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Double coverings of octic arrangements with isolated singularities

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Abstract

In this paper we construct 206 examples of Calabi–Yau manifolds with different Euler numbers. All constructed examples are smooth models of double coverings of $\mathbb{P}^3$ branched along an octic surface. We allow 11 types of (not necessary isolated) singularities in the branch locus. Thus we broaden the class of examples studied in [7]. For every considered example we compute the Euler number and give a precise description of a resolution of singularities.

1 Introduction

In this paper we study a class of Calabi–Yau manifolds. By a Calabi–Yau manifold we mean a kähler, smooth threefold with trivial canonical bundle and no global 1-forms. One method of constructing Calabi–Yau manifolds is to study a double covering of $\mathbb{P}^3$ branched along an octic surface.

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Let B be a surface in $\mathbb{P}^3$ of degree eight. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(4)$, $\mathcal{L}$ is a line bundle on $\mathbb{P}^3$ such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_{\mathbb{P}^3}(B)$. Then $\mathcal{L}$ defines a double covering of $\mathbb{P}^3$ branched along B and the singularities of the double cover are in one–to–one correspondence with singularities of B.

If B is smooth then the resulting double covering is a Calabi–Yau threefold, if B has only nodes (ordinary double points) then the double cover has also only nodes, and these nodes can be resolved by the mean of small resolution. In this case again the resulting threefold is Calabi–Yau. This construction was precisely described in [5].

In [7] a case of B being an octic arrangement (i.e. a surface which locally looks like a plane arrangement) was studied. In this paper we shall use methods introduced in [7] to study a bigger class of octic surfaces, namely we shall consider arrangements with ordinary multiple points of multiplicity 2, 4 and 5. Altogether we allow 11 types of singularities of the branch locus.

Our main result is the following theorem

Theorem 1.1 *If an octic arrangement B contains only*

- *double and triple curves,*
- *arrangement q–fold points, $q =$ 2, 3, 4, 5,*
- *isolated q–fold points $q =$ 2, 4, 5*

then the double covering of $\mathbb{P}^3$ branched along B has a non–singular model $\hat{X}$ which is a Calabi–Yau threefold.

Moreover if B contains no triple elliptic curves then

$$\mathrm{e}(\hat{X}) = 8 - \sum_i (d_i^3 - 4d_i^2 + 6d_i) + 2\sum_{i<j}(4 - d_i - d_j)d_i d_j - \sum_{i<j<k} d_i d_j d_k$$
$$+ 4p_4^0 + 3p_4^1 + 16p_5^0 + 18p_5^1 + 20p_5^2 + l_3 + 2m_2 + 36m_4 + 56m_5.$$

The idea of the proof of this theorem is to give a resolution of singularities of X by a sequence of admissible blowing-ups (i.e. blowing-ups that do not affect the first Betti number and the canonical divisor of the double covering, cf. [7]). We apply Theorem 1.1 to give examples of Calabi–Yau manifolds with 206 different Euler number (we realize any even number from the interval $\langle -296, 104\rangle$ as an Euler number of a Calabi–Yau manifold).

2 Surfaces with ordinary multiple points

Let B be a surface in $\mathbb{P}^3$ with only ordinary multiple points. That means that if we consider the blowing–up $\sigma : \tilde{\mathbb{P}}^3 \longrightarrow \mathbb{P}^3$ of $\mathbb{P}^3$ at all singular points of B then the strict transform $\tilde{B}$ of B is smooth and intersect the exceptional divisor of σ transversally. Let m_p denotes the number of p–fold points on B. The following Proposition contains the numerical data of B and $\tilde{B}$.

Proposition 2.1

$$c_1^2(\tilde{B}) = d(d-4)^2 - \sum_p (p-2)^2 p m_p \quad (1)$$

$$c_2(\tilde{B}) = d^3 - 4d^2 + 6d - \sum_p (p-2)p^2 m_p \quad (2)$$

$$e(B) = d^3 - 4d^2 + 6d - \sum_p (p-1)^3 m_p \quad (3)$$

$$p_a(\tilde{B}) = \binom{d-1}{3} - \sum_p \binom{p}{3} m_p \quad (4)$$

3 Octic arrangements with isolated singularities

Definition 3.1 *An* octic arrangement with isolated singularities *is a surface $B \subset \mathbb{P}^3$ of degree 8 which is a sum of irreducible surfaces $B_1, \dots, B_r$ with only ordinary multiple points which satisfies the following conditions:*

1. *For any $i \neq j$ the surfaces B_i and B_j intersects transversally along a smooth irreducible curve $C_{i,j}$ or they are disjoint,*
2. *The curves $C_{i,j}$ and $C_{k,l}$ are either equal or disjoint or they intersect transversally.*

This definition is a generalization of the notion of octic arrangement introduced in [7] (where the surfaces S_i are assumed to be smooth). Observe that from (1) $\mathrm{Sing}B_i \cap B_j = \emptyset$ for $i \neq j$. We shall denote $d_i := \deg B_i$.

