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Higher genus Gromov–Witten invariants of the Grassmannian, and the Pfaffian Calabi–Yau 3-folds

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Abstract

We solve Bershadsky–Cecotti–Ooguri–Vafa (BCOV) holomorphic anomaly equation to determine the higher genus Gromov–Witten invariants ($g \leq 5$) of the derived equivalent Calabi–Yau 3-folds, which are of the appropriate codimensions in the Grassmannian Gr$(2, 7)$ and the Pfaffian Pf(7).

1 Introduction

Since the first successful application of the mirror symmetry to the Gromov–Witten theory of the quintic hypersurface in $\mathbf{P}^4$ [1], and its highly non-trivial generalization to higher genus $g \geq 1$ in [2, 3], the mirror symmetry of Calabi–Yau manifolds has been attracting attentions in both mathematics

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and physics. Now, according to Kontsevich's homological mirror symmetry [4], we consider that two Calabi–Yau manifolds X and Y are mirror symmetric to each other when the derived category of the coherent sheaves on X, $D^b(\mathrm{Coh}(X))$, is equivalent to the derived Fukaya category $\mathrm{DFuk}(Y, \beta)$, and vice versa. In this homological viewpoint, it is clear that Calabi–Yau manifolds X, X', which are derived equivalent $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(X'))$, are of considerable interest.

For a smooth projective variety X, the projective varieties having equivalent derived category to X are called Fourier–Mukai partners of X, and the set of their isomorphism classes is denoted by $\mathrm{FM}(X)$. In dimension two, the set $\mathrm{FM}(X)$ has been studied in detail in [5] and it has been shown that the number of Fourier–Mukai partners of a smooth minimal projective surface X is finite, i.e., $|\mathrm{FM}(X)| < \infty$. In particular, for a K3 surface X, a necessary and sufficient condition for a K3 surface X' to be a partner of X is known in terms of the Hodge isometry in the Mukai lattice [6]. Based on the result in [6], a precise counting formula of the number of Fourier–Mukai partners has been given in [7, 8].

In dimension three, however, since birational Calabi–Yau 3-folds share the equivalent derived category [9], the counting problem should be considered under the birational equivalences. This contrasts with the fact in two dimensions that two birational K3 surfaces are biholomorphic to each other. Recently, an example of Calabi–Yau 3-folds which share the equivalent derived category but are non-birational to each other has been constructed in [10, 11] based on the earlier observation in [12]. This example is of our interest in this article.

In this paper, we apply the mirror symmetry to the derived equivalent Calabi–Yau 3-folds X and X', that appeared in [10, 12], of appropriate codimensions in the Grassmannian $\mathrm{Gr}(2, 7)$ and the Pfaffian $\mathrm{Pf}(7)$, respectively. In particular, we determine the higher genus Gromov–Witten invariants ($g \leq 5$) integrating the holomorphic anomaly equation in [3] recursively. The Gromov–Witten invariants at genus zero were determined earlier in [13] and [12] following the method in [1]. See [14, 15, 16] for mathematical proofs of the invariants. For the higher genus calculations, we solve the BCOV holomorphic anomaly equation [2, 3]. In particular we utilize the gap condition at the conifold singularities, which has been found recently in [17], with slight improvements in the estimate of the unknown parameters in the holomorphic ambiguities.

Both the Calabi–Yau manifolds X and X' have Picard number $\rho = 1$. Let $N_g^X(d)$, $N_g^{X'}(d)$ be the Gromov–Witten invariant of degree d with respect to the respective generator H of the Picard group. We denote by $\mathtt{F}_g(t)$

the generating functions of the Gromov–Witten invariants (the so-called Gromov–Witten potential), which have the following form for $g \geq 2$ in general,

$$\mathtt{F}_g(t) = \frac{\chi}{2}\,(-1)^g\,\frac{|B_{2g}B_{2g-2}|}{2\,g\,(2g-2)\,(2g-2)!} + \sum_{d>0} N_g(d)\,q^d, (q = \mathrm{e}^{2\pi i t}) \qquad (1.1)$$

where χ is the Euler number of a Calabi–Yau manifold and B_g is the Bernoulli number. The constant term above represents the Gromov–Witten invariant $N_g(0)$ of degree zero, and it represents the contribution from the

Table 1: Gopakumar–Vafa invariants $n_g^X(d)$ $(g \leq 5)$ of the Grassmannian Calabi–Yau 3-fold $X = \mathrm{Gr}(2,7)_{1^7}$

d	g=0	g=1	g=2
1	196	0	0
2	1225	0	0
3	12740	0	0
4	198058	0	0
5	3716944	588	0
6	79823205	99960	0
7	1877972628	8964372	0
8	47288943912	577298253	99960
9	1254186001124	31299964612	47151720
10	34657942457488	1535808070650	7906245550
11	990133717028596	70785403788680	858740761340
12	29075817464070412	3129139504135680	73056658523632
13	873796023687033916	134357808679487260	5317135023839604
14	26782042513523921505	5648906799029453044	347478656042915187
15	834938101511448746224	233816422635171601176	20996780173465726448
16	26417440686921151630504	9563588497688111378163	1195726471411561809370
17	846787615783681427068332	387581693402348794414352	65017598161994032437484

d	g=3	g=4	g=5
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	-1176	0	0
10	325409	0	0
11	956485684	-25480	0
12	301227323110	27885116	3675
13	52490228133616	67509270780	73892
14	6617949361316377	28917316111159	9783073244
15	676939616238018840	6764898614128228	13255130550228
16	59768711735781062098	1117634949252974670	6169573531612148
17	4730781899004364783412	146451269357268794212	1690718304511081104
18	344157075745064476608707	16239378567823605642392	332432097873830811843

constant maps [3, 18]. We determine the potential $\mathtt{F}_g(t)$ for $g \leq 5$ for X and X'. Also to see some implications of our results to the enumerative problem of holomorphic curves and/or the moduli problem related to Donaldson–Thomas invariants [19], we list the so-called Gopakumar–Vafa "invariants" $n_g(d)$ [20] which are determined from $N_g(d)$ by

$$\sum_{g\geq 0} \lambda^{2g-2}\mathtt{F}_g(t) = \sum_{g\geq 0}\sum_{k\geq 1, d\geq 0} n_g(d)\,\frac{1}{k}\left(2\,\sin\frac{k\lambda}{2}\right)^{2g-2} q^{kd}. \tag{1.2}$$

The organization of this paper is as follows: In Section 2, we summarize the constructions of the Grassmannian and the Pfaffian Calabi–Yau 3-folds, and their mirror orbifolds given in [12]. After introducing the Picard–Fuchs differential equation of the period integrals, we determine the $g = 0, 1$ Gromov–Witten prepotentials. We will also make a comment on a similarity of the Picard–Fuchs differential equation to the corresponding differential equation studied for a K3 surface with a non-trivial Fourier–Mukai partner in [8]. In Section 3, we briefly introduce the BCOV holomorphic

Table 2: Gopakumar–Vafa invariants $n_g^{X'}(d)$ $(g \leq 5)$ of the Pfaffian Calabi–Yau 3-fold X'

d	g=0	g=1	g=2
1	588	0	0
2	12103	0	0
3	583884	196	0
4	41359136	99960	0
5	3609394096	34149668	12740
6	360339083307	9220666238	25275866
7	39487258327356	2163937552736	21087112172
8	4633258198646014	466455116030169	11246111235996
9	572819822939575596	95353089205907736	4601004859770928
10	73802503401477453288	18829753458134112872	1586777390750641117
11	9831726718738661469404	3632247018393524104896	486768262807329916336
12	1346383795156980043546418	689243453496908009355852	137262882246594110683614

d	g=3	g=4	g=5
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	1225	0	0
7	22409856	0	0
8	58503447590	25371416	3675
9	67779027822044	216888021056	33575388
10	50069281882780727	521484626374894	1111788286385
11	27893405899311185184	660609023799091444	5358750700883104
12	12822179880173592308422	568693999386204794172	11048054952421812976
13	5131002509749249793297316	377653013301230457157640	14053721920121779703948

anomaly equation for $g \geq 2$ and its solutions given in [3]. According to [2, 3] we define the topological limit of the solutions. We also summarize the recent results found in [21] about some polynomiality of the solutions. Then, we introduce a "gap condition", which has been found recently in [17], to fix the holomorphic ambiguities contained in the solutions. In Section 4, we present our calculations in some details to determine the Gromov–Witten potentials. We determine the potentials up to $g = 5$ and list the resultant Gopakumar–Vafa invariants in Tables 1 and 2.[1] The conclusion and discussions are given in Section 5.

2 Gromov–Witten invariants at $g = 0$ and $g = 1$

In this section, we briefly summarize the constructions of Calabi–Yau manifolds, X and X', and their orbifold mirror construction following [12]. We summarize the genus zero and one Gromov–Witten invariants using the solutions of the Picard–Fuchs differential equation of the mirror family.

2.1 The Grassmannian and the Pfaffian Calabi–Yau 3-folds

Let us first summarize the construction of the Grassmannian Calabi–Yau 3-fold and its topological invariants. Let $\mathrm{Gr}(2,7)$ be the Grassmannian of the 2-planes in $\mathbf{C}^7$, and Q be the universal quotient bundle. The line bundle $\wedge^5 Q$ determines the Plücker embedding $i : \mathrm{Gr}(2,7) \hookrightarrow \mathbf{P}^{20}$, hence $\sigma_1 = c_1(Q)$ represents the class of a hyperplane section. Then $\int_{\mathrm{Gr}(2,7)} \sigma_1^{10} = 42$ gives the degree of the Grassmannian in the projective space. We denote by $\mathrm{Gr}(2,7)_{1^7}$ the complete intersection of $\mathrm{Gr}(2,7)$ with seven hyperplanes in $\mathbf{P}^{20}$. Then $X = \mathrm{Gr}(2,7)_{1^7}$ defines a Calabi–Yau 3-fold since $c_1(\mathrm{Gr}(2,7)) = 7\sigma_1$. In fact, the Chern class of $\mathrm{Gr}(2,7)$ is expressed by

$$\begin{aligned} c(\mathrm{Gr}(2,7)) = 1 &+ 7c_1(Q) + (25c_1(Q)^2 - 3c_2(Q)) \\ &+ 14(4c_1(Q)^3 - c_1(Q)c_2(Q)) + \cdots, \end{aligned}$$

see e.g., [22], and for the complete intersection, we have

$$c(X) = \frac{c(\mathrm{Gr}(2,7))}{(1+c_1(Q))^7} = 1 + (4c_1(Q)^2 - 3c_2(Q)) - 7(c_1(Q)^3 - c_1(Q)c_2(Q)).$$

[1] One may further continue the calculations for $g \geq 5$. The data $n_g^X(d)$, $n_g^{X'}(d)$ $g \leq 9$ is available upon request to the first named author.

