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The inverse mean curvature flow in cosmological spacetimes

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Abstract

We prove that the leaves of an inverse mean curvature flow provide a foliation of a future end of a cosmological spacetime N under the necessary and sufficient assumptions that N satisfies a future mean curvature barrier condition and a strong volume decay condition. Moreover, the flow parameter t can be used to define a new physically important time function.

1 Introduction

The inverse mean curvature flow has already been considered in Euclidean space [3] or in asymptotically flat Riemannian spaces [12]. In the latter case Huisken and Ilmanen used it to prove the Penrose inequality. One major difficulty in their proof was that jumps might occur during the flow, i.e., the mean curvature of the flow hypersurfaces might vanish even though the initial hypersurface has positive mean curvature.

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The Lorentzian geometry is much more favourable for curvature flows, cf. [1, 5, 6, 8], so that no jumps should occur in the case of the inverse mean curvature flow. We shall show that this is indeed the case, if the ambient space is a *globally hyperbolic* $(n+1)$-dimensional Lorentzian manifold N with a compact Cauchy hypersurface satisfying the time-like convergence condition

$$\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \ge 0 \qquad \forall\langle\nu,\nu\rangle = -1. \tag{1.1}$$

Such spaces are called *cosmological spacetimes*, a terminology due to Bartnik.

Let $M_0 \subset N$ be a space-like hypersurface, the mean curvature of which is either strictly positive or negative. Then we consider the inverse mean curvature flow (IMCF)

$$\dot{x} = -H^{-1}\nu \tag{1.2}$$

with initial hypersurface M_0. Here, ν is the past-directed normal of the flow hypersurfaces $M(t)$, and $H = H_{|_{M(t)}}$ the corresponding mean curvature, i.e., the trace of the second fundamental form.

If $H_{|_{M_0}}$ is positive respectively negative, then the flow moves to the future respectively past of M_0. Furthermore, $H_{|_{M(t)}}$ will uniformly tend to ∞ respectively $-\infty$, if the flow exists for all time.

In former papers we referred to this latter phenomenon by saying that there were *crushing singularities* in the future respectively past, erroneously assuming that only big crunch or big bang type singularities could produce space-like hypersurfaces, the mean curvatures of which become unbounded if the hypersurfaces approached the singularities.

But a behaviour like that could also be caused by a *null hypersurface* $\mathcal{H}$, e.g., by the event horizon of a black hole, if the spacetime can be viewed as having a past or future boundary component $\mathcal{H}$ that can be identified with a compact null hypersurface representing a *non-crushing* singularity, i.e., the Riemannian curvature tensors remain uniformly bounded near $\mathcal{H}$

$$\bar{R}_{\alpha\beta\gamma\delta}\bar{R}^{\alpha\beta\gamma\delta} \le \text{const.} \tag{1.3}$$

An example of such a spacetime is given in Section 1.

We therefore define

Definition 1.1. *Let N be a globally hyperbolic spacetime with compact Cauchy hypersurface $\mathcal{S}_0$, so that N can be written as a topological product*

$N = \mathbb{R} \times \mathcal{S}_0$, and its metric expressed as

$$d\bar{s}^2 = e^{2\psi}(-(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j). \tag{1.4}$$

Here, x^0 is a globally defined future-directed time function and (x^i) are local coordinates for $\mathcal{S}_0$. N is said to have a future mean curvature barrier respectively past mean curvature barrier, if there are sequences M_k^+ respectively M_k^- of closed space-like hypersurfaces such that

$$\lim_{k\to\infty} H_{|_{M_k^+}} = \infty \quad \text{respectively} \quad \lim_{k\to\infty} H_{|_{M_k^-}} = -\infty \tag{1.5}$$

and

$$\limsup \inf_{M_k^+} x^0 > x^0(p) \quad \forall p \in N \tag{1.6}$$

respectively

$$\liminf \sup_{M_k^-} x^0 < x^0(p) \quad \forall p \in N. \tag{1.7}$$

Remark 1.2. *Let N be a cosmological spacetime with future and past mean curvature barriers; then it can be foliated by closed hypersurfaces of constant mean curvature; cf. [2]. Moreover, the mean curvature function τ is continuous in N and smooth in $\{\tau \neq 0\}$ with non-vanishing gradient, hence it can be used as a time function; cf. [9]. These results are also valid in future respectively past ends.*

We shall assume in the following that N has a future mean curvature barrier. By reversing the time direction this configuration also comprises the case that N has a past mean curvature barrier.

Under this assumption we shall prove that, for a given compact space-like hypersurface M_0 with $H_{|_{M_0}} > 0$, the future of M_0 can be foliated by the leaves of an IMCF starting at M_0, provided a so-called future *strong volume decay condition* is satisfied; cf. Definition 2.2. A strong volume decay condition is both necessary and sufficient in order that the IMCF exists for all time.

The main result of this paper can be summarized in the following theorem.

Theorem 1.3. *Let N be a cosmological spacetime with compact Cauchy hypersurface $\mathcal{S}_0$ and with a future mean curvature barrier. Let M_0 be a closed space-like hypersurface with positive mean curvature, and assume furthermore that N satisfies a future volume decay condition. Then the IMCF* (1.2) *with initial hypersurface M_0 exists for all time and provides a foliation of the future $D^+(M_0)$ of M_0.*

The evolution parameter t can be chosen as a new time function. The flow hypersurfaces $M(t)$ are the slices $\{t = \text{const}\}$ and their volume satisfies

$$|M(t)| = |M_0|e^{-t}. \tag{1.8}$$

Defining an almost proper time function τ by choosing

$$\tau = 1 - e^{-(1/n)t} \tag{1.9}$$

we obtain $0 \le \tau < 1$,

$$|M(\tau)| = |M_0|(1-\tau)^n, \tag{1.10}$$

and the future singularity corresponds to $\tau = 1$.

Moreover, the length $L(\gamma)$ of any future-directed curve γ starting from $M(\tau)$ is bounded from above by

$$L(\gamma) \le c(1-\tau), \tag{1.11}$$

where $c = c(n, M_0)$. Thus, the expression $1-\tau$ can be looked at as the radius of the slices $\{\tau = \text{const}\}$ as well as a measure of the remaining life span of the universe.

