

E_7 groups from octonionic magic square

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Abstract

In this paper, we continue our program, started in [2], of building up explicit generalized Euler angle parameterizations for all exceptional compact Lie groups. Here we solve the problem for E_7 , by first providing explicit matrix realizations of the Tits construction of a Magic Square product between the exceptional octonionic algebra \mathfrak{J} and the quaternionic algebra \mathbb{H} , both in the adjoint and the 56-dimensional representations. Then, we provide the Euler parametrization of E_7 starting from its maximal subgroup $U = (E_6 \times U(1))/\mathbb{Z}_3$. Next, we give the constructions for all the other maximal compact subgroups.

1 Introduction

Simple Lie groups are well understood, and find application in a huge number of mathematical and physical fields. In particular, the role of compact exceptional Lie groups in grand unification gauge theories and in string theories, and that of the noncompact forms in supergravity theories is well known. Many properties of these groups can be already inferred from abstract theoretical considerations; however, it can be useful to have available explicit concrete realizations of such groups in term of matrices, for testing conjectures related to confinement in non abelian gauge theories, doing explicit nonperturbative computations in exceptional lattice GUT theories and in random matrix theories. Beyond these, there are other useful applications in physics or mathematical physics of an explicit matrix realization of the E_6 , E_7 and E_8 exceptional Lie groups. For example: sigma models based on exceptional Lie group quotients are of interest for string theory and conformal field theory applications; the study of the properties of the magnetic material Cobalt Niobate is also based on exceptional Lie groups of type E [1]. The connection to explicit realizations and special function theory would permit to perform calculations of matrix elements. These applications are also directly interesting in integrable models. In particular there is a specific motivations from physics to be interested to E_7 among all exceptional groups: recently a strict relation between cryptography and black hole physics based on E_7 exceptional supergravity has been discovered. However, actual computation of entangled expectation values require again explicit determination of the Haar measure and range of parameters. Moreover, the most general structure of the attractor mechanism of black holes in $N = 2$, four-dimensional supergravity arises in E_7 exceptional supergravity. The orbit of the U-duality group are only partially known just because a suitable explicit realization of the group E_7 is lacking.

In this paper, we will focus on the compact form of E_7 . In this case, the main difficulty consists in finding a realization admitting a simple characterization of the range of parameters. A way, mainly inspired by [4, 5], to solve this problem has been introduced in [2] for the exceptional Lie group G_2 (see also [3]), then applied to the $SU(N)$ groups in [6] and to the exceptional Lie groups F_4 and E_6 in [7, 8], respectively. In this paper, we continue our program of building up the generalized Euler parametrization for all the exceptional Lie groups. There are many possible realizations of the Euler parametrization on a given group (see [9] for a review), depending on the choice of the maximal subgroup one starts with. In any case, the simplest one is that based on the higher dimensional compact subgroup. For E_7 , this is the group $U = (E_6 \times U(1))/\mathbb{Z}_3$, where \mathbb{Z}_3 is the center of E_6 , which is indeed the first case we consider here. The other possible maximal compact

subgroups of E_7 are $SU(8)/\mathbb{Z}_2$ and $(\text{Spin}(12) \times SU(2))/\mathbb{Z}_2$ associated to the real forms $E_{7(7)}$ and $E_{7(-5)}$, respectively. We will provide a construction of the compact real form for each choice of the maximal compact subgroup. Indeed, in practical applications it can be necessary to select a specific subgroup among the others. Moreover, in other applications, as for example in extended supergravities, it becomes important to determine the various real forms and the corresponding symmetric spaces. In our constructions, these can be easily realized by applying the unitary Weyl trick.

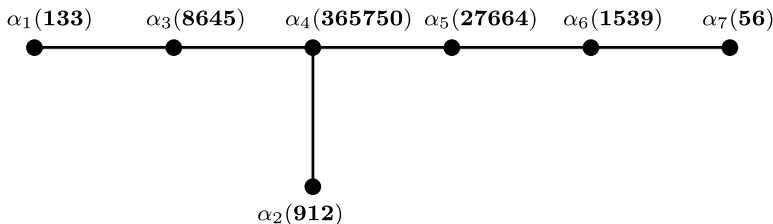
The plan of the paper is as follows. In Section 2, we present the main features of the Lie group E_7 . In Section 3, we review the Tits construction for the Lie algebra of E_7 . In particular, we derive from it the main properties we will need to apply the generalized Euler angles method. We will present both the adjoint representation **133** and the smallest fundamental representation **56**. We build up the group, presenting a very careful exposition of details in the appendices. In Section 4, we give a second construction associated to the split form $E_{7(7)}$, and in Section 5, a third construction based on the subgroup $(\text{Spin}(12) \times SU(2))/\mathbb{Z}_2$. In these two cases, we will not present all necessary checks, as, for example, the explicit computation of the volumes and the related integrals, but we limit ourselves to indicate the main steps. Indeed, the lacking details can be included in a more general framework, which deserves to be presented apart [20].

Remark. In www.dfm.uninsubria.it/E7/, one can find the Mathematica programs providing the constructions of the matrix realizations of the **133** and **56** algebra representations for the first construction, and the **56** for the other ones.

2 The exceptional Lie group E_7

As a complex Lie algebra, E_7 is the unique exceptional Lie algebra of rank 7. It is characterized by the Dynkin diagram drawn in figure 1. Recall that to each dot corresponds a simple root α_i . These are free generators of the root lattice $\Lambda_R = \sum_i \mathbb{Z}\alpha_i$. The space $H^* = \Lambda_R \otimes \mathbb{R}$ is endowed with a positive definite inner product (\cdot) . The weight lattice Λ_W is the dual of Λ_R w.r.t. the hooked product, which means that it is freely generated over \mathbb{Z} by the fundamental weights $\lambda^i \in H^*$, $i = 1, \dots, 7$ defined by $\langle \alpha_i, \lambda^j \rangle = \delta_i^j$, with

$$\langle \alpha, \lambda \rangle := 2 \frac{(\alpha|\lambda)}{(\alpha|\alpha)}. \tag{2.1}$$

Figure 1: Dynkin diagram for E_7 .

To each fundamental weight corresponds univocally a fundamental representation, and all irreducible finite-dimensional representations can be generated from the fundamental ones. In figure 1, the fundamental representations are indicated in parenthesis. Here, we will deal with the two lower dimensional ones.

We are interested in constructing the compact form of the group E_7 . For this reason it is worth to mention some further facts about both the E_7 algebras and groups. As we said, the complex algebra E_7 is completely characterized by the Dynkin diagram, from which one can reconstruct the adjoint representation that, being faithful, is isomorphic to the algebra itself. This is a 133-dimensional algebra that, indeed, correspond to the fundamental representation **133**. This algebra contains four distinct real forms. This means that there are four inequivalent ways to select a 133-dimensional real subspace of the 266-dimensional real space underlying the complex algebra E_7 , in such a way that the selected subspace inherited with the Lie product is itself a (real) Lie algebra. For each simple Lie algebra A there is a unique simply connected Lie group G (up to isomorphisms) such that A is the corresponding Lie algebra. The complex Lie group E_7 contains a maximal compact subgroup which is a 133-dimensional real Lie group whose Lie algebra is then called the compact form and usually indicated¹ $E_{7(-133)}$, where in parenthesis the signature of the Killing form (number of positive eigenvalues minus number of the negative ones) is indicated. We will use the same notation to indicate both the group and the algebra. When referring to the group we will mean the unique simply connected group.