Using $\int_{\mathrm{Gr}(2,7)} \sigma_1^{10} = 42$, $\int_{\mathrm{Gr}(2,7)} \sigma_1^8 \sigma_2 = 28$ ($\sigma_2 = c_2(Q)$), and representing by H the hyperplane σ_1 on X, we have the following topological invariants

$$\chi(X) = -98, \;\; c_2(X) \cdot H = 84, \;\; H^3 = 42 \tag{2.1}$$

Also we see $h^{1,1}(X) = 1$ by Lefschetz hyperplane theorem, which implies $h^{2,1}(X) = 50$.

The construction of the second Calabi–Yau manifold X' is more involved and utilizes the Pfaffian variety in the projective space $\mathbf{P}^{20}$. Let $\mathcal{S}$ be a 7×7 skew symmetric matrix $\mathcal{S} = (s_{ij})$ with $[s_{ij}] \in \mathbf{P}^{20}$. The rank of $\mathcal{S}$ is less than or equal to 6, and in particular, the rank 4 locus ($\mathrm{rk}\mathcal{S} \le 4$) determines a codimension three variety in $\mathbf{P}^{20}$, the Pfaffian variety. Explicitly, this variety is determined by the ideal generated by the square roots of the diagonal minors of $\mathcal{S}$, $p_0(\mathcal{S}), \dots, p_6(\mathcal{S})$. Restricting this variety to a generic projective space $\mathbf{P}^6 \subset \mathbf{P}^{20}$, i.e., specializing the parameters $[s_{ij}]$ to lie on a generic $\mathbf{P}^6$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^6}(-7) \xrightarrow{{}^t p(\mathcal{S})} \mathcal{O}_{\mathbf{P}^6}(-4)^{\oplus 7} \xrightarrow{\mathcal{S}} \mathcal{O}_{\mathbf{P}^6}(-3)^{\oplus 7} \xrightarrow{p(\mathcal{S})} \mathcal{O}_{\mathbf{P}^6} \to \mathcal{O}_{X'} \to 0,$$

where we set $p(\mathcal{S}) = (\,(-1)^{i+1} p_i(\mathcal{S})\,)_{i=0,\dots,6}$ to be a row vector and use $p(\mathcal{S})\,\mathcal{S} = 0$, since $(-1)^{i+j} p_i(\mathcal{S}) p_j(\mathcal{S})$ represents the ij-minor of $\mathcal{S}$ and $\det(\mathcal{S}) = 0$. From this exact sequence, we see that the canonical sheaf of X' is trivial, $\omega_{X'} \cong \mathcal{E}xt^3(\mathcal{O}_{X'}, \omega_{\mathbf{P}^6}) \cong \mathcal{O}_{X'}$ and therefore X' is a Calabi–Yau 3-fold. The degree of X' in $\mathbf{P}^6$ is 14 and $H^3 = 14$ for the hyperplane section H. Other topological invariants are determined by the general formulas $c_2(X')H = 84 - 2d, c_3(X') = -d^2 + 49d - 588$ ($d = H^3$) valid for codimension 3 smooth Calabi–Yau varieties, see [23] for example. Thus we have the following topological invariants

$$\chi(X') = -98, \;\; c_2(X') \cdot H = 56, \;\; H^3 = 14. \tag{2.2}$$

We also have the Hodge numbers $h^{1,1}(X') = 1$ and $h^{2,1}(X') = 50$.

As noted in the reference [12], the construction of X and X' are dual in the following sense,

$$X = \mathrm{Gr}(2,7) \cap \mathbf{P}^{13} \subset \mathbf{P}^{20}, \;\; X' = Pf(7) \cap \check{\mathbf{P}}^6 \subset \check{\mathbf{P}}^{20},$$

where $\check{\mathbf{P}}^{20}$ is the dual projective space to $\mathbf{P}^{20}$ and $\check{\mathbf{P}}^6$ is the annihilator of $\mathbf{P}^{13}$ under the dual pairing. This duality has been utilized to prove the derived equivalence $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(X'))$ [10, 11].

2.2 The mirror manifolds and the Picard–Fuchs differential equations

In a similar way to the orbifold construction of the quintic hypersurface in $\mathbf{P}^4$ [24, 1], the mirror manifolds Y and Y' have been constructed, respectively, for X and X' in [12]. Following [12], we introduce the mirror family $\{Y'_x\}_{x\in\mathbf{P}^1}$ and the Picard–Fuchs differential equation for the period integral of a holomorphic three form.

Consider the skew symmetric matrices

$$E_k(y) = \sum_{i+j=k} y_{i-j} E_{ij} \ (k = 0, 1, \ldots, 6;\ \ y_i + y_{-i} = 0)$$

parametrized by $[y_1, y_2, y_3] \in \mathbf{P}^2$, where the index of y_i is understood modulo 7, and $E_{ij}(0 \le i, j \le 6)$ are the matrix units. We define

$$\check{\mathbf{P}}^6_{[y_1,y_2,y_3]} = \text{the projective span of } \{E_0(y), \ldots, E_6(y)\} \subset \check{\mathbf{P}}^{20},$$

and consider the group $G = \langle \tau, \sigma \rangle$ acting on $\check{\mathbf{P}}^6_{[y_1,y_2,y_3]}$ by

$$\tau : E_k(y) \mapsto \mathrm{e}^{2\pi ik/7} E_k(y), \quad \sigma : E_k(y) \mapsto E_{k+2}(y).$$

Then $X'_{[y_1,y_2,y_3]} = Pf(7) \cap \check{\mathbf{P}}^6_{[y_1,y_2,y_3]}$ is a special family of X', and its general member has 49 double points at the orbit $G \cdot [y_0, y_1, \ldots, y_6]$. When we further specialize $X'_{[y_1,y_2,y_3]}$ to a $\mathbf{P}^1$ family $X'_{[y_1,y_2,0]}$, we have additionally 7 double points at the fixed points of τ. These double points arise from the process collapsing S^3 to points, and may be blown up to $\mathbf{P}^1$ without affecting the Calabi–Yau condition. Blowing up these $49 + 7$ double points in total, we have $-98 + (49 + 7) \times 2 = 14$ for the Euler number of the resolved space $\tilde{X}'_{[y_1,y_2,0]}$. Now, consider the quotient $\tilde{X}'_{[y_1,y_2,0]}/\langle\tau\rangle$. This quotient has singularities which come from the 7×2 fixed points under the action of τ. These singularities can be resolved under the Calabi–Yau condition, and for the Euler number we have

$$\chi\left(\widehat{\frac{\tilde{X}'_{[y_1,y_2,0]}}{\langle\tau\rangle}}\right) = \frac{1}{|\langle\tau\rangle|} \sum_{g,h\in\langle\tau\rangle} \chi(\tilde{X}'_{[y_1,y_2,0]}|_{g,h}) = 98$$

where $\tilde{X}'_{[y_1,y_2,0]}|_{g,h}$ represents the fixed points under g and h (i.e., the 7×2 points for $(g, h) \neq (e, e)$). The Hodge numbers are determined in [12] by looking the blow-ups more closely. The results are $h^{1,1} = 50, h^{2,1} = 1$, justifying the claim that $Y'_y := \widehat{\tilde{X}'_{[y_1,y_2,0]}/\langle\tau\rangle}$ is the mirror family of the Calabi–Yau variety X'.

For the concrete description of Y'_y, we write in Appendix (A.1) the diagonal Pfaffians $p_k(\mathcal{S})$ of the skew symmetric matrix $\mathcal{S}(y) = \mathcal{S}(y,[u])$ for the spacial family $X'_{[1,y,0]}$ with $[u] = [u_0, u_1, \ldots, u_6] \in \mathbf{P}^6$. From the explicit form of the generators $p_k(\mathcal{S})$, we see that $Y'_{e^{2\pi i/7}y} \cong Y'_y$ and hence $x = y^7$ parametrizes the genuine mirror family. Then in terms of $p_i(\mathcal{S})$, the holomorphic 3-form of the family Y'_x may be given by

$$\Omega(x) = \mathrm{Res}\frac{(-1)^{\epsilon} Pf(\mathcal{S}_{i_3i_4i_5i_6})d\mu}{p_{i_0}p_{i_1}p_{i_2}}, \tag{2.3}$$

where $d\mu = du_0du_1\cdots du_6$ and $\mathcal{S}_{i_3i_4i_5i_6}$ is the 4×4 "diagonal" sub-matrix of $\mathcal{S}$ specified by the index set $\{i_3i_4i_5i_6\}$, and ϵ represents the parity of the order $i_0i_1\cdots i_6$. Evaluating the period integral over a torus cycle as a power series in x, the Picard–Fuchs differential operator $\mathcal{D}_x$ has been determined in [12],

$$\begin{aligned}\mathcal{D}_x = {} & 9\,\theta_x^4 - 3\,x(15 + 102\,\theta_x + 272\,\theta_x^2 + 340\,\theta_x^3 + 173\,\theta_x^4) \\ & - 2\,x^2(1083 + 4773\,\theta_x + 7597\,\theta_x^2 + 5032\,\theta_x^3 + 1129\,\theta_x^4) \\ & + 2\,x^3(6 + 675\,\theta + 2353\,\theta_x^2 + 2628\,\theta_x^3 + 843\,\theta_x^4) \\ & - x^4(26 + 174\,\theta_x + 478\,\theta_x^2 + 608\,\theta_x^3 + 295\,\theta_x^4) + x^5(\theta_x + 1)^4,\end{aligned} \tag{2.4}$$

where we define $\theta_x = x\frac{d}{dx}$. Using this differential operator, and normalizing the holomorphic three form suitably, we can determine the Yukawa coupling to be

$$C_{xxx} := -\int_{Y'_x} \Omega(x) \wedge \left(\frac{d}{dx}\right)^3 \Omega(x) = \frac{42 - 14x}{x^3(1 - 57x - 289x^2 + x^3)}. \tag{2.5}$$

A similar orbifold construction works for the Grassmannian Calabi–Yau variety X by taking the dual projective space $\mathbf{P}^{13}_{[1,y,0]}$ to $\check{\mathbf{P}}^6_{[1,y,0]}$. Then the mirror family $\{Y_y\}$ is given by a resolution of a suitable orbifold of $\mathrm{Gr}(2,7) \cap \mathbf{P}^{13}_{[1,y,0]}$. The remarkable observation made in [12] is that we obtain the same Picard–Fuchs differential operator as above, which has the property of the maximally degeneration [25] at both $x = 0$ and $x = \infty$. This indicates that the two Calabi–Yau varieties X and X' share the same the mirror family $\{Y_y\} = \{Y'_y\}$, and that the complexified Kähler moduli of the two Calabi–Yau varieties X and X' are unified in one complex structure moduli of the mirror family (Conjecture 10 in [12]). The structure of the singularities of the Picard–Fuchs equation (2.4) (cf. [26, 27]) may be summarized in the following Riemann's P scheme listing the indices ρ_k at each regular singular point;

$$\left\{\begin{array}{c|cccccc} x & 0 & \alpha_1 & \alpha_2 & \alpha_3 & 3 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \rho_2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \rho_3 & 0 & 1 & 1 & 1 & 3 & 1 \\ \rho_4 & 0 & 2 & 2 & 2 & 4 & 1 \end{array}\right\}, \tag{2.6}$$

where α_k are the roots of the "discriminant" $1 - 57x - 289x^2 + x^3 = 0$, for which Y'_x has double points.