Without any further structural assumptions it seems impossible to derive any convergence results for an appropriately rescaled IMCF. In [10] we look at the IMCF in asymptotically Robertson Walker spaces and prove that a properly rescaled flow converges indeed.

2 Notations and definitions

The main objective of this section is to state the equations of Gauss, Codazzi, and Weingarten for space-like hypersurfaces M in an $(n+1)$-dimensional Lorentzian manifold N. Geometric quantities in N will be denoted by $(\bar g_{\alpha\beta})$, $(\bar R_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N respectively M will be denoted by (x^α) respectively (ξ^i). Covariant differentiation will simply be indicated by indices; only in case of possible ambiguity they will be preceded by a semicolon, i.e., for a function u in N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian but, e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar R_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\bar R_{\alpha\beta\gamma\delta;i} = \bar R_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon \tag{2.1}$$

with obvious generalizations to other quantities.

Let M be a *space-like* hypersurface, i.e., the induced metric is Riemannian, with a differentiable normal ν which is time-like.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the space-like hypersurface M are connected through the following equations:

$$x^\alpha_{ij} = h_{ij}\nu^\alpha, \tag{2.2}$$

the so-called *Gauss formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$x^\alpha_{ij} = x^\alpha_{,ij} - \Gamma^k_{ij}x^\alpha_k + \bar{\Gamma}^\alpha_{\beta\gamma}x^\beta_i x^\gamma_j. \tag{2.3}$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to ν.

The second equation is the *Weingarten equation*

$$\nu^\alpha_i = h^k_i x^\alpha_k, \tag{2.4}$$

where we remember that ν^α_i is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k \tag{2.5}$$

and the *Gauss equation*

$$R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_i x^\beta_j x^\gamma_k x^\delta_l. \tag{2.6}$$

Now, let us assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. N is then a topological product $I \times \mathcal{S}_0$, where I is an open interval, $\mathcal{S}_0$ is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that the metric in N has the form

$$d\bar{s}^2_N = e^{2\psi}\{-dx^{0^2} + \sigma_{ij}(x^0, x)dx^i dx^j\}, \tag{2.7}$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the space-like components (x^i). We also assume that the coordinate system is *future-oriented*, i.e., the time coordinate x^0 increases on future-directed curves. Hence, the *contravariant* time-like vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future-directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \operatorname{graph} u_{|_{\mathcal{S}_0}}$ be a space-like hypersurface,

$$M = \{\, (x^0, x) \colon x^0 = u(x),\, x \in \mathcal{S}_0 \,\}, \tag{2.8}$$

then the induced metric has the form

$$g_{ij} = e^{2\psi}\{-u_i u_j + \sigma_{ij}\}, \tag{2.9}$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$g^{ij} = e^{-2\psi}\left\{\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\right\}, \tag{2.10}$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \sigma^{ij} u_j, \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2. \end{aligned} \tag{2.11}$$

Hence, $\operatorname{graph} u$ is space-like if and only if $|Du| < 1$.

The covariant form of a normal vector of a graph looks like

$$(\nu_\alpha) = \pm v^{-1} e^{\psi}(1, -u_i). \tag{2.12}$$

and the contravariant version is

$$(\nu^\alpha) = \mp v^{-1} e^{-\psi}(1, u^i). \tag{2.13}$$

Thus we have

Remark 2.1. *Let M be space-like graph in a future-oriented coordinate system. Then the contravariant future-directed normal vector has the form*

$$(\nu^\alpha) = v^{-1} e^{-\psi}(1, u^i), \tag{2.14}$$

and the past-directed

$$(\nu^\alpha) = -v^{-1} e^{-\psi}(1, u^i). \tag{2.15}$$

In the Gauss formula (2.2) we are free to choose the future- or past-directed normal, but we stipulate that we always use the past-directed normal for reasons that we have explained in [5, Section 2].

Look at the component $\alpha = 0$ in (2.2) and obtain in view of (2.15)

$$e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{ij}. \tag{2.16}$$

Here, the covariant derivatives are taken with respect to the induced metric of M, and

$$-\bar{\Gamma}^0_{ij} = e^{-\psi} \bar{h}_{ij}, \tag{2.17}$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const.}\}$.

An easy calculation shows

$$\bar{h}_{ij}e^{-\psi} = -\tfrac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij}, \tag{2.18}$$

where the dot indicates differentiation with respect to x^0.

Next we shall define the strong volume decay condition.

Definition 2.2. *Suppose there exists a time function x^0 such that the future end of N is determined by $\{\tau_0 \le x^0 < b\}$ and the coordinate slices $M_\tau = \{x^0 = \tau\}$ have positive mean curvature with respect to the past-directed normal for $\tau_0 \le \tau < b$. In addition, the volume $|M_\tau|$ should satisfy*

$$\lim_{\tau \to b} |M_\tau| = 0. \tag{2.19}$$

A decay like that is normally associated with a future singularity, and we simply call it volume decay. If (g_{ij}) is the induced metric of M_τ and $g = \det(g_{ij})$, then we have

$$\log g(\tau_0, x) - \log g(\tau, x) = \int_{\tau_0}^{\tau} e^{\psi} \bar{H}(s, x) \quad \forall x \in \mathcal{S}_0, \tag{2.20}$$

where $\bar{H}(\tau, x)$ is the mean curvature of M_τ in (τ, x). For a proof we refer to [7].

In view of (2.19) the left-hand side of this equation tends to infinity if τ approaches b for a.e. $x \in \mathcal{S}_0$, i.e.,

$$\lim_{\tau \to b} \int_{\tau_0}^{\tau} e^{\psi} \bar{H}(s, x) = \infty \quad \textit{for a.e. } x \in \mathcal{S}_0. \tag{2.21}$$

Assume now there exists a continuous, positive function $\varphi = \varphi(\tau)$ such that

$$e^{\psi} \bar{H}(\tau, x) \ge \varphi(\tau) \quad \forall\, (\tau, x) \in (\tau_0, b) \times \mathcal{S}_0, \tag{2.22}$$

where

$$\int_{\tau_0}^{b} \varphi(\tau) = \infty. \tag{2.23}$$

Then we say that the future of N satisfies a strong volume decay condition.