The compact forms are in correspondence with the maximal compact subgroups of $E_{7(-133)}$, the compact subgroups that are not properly contained in a proper subgroup. There are four of such subgroups and then four real forms, which we collect in table 1. For a given real form we can write the

¹The Killing form K on a complex Lie algebra is defined by $K(X, Y) := \text{tr}(ad(X)ad(Y))$ and is non degenerate for a simple algebra and on the corresponding real forms. In particular, for a compact form it is negative definite on the maximal compact subalgebra, the maximal Lie subalgebra that exponentiated generates a compact group.

Table 1: The real forms of E_7

Symbol	Real Form	Maximal compact subgroup
$E_{7(-133)}$	Compact	$E_{7(-133)}$
$E_{7(7)}$	Split	$SU(8)/\mathbb{Z}_2$
$E_{7(-5)}$	EVI	$(Spin(12) \times Sp(1))/\mathbb{Z}_2$
$E_{7(-25)}$	EVII	$(E_{6(-78)} \times U(1))/\mathbb{Z}_3$

corresponding Lie algebra as $A = T + P$, where T is the maximal compact subalgebra and P its complement. From this, one obtains the compact form by using the Weyl unitary trick $P \mapsto iP$. Here we are interested in the compact form only. Nevertheless we will construct it in three different ways, each one evidencing a different proper maximal subalgebra. In this way the interested reader can reobtain the corresponding noncompact real form directly by applying the Weyl unitary trick.

Finally, we stress that for the group $E_{7(-133)}$ we mean the simply connected group, which is the universal covering group. In general, all Lie groups having the same (finite-dimensional real) Lie algebra are obtained from the universal covering by quotienting with a discrete subgroup of the center. For E_7 the center is \mathbb{Z}_2 . Note that the adjoint representation of the Lie algebra is faithful, however, the same is not true for the Adjoint representation of the universal cover group G , since the center C is just the kernel of the Adjoint map. Then, in general, the Adjoint representation of the group realizes the group G/C in place of G . Instead, a faithful representation is obtained by exponentiating the lowest dimensional fundamental representation.² In our case, this means that we need to exponentiate the **56**. Nevertheless, to get more information in some case we will work out the adjoint representation also.

3 $E_{7(-25)}$ construction

3.1 The Tits construction and 133

We start with the construction of the adjoint representation of the algebra. In order to catch the idea, recall first that the exceptional Lie group $F_{4(-52)}$ can be realized as the group of automorphisms of the exceptional Jordan algebra $\mathfrak{J}_3\mathbb{O}$. Its Lie algebra is the Lie algebra of derivations over

²The unique simple Lie group whose lowest dimensional representation coincides with the adjoint one is the exceptional group E_8 . Since in this case the center is trivial, our statement remains true.

$\mathfrak{J} := \mathfrak{J}_3(\mathbb{O})$. Thanks to a result due to Chevalley and Schafer, this algebra can be extended to an E_6 type algebra (more precisely the $E_{6(6)}$ split form) by adding the action of the traceless part \mathfrak{J}' over \mathfrak{J} itself, naturally given by Jordan multiplication:

$$E_{6(6)} = D(\mathfrak{J}) \oplus \mathfrak{J}', \tag{3.1}$$

where the symbol $D()$ means “the linear derivations of”. To obtain the compact form from the split form one has to apply the Weyl trick to \mathfrak{J}' that is to say that we have to “complexify” the Jordan algebra by adding the imaginary part $i \otimes \mathfrak{J}'$.

This way to construct the algebra can be summarized by saying that E_6 is the Magic Square product between \mathfrak{J} and \mathbb{C} . This can be extended to the realization of the E_7 compact form as the Magic Square product between the exceptional octonionic algebra \mathfrak{J} and the quaternionic algebra \mathbb{H} , [10–12]. We will refer to it as the Tits construction. In our language, this means that we need to “quaternionize” the Jordan algebra so that the vector space underlying the resulting algebra will be

$$\mathfrak{g} = D(\mathbb{H}) \oplus D(\mathfrak{J}) \dot{+} (\mathbb{H}' \otimes \mathfrak{J}'), \tag{3.2}$$

where \mathfrak{J}' is the set of traceless octonionic Jordan matrices and \mathbb{H}' are the imaginary quaternions. Here we use the symbol \oplus to mean direct sum of algebras whereas $\dot{+}$ is a direct sum of vector spaces but not of algebras. This is in order to stress that we have not yet extended the Lie product to the last summand. Thus, $D(\mathbb{H}) \oplus D(\mathfrak{J})$ is a subalgebra of \mathfrak{g} . Moreover, there is a natural action of $D(\mathbb{H}) \oplus D(\mathfrak{J})$ over $\mathbb{H}' \otimes \mathfrak{J}'$ which defines the mixed product

$$\begin{aligned} [(H, J), h \otimes j] &= H(h) \otimes j + h \otimes J(j) \quad \forall (H, J) \in D(\mathbb{H}) \oplus D(\mathfrak{J}), \\ h \otimes j &\in \mathbb{H}' \otimes \mathfrak{J}'. \end{aligned} \tag{3.3}$$

To extend (3.2) to a Lie algebra one must define a product between elements of $\mathbb{H}' \otimes \mathfrak{J}'$. This requires the introduction of some notations and properties.

Geometry of quaternions

On \mathbb{H} an inner product is defined $\langle h_1, h_2 \rangle = \text{Re}(\bar{h}_1 h_2)$, where complex conjugation changes the sign of the imaginary units: if $h = A_0 + iA_1 + jA_2 +$

kA_3 , $A_i \in \mathbb{R}$, then $\bar{h} = A_0 - iA_1 - jA_2 - kA_3$. From h_1 and h_2 one defines the derivation $D_{h_1, h_2} \in D(\mathbb{H})$ by

$$D_{h_1, h_2} = [L_{h_1}, L_{h_2}] + [R_{h_1}, R_{h_2}], \tag{3.4}$$

where L and R are the usual left and right translations. It is convenient to fix the orthonormal basis $h_0 = 1$, h_i , $i = 1, 2, 3$, where $h_1 = i$, $h_2 = j$, $h_3 = k$ are the imaginary units of \mathbb{H} . A basis for $D(\mathbb{H})$ is thus given by $H_i = ad_{h_i}$, $i = 1, 2, 3$.