Making the instanton expansions at each degeneration point, we find that the expansion about $x = 0$ corresponds to the Kähler moduli of the Grassmannian Calabi–Yau X, and the expansion about $x = \infty$ to that of the Pfaffian Calabi–Yau X'. Our main objective in this paper is to extend the instanton calculations to higher genera.

2.3 A digression to K3 surfaces

It is clear that the property of the Picard–Fuchs differential operator $\mathcal{D}_x$ is closely related to the equivalence $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(X'))$. Here we remark that essentially the same property may be observed in the case of K3 surfaces.

Let us recall that the set of Fourier–Mukai partners for a smooth projective variety X is defined by

$$\mathrm{FM}(X) = \{Y | D^b(\mathrm{Coh}(Y)) \cong D^b(\mathrm{Coh}(X))\}/\sim,$$

where $\sim$ represents the isomorphisms. When X is a K3 surface, one may expect that the cardinality of $|\mathrm{FM}(X)|$ is finite since birational K3 surfaces are biholomorphic. In fact, it is known that the number of Fourier–Mukai partners is finite[5, 6]. In particular, for a K3 surface X of degree $2n$ and the Picard number $\rho(X) = 1$, it is found [7] that the number of the Fourier–Mukai partners has a simple form,

$$|\mathrm{FM}(X)| = 2^{p(n)-1},$$

where $p(n)$ is the number of the prime factors ($p(1) := 1$). The first nontrivial case arises from $p(6) = 2$, i.e., we have $\mathrm{FM}(X) = \{X, X'\}$ for a K3 surface X of degree 12. According to [28], the partner X' may be identified with a moduli space of the rank 2 stable sheaves with $c_1(\mathcal{E}) = H, \chi(\mathcal{E}) = 2+3$. Also explicit constructions of the K3 surfaces of degree 12 and the mirror K3 surfaces are known in detail, see [29] and references therein. There the modular group $\Gamma(6)_{0+}$ appears as the monodromy group of the Picard–Fuchs differential equation of the mirror (one-parameter) family. It is found in [29] that one of the generators of the group $\Gamma(6)_{0+}$ does not correspond to any element in $Auteq(D^b(\mathrm{Coh}(X)))$ under the mirror symmetry, and argued that this generator represents the Fourier–Mukai transform $\Phi^{\mathcal{P}}$ defined by the Poincaré bundle $\mathcal{P}$ over $X \times X'$. The rest of the generators defines the index two subgroup $\Gamma(6)_{0+6}$ of $\Gamma(6)_+$.

Changing the monodromy group to the smaller group $\Gamma(6)_{0+6}$ doubles the moduli space of the mirror family (or the fundamental domain in the upper half plane). This doubled mirror family may be found in the table of [30], and it has the Picard–Fuchs differential operator,

$$\mathcal{D} = \theta_x^3 - x(2\,\theta_x + 1)(17\,\theta_x^2 + 17\,\theta_x + 5) + x^2(\theta_x + 1)^3.$$

This differential operator shows exactly the same property as (2.4), i.e., it has two maximal degeneration points at $x = 0$ and $x = \infty$. One may pursue the similarity further in that the Fourier–Mukai partner X' has an explicit construction using the orthogonal Grassmannian [28]. Here, a naive construction of the Grassmannian K3 surface $X = \mathrm{Gr}(2,6)_{1^6}$, however, does not give $\deg(X) = 12$ but 14, and hence $|\mathrm{FM}(X)| = 1$.

2.4 $g = 0$ and $g = 1$ Gromov–Witten invariants

We summarize the calculations of the genus zero and one Gromov–Witten invariants of the Grassmannian and the Pfaffian Calabi–Yau varieties X, X'.

(2.4.a) Let us first introduce the so-called mirror map [1]. We will denote henceforth the local coordinate $z = \frac{1}{x}$ to analyze the local solutions of the Picard-Fuchs equation (2.4) about $x = \infty$. At each degeneration point, we have one regular series solution with other solutions having (higher) logarithmic singularities. We normalize the regular solution and choose the following linear-logarithmic solution

$$\begin{cases} w_0(x) = 1 + 5\,x + 109\,x^2 + 3317\,x^3 + 121501\,x^4 + \cdots, \\ w_1(x) = \log(x) w_0(x) + 14\,x + 357\,x^2 + \frac{35105}{3}\,x^3 + \frac{2669975}{6}\,x^4 + \cdots. \end{cases} \tag{2.7}$$

The choice of the linear-logarithmic solution $w_1(x)$ is up to the addition of arbitrary multiple of $w_0(x)$. Here we fix this ambiguity so that the complexified Kähler moduli $2\pi i t = \frac{w_1(x)}{w_0(x)}$ has a "nice" form of the q-expansion,

$$\frac{1}{x(q)} = \frac{1}{q} + 14 + 189\,q + 2534\,q^2 + 42826\,q^3 + 869162\,q^4 + \cdots,$$

where $q := \mathrm{e}^{2\pi i t}$. In a similar way, we fix the regular solution $\tilde{w}_0(z)$ and the linear-logarithmic solution $\tilde{w}_1(z)$ at $z = 0$,

$$\begin{cases} \tilde{w}_0(z) = z + 17\,z^2 + 1549\,z^3 + 215585\,z^4 + 36505501\,z^5 + \cdots, \\ \tilde{w}_1(z) = \log(z)\tilde{w}_0(z) + 70\,z^2 + 7413\,z^3 + \frac{3268573\,z^4}{3} + \frac{1138372375\,z^5}{6} + \cdots. \end{cases} \tag{2.8}$$

By defining $2\pi i\tilde{t} = \frac{\tilde{w}_1(x)}{\tilde{w}_0(x)}$, $\tilde{q} = \mathrm{e}^{2\pi i\tilde{t}}$, we have

$$\frac{1}{z(\tilde{q})} = \frac{1}{\tilde{q}} + 70 + 3773\,\tilde{q} + 232750\,\tilde{q}^2 + 18421802\,\tilde{q}^3 + 1781859058\,\tilde{q}^4 + \cdots.$$

The expansions $x = x(q)$ and $z = z(\tilde{q})$ are called mirror maps at the respective degeneration points, $x = 0$ and $z = 0$.

(2.4.b) Now, by the formula in [1], we determine the quantum corrected Yukawa coupling $K_{ttt}(t)$ at $x = 0$ by

$$\Big(\frac{1}{w_0(x)}\Big)^2 C_{xxx}\Big(\frac{dx}{dt}\Big)^3 = 42 + 196\,q + 9996\,q^2 + 344176\,q^3 + 12685708\,q^4 + \cdots.$$

For the expansion at $z = 0$, we transform the Yukawa coupling (2.5) by

$$C_{zzz}(z) = C_{xxx}(x)\Big(\frac{dx}{dz}\Big)^3 = \frac{14 - 42\,z}{z(1 - 289\,z - 57\,z^2 + z^3)}. \tag{2.9}$$

Then the quantum Yukawa coupling $K_{\tilde{t}\tilde{t}\tilde{t}}(\tilde{t})$ at $z = 0$ is given by

$$\Big(\frac{1}{\tilde{w}_0(x)}\Big)^2 C_{zzz}\Big(\frac{dz}{dt}\Big)^3 = 14 + 588\,q + 97412\,q^2 + 15765456\,q^3 + 2647082116\,q^4 + \cdots.$$

These Yukawa couplings are related to the Gromov–Witten potentials by

$$K_{ttt}(t) = \Big(q\frac{d}{dq}\Big)^3 \mathtt{F}_0(t), \quad K_{\tilde{t}\tilde{t}\tilde{t}}(\tilde{t}) = \Big(\tilde{q}\frac{d}{d\tilde{q}}\Big)^3 \tilde{\mathtt{F}}_0(\tilde{t}).$$

Comparing the topological data given in (2.1) and (2.2), the degenerations at $x=0$ and $z=0$ have been identified in [12], respectively, with the Grassmannian Calabi–Yau X and the Pfaffian Calabi–Yau X', i.e.,

$$\mathtt{F}_0(t)=\mathtt{F}_0^{X}(t),\;\; \tilde{\mathtt{F}}_0(\tilde t)=\mathtt{F}_0^{X'}(\tilde t).$$

We observe in (2.9) that the numerator of the Yukawa coupling, i.e., $42-14\,x$, explains the difference of the leading term between the q- and the $\tilde q$-expansions. This simple observation should be contrasted to the similar calculations done for the "topology changes" (i.e., flops) [31].

(2.4.c) For the genus one invariants, we apply the BCOV formula [2] of the holomorphic potential $F^{(1)}(x)$ to our case,

$$F^{(1)}(x)=\frac{1}{2}\log\Big\{\Big(\frac{f_1(x)}{w_0(x)}\Big)^{3+h^{1,1}-\chi/12}\Big(\frac{dx}{dt}\Big)\,\mathrm{dis}(x)^{-1/6}\,x^{-1-c_2.H/12}\Big\}, \quad (2.10)$$

where $\mathrm{dis}(x)=1-57\,x-289\,x^2+x^3$ and $f_1(x)$ is some holomorphic function which we fix to $f_1(x)=1$ by requiring the regularity of $F^{(1)}(x)$ at $x=0,\infty,3$. Exactly the same form as $F^{(1)}(x)$ applies to $\tilde F^{(1)}(z)$ with $\tilde w_0(z)$, $\widetilde{\mathrm{dis}}(z)=1-289\,z-57\,z^2+z^3$, $\tilde f_1(z)=z$ and the data (2.2). The holomorphic function $\tilde f_1(z)$ guarantees the regularity of $\tilde F^{(1)}(z)$ at $z=0$. Using the topological data (2.1), (2.2) and the mirror maps $x=x(q)$ and $z=z(\tilde q)$, we obtain the genus one Gromov–Witten potentials,

$$\mathtt{F}_1^{X}(t)=F^{(1)}(x(q)),\quad \mathtt{F}_1^{X'}(\tilde t)=\tilde F^{(1)}(z(\tilde q)).$$

Here we remark that, except that one has to replace $w_0(x)$ with $\tilde w_0(z)$ by hand, one can verify the equality

$$F^{(1)}(x)=\tilde F^{(1)}(z),$$

with $x=\frac{1}{z}$. This relation holds because, by taking the topological limits, the BCOV formulas (2.10) and $\tilde F^{(1)}(z)$ follow from the "Quillen's norm" function

$$\mathcal{F}^{(1)}(x,\bar x)=\frac{1}{2}\log\Big\{\mathrm{e}^{(3+h^{11}-\chi/12)\,K}\,G^{x\bar x}\Big|\mathrm{dis}(x)^{-1/6}\,x^{-1-c_2.H/12}\Big|^2\Big\}\,, \quad (2.11)$$

of a certain holomorphic bundle over the moduli space [3], see also [32, 33]. We will define the topological limits in (3.5) and come to this point in the next section, see Section 3.5.