A singular point of B_i we shall call an *isolated singular point of the arrangement.* A point $P \in B$ which belongs to p of surfaces $B_1, \dots, B_r$ we shall call an *arrangement p–fold point.* We say that irreducible curve $C \subset B$ is a q–fold curve if exactly q of surfaces $B_1, \dots, B_r$ pass through it.

We shall use the following numerical data for an arrangement:

p_q^i Number of arrangement q–fold points lying on exactly i triple curves,

l_3 Number of triple lines,

m_q number of isolated q–fold points.

4 Proof of Theorem 1.1

Let B be an octic arrangement satisfying assumptions of the Theorem and let X be a double covering of $\mathbb{P}^3$ branched along B. We shall find a sequence of admissible blowing–ups (i.e. blowing–ups of double or triple curves and 4–fold or 5–fold points) $\sigma : \mathbb{P}^* \longrightarrow \mathbb{P}^3$ and a reduced divisor $B^* \subset \mathbb{P}^*$ with ordinary double points (nodes) as the only singularities and such that

$\tilde{B} \leq B \leq B^*$ (where $\tilde{B}$ is a strict transform and B^* is a pullback of B by σ),

B^* is even as an element of the Picard group Pic $\mathbb{P}^*$.

Let us now describe an algorithm to obtain σ, the method is in fact a modification of the method introduced in [7]. We resolve all singularities of B except the nodes

1. Resolution of multiple curves and arrangement multiple points. In these cases we shall apply the method described in [7].

2. Resolution of isolated 4–fold points. We blow–up a 4–fold point, and then replace the branch locus by its strict transform.

3. Resolution of isolated 5–fold points. We blow–up a 5–fold point, and then replace the branch locus by its strict transform plus the exceptional divisor. The proper transform intersects exceptional divisor transversally along a smooth plane curve of degree 5. We treat this curve in the same way as as an arrangement double curve i.e. we blow-up this curve, and replace the branch locus by its strict transform.

The double covering of $\mathbb{P}^*$ branched along B^* has nodes (corresponding to nodes of B^*) as the only singularities.

4. Resolution of nodes. There are two possibilities for the resolution of a node on a 3–dimensional variety: blow–up or small-resolution (for details see [5])

We shall denote the non–singular model of X by $\tilde{X}$ if in step 4 we choose a blow–up and by $\hat{X}$ if we choose a small resolution. To any node on B we have associated a line on $\hat{X}$. $\tilde{X}$ is a blowing-up of $\hat{X}$ at all this lines. As a consequence we see that $e(\tilde{X}) = e(\hat{X}) + 2m_2$.

The blow-ups used in steps 1—3 are (according to [7]) admissible, i.e. they do not affect the first Betti number and canonical divisor of $\tilde{X}$. We see therefore that

$$K_{\tilde{X}} = E_2$$

where E_2 denotes the exceptional divisor on $\tilde{X}$ associated to all nodes of B hence

$$K_{\hat{X}} = 0.$$

In order two compute $e(\tilde{X})$ we compare our case with the one studied in [7]. From the Proposition 2.1 we see that in our case $e(\tilde{\mathbb{P}}^3)$ increases by $2m_2 + 2m_4 - 8m_5$ whereas $e(\tilde{B})$ decreases by $32m_4 + 72m_5$.

The Euler number $e(\hat{X})$ is hencefore greater by

$$2m_2 + 36m_4 + 56m_5$$

in comparison with the case with no isolated singular points. Using [7, Thm. 3.4] proves the theorem. $\square$

5 Examples

In this section we shall apply Theorem1.1 to study various examples of octic arrangement. As a result we obtain 206 examples of Calabi–Yau manifolds with different Euler numbers, we shall for instance realize every even number from the interval $\langle -296, 104 \rangle$ as an Euler number of a Calabi–Yau manifold.

We shall need information about the number of nodes allowed on a nodal surface of degree ≤ 8 in $\mathbb{P}^3$ Using results from [1, 3, 4] we can formulate the following proposition