Geometry of the Jordan algebra

On \mathfrak{J} we can define the inner product $\langle j_1, j_2 \rangle = \text{Tr}(j_1 \circ j_2)$, where \circ is the Jordan product $j_1 \circ j_2 = (j_1 j_2 + j_2 j_1)/2$. The subspace of matrices orthogonal to the 3×3 identity I_3 is thus \mathfrak{J}' . On it we can define the product $\star : \mathfrak{J}' \times \mathfrak{J}' \rightarrow \mathfrak{J}'$ defined by

$$\star : (j_1, j_2) \mapsto j_1 \star j_2 = j_1 \circ j_2 - \frac{1}{3} \langle j_1, j_2 \rangle I_3. \tag{3.5}$$

With these simple tools in mind one can complete the Lie product on (3.2) by defining

$$\begin{aligned} [h_1 \otimes j_1, h_2 \otimes j_2] &:= \frac{\sigma^2}{12} \langle j_1, j_2 \rangle D_{h_1, h_2} - \sigma^2 \langle h_1, h_2 \rangle [L_{j_1}, L_{j_2}] \\ &+ \frac{\sigma}{2} [h_1, h_2] \otimes (j_1 \star j_2) \end{aligned} \tag{3.6}$$

in order to obtain a Lie algebra. For simplicity, we will use the notation

$$E_{7(-133)} = D(\mathbb{H}) \oplus D(\mathfrak{J}) \oplus (\mathbb{H}' \otimes \mathfrak{J}'), \tag{3.7}$$

to indicate the resulting Lie algebra, and choose $\sigma = 1$.

A sketch of the proof of the validity of (3.6) can be found in Appendix A. From these rules one easily reconstruct explicitly a matrix realization of the adjoint representation. This is done in Appendix B.

Before going to the construction of the smallest representation **56** it is worth to include some comments on the subalgebras.

The F_4 and E_6 subalgebras and some useful relations

The F_4 Lie subalgebra is manifestly included as the algebra of derivation over the exceptional real octonionic Jordan algebra \mathfrak{J} . If we choose a basis $\{j_a\}_{a=1}^{26}$ of \mathfrak{J}' normalized by $\langle j_a, j_b \rangle = \tau \delta_{ab}$, with a suitable real τ , then it can be completed to a basis for \mathfrak{J} by adding $j_{27} = \sqrt{\frac{i}{2}} I_3$. After fixing this, we can determine a basis $\{D_I\}_{I=1}^{52}$ for $D(\mathfrak{J})$ as in [7]. A 27-dimensional matrix representation is then obtained as

$$(C_I)^\mu{}_\nu = \hat{j}^\mu(D_I(j_\nu)), \tag{3.8}$$

where $\{\hat{j}^\mu\}_{\mu=1}^{27}$ is the dual basis $\hat{j}^\mu(j_\nu) = \delta_\nu^\mu$. Since $D_I(j_{27}) = 0$, the last row and column of all matrices vanish, so that, this is a $\mathbf{26} \oplus \mathbf{1}$ representation. Its extension to the 27 fundamental representation of the E_6 algebra is obtained by adding the operators corresponding to the right multiplication by \mathfrak{J}' . This adds 26 27×27 matrices $(\tilde{C}_a)^\mu{}_\nu$ defined by

$$(\tilde{C}_a)^\mu{}_\nu = -i \hat{j}^\mu(R_{j_a}(j_\nu)), \quad a = 1, \dots, 26, \quad \mu, \nu = 1, \dots, 27. \tag{3.9}$$

The factor $-i$ has been introduced in order to implement the Weyl trick. Note that this factor consists in making a choice among all possible imaginary quaternions. Passing from one to another choice is done by the acting with the $SU(2)$ group symmetry. With our choice we break this symmetry down to the $U(1)$ subgroup that lives the imaginary unit invariant. Its Lie algebra is generated by the derivation D_i over the quaternions, which indeed commutes with the whole $E_6 = D(\mathfrak{J}) \oplus \mathfrak{J}'$. In this way we have recognized the maximal compact subalgebra $E_6 \times \mathbb{R} = \text{Lie}((E_6 \times U(1))/\mathbb{Z}_3)$.

It is interesting to note that, \mathfrak{J} being Abelian, we have $R_{j_a}(j_b) = R_{j_b}(j_a)$ which implies

$$(\tilde{C}_a)^\mu{}_b = (\tilde{C}_b)^\mu{}_a. \tag{3.10}$$

We will see in the next subsection that this symmetry relation is a particular case of a more general symmetry relation that has a deep geometrical meaning.

Other interesting relations are obtained from the Leibnitz property $D_I(j_a \circ j_b) = D_I(j_a) \circ j_b + j_a \circ D_I(j_b)$: it becomes

$$(\tilde{C}_a)^c{}_b (C_I)^d{}_c j_d = [(C_I)^c{}_a (\tilde{C}_c)^\mu{}_b + (C_I)^c{}_b (\tilde{C}_a)^\mu{}_c] j_\mu. \tag{3.11}$$

Applying the dual basis \hat{j}^ν we obtain

$$[C_I, \tilde{C}_a]_b^d = (C_I)_a^c (\tilde{C}_c)_b^d, \tag{3.12}$$

$$(C_I)_a^c (\tilde{C}_b)^{27}_c + (C_I)_b^c (\tilde{C}_a)^{27}_c = 0. \tag{3.13}$$

The first identity gives the very interesting relations

$$\alpha_{Ia}^c = (C_I)_a^c, \tag{3.14}$$

where α_{Ia}^c are some of the structure constants of E_6 , the ones defined by $[C_I, \tilde{C}_a]_b^d = \alpha_{Ia}^c (\tilde{C}_c)_b^d$. Note that (3.14) relates the structure constants of E_6 (directly computable from the adjoint representation) to the matrices of the smallest fundamental representation of F_4 .

Another interesting information comes from identity (3.13). With our normalization, $\text{Tr}(j_a \star j_b) = 0$ implies

$$(\tilde{C}_a)^{27}_b = -i \sqrt{\frac{7}{3}} \delta_{ab}. \tag{3.15}$$

Inserted in (3.13) it gives

$$(C_I)_a^c \delta_{cb} + (C_I)_b^c \delta_{ca} = 0. \tag{3.16}$$

Then, the matrices $(C_I)_a^c$ are antisymmetric!

3.2 The Yokota construction and 56

We can obtain the representation **56** by applying the method explained by Yokota in [13]. We will first consider the general complex realization and then we will specialize to the compact form. From our previous analysis we know that the Lie algebra can be written as

$$E_7 = E_6 \oplus i \otimes \mathbb{R} \oplus j \otimes \mathfrak{J} \oplus k \otimes \mathfrak{J}, \tag{3.17}$$

where we have included the remaining derivative generators D_j and D_k in the last two addends. As supporting space we take the 56-dimensional space

$$V_{56} = (\mathfrak{J} \oplus \mathbb{C})^2. \tag{3.18}$$

One has to define an action of E_7 on this space. To this end it is convenient to introduce some further geometry.

