that contain c equals $\bigcup_{i=1}^m Q_c^{(i)}$ and its cardinality is $h(m,2,n) = \sum_{i=0}^m 2^i \binom{m}{i} \delta_{(n-1-i)/2 \leq \min\{q,m-i\}}$, where $q = \lfloor \frac{n-1}{2} \rfloor$.

Let P be associated to $\mathrm{BHH}(m,2,1)^n$ and not contain c. If P does not contain $x_{i,1}x_{i,2}$, then P is associated to B^n if and only if it is associated to $B^n : (x_{i,1}x_{i,2})^\infty = (a_1^4, a_2^4)^n$, in which case P must be equal to (a_1,a_2), which is minimal over B and hence associated to all the powers of B.

Thus it remains to consider the associated primes P not containing c that contain $x_{i,1}x_{i,2}$ for all $i \in [m]$. Then P must be g-good, and in Q_1 if $n=1$ by Theorem 3.9 (which has cardinality 2^m) and in Q_2 if $n \geq 2$ by Theorem 3.8 (which has cardinality 3^m).

The assertion follows. □

Remark 3.12. The number of associated primes of $\mathrm{BHH}(m,2,s)^n$ can also be written as

$$(3-\delta_{1=n})^m + \left(\sum_{\ell=0}^m \sum_{t=b(\ell)}^m \binom{m}{\ell} \binom{\ell}{\ell+t-m} \right) + \begin{cases} 0, & \text{if } n \leq 2m \text{ and } n \text{ is even;} \\ 1, & \text{otherwise,} \end{cases}$$

where $b(\ell) = \max\{n-1-\ell, m-\ell\}$. Namely, the first summand in the display plus 1 is the number of associated primes not containing c. The maximal ideal is associated to $\mathrm{BHH}(m,2,1)^n$ if and only if $n \leq 2m+1$ and n is odd. These two counts account for the first and the last summand in the display. It remains to count the non-maximal associated primes P that contain c. We know that for all $i \in [m]$, $x_{i,1}x_{i,2} \in P$. Let ℓ be the number of $x_{i,1}$ in P and let t be the number of $x_{i,2}$ in P. Necessarily $\ell + t \geq m$.

Also, $\ell+t$ should be at least $2u+v=n-1$ as in the notation of Theorem 3.3. There are $\binom{m}{\ell}$ ways of choosing ℓ of the variables $x_{i,1}$, after which for the remaining $m-\ell$ rows in the matrix $[x_{i,j}]$, the elements $x_{i,2}$ must be in P. This leaves $t-(m-\ell)$ variables $x_{i,2}$ to be chosen from the ℓ rows with the $x_{i,1}$. This justifies the middle summand in the display.

We just proved the following combinatorial identity:

$$\begin{aligned}&\sum_{i=0}^{m} 2^i \binom{m}{i} \delta_{n-1\le\min\{2q+i,2m-i\}}\\ &\quad=\sum_{\ell=0}^{m}\sum_{t=b(\ell)}^{m}\binom{m}{\ell}\binom{\ell}{\ell+t-m}+\begin{cases}-1, & \text{if } n\le 2m \text{ and } n \text{ is even;}\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

where $b(\ell)=\max\{n-1-\ell, m-\ell\}$.

Theorem 3.13. *For $m\ge 1$, the function ϕ taking $n\mapsto \#\operatorname{Ass}(R/\mathrm{BHH}(m,2,s)^n)$ has exactly $\lceil\frac{m-1}{2}\rceil$ local maxima. The local maxima occur at $n=3,5,\ldots,2\lceil\frac{m-1}{2}\rceil+1$, and they are all equal to the global maximum $2\cdot 3^m+1$.*

Proof. We refer to the three summands in the display in Remark 3.12 as ϕ_1,ϕ_2,ϕ_3 (in the given order). Observe that ϕ_1 is constant for $n\ge 2$, that ϕ_2 is zero for all $n\ge 2m+2$, and that ϕ_3 is constant for $n\ge 2m+1$. Thus ϕ is constant for $n\ge 2m+2$.

In the range $n=1,\ldots,m+1$, ϕ_2 equals

$$\sum_{\ell=0}^{m}\sum_{t=m-\ell}^{m}\binom{m}{\ell}\binom{\ell}{\ell+t-m}=(1+1+1)^m=3^m,$$

after which it strictly decreases to 0 at $n=2m+2$. Thus $\phi(1)=2^m+3^m+1<2\cdot 3^m=\phi(2)<2\cdot 3^m+1=\phi(3)$, and this is equal to $\phi(n)$ for all odd $n\in\{3,\ldots,m+1\}$. In other words, $\phi(n)=\phi(3)$ for all $n=3,5,\ldots,2\lceil\frac{m-1}{2}\rceil+1$. This value is strictly larger than $2\cdot 3^m=\phi(4)=\phi(6)=\cdots=\phi(2\lceil\frac{m-1}{2}\rceil)$, and is also strictly larger than $\phi(2\lceil\frac{m-1}{2}\rceil+2)$.