Remark 2.3. (i) *By approximation we may — and shall — assume that the function φ above is smooth.*

(ii) *A similar definition holds for the past of N by simply reversing the time direction. Notice that in this case the mean curvature of the coordinate slices has to be negative.*

Lemma 2.4. *Suppose that the future of N satisfies a strong volume decay condition; then there exist a time function $\tilde{x}^0 = \tilde{x}^0(x^0)$, where x^0 is the time function in the strong volume decay condition, such that the mean curvature $\bar{H}$ of the slices $\tilde{x}^0 = \text{const}$ satisfies the estimate*

$$e^{\tilde{\psi}}\bar{H} \ge 1. \tag{2.24}$$

The factor $e^{\tilde{\psi}}$ is now the conformal factor in the representation

$$d\bar{s}^2 = e^{2\tilde{\psi}}(-(d\tilde{x}^0)^2 + \sigma_{ij}dx^i dx^j). \tag{2.25}$$

The range of $\tilde{x}^0$ is equal to the interval $[0,\infty)$, i.e., the singularity corresponds to $\tilde{x}^0 = \infty$.

Proof. Define $\tilde{x}^0$ by

$$\tilde{x}^0 = \int_{\tau_0}^{x^0} \varphi(\tau), \tag{2.26}$$

where φ is the function in (2.22) now assumed to be smooth.

The conformal factor in (2.25) is then equal to

$$e^{2\tilde{\psi}} = e^{2\psi}\frac{\partial x^0}{\partial \tilde{x}^0}\frac{\partial x^0}{\partial \tilde{x}^0} = e^{2\psi}\varphi^{-2}, \tag{2.27}$$

and hence

$$e^{\tilde{\psi}}\bar{H} = e^{\psi}\bar{H}\varphi^{-1} \ge 1, \tag{2.28}$$

in view of (2.22). □

As we mentioned in the introduction there are spacetimes which satisfy a mean curvature barrier condition but the resulting singularity is not crushing.

To construct an example let us start with an $\text{S-AdS}_{(n+2)}$ spacetime with metric

$$d\hat{s}^2 = -f dt^2 + f^{-1}dr^2 + r^2\sigma_{ij}dx^i dx^j, \tag{2.29}$$

where

$$f = \kappa - \frac{2}{n(n+1)}\Lambda r^2 - m r^{-(n-1)} \tag{2.30}$$

with constants Λ and $m > 0$; (σ_{ij}) is the metric of a compact n-dimensional spaceform of curvature $\kappa = 0, 1, -1$.

This spacetime satisfies the Einstein equations

$$G_{\alpha\beta} + \Lambda \bar{g}_{\alpha\beta} = 0. \tag{2.31}$$

Let us suppose for simplicity that $\kappa = 1$ and $\Lambda < 0$, though this is not important in our considerations. In $\{r = 0\}$ is a black hole singularity and the event horizon $\mathcal{H} = f^{-1}(0)$ is characterized by $r = r_0$.

The region $\{f < 0\}$ is the black hole region. In this region r is the time function and t is a spatial variable. Let us pick the black hole region.

Normally the variable t describes the real axis but, since it is a spatial variable, we are free to compactify it, and we shall suppose that t is a variable for S^1. By this compactification we have defined a globally hyperbolic spacetime N with compact Cauchy hypersurface $\mathcal{S}_0 = S^1 \times S^n$ which satisfies the time-like convergence condition since

$$\bar{R}_{\alpha\beta} = \frac{2}{n} \Lambda \bar{g}_{\alpha\beta} \tag{2.32}$$

and Λ is supposed to be negative.

N has a crushing singularity in $r = 0$ and, as we shall show in a moment, also a mean curvature barrier singularity in $r = r_0$, which is however not crushing, since the metric quantities were not changed by the compactifcation but only the topology.

Define

$$\tilde{f} = -f \quad \text{and} \quad \psi = -\tfrac{1}{2} \log \tilde{f}, \tag{2.33}$$

then the metric can be expressed as

$$\begin{aligned} d\bar{s}^2 &= e^{2\psi}(-dr^2 + \tilde{f}^2 dt^2 + \tilde{f} r^2 \sigma_{ij} dx^i dx^j) \\ &\equiv e^{2\psi}(-dr^2 + \tilde{\sigma}_{ab} dx^a dx^b). \end{aligned} \tag{2.34}$$

The second fundamental form of the hypersurfaces $\{r = \text{const}\}$ with respect to the past-directed normal is given by

$$e^{-\psi} \bar{h}_{ab} = \tfrac{1}{2} \dot{\tilde{\sigma}}_{ab} - \tfrac{1}{2} \tilde{f}^{-1} \dot{\tilde{f}} \tilde{\sigma}_{ab}, \tag{2.35}$$

where the dot indicates differentiation with respect to r, and where we note that the time function r is past-directed in contrast to the usual convention. Hence the mean curvature $\bar{H}$ is equal to

$$\bar{H} = \tilde{f}^{-1/2} (\tfrac{1}{2} \dot{\tilde{f}} + n \tilde{f} r^{-1}), \tag{2.36}$$

and we deduce that $\bar{H}$ tends to $-\infty$, if the hypersurfaces approach the horizon $\mathcal{H}$, and to ∞, if the hypersurfaces approach the black hole singularity $r = 0$.

Sometimes, we need a Riemannian reference metric, e.g., if we want to estimate tensors. Since the Lorentzian metric can be expressed as

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{2\psi}\{-dx^{0^2} + \sigma_{ij}dx^i dx^j\}, \tag{2.37}$$

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

$$\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{2\psi}\{dx^{0^2} + \sigma_{ij}dx^i dx^j\}, \tag{2.38}$$

and we abbreviate the corresponding norm of a vectorfield η by

$$|||\eta||| = (\tilde{g}_{\alpha\beta}\eta^\alpha\eta^\beta)^{1/2}, \tag{2.39}$$

with similar notations for higher-order tensors.

3 The evolution problem

The evolution problem (1.2) is a parabolic problem, hence a solution exists on a maximal time interval $[0, T^*)$, $0 < T^* \le \infty$ (cf. [4, Section 2]), where we apologize for the ambiguity of also calling the evolution parameter *time*.