The determinant form

The Jordan algebra is endowed with a trilinear form, the determinant form

$$\text{Det} : \mathfrak{J} \times \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathbb{C}, \tag{3.19}$$

defined by

$$\begin{aligned} \text{Det}(j_1, j_2, j_3) = & \frac{1}{3} \text{tr}(j_1 \circ j_2 \circ j_3) - \frac{1}{6} (\text{tr}(j_1) \text{tr}(j_2 \circ j_3) + \text{tr}(j_2) \text{tr}(j_1 \circ j_3) \\ & + \text{tr}(j_3) \text{tr}(j_1 \circ j_2)) + \frac{1}{6} \text{tr}(j_1) \text{tr}(j_2) \text{tr}(j_3). \end{aligned} \tag{3.20}$$

This is a fundamental ingredient in realizing exceptional Lie algebras. For example, it is left invariant by the action of E_6 on \mathfrak{J} . Indeed, the group E_6 can be defined as the group of endomorphisms of \mathfrak{J} that leave the determinant form invariant. The form Det is a completely symmetric tensor D , also called the cubic invariant of E_6 , with component $D_{\alpha\beta\gamma} = \text{Det}(j_\alpha, j_\beta, j_\gamma)$. Det is nondegenerate. This means that it induces an action \triangleright of \mathfrak{J} on itself

$$\triangleright : \mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}, \tag{3.21}$$

$$(j_1, j_2) \longmapsto j_1 \triangleright j_2, \tag{3.22}$$

defined by the relation $\text{Det}(j_1, j_2, j_3) =: \frac{1}{3} \text{tr}((j_1 \triangleright j_2) \circ j_3)$. This is called the Freudenthal product. More explicitly:

$$j_1 \triangleright j_2 = j_1 \circ j_2 - \frac{1}{2} \text{tr}(j_1) j_2 - \frac{1}{2} \text{tr}(j_2) j_1 + \frac{1}{2} \text{tr}(j_1) \text{tr}(j_2) I_3 - \frac{1}{2} \text{tr}(j_1 \circ j_2) I_3. \tag{3.23}$$

With this richer structure at hand we can define the action of E_7 on V_{56} . Given $X \in E_6$, let ϕ_X its image under the fundamental representation **27** (for example, constructed in [8]). Then, the image of X under **27'** (the second 27-dimensional representation of E_6) is $-\phi_X^t$. For $g = (X, \nu, j_1, j_2) \equiv X + i \otimes \nu + j \otimes j_1 + k \otimes j_2 \in E_7$ and $v := (\tilde{j}_1, \mu_1, \tilde{j}_2, \mu_2)^t \in V_{56}$, we define [13]

$$gv := \begin{pmatrix} \phi_X \tilde{j}_1 + \frac{i}{2} \sqrt{\frac{\tau}{3}} \nu \tilde{j}_1 + (j_2 + i \otimes j_1) \triangleright \tilde{j}_2 + \frac{1}{2} \mu_2 (-j_2 + i \otimes j_1) \\ \frac{1}{2} \langle (-j_2 + i \otimes j_1), \tilde{j}_2 \rangle - \frac{i}{2} \sqrt{3\tau} \nu \mu_1 \\ -\phi_X^t \tilde{j}_2 - \frac{i}{2} \sqrt{\frac{\tau}{3}} \nu \tilde{j}_2 + (-j_2 + i \otimes j_1) \triangleright \tilde{j}_1 + \frac{1}{2} \mu_1 (-j_1 + i \otimes j_1) \\ \frac{1}{2} \langle (j_2 + i \otimes j_1), \tilde{j}_1 \rangle + \frac{i}{2} \sqrt{3\tau} \nu \mu_2 \end{pmatrix}. \tag{3.24}$$

Note that w.r.t. [13] we have fixed a prescription $2B = j_2 + i \otimes j_1$, $2A = -j_2 + i \otimes j_1$ and $\nu \mapsto -\frac{i}{2}\sqrt{\frac{3}{2}}\nu$ in order to get a compact real form. τ is the normalization in the trace product of the Jordan basis: $\text{tr}(j_\alpha \circ j_\beta) = \tau\delta_{\alpha\beta}$. The remaining coefficient are chosen so that the corresponding matrices have all the same normalization w.r.t. the trace product. An explicit realization with $\tau = 2$ is given in Appendix C.

It is worth to look at the general form of the resulting matrices in the **56**. Let us choose the normalization $\tau = 2$:

$$Y = \begin{pmatrix} \phi + i\frac{\nu}{\sqrt{6}} & \bar{0}_{27} & -\sum_{\alpha=1}^{27} \bar{z}^\alpha A_\alpha & \frac{1}{\sqrt{2}}\sum_{\alpha=1}^{27} z^\alpha \bar{e}_\alpha \\ \bar{0}_{27}^t & -i\nu\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}}\sum_{\alpha=1}^{27} z^\alpha \bar{e}_\alpha^t & 0 \\ \sum_{\alpha=1}^{27} z^\alpha A_\alpha & -\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{27} \bar{z}^\alpha \bar{e}_\alpha & -\phi^t - i\frac{\nu}{\sqrt{6}} & \bar{0}_{27} \\ -\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{27} \bar{z}^\alpha \bar{e}_\alpha^t & 0 & \bar{0}_{27}^t & i\nu\sqrt{\frac{3}{2}} \end{pmatrix}, \quad (3.25)$$

where ϕ is in the **27** of E_6 , $\nu \in \mathbb{R}$ and $z^\alpha \in \mathbb{C}$. The matrices A_α have components

$$(A_\alpha)_{\beta\gamma} = \frac{1}{2}\text{tr}((j_\alpha \triangleright j_\gamma) \circ j_\beta) = \frac{3}{2}D_{\alpha\gamma\beta}. \quad (3.26)$$

Thus, the matrices A_α , $\alpha = 1, \dots, 27$ are determined by the invariant cubic tensor D and are then symmetric and totally symmetric by including the index α . Actually we can say more. Assume $a, b, c = 1, \dots, 26$. Then $\text{tr}j_a = \text{tr}j_b = \text{tr}j_c = 0$ and

$$D_{abc} = (A_c)_{ab} = \frac{1}{2}\text{tr}(j_c \circ j_b \circ j_a) \equiv i(\tilde{C}_c)^a_b. \quad (3.27)$$

Then, also the matrices \tilde{C}_a , when we drop the 27th row and column, are totally symmetric. This implies (3.10) as a particular case.

By applying (3.23) to compute the remaining components, we get

$$0 = \frac{3}{2}D_{a,27,27} = (A_a)_{27,27} = (A_{27})_{a,27} = (\tilde{C}_a)^{27}_{27} = (\tilde{C}_{27})^a_{27}; \quad (3.28)$$

$$-\frac{1}{2}\sqrt{\frac{2}{3}}\delta_{ab} = \frac{3}{2}D_{a,b,27} = (A_a)_{b,27} = (A_{27})_{ab} = -\frac{i}{2}(\tilde{C}_a)^b_{27} = -\frac{i}{2}(\tilde{C}_{27})^b_a; \quad (3.29)$$

$$\sqrt{\frac{2}{3}} = \frac{3}{2}D_{27,27,27} = (A_{27})_{27,27} = i(\tilde{C}_{27})^{27}_{27}. \quad (3.30)$$

These interesting relations provide a simple way to construct the matrices \tilde{C}_a which extend the F_4 algebra to the E_6 algebra.