Furthermore, for $n\in\{m+1,m+2,\ldots,2m\}$,

$$\phi_2(n)-\phi_2(n+1)=\sum_{\ell=0}^{m}\binom{m}{\ell}\binom{\ell}{\ell+(n-1-\ell)-m}=\sum_{\ell=n-1-m}^{m}\binom{m}{\ell}\binom{\ell}{n-1-m}\ge 2.$$

Thus $\phi(n)>\phi(n+1)$ for $n\in\{m+1,m+2,\ldots,2m\}$. Finally,

$$\phi(2m+1)-\phi(2m+2)=\sum_{\ell=0}^{m}\binom{m}{\ell}\binom{\ell}{\ell+(2m+1-1-\ell)-m}=\sum_{\ell=0}^{m}\binom{m}{\ell}\binom{\ell}{m}=1,$$

so that $\phi(n)>\phi(n+1)$ for $n\in\{m+1,m+2,\ldots,2m+1\}$. This finishes the proof. $\square$

4. Depth

The depth of quotients of powers of $\mathrm{BHH}(m,r,s)$ depend on s, so in this section we return to arbitrary s.

Lemma 4.1. *Set* $B=\mathrm{BHH}(m,r,s)$. *Let* $w=a_1^{e_n}a_1^4\cdots a_r^4\prod_{i,j}x_{i,j}$ *where* e_n *is defined as in the proof of Theorem 3.8. If* $n\geq 2$, *then* $w\notin B^n$ *and* w *multiplies* $(a_j,x_{i,j}: i\in[m], j\in[r])$ (*but not* $c_1,\dots,c_s$) *into* $(B_0+X)^n$.

Let $u_1,\dots,u_s$ *be linear forms with* u_j *of the form* c_j *minus a linear combination* d_j *in the variables* $x_{i,j'}$ *as* i,j' *vary in* $[m]$ *and* $[r]$, *respectively. Then* $w\notin B^n+(u_1,\dots,u_s)$ *and* $w(a_j,x_{i,j}: i\in[m], j\in[r])\in B^n+(u_1,\dots,u_s)$.

Proof. The first paragraph is an immediate consequence of the proof of Theorem 3.8.

For the second paragraph, it is still the case that w multiplies the a_j, $x_{i,j}$ (but not $c_1,\dots,c_s$) into the ideal $C=B^n+(u_1,\dots,u_s)$. It remains to prove that $w\notin C$. By Lemma 1.2 we can rewrite C as $(c_1,\dots,c_s)a_1^4\cdots a_r^4X^{n-1}+(B_0+X)^n+(u_1,\dots,u_s) = (d_1,\dots,d_s)a_1^4\cdots a_r^4X^{n-1}+(B_0+X)^n+(u_1,\dots,u_s)$. Without restriction, we can switch to the polynomial ring where $u_1,\dots,u_s$ are variables and, for $1\leq j\leq s$, c_j are the linear forms in u_j and d_j. Since $u_1,\dots,u_s$ do not appear in w or in any minimal generating set of $(d_1,\dots,d_s)a_1^4\cdots a_r^4X^{n-1}+(B_0+X)^n$, it follows that if w is in C, then $w\in(d_1,\dots,d_s)a_1^4\cdots a_r^4X^{n-1}+(B_0+X)^n$, so that w multiplies $c_1,\dots,c_s$ into B^n, which is a contradiction. $\square$

Theorem 4.2. *For any positive integers* r,m,s,e *with* $r\geq 2$, *there exists an ideal* I *in a polynomial ring* A *such that for all positive integers* n,

$$\operatorname{depth}\left(\frac{A}{I^n}\right)=\begin{cases} e-1, & \text{if } n=ru+1 \text{ with } u=0,\dots,m;\\ e, & \text{if } n\leq rm+1 \text{ and } n\not\equiv 1 \mod r;\\ s+e-1, & \text{otherwise, i.e., if } n>mr+1.\end{cases}$$

In particular, the depth function $n\mapsto\operatorname{depth}(A/I^n)$ *has* $m+1$ *local minima, it is periodic of period* r *when restricted to the domain* $[1,r(m+1)-1]$, *and it is constant afterwards.*