3 BCOV holomorphic anomaly equation

Here we introduce the BCOV holomorphic anomaly equation and its topological limits at the degeneration points in the moduli space.

3.1 The special Kähler geometry

The mirror family $\{Y_x\}_{x\in\mathbf{P}^1}$ defines the so-called special Kähler geometry on each neighborhood B_0 of $x_0(\neq 0, \alpha_1, \alpha_2, \alpha_3, \infty)$ on the moduli space $\mathbf{P}^1$. Let us denote $\mathcal{M}^{\rm cpl} = \mathbf{P}^1 \setminus \{0, \alpha_1, \alpha_2, \alpha_3, \infty\}$. To describe the geometry on $\mathcal{M}^{\rm cpl}$, let $\Omega(x) = \Omega(Y_x)$ $(x \in B_0)$ be the holomorphic three form (2.3), normalized by (2.5). Consider the middle cohomology $H^3_{x_0} = H^3(Y_{x_0}, \mathbf{Z})$, and define the period domain,

$$D = \{\omega \in \mathbf{P}(H^3_{x_0} \otimes \mathbf{C})|\ (\omega, \omega) = 0, \quad (\omega, \bar{\omega}) > 0\},$$

where $(\omega, \omega') := i\int_{Y_{x_0}} \omega \wedge \omega'$. Making an identification $H^3(Y_x, \mathbf{Z}) \cong H^3(Y_{x_0}, \mathbf{Z})$ for $x \in B_0$, the choice of the holomorphic 3-form $\Omega(x)$ determines the period map $\mathcal{P}_0 : B_0 \to D$. Let $\mathcal{U}$ be the restriction of the tautological line bundle of $\mathbf{P}(H^3_{x_0} \otimes \mathbf{C})$ to D. Then we have a holomorphic line bundle $\mathcal{L} = \mathcal{P}_0^* \mathcal{U}$ over B_0. Globalizing this local construction, we obtain a holomorphic line bundle $\mathcal{L}$ over a covering space $\tilde{\mathcal{M}}^{\rm cpl}$ with its covering group ("modular group") $\Gamma \subset \mathrm{Sp}(4, \mathbf{Z})$.

The special Kähler geometry on B_0 is defined by the Weil–Petersson metric $G_{x\bar{x}} = \partial_x \partial_{\bar{x}} K(x, \bar{x})$ with the Kähler potential $K(x, \bar{x}) = -\log(\Omega(x), \overline{\Omega(x)})$. Since $K(x, \bar{x})$ is monodromy invariant, we see that this local geometry naturally glues together on $\mathcal{M}^{\rm cpl}$. Consider the metric connection given by $\Gamma^{\ x}_{xx} = G^{x\bar{x}}\partial_x G_{x\bar{x}}$ and $\Gamma^{\ \bar{x}}_{\bar{x}\bar{x}} = G^{\bar{x}x}\partial_{\bar{x}} G_{\bar{x}x}$. This connection defines the covariant derivative on the sections of the tangent bundle $T\mathcal{M}^{\rm cpl} \otimes \mathbf{C} = T'\mathcal{M}^{\rm cpl} \oplus T''\mathcal{M}^{\rm cpl}$. Then we may write the so-called special Kähler geometry relation,

$$\partial_{\bar{x}} \Gamma^{\ x}_{xx} = 2G_{x\bar{x}} - C_{xxx} C_{\bar{x}\bar{x}\bar{x}} e^{2K} G^{x\bar{x}} G^{x\bar{x}}, \tag{3.1}$$

where K is the Kähler potential and C_{xxx} is the Yukawa coupling (2.5). It is known that this relation follows from a certain local system over $\mathcal{M}^{\rm cpl}$ associated to $H^3(Y_x, \mathbf{Z})$[34].

Now let us introduce "Kähler connection" by $K_x = \partial_x K$ and $K_{\bar{x}} = \partial_{\bar{x}} K$. We see that this connection defines the covariant derivative on the sections of $\mathcal{L}$ and also its complex conjugate $\bar{\mathcal{L}}$, and the tensor products thereof. We have $D_x \xi = \partial_x \xi + nK_x \xi + mK_{\bar{x}} \xi$ for a section $\xi \in \Gamma(\mathcal{L}^n \otimes \bar{\mathcal{L}}^m)$. Thus for a holomorphic tangent vector ξ^x taking a value in $\bar{\mathcal{L}}$, for example, we have $D_x\xi^x = (\partial_x + \Gamma_{xx}^{\,x})\xi^x$ and $D_{\bar{x}}\xi^x = (\partial_{\bar{x}} + K_{\bar{x}})\xi^x$.

3.2 BCOV anomaly equation and the general solutions $\mathcal{F}^{(g)}$

Using the special Kähler geometry and also the Griffiths transversality for the period map, we can show that there exist potential functions which express the Yukawa coupling (2.5) and its complex conjugate by

$$C_{xxx} = D_x D_x D_x \mathcal{F}^{(0)}(x,\bar{x}), \;\; C_{\bar{x}\bar{x}\bar{x}} = D_{\bar{x}} D_{\bar{x}} D_{\bar{x}} \bar{\mathcal{F}}^{(0)}(x,\bar{x}),$$

where $\mathcal{F}^{(0)}(x,\bar{x})$ and $\bar{\mathcal{F}}^{(0)}(x,\bar{x})$ are, respectively, a C^∞ section of $\mathcal{L}^2$ and a C^∞ section of $\bar{\mathcal{L}}^2$ [34]. The extension of $\mathcal{F}^{(0)}(x,\bar{x})$ to genus one was introduced in [2] by the t-t^* equation,

$$\partial_x\partial_{\bar{x}}\mathcal{F}^{(1)}(x,\bar{x}) = \frac{1}{2}C_{xxx}C_{\bar{x}\bar{x}\bar{x}}\mathrm{e}^{2K}G^{x\bar{x}}G^{x\bar{x}} - (\frac{\chi}{24} - 1)G_{x\bar{x}}.$$

Geometrically $\mathcal{F}^{(1)}(x,\bar{x})$ is understood to represent a certain Hermitian norm ("Quillen's norm" or analytic torsion) of a holomorphic line bundle [2] (see also [32, 33]) over the complex structure moduli space. The higher genus generalization $\mathcal{F}^{(g)}(x,\bar{x})$ $(g \geq 2)$ are defined by a kind of recursion relation, the BCOV holomorphic anomaly equation,

$$\partial_{\bar{x}}\mathcal{F}^{(g)} = \frac{1}{2}C_{\bar{x}\bar{x}\bar{x}}\mathrm{e}^{2K}G^{x\bar{x}}G^{x\bar{x}}\Big\{D_xD_x\mathcal{F}^{(g-1)} + \sum_{r=1}^{g-1} D_x\mathcal{F}^{(g-r)}D_x\mathcal{F}^{(r)}\Big\}, \quad (3.2)$$

for C^∞ sections $\mathcal{F}^{(g)}(x,\bar{x})$ of $\mathcal{L}^{2-2g}$.

Recent progresses made in [35, 36] clarify the meaning of the anomaly equation (3.2) using the wave function interpretation of the topological string amplitude [37]. In particular, in [35], modular property of $\mathcal{F}^{(g)}$ has been discussed in relation to the quasi-modular forms in elliptic curves [38].

The general solutions of the BCOV anomaly equation have been obtained by certain Feynman rules in [3]. To present the result, let us introduce the notation $\mathcal{F}_r^{(g)} = \underbrace{D_x \cdots D_x}_{r} \mathcal{F}^{(g)}$ and define $\mathcal{F}_{r;s}^{(g)}$ recursively by

$$\mathcal{F}_{r;s+1}^{(g)} = (2g-2+r+s)\mathcal{F}_{r;s}^{(g)} \quad (\mathcal{F}_{r;0}^{(g)} = \mathcal{F}_r^{(g)}),$$

with the conditions,

$$\mathcal{F}_{r;1}^{(0)} = 0 \ \ (r \le 2); \ \ \mathcal{F}_{0;1}^{(1)} = \frac{\chi}{24} - 1, \ \ \mathcal{F}_{0;0}^{(1)} = 0.$$

Define perturbative interaction function $P(J,\phi)$ and the source function $G(J,\phi)$ by

$$P(J,\phi) = \sum_{g\ge 0} \sum_{r,s\ge 0} \lambda^{2g-2} \mathcal{F}_{r;s}^{(g)} \frac{J^r}{r!}\frac{\phi^s}{s!}, \ \ G(J,\phi) = \mathrm{e}^{-\lambda^2(1/2\, S^{xx} J^2 - S^x J\phi - 1/2\, S\phi^2)},$$

where λ is a parameter (string coupling constant) and S^{xx}, S^x, S represent the propagators determined by integrating $\mathrm{e}^{2K} D_{\bar{x}} D^x D^x \bar{\mathcal{F}}^{(0)} = \partial_{\bar{x}} S^{xx}$ and similar relations for S^x and S, see Appendix (A.2). One may solve these propagators in the following form,

$$\begin{aligned} S^{xx} &= \frac{1}{C_{xxx}}(2\,K_x - \Gamma_{xx}^{\ x} + \frac{1}{v^x}\partial_x v^x), \quad S^x = \frac{1}{2} D_x S^{xx} + \frac{1}{2}(S^{xx})^2\, C_{xxx} + H_1^x, \\ S &= H_1^x K_x + \frac{1}{2} D_x S^x + \frac{1}{2} S^{xx} S^x C_{xxx} + H_2, \end{aligned} \tag{3.3}$$

where $v^x(x), H_1^x(x)$ represent some (rational) vector fields and $H_2(x)$ is a rational function on the moduli space. These propagators are C^∞ sections of $\mathcal{L}^{-2}$ with suitable tensor indices. We introduce the holomorphic (meromorphic) functions $f_g(x)$ on the moduli space to represent the "constants" of the integration of the anomaly equation (3.2). Then the solutions of the anomaly equation can be formulated in the following perturbative expansion;

$$\mathrm{e}^{-\sum_g \lambda^{2g-2} f_g} = e^{P(\partial/\partial J, \partial/\partial\phi)} G(J,\phi)\Big|_{J=\phi=0}.$$

The logarithm of the right hand side represents summing over connected Feynman diagrams with the interaction terms determined by $P(J,\phi)$, and we see the perturbative expansion of $\mathcal{F}^{(g)}$ at the coefficient of λ^{2g-2} (see (6.16) in [3]).