3.3 Construction of the group

From the Tits construction we can easily get the main features required to perform the Euler construction with respect to the maximal subgroup $(E_6 \times U(1))/\mathbb{Z}_3$. If \vec{n} is any normalized three vector, we see that setting $\vec{h} = (h_1, h_2, h_3)$ and $\vec{H} = (H_1, H_2, H_3)$,

$$\mathfrak{u} := \vec{n} \cdot \vec{H} \oplus \mathfrak{J} \dot{+} ((\vec{n} \cdot \vec{h}) \otimes \mathfrak{J}') \tag{3.31}$$

is a subalgebra of \mathfrak{g} which generates a maximal compact subgroup $U(1) \times E_6$. Indeed, note that $\vec{n} \cdot \vec{H}$ has vanishing Lie product with the whole \mathfrak{u} . The linear complement of $\vec{n} \cdot \vec{H}$ in \mathfrak{u} reproduces the same rules we used in [8] to extend F_4 to E_6 . In particular, restricted to $(\vec{n} \cdot \vec{h}) \otimes \mathfrak{J}'$, the product (3.6) is $[(\vec{n} \cdot \vec{h}) \otimes j, \vec{n} \cdot \vec{h} \otimes j'] := -[L_j, L_{j'}]$. The minus sign is exactly what we needed in [8] to go from the split form $E_{6(-26)}$ to the compact form. This is not surprising since we are starting from a compact group, but it can be considered as a consistency check. Note that we thus have a family of $E_6 \times U(1)$ subgroups parameterized by the choice of the vector \vec{n} . On this family there is the action of an $SU(2)$ subgroup changing \vec{n} , underlying the presence of a quaternionic structure. However, note also that in order to realize the quotient space $E_7/((E_6 \times U(1))/\mathbb{Z}_3)$ one has to fix the \vec{n} thus breaking the $SU(2)$ structure and consequently the quaternionic structure.

For definiteness, we will choose $\vec{n} = e_1$, where $e_i, i = 1, 2, 3$ is the canonical basis for \mathbb{R}^3 . Thus

$$\mathfrak{u} = \text{Lie}(U(1) \times E_6) = H_1 \oplus D(\mathfrak{J}) \dot{+} (h_1 \otimes \mathfrak{J}'). \tag{3.32}$$

Having selected the maximal subgroup $U = \exp \mathfrak{u}$, we look at the construction of the Euler parametrization [7]

$$E_7 = B e^V U. \tag{3.33}$$

Here V is a maximal subspace of the linear complement \mathfrak{p} of \mathfrak{u} in \mathfrak{g} such that $Ad_U(V) = \mathfrak{p}$, whereas $B = U/U_o$, where U_o is the kernel of the map:

$$\text{Ad}^o : U \rightarrow \text{Aut}(\text{Lie}(U)), \quad u \mapsto \text{Ad}^o(U) := \text{Ad}_u|_V,$$

the apex o means restriction to V . Let us begin with the selection of V . We take H_2 as a first generator of V . Acting on it with $ad_{\mathfrak{u}}$ we generate H_3 and $h_3 \otimes \mathfrak{J}'$. To generate $h_2 \otimes \mathfrak{J}'$ we need to add at least an element of the form $h_2 \otimes j_a$. Now, \mathfrak{J}' contains a particular two dimensional subspace W which corresponds to the vector space of diagonal traceless Jordan matrices. Let

$\{j_1, j_{18}\}$ be a basis for this subspace. Up to now, the indices we have chosen are arbitrary labels. However, we are now referring to [8] where we have chosen a well defined basis for \mathfrak{J}' . In that basis the diagonal generators are just j_1 and j_{18} which correspond to the matrices c_{53} and c_{70} of E_6 . Note that the elements of $h_2 \otimes W$ commute w.r.t. the product (3.6). If we fix, for example, the element $h_2 \otimes j_1$, acting on it with $ad_{D(\mathfrak{J})}$ we will generate all the basis elements $h_2 \otimes j_a$ but $h_2 \otimes j_{18}$. We conclude that

$$V = \mathbb{R}H_2 \oplus W = \mathbb{R}H_2 \oplus \mathbb{R}(h_2 \otimes j_1) \oplus \mathbb{R}(h_2 \otimes j_{18}). \tag{3.34}$$

At this point we have that $\dim U_o = 2\dim u + \dim V - \dim \mathfrak{g} = 28$. On the other hand, we already know what U_o is, it has, in fact, been studied in [8]. Indeed, if $u_o = \text{Lie}(U_o)$, then u_o is the subset of u of elements which commute with $H_2, h_2 \otimes j_1$ and $h_2 \otimes j_{18}$. But this is the subset of $D(\mathfrak{J})$, commuting with $h_2 \otimes j_1$ and $h_2 \otimes j_{18}$, which determine exactly the subgroup $SO(8)$ of F_4 commuting with W , studied in [8] (in that case W was generated by c_{53} and c_{70}).

This allows us to provide the final expression for the general element of the group. Indeed, if $\Psi_A, A = 1, \dots, 133$ is the basis of \mathfrak{g} defined above, and $c_s, s = 1, \dots, 78$ is the matrix representation of $\text{Lie}(E_6)$ given in [8], then the map

$$\psi : \text{Lie}(E_6) \hookrightarrow \mathfrak{g}, \quad c_s \mapsto \Psi_{s+3} \tag{3.35}$$

gives an embedding of E_6 in E_7 . This provides the generic element

$$U(x_1, \dots, x_{79}) = \exp(x_1 \Psi_1) \psi_*(E_6[x_2, \dots, x_{79}]),$$

where E_6 is the parametrization given in [8] and ψ_* is the push forward of ψ under the exponential map. Next, B is easily obtained from E_6 by dropping the last 28 factors on the right, obtaining

$$B[x_1, \dots, x_{51}] = U(x_1, \dots, x_{51}, 0, \dots, 0).$$

The basis of V being given by $\{\Psi_2, \Psi_{82}, \Psi_{99}\}$, we finally get

$$\begin{aligned} E_7[x_1, \dots, x_{133}] &= e^{x_1 \Psi_1} \psi_*(E_6[x_2, \dots, x_{51}, 0, \dots, 0]) e^{x_{52} \Psi_2 + x_{53} \Psi_{82} + x_{54} \Psi_{99}} \\ &\times e^{x_{55} \Psi_1} \psi_*(E_6[x_{56}, \dots, x_{133}]). \end{aligned} \tag{3.36}$$

Thus we are left with the problem of determining the range of the parameters. This can be done by means of the topological method developed in [7]. Concretely, this consist in choosing the range of parameters in order to define a 133-dimensional closed cycle. This will eventually cover the group

an integer number N of times, so that one must finally reduce the range to have $N = 1$. The explicit deduction of the range of parameters is done in Appendix D. We conclude this section with some further comments.