Next, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives. We refer to [5] for more general results, and to [4, Section 3], where proofs are given in a Riemannian setting, but these proofs are also valid in a Lorentzian environment.

Lemma 3.1. *The metric, the normal vector, and the second fundamental form of $M(t)$ satisfy the evolution equations*

$$\dot{g}_{ij} = -2H^{-1}h_{ij}, \tag{3.1}$$

$$\dot{\nu} = \nabla_M(-H^{-1}) = g^{ij}(-H^{-1})_i x_j, \tag{3.2}$$

and

$$\dot{h}_i^j = (-H^{-1})_i^j + H^{-1}h_i^k h_k^j + H^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj}, \tag{3.3}$$

$$\dot{h}_{ij} = (-H^{-1})_{ij} - H^{-1}h_i^k h_{kj} + H^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{3.4}$$

Lemma 3.2 (**Evolution of H^{-1}**)**.** *The term H^{-1} evolves according to the equation*

$$(H^{-1})' - H^{-2}\Delta H^{-1} = -H^{-2}(\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)H^{-1}, \tag{3.5}$$

where

$$(H^{-1})' = \frac{d}{dt}H^{-1} \tag{3.6}$$

and

$$\|A\|^2 = h_{ij}h^{ij}. \tag{3.7}$$

From (2.3) we deduce with the help of the Ricci identities and the Codazzi equations a parabolic equation for the second fundamental form.

Lemma 3.3. *The mixed tensor h_i^j satisfies the parabolic equation*

$$\begin{aligned}\dot{h}_i^j - H^{-2}\Delta h_i^j = &-H^{-2}\|A\|^2 h_i^j + 2H^{-1}h_i^k h_k^j - 2H^{-3}H_i H^j \\ &+ 2H^{-2}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h^{km}g^{rj} \\ &- H^{-2}g^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} \\ &- H^{-2}g^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^{mj} \\ &- H^{-2}\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta h_i^j + 2H^{-1}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\ &+ H^{-2}g^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\}.\end{aligned} \tag{3.8}$$

Since the time-like convergence condition is assumed to be valid we immediately deduce from (2.2)

Lemma 3.4. *There exists a positive constant $c_0 = c_0(M_0)$, such that the estimate*

$$H \geq c_0\, e^{(1/n)t} \tag{3.9}$$

is valid during the evolution.

Proof. Let $\varphi = H^{-1}e^{(1/n)t}$. Then φ satisfies the inequality

$$\dot{\varphi} - H^{-2}\Delta\varphi \leq -H^{-2}|A|^2\varphi + \frac{1}{n}\varphi \leq 0, \tag{3.10}$$

hence we conclude

$$\varphi \leq \sup_{M_0}\varphi = \sup_{M_0} H. \tag{3.11}$$

□

4 Lower order estimates

The evolution problem (1.2) exists on a maximal time interval $I = [0, T^*)$. We want to prove that $T^* = \infty$, and that the flow hypersurfaces $M(t)$ run into the future singularity, if t tends to infinity.

The latter property is a characteristicum of the inverse mean curvature flow under very weak assumptions: if the flow exists for all time, then it cannot stay in a compact region of N or, more precisely,

Lemma 4.1. *Let N be a cosmological spacetime with a future mean curvature barrier, and let M_0 be a compact space-like hypersurface with positive mean curvature. Suppose that $N = \mathbb{R} \times \mathcal{S}_0$ and that the metric is given as in (0.4). Assume that the inverse mean curvature flow with initial hypersurface M_0 exists for all time, and let the flow hypersurfaces $M(t)$ be expressed as graphs of a function u over $\mathcal{S}_0$:*

$$M(t) = \{ (x^0, x) \colon x^0 = u(t,x), x \in \mathcal{S}_0 \}. \tag{4.1}$$

Then there holds

$$\lim_{t\to\infty} \inf_{\mathcal{S}_0} u(t,\cdot) = \infty. \tag{4.2}$$

Proof. (i) Because of the barrier condition a future end of N, N_+, can be foliated by hypersurfaces of positive constant mean curvature and we can choose the mean curvature τ of that CMC foliation as a new time function $x^0 = \tau$ in N_+:

$$N_+ = \{ (\tau, x) \colon k \le \tau < \infty, x \in \mathcal{S}_0 \}, \tag{4.3}$$

cf. Remark (1.2), where k is a positive constant and where we used the same symbol $\mathcal{S}_0$ for the compact Cauchy hypersurface — indeed, we could use the original Cauchy hypersurface $\mathcal{S}_0$, since it need not be a level hypersurface.

Let t_0 be such that

$$c_0 e^{(1/n)t_0} > 2k, \tag{4.4}$$

where c_0 is the constant in inequality (2.9). Then we claim that

$$M(t) \subset N_+ \qquad \forall t \ge t_0. \tag{4.5}$$

To prove this claim we shall apply the Synge's lemma. Denote the coordinate slices $x^0 = \tau$ by M_τ, i.e., M_τ has constant mean curvature $\bar{H} = \tau$.

It suffices to show that all $M(t)$ with $t \ge t_0$ lie in the future of M_k. Suppose this were not the case for some $M(t)$; then the Lorentzian distance between $M(t)$ and M_k would be positive:

$$d = d(M(t), M_k) > 0, \tag{4.6}$$

and hence there would exist a maximal future-directed geodesic γ from $M(t)$ to M_k. Synge's lemma would then yield

$$H_{|_{M_k}}(\gamma(d)) \ge H_{|_{M(t)}}(\gamma(0)) + \int_0^d \bar{R}_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta, \tag{4.7}$$

a contradiction in view of (4.4) and the time-like convergence condition.

(ii) Thus, the flow hypersurfaces $M(t)$ are covered by the new coordinate system for $t \ge t_0$. The metric of N has again the form as in (0.4).

Now, the mean curvature $\bar{H}$ of the coordinate slices satisfies the evolution equation

$$\dot{\bar{H}} = -\Delta e^{\psi} + (|\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^{\alpha}\nu^{\beta})e^{\psi}, \tag{4.8}$$

where the dot indicates differentiation with respect to x^0, the Laplace operator is the Laplace Beltrami operator of the slice, $|\bar{A}|^2$ the square of the second fundamental form and ν the past-directed normal and e^{ψ} the conformal factor of the metric.