$(d_1, d_2, \ldots, d_r)$	p_4^0	p_4^1	p_5^0	p_5^1	p_5^2	l_3	m_2	$\mathbf{e}(\hat{X})$
8								-296
8							1	-294
⋮	⋮						⋮	⋮
8							107	-82
(1,1,2,4)	1							-80
(1,1,2,4)	1						1	-78
⋮	⋮						⋮	⋮
(1,1,2,4)	1						16	-48
(1,1,1,1,4)			1				1	-46
⋮	⋮						⋮	⋮
(1,1,1,1,4)			1				16	-16
(1,1,1,1,1,3)							1	-14
⋮	⋮						⋮	⋮
(1,1,1,1,1,3)							4	-8
(1,1,1,1,1,3)	2						1	-6
⋮	⋮						⋮	⋮
(1,1,1,1,1,3)	2						4	0
(1,1,1,1,1,3)		5				1	1	2
⋮	⋮						⋮	⋮
(1,1,1,1,1,3)		5				1	4	8
(1,1,1,1,2,2)	2						1	10
(1,1,1,1,1,3)		3		1		1		12
⋮	⋮						⋮	⋮
(1,1,1,1,1,3)		3		1		1	4	20
(1,1,1,1,1,1,2)							1	22
(1,1,1,1,1,3)		1		2		1		24
⋮	⋮						⋮	⋮
(1,1,1,1,1,3)		1		2		1	4	32
(1,1,1,1,1,1,2)	3						1	34
(1,1,1,1,1,1,2)			1					36
(1,1,1,1,1,1,2)			1				1	38
(1,1,1,1,1,1,1,1)								40
(1,1,1,1,1,1,2)	1	5				1	1	42

$(d_1, d_2, \ldots, d_r)$	p_4^0	p_4^1	p_5^0	p_5^1	p_5^2	l_3	m_2	$\mathbf{e}(\hat{X})$
(1,1,1,1,1,1,1,1)	1							44
(1,1,1,1,2,2)	1	1		2		1	1	46
(1,1,1,1,1,1,1,1)	2							48
(1,1,1,1,1,1,2)		3		1		1	1	50
(1,1,1,1,1,1,1,1)	3							52
(1,1,1,1,1,1,2)	1	3		1		1	1	54
(1,1,1,1,1,1,1,1)	4							56
(1,1,1,1,1,1,2)	2	3		1		1	1	58
(1,1,1,1,1,1,1,1)	5							60
(1,1,1,1,1,1,2)		6			1	2	1	62
(1,1,1,1,1,1,1,1)	6							64
(1,1,1,1,1,1,2)	1	6			1	2	1	66
(1,1,1,1,1,1,1,1)	7							68
(1,1,1,1,1,1,2)	2	6			1	2	1	70
(1,1,1,1,1,1,1,1)	8							72
(1,1,1,1,1,1,2)		4		1	1	2	1	74
(1,1,1,1,1,1,1,1)	9							76
(1,1,1,1,1,1,2)	1	4		1	1	2	1	78
(1,1,1,1,1,1,1,1)		1		2		1		80
(1,1,1,1,1,1,2)	2	4		1	1	2	1	82
(1,1,1,1,1,1,1,1)		8		1		2		84
(1,1,1,1,1,1,2)		2		2	1	2	1	86
(1,1,1,1,1,1,1,1)	12							88
(1,1,1,1,1,1,2)	1	2		2	1	2	1	90
(1,1,1,1,1,1,1,1)		4		1	1	2		92
(1,1,1,1,1,1,2)	2	2		2	1	2	1	94
(1,1,1,1,1,1,1,1)		6		2		2		96
(1,1,1,1,1,1,2)	3	2		2	1	2	1	98
(1,1,1,1,1,1,2)	4	2		2	1	2		100
(1,1,1,1,1,1,2)	4	2		2	1	2	1	102
(1,1,1,1,1,1,1,1)		7			2	3		104
(1,1,1,1,1,1,1,1)		9		1	1	3		108
(1,1,1,1,1,1,1,1)		3			3	3		112
(1,1,1,1,1,1,1,1)	1	3			3	3		116
(1,1,1,1,1,1,1,1)	2	3			3	3		120
(1,1,1,1,1,1,1,1)		4			4	4		136

Proposition 5.1

a. For $m_2 = 0, 1, 2, 3, 4$ *there exists a nodal cubic surface with exactly* m_2 *nodes,*

b. For $m_2 = 0, 1, \dots, 16$ *there exists a nodal quartic surface with exactly* m_2 *nodes,*

c. For $m_2 = 0, 1, \dots, 65$ *there exists a nodal sextic surface with exactly* m_2 *nodes,*

d. For $m_2 = 0, 1, \dots, 107$ *there exists a nodal octic surface with exactly* m_2 *nodes.*

Using the above Proposition and Theorem 1.1 we can compile a table containing numerical data of octic arrangements and corresponding Euler numbers. Most of Euler numbers can be obtained from several different arrangements, in the table we give one example per number. In the table we avoid arrangements with 4–fold and 5–fold points, they do not leave to new Euler numbers. On the other hand arrangements with 4–fold and 5–fold points usually have higher Picard number then the ones with only nodes.

Most of the examples in the table are modification of the ones given in [7] obtained by adding isolated singularities. In many cases it is easy to write down explicit equation of the branch locus. Proof of Theorem 1.1 gives a detailed description of a resolution of singularities. Although the resolution of singularities is not uniquely determined, the different resolutions of the same double solid differ only by flop. Consequently most of the numerical data (like f.i. Euler number) are uniquely determined (cf. [11]).

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