3.3.1 Remarks

To realize concretely the group we could use either the adjoint representation or the **56**. However, the kernel of the Adjoint representation of the group is its center, which is \mathbb{Z}_2 for E_7 , so that using it in (3.36) will provide the group E_7/\mathbb{Z}_2 in place of E_7 . Indeed, $\ker Ad$ is, by construction, the subset of G which commute with all G . On the contrary, the representation **56 \mathbf{G}** of the group (we add a suffix G to distinguish the group from the algebra) is faithful so that we can use **56** to construct E_7 . This is a well known fact, but we can check it directly from our construction. Let M_A be the basis of the algebra E_7 in the adjoint representation, and Y_A the corresponding basis in the **56**. They have exactly the same structure constants, so that they can be thought of as representing the same elements of the algebra. However, **56** contains the nontrivial generator of the center

$$-I_{56} = \exp(\sqrt{6}\pi Y_1). \tag{3.37}$$

Obviously the Adjoint representation cannot contain $-I_{133}$ and, indeed, $\exp \sqrt{6}\pi M_1 = +I_{133}$. Thus, the correspondence

$$\xi : \mathbf{56} \longrightarrow \mathbf{133}, \quad Y_A \longmapsto M_A, \quad A = 1, \dots, 133, \tag{3.38}$$

defines a surjective homomorphism

$$\Xi : \mathbf{56}\mathbf{G} \longrightarrow \mathbf{133}\mathbf{G}, \quad \exp \sum_{A=1}^{133} \lambda^A Y_A \longmapsto \exp \sum_{A=1}^{133} \lambda^A M_A, \tag{3.39}$$

which has kernel $\ker \Xi = \mathbb{Z}_2 = \{I_{56}, -I_{56}\}$. Then, **56 \mathbf{G}** is a double covering of **133 \mathbf{G}** .

In particular, let us consider the corresponding one parameter subgroups:

$$h_A(t) = \exp(tM_A), \quad A = 1, \dots, 133, \quad t \in \mathbb{R}, \tag{3.40}$$

$$g_A(t) = \exp(tY_A), \quad A = 1, \dots, 133, \quad t \in \mathbb{R}. \tag{3.41}$$

We then note that for $A > 3$ all of them have period 4π , apart from $A = 73, 99$ and 125 , which have period $4\pi\sqrt{3}$, whereas for $A = 1, 2, 3$, h_A has period $T_h = \sqrt{6}\pi$ and g_A has period $T_g = \sqrt{6} 2\pi = 2T_h$ as a consequence of the double covering.

Finally, we remark that the maximal subgroup of E_7 is $(E_6 \times U(1))/\mathbb{Z}_3$. Indeed, the generator

$$\omega := \exp\left(4\frac{\pi}{\sqrt{3}}Y_{73}\right) = \exp\left(2\sqrt{\frac{2}{3}}\pi Y_1\right) \in E_6 \cap U(1) \tag{3.42}$$

satisfy $\omega^3 = I_{56}$ so that it is a generator of \mathbb{Z}_3 common to both E_6 and $U(1)$.

4 The $E_{7(7)}$ construction

We want to construct the E_7 compact form related to the split form by means of the Weyl unitary trick. In this case the maximal compact subgroup that is privileged is $SU(8)/\mathbb{Z}_2$. To this aim we will follow the paper [15], chapter 12; see also [16].

Let V be an eight dimensional real vector space and V^* be its dual. Let $\wedge^i V$ be the i th external power of V . We can fix an isomorphism $\wedge^8 V \simeq \mathbb{R}$. $SL(V)$ is the group of automorphisms preserving such isomorphism. Let $L := \mathfrak{sl}(V)$ be its Lie algebra. We will construct the representation **56** of E_7 by extending the representation of L on $W := \wedge^2 V \oplus \wedge^2 V^*$ to a representation of E_7 . Set $A := L \otimes \wedge^4 V$. We wish to see A acting as a Lie algebra of linear maps $W \rightarrow W$. The action of L on W is as usual:

$$L(W) = L(V) \wedge V \oplus L(V^*) \wedge V^* + V \wedge L(V) \oplus V^* \wedge L(V^*), \tag{4.1}$$

where $L(V^*)$ is the adjoint action. If $i + j = 8$ the pairing $\wedge^i V \otimes \wedge^j V \rightarrow \wedge^8 V \simeq \mathbb{R}$, given by the wedge product, defines an isomorphism $\wedge^i V \simeq \wedge^j V^*$. This isomorphism can be used to define the second component of the action of A . Actually, an action of $\lambda^4 \equiv \wedge^4 V$ on W can be obtained as follows:

$$\begin{aligned} \lambda^4 \otimes \wedge^2 V &\xrightarrow{\wedge} \wedge^6 V \simeq \wedge^2 V^*, \\ \lambda^4 \otimes \wedge^2 V^* &\simeq \wedge^4 V^* \otimes \wedge^2 V^* \xrightarrow{\wedge} \wedge^6 V^* \simeq \wedge^2 V, \end{aligned} \tag{4.2}$$

where \wedge is the usual multiplication in the exterior algebra $\wedge V$. Thus, A is a 133-dimensional real vector space of operators acting on W , that indeed realizes an E_7 Lie algebra representation, more precisely the $E_{7(7)}$ split form. In order to see this and realize the compact form, let us look more carefully at the explicit matrix realization.

4.1 Matrix realization

We identify V with \mathbb{R}^8 . The action of L on V is generated by the action of all traceless matrices in $M(8, \mathbb{R})$. Fix a basis $\{e_i\}_{i=1}^8$ of V and define a basis $\{A_{kl}, S_{kl}, D_\alpha\}$ for $M(8, \mathbb{R})$, where $1 \leq k < l \leq 8$, $\alpha = 1, \dots, 7$ and $A_{kl}e_i = \delta_{li}e_k - \delta_{ki}e_l$, $S_{kl}e_i = \delta_{li}e_k + \delta_{ki}e_l$, and D_α form a basis of diagonal traceless matrices $D_\alpha = \text{diag}(D_\alpha^1, \dots, D_\alpha^8)$. We can normalize them as $\text{Tr}(D_\alpha D_\beta) = 2\delta_{\alpha\beta}$, so that all matrices are orthogonal, the symmetric matrices are normalized to 2 and the antisymmetric ones to -2 w.r.t. the trace product.