This relation is valid for the slices of any time function x^0 for which the metric has the form as in (1.4), since the slices are solutions of the evolution equation

$$\dot{x} = -e^{\psi}\nu, \tag{4.9}$$

from which the relation (4.8) can be easily deduced: apply the general formula (4.8) in [5] with $\Phi = -e^{\psi}$.

For the special time function $x^0 = \tau$ we therefore obtain

$$1 = \dot{\bar{H}} \geq -\Delta e^{\psi} + \frac{1}{n}\tau^2 e^{\psi}. \tag{4.10}$$

Moreover, let $x_0 \in \mathcal{S}_0$ be a point where, for fixed τ,

$$\sup_{\mathcal{S}_0} e^{\psi(\tau,\cdot)} = e^{\psi(\tau,x_0)}. \tag{4.11}$$

Then the maximum principle implies

$$1 \geq \frac{1}{n}\tau^2 e^{\psi(\tau,x_0)} \geq \frac{1}{n}\tau^2 e^{\psi(\tau,x)} \qquad \forall x \in \mathcal{S}_0 \tag{4.12}$$

and hence

$$\bar{H}e^{\psi} \leq n\bar{H}^{-1} \tag{4.13}$$

for all slices M_τ.

This inequality will be the key ingredient to prove the limit relation (4.2).

(iii) Define the function φ on $t \geq t_0$ by

$$\varphi(t) = \inf_{\mathcal{S}_0} u(t,\cdot). \tag{4.14}$$

Then φ is Lipschitz continuous and for a.e. t there holds

$$\dot{\varphi}(t) = \dot{u}(t, x_t), \tag{4.15}$$

where x_t is such that the infimum is attained in x_t. This result is well known; we shall give a short proof in Lemma (3.2) below for the sake of completeness.

Now, from (1.2), looking at the component $\alpha = 0$, we deduce that u satisfies the evolution equation

$$\dot{u} = \frac{\tilde{v}}{He^{\psi}}, \tag{4.16}$$

where $\tilde{v} = v^{-1}$ and where the time derivative is the total derivative, i.e.,

$$\dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i, \tag{4.17}$$

and hence

$$\frac{\partial u}{\partial t} = \frac{v}{He^{\psi}}. \tag{4.18}$$

From (2.16) we infer

$$e^{-\psi}\tilde{v}H = -\Delta u - \bar{\Gamma}^0_{00}\|Du\|^2 - 2\bar{\Gamma}^0_{0i}u^i + e^{-\psi}\bar{H}, \tag{4.19}$$

and conclude further, with the help of the maximum principle, that in x_t

$$H \le \bar{H}, \tag{4.20}$$

and thus

$$\frac{\partial u}{\partial t} \ge \frac{1}{\bar{H}e^{\psi}} \tag{4.21}$$

in x_t.

Therefore, φ satisfies

$$\dot{\varphi} \ge \frac{1}{\bar{H}e^{\psi}} \quad \text{for} \quad \text{a.e. } t \ge t_0; \tag{4.22}$$

hence

$$\dot{\varphi} \ge \frac{1}{n}\bar{H} = \frac{1}{n}\varphi \tag{4.23}$$

in view of (4.13) and the fact that the slices M_τ have mean curvature τ.

From this inequality we deduce immediately

$$\varphi(t) \ge \varphi(t_0)e^{(1/n)(t-t_0)} \quad \forall t \ge t_0, \tag{4.24}$$

proving the lemma. □

Lemma 4.2. *Let $\mathcal{S}_0$ be compact and $f \in C^1(J \times \mathcal{S}_0)$, where J is any open interval. Then*

$$\varphi(t) = \inf_{\mathcal{S}_0} f(t, \cdot) \tag{4.25}$$

is Lipschitz continuous and there holds a.e.

$$\dot{\varphi} = \frac{\partial f}{\partial t}(t, x_t), \tag{4.26}$$

where x_t is a point in which the infimum is attained.

A corresponding result is also valid if φ is defined by taking the supremum instead of the infimum.

Proof. φ is obviously Lipschitz continuous and thus a.e. differentiable by Rademacher's theorem.

For arbitrary $t_1, t_2 \in J$ we have

$$\varphi(t_1) - \varphi(t_2) = f(t_1, x_{t_1}) - f(t_2, x_{t_2}) \geq f(t_1, x_{t_1}) - f(t_2, x_{t_1}). \tag{4.27}$$

Now, let φ be differentiable in t_1; then, by choosing $t_2 > t_1$, and looking at the difference quotients of both sides, we conclude

$$\dot{\varphi}(t_1) \leq \frac{\partial f}{\partial t}(t_1, x_{t_1}). \tag{4.28}$$

Choosing $t_2 < t_1$ we obtain the opposite inequality, completing the proof of the lemma. □

We have proved that the flow hypersurfaces run straight in the singularity, if the flow exists for all time. However, it might happen that the flow runs into the future singularity in finite time.

To exclude this possibility we have imposed the strong volume decay condition.

Lemma 4.3. *Let N satisfy a strong volume decay condition with respect to the future. Then, for any finite T, $0 < T \leq T^*$, the flow stays in a precompact region Ω_T for $0 \leq t < T$.*

Proof. According to Lemma 1.4 we may choose a time function x^0 such that the relation (2.24) is valid for the coordinate slices $x^0 = \text{const}$.

Let $M(t) = \operatorname{graph} u$ be the flow hypersurfaces, and set

$$\varphi(t) = \sup_{\mathcal{S}_0} u(t, \cdot). \tag{4.29}$$

Then, similarly as in the proof of Lemma 3.1, we deduce that for a.e. t

$$\dot{\varphi} = \frac{1}{He^{\psi}} \leq \frac{1}{\bar{H}e^{\psi}} \leq 1, \tag{4.30}$$

in view of (2.24).

Hence we infer

$$\varphi \leq \varphi(0) + t \quad \forall 0 \leq t < T^*, \tag{4.31}$$

which proves the lemma since the singularity corresponds to $x^0 = \infty$. □

5 C^1-Estimates

We consider a smooth solution of the evolution equation (1.2) in a maximal time interval $[0, T^*)$ and shall prove a priori estimates for

$$\tilde{v} = v^{-1} = \frac{1}{\sqrt{1 - |Du|^2}} \tag{5.1}$$

in $Q_T = [0, T] \times \mathcal{S}_0$ for any $0 < T < T^*$.