Proof. Set $B=\mathrm{BHH}(m,r,s)$ in the ambient polynomial ring R. Let A be the polynomial ring obtained from R by replacing the variable c_1 with variables $d_1,\dots,d_e$. Let $\varphi\colon R\to A$ be the algebra homomorphism taking c_1 to the product $d_1\cdots d_e$ and all other variables to themselves. Let $I=\varphi(B)A$. Since φ is a free and hence a flat map by [8, Theorem 1.2] (such maps are called splittings there), we have that φ takes a free resolution of R/B^n to a free resolution of A/I^n, and it preserves minimality of the resolution. Thus the projective dimensions of R/B^n and A/I^n are the same, and by the Auslander-Buchsbaum formula,

$$\begin{aligned}\operatorname{depth}(A/I^n) &= \dim(A) - \operatorname{pd}(A/I^n)\\ &= \dim(R) + e - 1 - \operatorname{pd}(R/B^n)\\ &= \dim(R) + e - 1 - (\dim(R) - \operatorname{depth}(R/B^n))\\ &= \operatorname{depth}(R/B^n) + e - 1.\end{aligned}$$

So it suffices to prove that

$$\operatorname{depth}\left(\frac{R}{B^n}\right) = \begin{cases} 0, & \text{if } n = ru+1 \text{ with } u = 0, \dots, m;\\ 1, & \text{if } n \le rm+1 \text{ and } n \not\equiv 1 \mod r;\\ s, & \text{otherwise, i.e., if } n > mr+1.\end{cases}$$

By Theorems 1.4 and 3.3, the maximal ideal of R is associated to R/B^n exactly for the n of the form $ru+1$ with $u=0,\dots,m$. Thus the depth of R/B^n equals 0 exactly for all such n.

So we may assume that either $n \not\equiv 1 \mod r$ or that $n > mr+1$.

No $c_1,\dots,c_s$ appear in any generator of a minimal generating set of $B^n : c_1 = (B + (a_1^4 \cdots a_r^4))^n$. This means that $\operatorname{depth}(R/(B^n : c_1)) \ge s$. By Theorem 3.8 (and Theorem 1.4) we then have $\operatorname{depth}(R/(B^n : c_1)) = s$. Also, $B^n + (c_1) = B(m,r,s-1)^n + (c_1)$ with $B(m,r,s-1)$ defined using variables $a_j, x_{i,j}$ and $c_2,\dots,c_s$ only. When $s=1$, by Lemma 4.1, $\operatorname{depth}(R/(B^n + (c_1)))$ is 0 for all n, and for $s \ge 2$, by induction on s, $\operatorname{depth}(R/(B^n + (c_1)))$ is 1 or $s-1$, depending on n. We will use the short exact sequence

$$(*) \qquad 0 \longrightarrow \frac{R}{B^n : c_1} \longrightarrow \frac{R}{B^n} \longrightarrow \frac{R}{B^n + (c_1)} \longrightarrow 0.$$

This short exact sequence induces a long exact sequence on $\operatorname{Ext}_R(R/M, \_)$, where M is the maximal homogeneous ideal of R. We use the fact that for any finitely generated R-module U, $\operatorname{depth}(U) = \min\{\ell : \operatorname{Ext}_R^\ell(R/M, U) \ne 0\}$.

Let $\ell = \operatorname{depth}(R/(B^n + (c_1)))$. By induction on s, we have that $\ell = 0$ if $s=1$, and otherwise that $\ell = 1$ if $n \le mr+1$ (and $n \not\equiv 1 \mod r$), and $\ell = s-1$ otherwise, and so since the depth of $R/(B^n : c_1) = s$, the relevant part of the long exact sequence equals:

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Ext}_R^\ell\left(\frac{R}{M}, \frac{R}{B^n}\right) \longrightarrow \operatorname{Ext}_R^\ell\left(\frac{R}{M}, \frac{R}{B^n + (c_1)}\right) \longrightarrow \operatorname{Ext}_R^{\ell+1}\left(\frac{R}{M}, \frac{R}{B^n : c_1}\right) \longrightarrow \cdots.$$

We need to establish that $\operatorname{Ext}_R^\ell(\frac{R}{M}, \frac{R}{B^n})$ is non-zero if $n \le mr+1$ (and $n \not\equiv 1 \mod r$) and is zero if $n > mr+1$.