In order to write the action of this basis on W let us introduce the following notations:

- we select a basis $e_{ij} := e_i \wedge e_j$, $i < j$ of $V \wedge V$ and the canonical dual basis ε^{ij} . As usual we will extend the range of the indices i, j as running independently from 1 to 8, by assuming antisymmetry (so $e_{ij} = -e_{ji}$ and so on).
- A vector $v \in V \wedge V$ can be then written as

$$v = \frac{1}{2} \sum_{i,j} v^{ij} e_{ij} = \sum_{i < j} v^{ij} e_{ij},$$

and a vector $w \in V^* \wedge V^*$ as

$$w = \frac{1}{2} \sum_{i,j} w_{ij} \varepsilon^{ij} = \sum_{i < j} w_{ij} \varepsilon^{ij}.$$

- A linear operator $M : V \wedge V \rightarrow V \wedge V$ acts on the components as

$$(Mv)^{ij} = \sum_{i < j} M^{ij}_{kl} v^{kl},$$

and similar notations for the other possibilities $[V^* \wedge V^* \rightarrow V^* \wedge V^*]$, $[V^* \wedge V^* \rightarrow V \wedge V]$ and $[V \wedge V \rightarrow V^* \wedge V^*]$.

We can then easily write down the explicit matrix action of L over $\mathbb{R}^{56} \simeq V \wedge V \oplus V^* \wedge V^*$:

$$A_{kl} = \begin{pmatrix} (A_{kl}^u)^{ij} & 0 \\ 0 & (A_{kl}^d)_{ij} \end{pmatrix}$$

$$= \begin{pmatrix} (\delta_{ki'}\delta_{li} - \delta_{ki}\delta_{li'})\delta_{jj'} + (\delta_{kj'}\delta_{lj} - \delta_{kj}\delta_{lj'})\delta_{ii'} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \tag{4.3}$$

$$S_{kl} = \begin{pmatrix} (S_{kl}^u)^{ij} & & 0 \\ & i'j' & \\ 0 & & (S_{kl}^d)_{ij} \end{pmatrix} \\ = \begin{pmatrix} i(\delta_{ki'}\delta_{li} + \delta_{ki}\delta_{li'})\delta_{jj'} + i(\delta_{kj'}\delta_{lj} + \delta_{kj}\delta_{lj'})\delta_{ii'} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \tag{4.4}$$

$$D_\alpha = \begin{pmatrix} (D_\alpha^u)^{ij} & & 0 \\ & i'j' & \\ 0 & & (D_\alpha^d)_{ij} \end{pmatrix} = \begin{pmatrix} i(D_\alpha^i + D_\alpha^j)\delta^{ij} & & 0 \\ & 0 & \\ & & -i(D_\alpha^i + D_\alpha^j)\delta^{i'j'} \end{pmatrix}, \tag{4.5}$$

where $\delta^{ij}_{i'j'}$ is the identity over $V \wedge V$. Note that, in order to obtain the compact form, we have multiplied by i the symmetric matrices.

For the remaining 70 generators we have to consider the action of $\{\lambda_{i_1 i_2 i_3 i_4} = e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}\}_{i_1 < i_2 < i_3 < i_4}$ on W . This is easily realized by implementing the identifications (4.2):

$$(\lambda_{i_1 i_2 i_3 i_4}) \otimes (e_{j_1 j_2}) \mapsto \frac{1}{2} \epsilon_{i_1 i_2 i_3 i_4 j_1 j_2 k_1 k_2} \epsilon^{k_1 k_2}, \\ (\lambda_{i_1 i_2 i_3 i_4}) \otimes (\epsilon_{j_1 j_2}) \mapsto \frac{1}{2} \delta_{i_1 i_2 i_3 i_4}^{j_1 j_2 k_1 k_2} e_{k_1 k_2}, \tag{4.6}$$

where ϵ is the standard eight-dimensional Levi-Civita tensor and

$$\delta_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} = \sum_{\sigma \in \mathcal{P}} \epsilon_\sigma \delta_{i_\sigma(1)}^{j_1} \delta_{i_\sigma(2)}^{j_2} \delta_{i_\sigma(3)}^{j_3} \delta_{i_\sigma(4)}^{j_4} \tag{4.7}$$

with \mathcal{P} the set of permutations and ϵ_σ is the parity of σ . The action of $\lambda_{i_1 i_2 i_3 i_4}$ in the block matrix form, with respect to the decomposition $W = \wedge^2 V \oplus \wedge^2 V^*$, is then

$$\lambda_{i_1 i_2 i_3 i_4} = \begin{pmatrix} 0 & (\lambda_{i_1 i_2 i_3 i_4}^u)_{ij,kl} \\ (\lambda_{i_1 i_2 i_3 i_4}^d)_{ij,kl} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_{i_1 i_2 i_3 i_4} ijkl \\ \delta_{i_1 i_2 i_3 i_4}^{ijkl} & 0 \end{pmatrix}. \tag{4.8}$$

Note that the matrices λ^u and λ^d are both symmetric. Let us introduce the ordered tetra-indices $I \equiv \{i_1 i_2 i_3 i_4\}$ with the rule $i_1 < i_2 < i_3 < i_4$. Its

complementary is the unique ordered tetra-index \tilde{I} such that $\epsilon_{I\tilde{I}} \neq 0$. Then

$${}^t\lambda_I = \epsilon_{I\tilde{I}}\lambda_{\tilde{I}}. \tag{4.9}$$

Thus, we can change basis for λ^4 introducing symmetric matrices

$$\mathcal{S}_I := \frac{i}{\sqrt{2}}(\lambda_I + \epsilon_{I\tilde{I}}\lambda_{\tilde{I}}) \tag{4.10}$$

and antisymmetric matrices

$$\mathcal{A}_I := \frac{1}{\sqrt{2}}(\lambda_I - \epsilon_{I\tilde{I}}\lambda_{\tilde{I}}). \tag{4.11}$$

Again, we have included the imaginary unit for the symmetric matrices. The cardinality of the set of tetra indices is 70 so that only half of the \mathcal{S}_I (and of the \mathcal{A}_I) can be linearly independent. Indeed, we have $\mathcal{S}_I = \mathcal{S}_{\tilde{I}}$ ($\mathcal{A}_I = -\mathcal{A}_{\tilde{I}}$). To avoid this double over-counting we can restrict ourselves to the subset \mathcal{I}_0 of tetra-indices defined in Appendix H. Thus, a basis for $\wedge^4 V$ (as linear operators over W) is

$$\{\mathcal{S}_I, \mathcal{A}_I\}_{I \in \mathcal{I}_0}. \tag{4.12}$$

A basis for $A \equiv E_7$ is then

$$\{A_{kl}, \mathcal{A}_I, D_\alpha, S_{kl}, \mathcal{S}_I\}_{1 \leq k < l \leq 8; 1 \leq \alpha \leq 8; I \in \mathcal{I}_0}. \tag{4.13}$$

All matrices are orthogonal. The antisymmetric matrices $A_\mu \equiv \{A_{kl}, \mathcal{A}_I\}$ are normalized by $\text{Tr}(A_\mu A_\nu) = -2\delta_{\mu\nu}$ and have cardinality $28 + 35 = 63$ so that generate the maximal compact subgroup $\text{SU}(8)/\mathbb{Z}_2$. The remaining 70 symmetric generators $S_\Lambda \equiv \{D_\alpha, S_{kl}, \mathcal{S}_I\}$ are normalized by $\text{Tr} S_\Lambda S_M = 2\delta_{\Lambda M}$, so that are the noncompact part of the algebra. In particular, the 7 diagonal matrices D_α generate a Cartan subalgebra.