For example, we may write the resulting expression at the coefficient $\lambda^{2g-2} = \lambda^2$,

$$\mathcal{F}^{(2)} = \frac{5}{24}(S^{xx})^3(\mathcal{F}_3^{(0)})^2 - \frac{1}{8}(S^{xx})^2\mathcal{F}_4^{(0)} - \frac{1}{2}(S^{xx})^2\mathcal{F}_3^{(0)}\mathcal{F}_1^{(1)} + \frac{1}{2}S^{xx}(\mathcal{F}_1^{(1)})^2 + \frac{1}{2}S^{xx}\mathcal{F}_2^{(1)} + \frac{\chi}{24}S^x\mathcal{F}_1^{(1)} - \frac{\chi}{48}S^xS^{xx}\mathcal{F}_3^{(0)} + \frac{\chi}{24}(\frac{\chi}{24} - 1)S + f_2\,,$$

where by definition $\mathcal{F}_3^{(0)} = C_{xxx}$, and $f_2 = f_2(x)$ is the holomorphic ambiguity. In general, $\mathcal{F}^{(g)}$ is an element of $\Gamma_\infty(\mathcal{L}^{2-2g})$ and may be expressed by

$$\mathcal{F}^{(g)}(x,\bar{x}) = \Gamma(S^{xx}, S^x, S; \mathcal{F}_r^{(h<g)}(x,\bar{x})) + f_g(x), \tag{3.4}$$

where Γ represents symbolically the summation over the Feynman diagrams.

3.3 $\mathtt{F}_g(t)$ from the topological limit

Following [2], we define the "topological limit" of (3.4). First, the data of the topological limit consists of the normalized solutions $w_0(x)$ and $w_1(x)$ at the degeneration point, which determines the mirror map $t = t(x)$, and also the initial data for $g = 0, 1$,

$$F_3^{(0)}(x) = C_{xxx},\;\; F_1^{(1)}(x) = \partial_x F^{(1)}(x),$$

where C_{xxx} is the Yukawa coupling (2.5) and $F^{(1)}(x)$ is the BCOV formula (2.10). Then the topological "limit" is defined by the following replacements,

$$G_{x\bar{x}} \to \frac{dt}{dx}\frac{d\bar{t}}{d\bar{x}},\;\; K_x \to -\partial_x \log w_0(x),\;\; \mathcal{F}^{(g)}(x,\bar{x}) \to F^{(g)}(x), \tag{3.5}$$

in the solution (3.4), which gives

$$F^{(g)}(x) = \Gamma(S^{xx}(x), S^x(x), S;\;\; F_r^{(h<g)}(x)) + f_g(x). \tag{3.6}$$

This is a recursion relation that determines the holomorphic prepotentials $F^{(g)}(x)$ as the holomorphic sections of $\mathcal{L}^{2-2g}$ starting with the initial data $F_3^{(0)}(x)$ and $F_1^{(1)}(x)$ above. Leaving aside the holomorphic ambiguity $f_g(x)$, the holomorphic prepotential gives the Gromov–Witten potential by

$$\begin{aligned} \mathtt{F}_g(t) &= (w_0(x))^{2g-2}\, F^{(g)}(x) \\ &= (w_0(x))^{2g-2}\, \Gamma(S^{xx}, S^x, S; F_r^{(h<g)}(x)) + (w_0(x))^{2g-2}\, f_g(x). \end{aligned} \quad (3.7)$$

The meaning of the topological limit has been discussed recently [35, 36] in terms of the wave function interpretation of $\exp(\sum_{g\geq 0} \lambda^{2g-2}\mathcal{F}^{(g)})$ in [37], however the connection of the holomorphic potential $F^{(g)}(x)$ to the Gromov–Witten potential $\mathtt{F}_g(t)$ above is still open mathematically (cf. the so-called "mirror theorem" by [39, 40] for $g = 0$). To determine the ambiguity $f_g(x)$, we have to invoke some regularity arguments for $\mathtt{F}_g(t)$. This restricts the possible form of $f_g(x)$. Although the regularity arguments put rather strong constraints on the possible forms of the ambiguities, we need some "boundary" conditions to fix them completely. We will describe in Section 3.6 the gap conditions at the conifolds which has been recently introduced in [17].

3.4 Solving BCOV equation recursively

The general form (3.4) or its topological limit (3.6) is not so useful for higher genus calculations, since it contains the contributions from the large number of connected Feynman diagrams, even for $g = 4$ or $g = 5$. On this respect, Yamaguchi and Yau [21] found a nice way to improve the situation. Their idea is to formulate a recursion relation for the sections $\{\mathcal{F}^{(g)}(x, \bar{x})\}$ in the form of a differential equation. This avoids the large summation over the Feynman diagrams.

(3.4.a) Following Yamaguchi and Yau [21], let us introduce the following expressions,

$$A_k = G^{x\bar{x}}\, \theta_x^k\, G_{x\bar{x}}, \;\; B_k = \mathrm{e}^{K(x,\bar{x})}\, \theta_x^k\, \mathrm{e}^{-K(x,\bar{x})} (k = 1, 2, \ldots), \quad (3.8)$$

where $\theta_x = x\frac{d}{dx}$. By definition, these satisfy

$$\theta_x A_k = A_{k+1} - A_1 A_k, \;\; \theta_x B_k = B_{k+1} - B_1 B_k.$$

Also, since $\mathrm{e}^{-K(x,\bar{x})} = (\Omega(x), \overline{\Omega(x)})$ satisfies the (holomorphic) Picard–Fuchs equation (2.4) of the fourth order, there is a linear relation

$$B_4 + r_1(x)\, B_3 + r_2(x)\, B_2 + r_3(x)\, B_1 + r_4(x) = 0,$$

with the rational functions $r_k(x)$ which follow from (2.4). Similarly for $A_2(x)$, but from a non-trivial reasoning, we have [21]

$$A_2 = -4\,B_2 - 2\,B_1(A_1 - B_1 - 1) + \theta_x \log(xC_{xxx})\,(A_1 + 2\,B_1 + 4) + r(x), \tag{3.9}$$

with a rational function $r(x)$, see Appendix (A.3). These relations (3.8) and (3.9) entail an important property,

$$\theta_x : \mathbf{C}(x)[A_1, B_1, B_2, B_3] \to \mathbf{C}(x)[A_1, B_1, B_2, B_3], \tag{3.10}$$

i.e., θ_x acts on the polynomial ring of A_1, B_1, B_2, B_3 with the coefficients over the rational functions $\mathbf{C}(x)$.

(3.4.b) As for the $\partial_{\bar{x}}$ operation, it is easy to see

$$\partial_{\bar{x}} : \mathbf{C}(x)[A_1, B_1, B_2, B_3] \to \mathbf{C}(x)[A_1, B_1, B_2, B_3][\partial_{\bar{x}} A_1, \partial_{\bar{x}} B_1].$$

To show this property, let us note the relations $B_2 = \theta_x B_1 + B_1^2$ and $\partial_{\bar{x}} B_1 = -x\,G_{x\bar{x}}$. Then for $\partial_{\bar{x}} B_2$, we have

$$\partial_{\bar{x}} B_2 = -\theta_x(xG_{x\bar{x}}) + 2\,B_1 \partial_{\bar{x}} B_1 = (1 + A_1 + 2\,B_1)\,\partial_{\bar{x}} B_1,$$

where we use $\theta_x G_{x\bar{x}} = A_1\,G_{x\bar{x}} = -\frac{1}{x} A_1 \partial_{\bar{x}} B_1$. Applying θ_x to this result and using $B_3 = \theta_x B_2 + B_1 B_2$, we have

$$\partial_{\bar{x}} B_3 = (A_2 + 2\,A_1 + 3\,B_1 + 3\,B_2 + 3\,A_1 B_1 + 1)\,\partial_{\bar{x}} B_1.$$

This shows the claim above.

(3.4.c) Now let us focus on the recursion relation (3.6) for $\mathcal{F}^{(g)}(x, \bar{x})$ with the results obtained in (3.4.a) and (3.4.b) above. First, we note that the initial conditions are given in the ring $\mathbf{C}(x)[A_1, B_1, B_2, B_3]$ since $\mathcal{F}_3^{(0)}(x) = C_{xxx}$ and we have, from (2.11),

$$\mathcal{F}_1^{(1)}(x, \bar{x}) = \frac{1}{2\,x}\Big\{ -A_1 - (3 + h^{11} - \frac{\chi}{12})B_1 - 1 - \frac{c_2.H}{12} + \frac{x(57 + 578\,x - 3\,x^2)}{6\,\mathrm{dis}(x)}\Big\}.$$

Also for the propagators we see that S^{xx}, S^x, S belong to the ring $\mathbf{C}(x)[A_1, B_1, B_2, B_3]$, see (3.3). For example, we have

$$S^{xx} = -\frac{1}{x\,C_{xxx}}(A_1 + 2\,B_1 + 4), \quad S^x = \frac{1}{x^2\,C_{xxx}}(3\,B_1 + B_2 + 2). \tag{3.11}$$

The recursion relation (3.4) contains the covariant derivatives D_x to define $\mathcal{F}_r^{(h<g)} = D_x \cdots D_x \mathcal{F}^{(h<g)}(x,\bar{x})$. Note that these covariant derivations act inside the ring due to the property (3.10). Therefore, by induction, we may conclude that the prepotentials $\mathcal{F}^{(g)}(x,\bar{x})$ are in the ring $\mathbf{C}(x)[A_1, B_1, B_2, B_3]$ for all $g \geq 2$. This is the polynomiality found in [21].