First let $n>mr+1$. We need to show that the depth of R/B^n is s. By the long exact sequence we first prove that $\operatorname{Ext}_R^{s-1}\big(\frac{R}{M},\frac{R}{B^n}\big)$ is zero, i.e., that

$$\operatorname{Ext}_R^{s-1}\left(\frac{R}{M},\frac{R}{B^n+(c_1)}\right)\longrightarrow \operatorname{Ext}_R^{s}\left(\frac{R}{M},\frac{R}{B^n:c_1}\right)$$

is injective. By faithful flatness we may assume that the base field is infinite. By prime avoidance we can find linear forms $u_2,...,u_s,u_1$ that form a regular sequence modulo $B^n{:}c_1$ and for which $u_2,...,u_s$ is a regular sequence modulo $B^n+(c_1)$. Since $a_1,...,a_r$ are in the radical of B, we may assume that the u_i are forms in the variables c_j and $x_{i,j}$ only. We claim that for $\ell=2,...,s$, we may take $u_\ell=c_\ell-d_\ell$, where $d_\ell=\sum_{i,j}\alpha_{\ell,i,j}x_{i,j}$ for some (generic) scalars $\alpha_{\ell,i,j}$. Certainly any such $u_2,...,u_s,u_1$ form a regular sequence modulo $B^n{:}c_1$ since $c_1,...,c_s$ do not appear in any generators of this ideal. Suppose that we have proved for some $\ell\in\{1,...,s-1\}$ that $u_2,...,u_\ell$ form a regular sequence modulo $B^n+(c_1)$. Then

$$B^n+(c_1,u_2,...,u_\ell)=\big((d_2,...,d_\ell,c_{\ell+1},...,c_s)a_1^4\cdots a_r^4+B_0+X\big)^n+(c_1,u_2,...,u_\ell),$$

and since $(a_1^4\cdots a_r^4)^2\in B_0^2$, by Lemma 1.2, modulo the variables $c_1,u_2,...,u_\ell$,

$$B^n=(c_{\ell+1},...,c_s)a_1^4\cdots a_r^4X^{n-1}+\big((d_2,...,d_\ell)a_1^4\cdots a_r^4+B_0+X\big)^n.$$

Thus by Lemma 1.3, each associated prime of $B^n+(c_1,u_2,...,u_\ell)$ either contains all $c_{\ell+1},...,c_s$ or it contains none of them. Thus $c_{\ell+1}-d_{\ell+1}$ for sufficiently general $\alpha_{\ell+1,i,j}$ is a non-zerodivisor modulo $B^n+(u_2,...,u_\ell)$. This proves the stated forms of $u_2,...,u_s$ and we may also take $u_1=c_1$.

By a theorem of Rees (see [9, Lemma 2 (i)]), due to natural isomorphisms, it suffices to prove that the natural map

$$\operatorname{Hom}_R\left(\frac{R}{M},\frac{R}{B^n+(c_1,u_2,...,u_s)}\right)\longrightarrow \operatorname{Hom}_R\left(\frac{R}{M},\frac{R}{(B^n:c_1)+(c_1,u_2,...,u_s)}\right)$$

is injective. In other words, we need to show that the natural map $\frac{L_1:M}{L_1}\to\frac{L_2:M}{L_2}$ is injective, where $L_1=B^n+(c_1,u_2,...,u_s)$ and $L_2=(B^n{:}c_1)+(c_1,u_2,...,u_s)$. Let $w\in(L_1{:}M)\cap L_2$. We have to prove that $w\in L_1$.

By subtracting elements of L_1 from w, by Lemma 1.2, $w\in a_1^4\cdots a_r^4X^{n-1}$, and since $(a_1,...,a_r,c_1,...,c_s)a_1^4\cdots a_r^4X^{n-1}\subseteq B^n\subseteq L_1$, we may assume that

$$w=\sum_\nu e_\nu a_1^4\cdots a_r^4\left(\prod_{i,j}x_{i,j}^{v_{\nu,i,j}}\right)\left(\prod_{i,j}h_{i,j}^{u_{\nu,i,j}}\right),$$