The proof is a slight modification of the proof of the corresponding result for the mean curvature flow in [6]. We note that the time-like convergence condition is not necessary for this estimate.

Let us first state an evolution equation for $\tilde{v}$.

Lemma 5.1 (**Evolution of** $\tilde{v}$)**.** *The quantity* $\tilde{v}$ *satisfies the evolution equation*

$$\begin{aligned} \dot{\tilde{v}} - H^{-2}\Delta\tilde{v} = &-H^{-2}\|A\|^2\tilde{v} - 2H^{-1}\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &- 2H^{-2}h^{ij}x_i^\alpha x_j^\beta\eta_{\alpha\beta} - H^{-2}g^{ij}\eta_{\alpha\beta\gamma}x_i^\beta x_j^\gamma\nu^\alpha \\ &- H^{-2}\bar{R}_{\alpha\beta}\nu^\alpha x_k^\beta\eta_\gamma x_l^\gamma g^{kl}, \end{aligned} \tag{5.2}$$

where η *is the covariant vector field* $(\eta_\alpha) = e^\psi(-1, 0, \dots, 0)$.

Proof. We have $\tilde{v} = \langle \eta, \nu \rangle$. Let (ξ^i) be local coordinates for $M(t)$. Differentiating $\tilde{v}$ yields the result; cf. [6, Lemma 3.2] for details. □

Lemma 5.2. *Consider the flow in a precompact region* Ω*. Then there exists a constant* $c = c(\Omega)$ *such that for any positive function* $0 < \epsilon = \epsilon(x)$ *on* $\mathcal{S}_0$ *and any hypersurface* $M(t) \subset \Omega$ *of the flow we have*

$$|||\nu||| \le c\tilde{v}, \tag{5.3}$$

$$g^{ij} \le c\tilde{v}^2\sigma^{ij}, \tag{5.4}$$

and

$$|h^{ij}\eta_{\alpha\beta}x_i^\alpha x_j^\beta| \le \frac{\epsilon}{2}\|A\|^2\tilde{v} + \frac{c}{2\epsilon}\tilde{v}^3, \tag{5.5}$$

where (η_α) *is the vector field in Lemma 4.1.*

Confer [6, Lemma 3.3] for a proof.

Combining the preceding lemmata we infer

Lemma 5.3. *Consider the flow in a precompact region Ω, then there exists a constant $c = c(\Omega)$ such that for any positive function $\epsilon = \epsilon(x)$ on $\mathcal{S}_0$ the term $\tilde{v}$ satisfies a parabolic inequality of the form*

$$\dot{\tilde{v}} - H^{-2}\Delta\tilde{v} \le -(1-\epsilon)H^{-2}\|A\|^2\tilde{v} + cH^{-2}[1+\epsilon^{-1}]\tilde{v}^3. \tag{5.6}$$

Proof. The terms on the right-hand side of (5.2) having a factor H^{-2} can obviously be estimated as claimed.

The remaining term can be estimated by

$$\begin{aligned} 2H^{-1}|\eta_{\alpha\beta}\nu^\alpha\nu^\beta| &\le 2cH^{-1}\tilde{v}^2 \\ &\le \frac{\epsilon}{2}\frac{1}{n}\tilde{v} + 2nc^2\epsilon^{-1}H^{-2}\tilde{v}^3. \end{aligned} \tag{5.7}$$

The claim then follows from the relation

$$\frac{1}{n}H^2 \le |A|^2, \tag{5.8}$$

i.e.,

$$-H^{-2}|A|^2\tilde{v} \le -\frac{1}{n}\tilde{v}. \tag{5.9}$$

□

We further need the following two lemmata.

Lemma 5.4. *Let $M(t) = \operatorname{graph} u(t)$ be the flow hypersurfaces; then we have*

$$\begin{aligned} \dot{u} - H^{-2}\Delta u = 2e^{-\psi}\tilde{v}H^{-1} - H^{-2}e^{-\psi}g^{ij}\bar{h}_{ij} \\ + H^{-2}\bar{\Gamma}^0_{00}\|Du\|^2 + 2H^{-2}\bar{\Gamma}^0_{0i}u^i, \end{aligned} \tag{5.10}$$

where the time derivative is a total derivative.

Proof. We use the relation

$$\dot{u} = e^{-\psi}\tilde{v}H^{-1} \tag{5.11}$$

together with (2.16). □

Lemma 5.5. *Let $\Omega \subset N$ be precompact and $M \subset \Omega$ be a space-like graph over $\mathcal{S}_0$, $M = \operatorname{graph} u$. Then*

$$|\tilde{v}_i u^i| \le c\tilde{v}^3 + \|A\|e^{\psi}\|Du\|^2, \tag{5.12}$$

where $c = c(\Omega)$.

Proof. Confer the proof of [6, Lemma 3.6]. □

We are now ready to prove the a priori estimate for $\tilde{v}$.

Lemma 5.6. *Let $\Omega \subset N$ be precompact. Then, as long as the flow stays in Ω, the term $\tilde v$ is a priori bounded*

$$\tilde v \le c = c(\Omega, \sup_{M_0} \tilde v). \tag{5.13}$$

In particular, we do not have to assume that the time-like convergence is valid, and we note that c does not depend explicitly on T.

Proof. Let μ, λ be positive constants, where μ is supposed to be small and λ large, and define

$$\varphi = e^{\mu e^{-\lambda u}}, \tag{5.14}$$

where we assume without loss of generality that $u \le -1$; otherwise replace in (5.14) u by $(u - c)$, c large enough.

We shall show that

$$w = \tilde v \varphi \tag{5.15}$$

is a priori bounded as indicated in (5.13) if μ, λ are chosen appropriately.