Now we proceed to combine the polynomiality with the integration of the BCOV anomaly equation (3.2). Following [21], let us introduce $P_n^{(g)} \in \mathbf{C}(x)[A_1, B_1, B_2, B_3]$ ($P_0^{(g)} = P^{(g)}$) by

$$P_n^{(g)} = (x^3\, C_{xxx})^{g-1}\, x^n D_x^n \mathcal{F}^{(g)} \ \ (n = 0, 1, 2, \ldots). \tag{3.12}$$

Then it is straightforward to rewrite the BCOV equation as

$$\partial_{\bar{x}} P^{(g)} = \frac{1}{2} \partial_{\bar{x}}(x\, C_{xxx} S^{xx}) \Big\{ P_2^{(g-1)} + \sum_{r=1}^{g-1} P_1^{(g-r)}\, P_1^{(r)} \Big\}.$$

Both sides of this equation are linear in $\partial_{\bar{x}} A_1, \partial_{\bar{x}} B_1$, and if we *assume* these two are linearly independent, then we have

$$\begin{aligned} &2\, \frac{\partial P^{(g)}}{\partial A_1} - \Big(\frac{\partial P^{(g)}}{\partial B_1} + \frac{\partial_{\bar{x}} B_2}{\partial_{\bar{x}} B_1} \frac{\partial P^{(g)}}{\partial B_2} + \frac{\partial_{\bar{x}} B_3}{\partial_{\bar{x}} B_1} \frac{\partial P^{(g)}}{\partial B_3} \Big) = 0, \\ &\frac{\partial P^{(g)}}{\partial A_1} = -\frac{1}{2} \Big\{ P_2^{(g-1)} + \sum_{r=1}^{g-1} P_1^{(g-r)}\, P_1^{(r)} \Big\}. \end{aligned} \tag{3.13}$$

The first equation implies that $P^{(g)}$ is a polynomial of essentially three variables. This suppresses the length of the polynomial $P^{(g)}$ when g becomes large. A nice choice of variables that respects the first equation is given in [21] by

$$\begin{aligned} &B_1 = u, \ \ A_1 = v_1 - 2\,u - 1, \ \ B_2 = v_2 + u\, v_1, \\ &B_3 = v_3 + u\Big(2\, v_1 + \theta_x \log(x C_{xxx})\, v_1 - v_2 + 3\theta_x \log(x C_{xxx}) + r(x) - 1 \Big). \end{aligned} \tag{3.14}$$

Note that the inverse relation to this may be found easily because the above relation is of "upper triangular form". Using the new variables for the first equation of (3.13), we have $\frac{\partial}{\partial u} P^{(g)} = 0$ and conclude,

$$P^{(g)} \in \mathbf{C}(x)[v_1, v_2, v_3] \ \subset \ \mathbf{C}(x)[u, v_1, v_2, v_3] = \mathbf{C}(x)[A_1, B_1, B_2, B_3].$$

Furthermore, note that the both sides of the second equation in (3.13) are polynomial in u of degree less than three. Then, writing $\frac{1}{2}\{P_2^{(g-1)} + \sum_{r=1}^{g-1}$

$P_1^{(g-r)}\,P_1^{(r)}\} =: Q_0 + u\,Q_1 + u^2\,Q_2$, we have

$$\begin{cases} \dfrac{\partial P^{(g)}}{\partial v_1} = -Q_0, \qquad \dfrac{\partial P^{(g)}}{\partial v_3} = Q_2 \\ \dfrac{\partial P^{(g)}}{\partial v_2} = Q_1 + (2 + \theta_x \log(x C_{xxx}))\,Q_2 \end{cases} \tag{3.15}$$

This is the equation we can solve recursively with the initial data $P_3^{(0)} = 1$ and $P_1^{(1)}$.

(3.4.d) The holomorphic ambiguity f_g in (3.4) corresponds to the 'constants' of integration of the differential equation (3.15). To make the correspondence more precise, we note that f_g in (3.4) may be identified by the vanishing limit of the propagators, i.e., $\mathcal{F}^{(g)} \to f_g$ when $S^{xx}, S^x, S \to 0$. Now assume that $P^{(g)} \in \mathbf{C}(x)[v_1, v_2, v_3]$ is a solution of the differential equation (3.15). We substitute in $(x^3 C_{xxx})^{1-g}\,P^{(g)}(v_1, v_2, v_3)$ the expressions for v_1, v_2, v_3 in terms of the propagators, which follow from (3.11) and (3.14). Then the vanishing limit of the propagators gives the holomorphic ambiguity f_g. In other words, we may write

$$\mathcal{F}^{(g)} = (x^3 C_{xxx})^{1-g} P^{(g)} + f_g(x), \tag{3.16}$$

where we fix the integration "constant" in $P^{(g)}$ by the property $P^{(g)}$ $(v_1, v_2, v_3) \to 0$ when $S^{xx}, S^x, S \to 0$.

3.5 Relating the topological limits

Let us note that the topological limit (3.5) with the data $w_0(x), w_1(x), t = t(x)$ corresponds to the replacements

$$A_1 \to \Big(\frac{dx}{dt}\Big)\theta_x\Big(\frac{dt}{dx}\Big), \quad B_k \to \frac{1}{w_0(x)}\theta_x^k w_0(x) \ \ (k = 1, 2, 3),$$

in the polynomial solutions $\mathcal{F}^{(g)} = \mathcal{F}^{(g)}(A_1(x, \bar{x}), B_k(x, \bar{x}), x)$. We denote the resulting holomorphic potential $F^{(g)}(x)$.

Now we define $\tilde{\mathcal{F}}^{(g)}(z, \bar{z})$ to be the solutions of the BCOV equation in z-coordinate with the initial conditions $\tilde{\mathcal{F}}_1^{(1)}(z, \bar{z})$ and $\tilde{\mathcal{F}}_3^{(0)} = D_z D_z D_z \tilde{\mathcal{F}}^{(0)}$.

Since the initial data, in particular for $g=0$, are related by

$$\tilde{\mathcal{F}}_3^{(0)}(z,\bar z)=C_{zzz}(z)=C_{xxx}(\tfrac{1}{z})\Big(\frac{dx}{dz}\Big)^3=\mathcal{F}_3^{(0)}(\tfrac{1}{z},\tfrac{1}{\bar z})\Big(\frac{dx}{dz}\Big)^3,$$

we see that $\tilde{\mathcal{F}}^{(g)}(z,\bar z)$ and $\mathcal{F}^{(g)}(x,\bar x)$ are in the same coordinate patch of a trivialization of the line bundle $\mathcal{L}$. Hence we have

$$\tilde{\mathcal{F}}^{(g)}(z,\bar z)=\mathcal{F}^{(g)}(\tfrac{1}{z},\tfrac{1}{\bar z}), \tag{3.17}$$

for the C^∞ sections of $\mathcal{L}^{2-2g}$. Then, by the data $\tilde w_0(z),\tilde w_1(z),\tilde t=\tilde t(z)$ given in (2.8), the topological limit of $\tilde{\mathcal{F}}^{(g)}(z,\bar z)=\mathcal{F}^{(g)}(A_1(\frac{1}{z},\frac{1}{\bar z}),B_k(\frac{1}{z},\frac{1}{\bar z}),\frac{1}{z})$ may be achieved by

$$\begin{aligned}
A_1(\tfrac{1}{z},\tfrac{1}{\bar z})&=\Big(\frac{dx}{dz}\frac{d\bar x}{d\bar z}G^{z\bar z}\Big)(-\theta_z)\Big(\frac{dz}{dx}\frac{d\bar z}{d\bar x}G_{z\bar z}\Big)\to-\Big(\frac{dz}{d\tilde t}\Big)\theta_z\Big(\frac{d\tilde t}{dz}\Big)-2\,,\\
B_k(\tfrac{1}{z},\tfrac{1}{\bar z})&=\mathrm{e}^{\tilde K(z,\bar z)}(-\theta_z)^k\mathrm{e}^{-\tilde K(z,\bar z)}\to\frac{1}{\tilde w_0(z)}(-\theta_z)^k\tilde w_0(z),\quad(k=1,2,3),
\end{aligned}$$

where the relations $G_{x\bar x}(\frac{1}{z},\frac{1}{\bar z})=\frac{dz}{dx}\frac{d\bar z}{d\bar x}G_{z\bar z}(z,\bar z)$, $K(\frac{1}{z},\frac{1}{\bar z})=\tilde K(z,\bar z)$ have been used. We denote the resulting holomorphic potential $\tilde F^{(g)}(z)$.

According to [3], we finally obtain the Gromov–Witten potentials for X and X' by

$$\mathtt{F}_g(t)=(w_0(x))^{2g-2}\,F^{(g)}(x),\ \ \tilde{\mathtt{F}}_g(\tilde t)=(\tilde w_0(z))^{2g-2}\,\tilde F^{(g)}(z), \tag{3.18}$$

with the mirror maps $t=t(x)$ and $\tilde t=\tilde t(z)$, respectively.

We remark that if we require $\mathtt{F}_g(t)$ and $\tilde{\mathtt{F}}_g(\tilde t)$ are regular at $x=0$ and $z=0$, respectively, then the relation (3.17) restricts possible behaviors of the holomorphic (rational) function $f_g(x)$, near $x=0$ and ∞. Taking these reguarlity constraints into accounts, following [3], we may set the following anzatz for f_g,

$$\begin{aligned}
f_g(x)&=a_0+a_1x+\cdots+a_{2g-2}x^{2g-2}\\
&\qquad+\frac{b_0+b_1x+\cdots+b_{2g-3}x^{2g-3}}{(x-3)^{2g-2}}+\frac{c_0+c_1x+\cdots+c_{6g-7}x^{6g-7}}{\mathrm{dis}(x)^{2g-2}},
\end{aligned} \tag{3.19}$$

where $\mathrm{dis}(x) = 1 - 57\,x - 289\,x^2 + x^3$. Alghough $x = 3$ does not corresponds to any degeneration of the mirror family, we introduce $b_0, \ldots, b_{2g-3}$ in this general form (see Section 5 for more detailed analysis on this). In this form, we see $10(g-1)+1$ unknown parameters which grow linearly in g.

3.6 The gap conditions at conifolds

One of the most subtle parts in solving the BCOV anomaly equation is to fix the holomorphic ambiguities $f_g(x)$ whose general form has been argued in (3.19). To determine the unknown constants contained in $f_g(x)$, we may use the first few terms of $N_g(d)$ in the expansion (1.1) if they are known from other methods, e.g., enumerative geometry. In many cases, one may expect $n_g(d) = 0$ for lower d assuming that $n_g(d)$ counts the number of genus g curves in X of degree d and also some genus formula for curves, see e.g., [43]. However these conditions are not sufficient to determine $f_g(x)$ in general, and this fact reduces the predictive power of the BCOV equation for determining the Gromov–Witten potentials $\mathtt{F}_g(t)$. Recently, on this problem, Huang, Klemm and Quackenbush [17] have found that a certain vanishing property (the gap condition) at conifolds provides considerably strong conditions for $f_g(x)$. The gap condition has been tested for quintic hypersurface in $\mathbf{P}^4$ and other cases that have the mirror family over $\mathbf{P}^1$ with only one conifold singularity.

The gap condition in [17] arises from the topological limit made around a conifold singularity. Let $x = c$ be a conifold singularity of the mirror family, or the corresponding singularity of the Picard–Fuchs differential equation. In our case, c may be one of the three singularities $\alpha_1, \alpha_2, \alpha_3$ in (2.6). As we observe in (2.6), the indices ρ_k at the conifold are all integral but have one degeneracy, which indicates there exists one solution with logarithmic singularity.

Assume $(\rho_1, \rho_2, \rho_3, \rho_4) = (0, 1, 1, 2)$, and normalize the logarithmic solution $\log(s) w_1^c(s) + O(s^1)$ by requiring $w_1^c(s) = s + O(s^2)$ $(s = (x-c))$. Then, according to the Picard–Lefschetz theory, the series $w_1^c(s)$ represents the (normalized) period integral of the vanishing cycle. $w_1^c(s)$ together with the logarithmic solution corresponds to the indices $\rho_2 = \rho_3 = 1$. For the index $\rho_4 = 2$ we have the solution of the form $w_2^c(s) = s^2 + O(s^3)$. Then, making a suitable linear combination with $w_1^c(s)$ and $w_2^c(s)$, we may fix the solution for the index $\rho_1 = 0$ by the property

$$w_0^c(s) = 1 + O(s^3).$$

By the data of the topological limit at the conifold $x = c$, we mean the series data $w_0^c(s), w_1^c(s)$ with the "mirror map" $s = s(U)$ defined by

$$k_U U = \frac{w_1^c(s)}{w_0^c(s)},$$

where k_U is a constant characterized below.