where for all ν, $e_\nu\in k$ and $\sum_{i,j}u_{\nu,i,j}=n-1$. Since L_1,L_2 are not monomial ideals, w need not be a monomial. Nevertheless, we claim that each $x_{i,j}$ multiplies each summand in w into L_1. Proof of the claim: Fix (i,j). We know that $x_{i,j}w\in L_1$. This means that in at least one monomial summand w_0 of w, $x_{i,j}$ must be incorporated into some new factor of B while at the same time possibly breaking up some of the $n-1$ factors that are generators of B. Then by Corollary 2.5 (1), $x_{i,j}w_0\in B^n\subseteq L_1$. Hence $x_{i,j}(w-w_0)$ is also in L_1, and $w-w_0$ has fewer summands in it, which proves the claim for (i,j) by induction on the number of monomial summands in w. This proves that every monomial appearing in w is multiplied by $(x_{i,j}: i\in[m], j\in[r])$ into B^n. Thus by Corollary 2.5 (2), each monomial appearing in w is in L_1, so that $w\in L_1$. This proves that $\mathrm{Ext}_R^{s-1}(R/M, R/B^n)=0$, which means that the depth of R/B^n is at least s. By the same reasoning as before, there exists a regular sequence $u_1,\dots,u_s$ on R/B^n of the form $u_j=c_j-d_j$ for some generic linear combinations $d_1,\dots,d_s$ in the $x_{i,j}$. Consider the element $w=a_1^{e_n}a_1^4\cdots a_r^4\prod_{i,j}x_{i,j}$, with e_n as defined in the proof of Theorem 3.8. By Lemma 4.1, $Mw\in B^n+(u_1,\dots,u_s)$ and $w\notin B^n+(u_1,\dots,u_s)$, so that the depth of R/B^n is exactly s.

Finally, let $n\le mr+1$ and $n\not\equiv 1 \mod r$. We need to prove that the connecting homomorphism in the displayed long exact sequence is not injective. If $s\ge 3$, then $\mathrm{Ext}_R^{\ell+1}(\frac{R}{M},\frac{R}{B^n:c_1})=0$ and we are done. So, let $s\le 2$. As in in the proof for $n>mr+1$ there exists a linear form $u_2=c_2-d_2\in M$ that is a non-zerodivisor modulo $B^n:c_1$ and modulo $B^n+(c_1)$, and by the same theorem of Rees, due to natural isomorphisms, it suffices to prove that the natural homomorphism $\frac{L_1:M}{L_1}\to\frac{L_2:M}{L_2}$ is not injective, where $L_1=B^n+(c_1,u_2)$ and $L_2=(B_n:c_1)+(c_1,u_2)$. Write $n-2=ur+v$ for some non-negative integers u,v with $v<r$. Since $n\le mr+1$ and $n\not\equiv 1 \mod r$, it follows that $u<m$ and $v\not\equiv 0 \mod r$. Let $w_h=h_{1,1}\cdots h_{1,v}\prod_{i>m-u,j}h_{i,j}\in B^{n-2}$, $w_x=\prod_{i\le m-u,j}x_{i,j}$, $w_a=a_1^4\cdots a_{r-1}^4a_r^8\in B$, and $w=x_{1,1}w_hw_aw_x$. Thus clearly $w\in B^{n-2}(a_r^4x_{1,r}x_{1,1}^2)(a_1^4\cdots a_r^4)\in L_2$. However, $w\notin L_1$. We next prove that $Mw\subseteq L_1$:

$$
\begin{aligned}
&x_{i,j}w\in\frac{w_h}{h_{i,j-2}h_{i,j}}h_{i,j-1}(a_{j-2}^6)(a_j^6)(h_{1,r})\subseteq(B_0+X)^n,\ \text{if } i>m-u;\\
&x_{i,j}w\in(w_h)(a_{j-1}^4x_{i,j-1}x_{i,j}^2)(a_r^6)\subseteq(B_0+X)^n,\ \text{if } j\ne 1 \text{ and } i\le m-u;\\
&x_{i,1}w\in(w_h)(a_r^4x_{i,r-1}x_{i,1}^2)(a_r^4x_{1,r-1}x_{1,1}^2)\subseteq(B_0+X)^n,\ \text{if } 2\le i\le m-u;\\
&x_{1,1}w\in(c_2a_1^4\cdots a_r^4)w_h(a_r^4x_{1,r}x_{1,1}^2)+(u_2)+\sum_{(i,j)\ne(1,1)}x_{i,j}w\in L_1;\\
&\quad a_rw\in(w_h)(a_r^5a_1)(a_r^4x_{1,r}x_{1,1}^2)\in L_1;\\
&\quad a_jw\in(w_h)(a_j^5a_{j+1})(a_r^4x_{1,r}x_{1,1}^2)\in L_1, \text{if } j\ne 1.
\end{aligned}
$$

This proves that $w\in(L_1:M)\cap L_2$ and $w\notin L_1$, which proves that the map $\frac{L_1:M}{L_1}\to\frac{L_2:M}{L_2}$ is not injective. Thus the depth of R/B^n is 1 if $n\le rm+1$ and $n\not\equiv 1 \mod r$. $\square$

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