In view of Lemmas 4.2 and 4.4 we have

$$\dot\varphi - H^{-2}\Delta\varphi \le c\mu\lambda e^{-\lambda u} H^{-2}\tilde v^2 \varphi - \mu\lambda^2 e^{-\lambda u}[1 + \mu e^{-\lambda u}] H^{-2}\|Du\|^2\varphi, \tag{5.16}$$

since $0 < H$, from which we further deduce, taking Lemmas 4.3 and 4.5 into account,

$$\begin{aligned} \dot w - H^{-2}\Delta w \le &-(1-\epsilon)H^{-2}\|A\|^2\tilde v\varphi + cH^{-2}[1+\epsilon^{-1}]\tilde v^3\varphi \\ &- \mu\lambda^2 e^{-\lambda u}[1+\mu e^{-\lambda u}]H^{-2}\tilde v\|Du\|^2\varphi \\ &+ c\mu\lambda e^{-\lambda u}H^{-2}\tilde v^3\varphi + 2\mu\lambda e^{-\lambda u}H^{-2}\|A\|e^{\psi}\|Du\|^2\varphi. \end{aligned} \tag{5.17}$$

We estimate the last term on the right-hand side by

$$\begin{aligned} 2\mu\lambda e^{-\lambda u}H^{-2}\|A\|e^{\psi}\|Du\|^2\varphi \le &(1-\epsilon)H^{-2}\|A\|^2\tilde v\varphi \\ &+ \frac{1}{1-\epsilon}\mu^2\lambda^2 e^{-2\lambda u}H^{-2}\tilde v^{-1}e^{2\psi}\|Du\|^4\varphi, \end{aligned} \tag{5.18}$$

and conclude

$$\begin{aligned} \dot w - H^{-2}\Delta w \le &c[1+\epsilon^{-1}]H^{-2}\tilde v^3\varphi + \left[\frac{1}{1-\epsilon} - 1\right]\mu^2\lambda^2 e^{-2\lambda u}H^{-2}\|Du\|^2\tilde v\varphi \\ &- \mu\lambda^2 e^{-\lambda u}H^{-2}\|Du\|^2\tilde v\varphi, \end{aligned} \tag{5.19}$$

where we have used that

$$e^{2\psi}\|Du\|^2 \le \tilde v^2. \tag{5.20}$$

Setting $\epsilon = e^{\lambda u}$, we then obtain

$$\begin{aligned} H^2(\dot{w} - H^{-2}\Delta w) \le{} & ce^{-\lambda u}\tilde{v}^3\varphi + c\mu\lambda e^{-\lambda u}\tilde{v}^3\varphi \\ & + \left[\frac{\mu}{1-\epsilon} - 1\right]\mu\lambda^2 e^{-\lambda u}\|Du\|^2\tilde{v}\varphi. \end{aligned} \tag{5.21}$$

Now, we choose $\mu = \frac{1}{2}$ and λ_0 so large that

$$\frac{\mu}{1 - e^{\lambda u}} \le \frac{3}{4} \quad \forall \lambda \ge \lambda_0, \tag{5.22}$$

and infer that the last term on the right-hand side of (5.21) is less than

$$-\tfrac{1}{8}\lambda^2 e^{-\lambda u}\|Du\|^2\tilde{v}\varphi, \tag{5.23}$$

which in turn can be estimated from above by

$$-c\lambda^2 e^{-\lambda u}\tilde{v}^3\varphi \tag{5.24}$$

at points where $\tilde{v} \ge 2$.

Thus we conclude that for

$$\lambda \ge \max(\lambda_0, 4) \tag{5.25}$$

the parabolic maximum principle, applied to w, yields

$$w \le \text{const}(|w(0)|_{\mathcal{S}_0}, \lambda_0, \Omega). \tag{5.26}$$

□

6 C^2-Estimates

We want to prove that, as long as the flow stays in a precompact set $\Omega \subset N$, the principal curvatures of the flow hypersurfaces are a priori bounded by a constant depending only on Ω and the initial hypersurface M_0. Again we do not need the time-like convergence condition for this estimate.

Let us first prove an a priori estimate for H.

Lemma 6.1. *Let $\Omega \subset N$ be precompact, and assume that the flow (1.2) stays in Ω for $0 \le t \le T < T^*$. Then the mean curvature of the flow hypersurfaces is bounded by*

$$0 < H \le c(\Omega, \sup_{M_0} H). \tag{6.1}$$

Proof. From Lemma 3.2 we immediately deduce that $\varphi = \log H$ satisfies the evolution equation

$$\dot{\varphi} - H^{-2}\Delta\varphi = H^{-2}(|A|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta) - H^{-2}\|D\varphi\|^2. \tag{6.2}$$

Let λ be large and set

$$w = \varphi + \lambda\tilde{v}. \tag{6.3}$$

Then we conclude from (5.6) that w satisfies the parabolic inequality

$$\dot{w} - H^{-2}\Delta w \le -\frac{\lambda}{2}H^{-2}|A|^2 + c\lambda H^{-2}, \tag{6.4}$$

if λ is large enough, $\lambda \ge \lambda(\Omega)$. Hence the parabolic maximum principle yields the result in view of the relation

$$\frac{1}{n}H^2 \le |A|^2. \tag{6.5}$$

□

Lemma 6.2. *Under the assumptions of Lemma 5.1 the principal curvatures* κ_i*,* $1 \le i \le n$*, of the flow hypersurfaces are a priori bounded in* Ω

$$|\kappa_i| \le c(\Omega, \sup_{M_0}|A|). \tag{6.6}$$

Proof. Since $0 \le H$, it suffices to estimate

$$\sup_i \kappa_i \le c(\Omega, \sup_{M_0}|A|). \tag{6.7}$$

Let φ be defined by

$$\varphi = \sup\{\, h_{ij}\eta^i\eta^j \colon \|\eta\| = 1 \,\}. \tag{6.8}$$

We claim that φ is a priori bounded in Ω.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0}\varphi < \sup\{\, \sup_{M(t)}\varphi \colon 0 < t \le T \,\} = \varphi(x_0). \tag{6.9}$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h^n_n. \tag{6.10}$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$\tilde{\eta} = (0, \dots, 0, 1), \tag{6.11}$$

and set

$$\tilde{\varphi} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}. \tag{6.12}$$

$\tilde{\varphi}$ is well defined in the neighbourhood of (t_0, ξ_0), and $\tilde{\varphi}$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$\dot{\tilde{\varphi}} = \dot{h}_n^n, \tag{6.13}$$

and the spatial derivatives do also coincide; in short, at $(t_0, \xi_0)\tilde{\varphi}$ satisfies the same differential equation (3.8) as h_n^n. For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that $\varphi = h_n^n$.