The gap condition arises from the topological limit $\mathcal{F}_c^{(g)}(s,\bar{s}) \to F_c^{(g)}(s)$ at each conifold. We define this topological limit, in the exactly same way as described in Sections (3.3) to (3.5), by the replacements

$$A_1(s+c,\bar{s}+\bar{c}) \to (s+c)\frac{d}{ds}\log\frac{dU}{ds}, \quad B_k \to \frac{1}{w_0^c(s)}\Big((s+c)\frac{d}{ds}\Big)^k w_0^c(s).$$

in the relation $\mathcal{F}_c^{(g)}(s,\bar{s}) = \mathcal{F}^{(g)}(A_1(x,\bar{x}), B_k(x,\bar{x}), x)$.

The observation made in [17] based on the physical interpretation of the vanishing cycles [41] is the following: *There exists a choice of the constant k_U, under which we have*

$$\mathtt{F}_c^{(g)}(U) = (w_0^c(s))^{2g-2}\, F_c^{(g)}(s) = \frac{|B_{2g}|}{2\,g\,(2\,g-2)}\frac{1}{U^{2g-2}} + O(U^0), \qquad (3.20)$$

for $g \geq 2$ (and $\mathtt{F}_c^{(1)}(s) = -\frac{1}{12}\log U + O(U^0)$). Since the leading behavior $F_c^{(g)}(s) \sim \frac{\text{const.}}{U^{2g-2}} + \cdots$ can be verified in general, the above equation provides $(2g-2)-1$ vanishing conditions for the coefficients of $\frac{1}{U^k}$ $(1 \leq k \leq 2g-3)$, the *gap condition.* Note that once we find k_U at some g, then the leading term in (3.20) provides an additional condition for each other value of g. It has been observed for the quintic and similar Calabi–Yau 3-folds [17] that these vanishing conditions provides an efficient way to determine the holomorphic ambiguity $f_g(x)$ for higher values of g.

4 Calculations

Here we present some details of our calculations of the Gromov–Witten potentials $\mathtt{F}_g^X(t) = \mathtt{F}_g(t)$ and $\mathtt{F}_g^{X'}(\tilde{t}) = \tilde{\mathtt{F}}_g(\tilde{t})$, and list the resultant Gopakumar–Vafa invariants, $n_g^X(d)$ and $n_g^{X'}(d)$ for $g \leq 5$ in tables 1 and 2.

4.1 Expansions about the conifolds

The evaluations of the Gromov–Witten potentials

$$\mathtt{F}_g(t) = (w_0(x))^{2g-2} F^{(g)}(x), \;\; \tilde{\mathtt{F}}_g(\tilde{t}) = (\tilde{w}_0(z))^{2g-2} \tilde{F}^{(g)}(z),$$

are straightforward with the topological data $w_0(x), w_1(x), t = t(x)$ and $\tilde{w}_0(z), \tilde{w}_1(z), \tilde{t} = \tilde{t}(z)$ as described precisely in the previous sections. For the expansion about the conifolds, however, we need to make the series expansions about $x = \alpha_k$ $(k = 1, 2, 3)$ given by the algebraic equation $1 - 57\,x - 289\,x^2 + x^3 = 0$. To achieve this, we first write the Picard–Fuchs equation

$$\sum_{k=0}^{4} p_k(\alpha, s)\Big(\frac{d}{ds}\Big)^k w^c(s) = 0 \tag{4.1}$$

in the coordinate $s = x - \alpha$ with some polynomials $p_k(\alpha, s)$. Note that α may be taken to be any of α_k since we only need the relation $1 - 57\,\alpha - 289\,\alpha^2 + \alpha^3 = 0$ in the derivation. Now we try to find the solutions of the form

$$w^c(\alpha, s) = \sum_{n \geq 0} c_n(\alpha)\, s^{n+\rho},$$

for each choice of the index $\rho = 0, 1, 1, 2$. Namely we solve the differential equation over the ring $\mathcal{R}_\alpha = \mathbf{C}[\alpha]/(\alpha^3 - 289\,\alpha^2 - 57\,\alpha + 1)$. Solving the Picard–Fuchs equation (4.1) over $\mathcal{R}_\alpha$ is rather technical, but turns out quite useful since we can impose the gap conditions at the three conifold points α_k at one time.

Recall that the gap conditions may be imposed by making the data $w_0^c(\alpha, s)$, $w_1^c(\alpha, s)$ and $s = s(U)$ as defined in the Section 3.6. After some calculations, for the solutions, we obtain

$$w_0^c(\alpha, s) = 1 - \Big(\frac{82833753}{33614} + \frac{1555547739}{134456}\,\alpha - \frac{16148435}{403368}\,\alpha^2\Big) s^3 + \cdots$$
$$w_1^c(\alpha, s) = s - \Big(\frac{64163}{1372} + \frac{83161}{343}\,\alpha - \frac{1151}{1372}\,\alpha^2\Big) s^2 + \cdots,$$

and also, inverting the defining relation $k_U U = \frac{w_1^c(\alpha,s)}{w_0^c(\alpha,s)}$, we have

$$s(U) = k_U U + \Big(\frac{64163}{1372} + \frac{83161}{343}\,\alpha - \frac{1151}{1372}\,\alpha^2\Big)(k_U U)^2 + \cdots$$

Using the above data, we can evaluate the holomorphic potential $F_c^{(g)}(s)$ in the following form,

$$\mathrm{F}_c^{(g)}(U) = \frac{R_{2g-2}(\alpha)}{(k_U U)^{2g-2}} + \frac{R_{2g-1}(\alpha)}{(k_U U)^{2g-3}} + \cdots + \frac{R_1(\alpha)}{(k_U U)} + O(U^0),$$

with $R_k(\alpha) = c_{k,2}\,\alpha^2 + c_{k,1}\,\alpha + c_{k,0}$. Since $1, \alpha, \alpha^2$ are linearly independent, the gap condition (3.20) entails $3(2g-3)$ conditions, or $3(2g-2)$ conditions once k_U is fixed. Thus we can impose the gap conditions at the three conifold points at once in this algebraic manipulation.

4.2 Examples ($g = 2,\ 3$)

We use the gap condition above extensively together with some natural vanishing assumptions to fix the $10\,g-9$ unknown parameters in $f_g(x)$, see (3.19). Here we illustrate how we impose the additional vanishing conditions using the cases $g=2$ and $g=3$. For $g=2$, we have to fix $10\,g-9=11$ unknown parameters among which $3\,(2g-3)=3$ may be determined from the gap conditions. To fix the remaining 8 parameters, we note the following $g=1$ Gopakumar–Vafa invariants which follow from the BCOV formula (2.10);

	1	2	3	4	5	6
$n_1^X(d)$	0	0	0	0	588	$\cdots$
$n_1^{X'}(d)$	0	0	196	99960	34149668	$\cdots$

From the higher genus calculations done in several examples, see [42, 17] for example, we observe that the vanishing $n_{g-1}(d)=0$ indicates $n_g(d)=0$. This observation seems to be a natural consequence of the geometrical meaning of the Gopakumar–Vafa invariants that $n_h(d)$ is evaluating the Euler numbers of the degeneration loci in the genus g curve of degree d [20, 43, 44]. Assuming that this vanishing condition holds in our case, we have

$$n_2^X(d) = 0(d=1,\ldots,4),\ \ n_2^{X'}(d) = 0\ \ (d=1,2),\ \ n_2^X(0) = n_2^{X'}(0) = \frac{\chi}{5760},$$

which provide 8 conditions sufficient to fix $f_2(x)$. Using these conditions we obtain for the holomorphic potential $F^{(2)}(x)$,

$$F^{(2)}(x) = (x^3\,C_{xxx})^{-1}\Big(\frac{2989}{288}\,v_3 + \frac{49}{24}\,v_1\,v_2 - \frac{5}{24}v_1^3 + \frac{p_2(x)}{(x-3)\mathrm{dis}(x)}\,v_2 + \frac{p_{1,1}(x)}{(x-3)\,\mathrm{dis}(x)}\,v_1^2 + \frac{p_1(x)}{(x-3)\,\mathrm{dis}(x)^2}v_1 + \frac{p_3(x)}{(x-3)\,\mathrm{dis}(x)^2}\Big) + f_2(x),$$

with some polynomials $p_1(x), p_2(x), p_3(x), p_{1,1}(x)$, which we leave implicit, and

$$\begin{aligned} f_2(x) = &-\frac{359293}{2520} + \frac{1850909\,x}{20160} - \frac{2081\,x^2}{6720} - \frac{15739}{24\,(x-3)^2} + \frac{38147}{84\,(x-3)} \\ &+ \frac{1}{\mathrm{dis}(x)^2}\Big(\frac{264137}{720} - \frac{1881913}{45}x + \frac{39189063}{40}x^2 + \frac{72541963}{6}x^3 \\ &+ \frac{7353789043}{240}x^4 - \frac{8892629}{90}x^5\Big). \end{aligned}$$

Also the leading term of the conifold expansion $F_c^{(2)}(s) = \frac{1}{240}\frac{1}{U^2} + \cdots$ determines the constant k_U by

$$k_U^2 = 240\left(\frac{1183163}{1120}\,\alpha^2 + \frac{58293}{280}\,\alpha - \frac{4091}{1120}\right).$$

The resultant Gopakumar–Vafa invariants $n_2^X(d)$ and $n_2^{X'}(d)$ are listed in Tables 1 and 2.

For $g = 3$ calculation, since k_U has been fixed as above, we have $3\,(2g-2) = 12$ constraints from the gap condition to fix $10\,g - 9 = 21$ parameters in $f_3(x)$. Fortunately, we have enough additional vanishing conditions from the $g = 2$ results; $n_2^X(d) = 0 \ \ (d = 1, \ldots, 7)$, $n_2^{X'}(d) = 0 \ \ (d = 1, \ldots, 4)$, see Tables 1 and 2, We may adopt the following 9 conditions

$$\begin{aligned} &n_3^X(d) = 0 \ (d = 1, \ldots, 5), \quad n_3^{X'}(d) = 0 \ \ (d = 1, 2), \\ &n_3^X(0) = n_3^{X'}(0) = \frac{-\chi}{1451520}, \end{aligned}$$

to fix $f_3(x)$.

We have continued this process up to $g = 5$. Although we may continue this further to higher g, the exact value of g where this process might break down is not clear to us (see the discussion in the next section).

5 Conclusion and discussions

We have determined the Gromov–Witten potentials $\mathtt{F}_g^X$ and $\mathtt{F}_g^{X'}$, up to $g = 5$, of the Grassmannian and the Pfaffian Calabi–Yau 3-folds using the mirror symmetry. Our calculations are based on the original BCOV holomorphic anomaly equation [2, 3] and the polynomiality in the solutions found in [21].

In particular, following [17], we used extensively the gap conditions at the conifold singularities to determine the holomorphic ambiguities f_g.

Apart from these computational aspects of the Gromov–Witten invariants, we have also remarked that the (mirror) Picard–Fuchs differential equation has a similar property to that appeared in the mirror symmetry of a K3 surface of degree 12. For a K3 surface of degree 12, the number of the Fourier–Mukai partners is two, i.e., $|\mathrm{FM}(X)| = 2$ [7, 29]. One may expect a similar result for the Grassmannian and the Pfaffian Calabi–Yau manifolds, i.e., there is no more variety which is derived equivalent to these up to isomorphisms. Also one may expect that X' appears as a suitable moduli space of stable sheaves on X, which is the case for the K3 surfaces of degree 12.

Finally we comment on the singularity we see at $x = 3$ in (2.6). This point does not corresponds to a singularity of the mirror manifold Y_x in (2.1), see [12] for more details. In fact, we see from the indices at $x = 3$, there is no local monodromy around this point. However, we can formulate additional "gap condition" which may be used to determine the holomorphic ambiguity f_g. Let us fix the local solutions corresponding to $\rho = 0, 1, 3, 4$, respectively, by the following properties;

$$w_0(s) = 1 - \frac{s^2}{42} + O(s^5), \quad w_1(s) = s - \frac{8}{21}\, s^2 + O(s^5),$$
$$w_2(s) = s^3 - \frac{191}{210}\, s^4 + O(s^5), \quad w_3(s) = s^4 + O(s^5),$$

where $s = x - 3$. Then similarly to the conifold points, one may define the topological limit with the data $w_0(s), w_1(s)$ and the mirror map $U = \frac{w_1(s)}{w_0(s)}$. Then corresponding to the gap condition (3.20) at the conifolds, we *observe* that the following vanishing property,

$$\mathtt{F}^{(g)}(U) = (w_0(s))^{2g-2}\, F^{(g)}(s) = 0\, \frac{1}{U^{2g-2}} + \cdots + 0\, \frac{1}{U} + O(U^0),$$

holds for $g \leq 5$. Note that by the form f_g in (3.19) this expansion can start from $\frac{1}{U^{2g-2}}$ in general. However the $\mathtt{F}^{(g)}(U)$ is regular as above since there does not appear any massless state (or vanishing cycle) at $x = 3$. We may utilize this property to determine the unknown constants in f_g. Thus, together with the gap conditions at the conifolds, we have $8(g-1)$ conditions in total, and hence in order to fix f_g completely we need additionally $2\,g - 1$ vanishing conditions, $n_g^X(d) = 0$, $n_g^{X'}(d') = 0$ for lower degrees d and d'. From the results at $g = 5$, one may expect that the calculations done in

Section 4 may be continued to considerably higher value of g, like the case of the quintic [17].

Appendix A

A.1 The Pfaffians of $\mathcal{S}(y)$

The 7×7 skew symmetric matrix $\mathcal{S}(y)$ parametrized by $[1, y, 0]$ in the Section 2.2 has the following form,

$$\mathcal{S} = \begin{pmatrix} 0 & -u_3 & -y\,u_4 & 0 & 0 & y\,u_0 & u_1 \\ u_3 & 0 & -u_5 & -y\,u_6 & 0 & 0 & y\,u_2 \\ y\,u_4 & u_5 & 0 & -u_0 & -y\,u_1 & 0 & 0 \\ 0 & y\,u_6 & u_0 & 0 & -u_2 & -y\,u_3 & 0 \\ 0 & 0 & y\,u_1 & u_2 & 0 & -u_4 & -y\,u_5 \\ -y\,u_0 & 0 & 0 & y\,u_3 & u_4 & 0 & -u_6 \\ -u_1 & -y\,u_2 & 0 & 0 & y\,u_5 & u_6 & 0 \end{pmatrix},$$

where $[u_0, \ldots, u_6] \in \check{\mathbf{P}}^6$. Then the explicit form of the Pfaffians, $p_k(\mathcal{S})$ are

$$\begin{aligned}
p_0(\mathcal{S}) &= y^3\, u_1u_2u_3 - y^2\,(u_3u_5^2 + u_1u_6^2) - y\,u_0u_2u_4 + u_2u_5u_6, \\
p_1(\mathcal{S}) &= y^3\, u_3u_4u_5 - y^2\,(u_5u_0^2 + u_3u_1^2) - y\,u_2u_4u_6 + u_0u_1u_4, \\
p_2(\mathcal{S}) &= y^3\, u_0u_5u_6 - y^2\,(u_5u_3^2 + u_0u_2^2) - y\,u_1u_4u_6 + u_2u_3u_6, \\
p_3(\mathcal{S}) &= y^3\, u_0u_1u_2 - y^2\,(u_2u_4^2 + u_0u_5^2) - y\,u_1u_3u_6 + u_1u_4u_5, \\
p_4(\mathcal{S}) &= y^3\, u_2u_3u_4 - y^2\,(u_2u_0^2 + u_4u_6^2) - y\,u_1u_3u_5 + u_0u_3u_6, \\
p_5(\mathcal{S}) &= y^3\, u_4u_5u_6 - y^2\,(u_6u_1^2 + u_4u_2^2) - y\,u_0u_3u_5 + u_1u_2u_5, \\
p_6(\mathcal{S}) &= y^3\, u_0u_1u_6 - y^2\,(u_1u_3^2 + u_6u_4^2) - y\,u_0u_2u_5 + u_0u_3u_4.
\end{aligned}$$

A.2 Propagators S^{xx}, S^x, S

These propagators are defined in [3] by integrating

$$e^{2K} D_{\bar{x}} D^x D^x \bar{\mathcal{F}}^{(0)} = \partial_{\bar{x}} S^{xx}, \;\; G_{\bar{x}x} S^{xx} = \partial_{\bar{x}} S^x, \;\; G_{\bar{x}x} S^x = \partial_{\bar{x}} S.$$

Using the special Kähler geometry relation (3.1), one may easily verify that (3.3) solves these equations. The explicit forms $v^x(x), H_1^x(x), H_2(x)$ are determined following [3],

$$v^x(x) = \frac{1}{x^4}, \ \ H_1^x(x) = -\frac{1}{2}\frac{1}{x^2\, C_{xxx}}(12 - r(x)), \ \ H_2(x) = -\frac{1}{x}H_1^x(x),$$

where $r(x)$ is the rational function in (3.9), see also (A.2) below. The topological limits of these propagators in the z coordinate have similar forms to those found in [3] for the quintic,

$$\begin{aligned} S^{zz} &= \frac{1}{C_{zzz}}\partial_z \log\Big\{\Big(\frac{f(z)}{\tilde{w}_0(z)}\Big)^2 \frac{dz}{d\tilde{t}}\Big\}, \\ S^z &= \frac{1}{C_{zzz}}\Big\{\big(\partial_z \log \frac{f(z)}{\tilde{w}_0(z)}\big)^2 - \partial_z^2 \log \frac{f(z)}{\tilde{w}_0(z)}\Big\}, \\ S &= \Big\{S^z - \frac{1}{2}D_z S^{zz} - \frac{1}{2}\big(S^{zz}\big)^2 C_{zzz}\Big\}\partial_z \log\frac{f(z)}{\tilde{w}_0(z)} + \frac{1}{2}D_z S^z + \frac{1}{2}S^{zz}S^z C_{zzz}, \end{aligned}$$

where $f(z) = z$. Rather complicated forms of v^x, H_1^x, H_2 above have been found from the latter expressions of S^{zz}, S^z, S.

A.3 The derivation of A_2 in (3.9)

The relation (3.9) follows from the definitions

$$\partial_{\bar{x}} S^{xx} = \mathrm{e}^{2K}(G^{x\bar{x}})^2 C_{\bar{x}\bar{x}\bar{x}}, \ \ \partial_x C_{\bar{x}\bar{x}\bar{x}} = 0,$$

where $C_{\bar{x}\bar{x}\bar{x}} = D_{\bar{x}}D_{\bar{x}}D_{\bar{x}}\bar{\mathcal{F}}^{(0)}(x,\bar{x})$ is the anti-holomorphic Yukawa coupling. From these relations, after some algebra, we have

$$\partial_{\bar{x}}(xC_{xxx}\,\theta_x S^{xx}) = 2x\,\{K_x - \Gamma_{xx}^{\ x}\}\partial_{\bar{x}}(xC_{xxx}S^{xx}).$$

Now from the special geometry relation (3.1), we have $\partial_{\bar{x}}(K_x - \Gamma_{xx}^{\ x}) = -G_{x\bar{x}} + C_{xxx}\partial_{\bar{x}}S^{xx}$. Using this relation for $\partial_{\bar{x}}(xC_{xxx}S^{xx})$ in the right hand side, we obtain

$$\begin{aligned} \partial_{\bar{x}}(xC_{xxx}\,\theta_x S^{xx}) &= 2\,x\,(K_x - \Gamma_{xx}^{\ x})\Big\{\partial_{\bar{x}}\big(x(K_x - \Gamma_{xx}^{\ x})\big) + x\,G_{x\bar{x}}\Big\} \\ &= \partial_{\bar{x}}\Big\{(xK_x - \Gamma_{xx}^{\ x})^2 + (xK_x)^2 - 2(\theta_x - 1)(xK_x)\Big\}. \quad \text{(A.1)} \end{aligned}$$

We may express this relation in terms of A_1, B_1, B_2 and B_3 as follws,

$$\begin{aligned}&\partial_{\bar{x}}(xC_{xxx}\theta_x S^{xx})\\ &\quad = \partial_{\bar{x}}\Big(- A_2 + A_1^2 - 2\,B_2 + 2\,B_1^2 + \theta_x \log(xC_{xxx})\,(A_1 + 2\,B_1 + 4)\Big)\\ &\quad = \partial_{\bar{x}}(A_1^2 + 2\,A_1 B_1 + 2\,B_2 - 2\,B_1),\end{aligned}$$

where, for the first line, we use the expression $S^{xx} = \frac{-1}{xC_{xxx}}(A_1 + 2\,B_1 + 4)$ in (3-4.c) and the relation $\theta_x A_1 = A_2 - A_1^2$. This determines the form $A_2(x)$ up to a holomorphic (rational) function. Substituting the series data (2.7) under the topological limit (3.5), we finally find

$$r(x) = 11 - \frac{36}{7\,(x-3)} - \frac{4\,\left(10 - 331\,x - 751\,x^2\right)}{7\,\mathrm{dis}(x)},$$

in the relation (3.9).

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