At (t_0, ξ_0) we have $\dot{\varphi} \geq 0$ and, in view of the maximum principle, we deduce from Lemma 2.3

$$0 \leq H^{-2}(-\|A\|^2 h_n^n + c|h_n^n|^2 + c). \tag{6.14}$$

Thus φ is a priori bounded in Ω by a constant c depending only on Ω and the initial hypersurface M_0. □

7 Longtime existence

Let us look at the scalar version of the flow as in (4.18)

$$\frac{\partial u}{\partial t} = e^{-\psi} v H^{-1} \tag{7.1}$$

defined in the cylinder

$$Q_{T^*} = [0, T^*) \times \mathcal{S}_0 \tag{7.2}$$

with initial value $u(0) \in C^\infty(\mathcal{S}_0)$.

Suppose that $T^* < \infty$. Then, from Lemma 3.3, we conclude that the flow stays in a compact region of N. Furthermore, in view of Lemma 4.6 and the C^2-estimates of Section 5, we obtain uniform C^2-estimates for u.

Thus, the differential operator on the right-hand side of (7.1) is uniformly elliptic in u independent of t, since there are constants c_1, c_2 such that

$$0 < c_1 \leq H \leq c_2 \quad \forall 0 \leq t < T^*, \tag{7.3}$$

in view of Lemma 2.4.

Hence, we can apply the known regularity results (cf. e.g., [13, Chap. 5.5]) to conclude that uniform $C^{2,\alpha}$-estimates are valid, leading further to uniform $C^{m,\alpha}$-estimates for any $m \in \mathbb{N}$, due to the regularity result for linear operators. But this will contradict the maximality of T^*.

Therefore, $T^* = \infty$, i.e., the flow exists for all time, and for any finite T we have a priori estimates in $C^m([0, T] \times \mathcal{S}_0)$ for any $m \in \mathbb{N}$.

8 A new time function

We know that the flow exists for all time and hence we conclude from Lemmas 3.1 and 3.3 that the flow hypersurfaces provide a foliation of the future of M_0, i.e., the flow parameter t could be used as a new time function in $D^+(M_0)$, if Dt is time-like.

Lemma 8.1. *The flow parameter t can be used as future-directed time function in $D^+(M_0)$.*

Proof. Let (x^α) be a future-directed coordinate system such that the relation (1.4) is valid. Then look at the scalar version of the flow, equation (7.1). If we can show that $(\tilde{x}^\alpha)$ with

$$\tilde{x}^0 = t, \quad \tilde{x}^i = x^i, \tag{8.1}$$

represents a regular coordinate transformation with positive Jacobi determinant, then the lemma is proved.

Now, the inverse coordinate transformation $x = x(\tilde{x})$, which exists, since we already know that the flow hypersurfaces provide a foliation, has the form

$$x^0 = u(t,x) \equiv u(\tilde{x}), \quad x^i = \tilde{x}^i, \tag{8.2}$$

where we apologize for using the same symbol x to represent an $(n+1)$-tuple as well as the space coordinates (x^i).

We immediately deduce

$$\left|\frac{\partial x}{\partial \tilde{x}}\right| = \frac{\partial u}{\partial t} > 0, \tag{8.3}$$

hence the result in view of the inverse function theorem. □

The strong volume decay condition is not only sufficient to prove the long time existence of the inverse mean curvature flow, but also necessary.

Proposition 8.2. *Let N be a cosmological spacetime, $M_0 \subset N$ a compact, space-like hypersurface with positive mean curvature, and suppose that the inverse mean curvature flow with initial hypersurface M_0 exists for all time and provides a foliation of $D^+(M_0)$. Then N satisfies a future strong volume decay condition as well as a future mean curvature barrier condition.*

Proof. Choose $x^0 = t$ as a new time function, and let the metric of N be expressed as

$$d\bar{s}^2 = e^{2\psi}(-(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j). \tag{8.4}$$

M_0 now replaces the Cauchy hypersurface $\mathcal{S}_0$, and the flow hypersurfaces $M(t)$ are given as graphs of functions u with

$$u(t, x) = t. \tag{8.5}$$

Thus we conclude from (7.1) that

$$1 = \frac{\partial u}{\partial t} = e^{-\psi} H^{-1} \tag{8.6}$$

or equivalently,

$$He^{\psi} = 1 \quad \forall x \in M(t), \tag{8.7}$$

i.e., the strong volume decay condition is satisfied.

The mean curvature of the leaves $M(t)$ tends to ∞ in view of Lemma 2.4, hence N satisfies a future mean curvature barrier condition. □

From now on, let us assume that $x^0 = t$ is the time function. Set

$$\tau = 1 - e^{-(1/n)t}, \tag{8.8}$$

then the future spacetime singularity corresponds to $\tau = 1$, and there holds

Theorem 8.3. *The quantity* $1 - \tau$ *can be looked at as the radius of the slices* $\tau = \text{const}$ *as well as a measure of the remaining life span of the spacetime, since we have*

$$|M(\tau)| = |M_0|(1 - \tau)^n, \tag{8.9}$$

and the length $L(\gamma)$ *of any future-directed curve starting from* $M(\tau)$ *is estimated from above by*

$$L(\gamma) \leq c(1 - \tau), \tag{8.10}$$

where

$$c = \frac{n}{\inf_{M_0} H}. \tag{8.11}$$

Proof. Let $g = \det(g_{ij})$, where (g_{ij}) is the induced metric of $M(t) \equiv M(\tau)$; then

$$\frac{d}{dt}\sqrt{g} = -\sqrt{g} \tag{8.12}$$

in view of (3.1), and hence

$$|M(t)| = |M_0|e^{-t} = |M_0|(1-\tau)^n. \tag{8.13}$$

To prove (8.10), we first note that in view of Lemma 2.4

$$H \ge \inf_{M_0} He^{(1/n)t} = \frac{n}{c}(1-\tau)^{-1}, \tag{8.14}$$

where c is the constant in (8.11). One of Hawking's singularity theorems then asserts that

$$L(\gamma) \le c(1-\tau), \tag{8.15}$$

cf. [14, Prop. 37, p. 288]. □

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