Summarizing

The above construction furnishes the Lie algebra of $E_{7(-133)}$, but the split form can be recovered simply by dropping the imaginary unit i from the symmetric matrices.

Let us summarize the structure of the matrices. 63 of them are block diagonal

$$M_i = \begin{pmatrix} \mathfrak{sl}(8) & 0 \\ 0 & \mathfrak{sl}(8) \end{pmatrix}, \quad i = 1, \dots, 63, \tag{4.14}$$

where the two diagonal block are the 28-dimensional representation of $\mathfrak{sl}(8)$ on $\wedge^2 V$ and $\wedge^2 V^*$ respectively. A basis for $\mathfrak{sl}(8)$ is composed by 35 symmetric matrices and 28 antisymmetric matrices. $\mathrm{SL}(8)$ contains the maximal compact subgroup $\mathrm{SO}(8)$. The 28 antisymmetric matrices generate its Lie algebra in the compact form. Of the other 35 symmetric matrices 7 are diagonal with vanishing trace and 28 are symmetric with all diagonal elements equal to zero. The 7 diagonal elements generate a Cartan subalgebra of $\mathfrak{sl}(8)$ and, obviously, also of E_7 . The remaining 70 matrices have the structure

$$M_i = \begin{pmatrix} 0 & \wedge^4 V \\ \wedge^4 V & 0 \end{pmatrix}, \quad i = 70, \dots, 133. \tag{4.15}$$

Among these matrices 35 are symmetric and 35 antisymmetric. The group $\mathrm{SU}(8)/\mathbb{Z}_2$ is a maximal compact subgroup of E_7 and its Lie algebra is generated by the 63 antisymmetric matrices. Thus we can write $e_7 = \mathfrak{su}(8) \oplus \mathfrak{p}$, where \mathfrak{p} is the complement of $\mathfrak{su}(8)$ in e_7 .

4.2 Construction of the group

The strategy for constructing the generalized Euler parametrization of the group is the same as for the previous construction so that we will only sketch the main steps. In this case the reference subgroup is the smallest maximal compact subgroup $\mathrm{SU}(8)/\mathbb{Z}_2$. Let $\mathfrak{u} = \mathrm{Lie}(\mathrm{SU}(8))$ and, as before, $\mathfrak{g} = \mathrm{Lie}(E_7) \equiv e_7$ and \mathfrak{p} the complement of \mathfrak{u} in \mathfrak{g} . Having selected the maximal subgroup $U = \exp \mathfrak{u}$ we look at the construction of the Euler parametrization [7]:

$$E_7 = B e^V U. \tag{4.16}$$

Here V is a maximal subspace of the linear complement \mathfrak{p} of \mathfrak{u} in \mathfrak{g} such that $\mathrm{Ad}_U(V) = \mathfrak{p}$, whereas $B = U/U_0$, where U_0 is the kernel of the map

$$\mathrm{Ad}^0 : U \rightarrow \mathrm{Aut}(\mathrm{Lie}(U)), \quad u \mapsto \mathrm{Ad}^0(U) := \mathrm{Ad}_u|_V, \tag{4.17}$$

the apex 0 means restriction to V . As V we can choose the 7 diagonal matrices of a Cartan subalgebra of e_7 : they form an Abelian subalgebra that

can be used to generate \mathfrak{p} by means of the adjoint action of U (a general proof of these statements will appear in [20]). For dimension reasons it follows that U_0 , the kernel of Ad^0 , is at most a discrete subgroup. Indeed, it can be shown that $U_0 = \mathbb{Z}_2^7$, [20].

Thus, we can introduce coordinates $\underline{x} = (x_1, \dots, x_{63})$, $\underline{z} = (z_1, \dots, z_{63})$ and $\vec{y} = (y_1, \dots, y_7)$ so that³ $E_7 = U[\underline{x}]e^{V[\vec{y}]}U[\underline{z}]$, where $U[\underline{z}]$ is a parametrization of $\text{SU}(8)/\mathbb{Z}_2$, the range of the parameters \underline{x} is reduced by the action of U_0 , and

$$V[\vec{y}] = \sum_{\alpha=1}^7 y^\alpha D_\alpha.$$

We will now focus on the determination of the range for \vec{y} . The ranges for the coordinates \underline{x} and \underline{z} , can be easily determined, for example, as in [6].

Let \mathfrak{t} be the complement of V in \mathfrak{p} . The general strategy developed in [7] shows that the invariant measure over E_7 is

$$d\mu_{E_7} = d\mu_U[\underline{x}] d\mu_U[\underline{z}] |f(\vec{y})| d\vec{y}^7, \tag{4.18}$$

$$f(\vec{y}) := \det[\Pi \circ \text{Ad}_{e^{-V}} : \mathfrak{u} \rightarrow \mathfrak{t}], \tag{4.19}$$

where Π is the orthogonal projection on \mathfrak{t} . We assume that one always works with orthonormal bases, so that the determinant function is well defined up to an irrelevant sign (we need only the modulus). The function f is determined in Appendix H, the final expression is

$$|f(\vec{y})| = \prod_{\beta \in \text{Rad}^+} \sin \left(\left| \sum_{a=1}^7 y^a \beta(D_a) \right| \right), \tag{4.20}$$

where Rad^+ is the set of positive roots w.r.t. V . Thus, the equations for the range of \vec{y} are

$$0 < \left| \sum_{a=1}^7 y^a \beta(D_a) \right| < \pi, \quad \beta \in \text{Rad}^+. \tag{4.21}$$

This is a set of 63 double inequalities, which, however, can be quickly reduced to a set of eight equations as follows. Indeed, all positive roots can be

³Concretely, $B[\underline{x}] = U[\underline{x}]$ and the difference is only in the range.

obtained as nonnegative integer linear combinations of the simple roots. In particular, there exists a unique longest root

$$\beta_{\max} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7,$$

whose coefficients are the highest ones. From this it follows that all inequalities are a consequence of the ones corresponding to the simple roots plus the one associated to the longest one. These are

$$0 < \frac{1}{2}(y^1 - y^2 - y^3 - y^4 - y^5 - y^6 + \sqrt{2}y^7) < \pi, \tag{4.22}$$

$$0 < y^1 + y^2 < \pi, \tag{4.23}$$

$$0 < -y^1 + y^2 < \pi, \tag{4.24}$$

$$0 < -y^2 + y^3 < \pi, \tag{4.25}$$

$$0 < -y^3 + y^4 < \pi, \tag{4.26}$$

$$0 < -y^4 + y^5 < \pi, \tag{4.27}$$

$$0 < -y^5 + y^6 < \pi, \tag{4.28}$$

$$0 < \sqrt{2}y^7 < \pi, \tag{4.29}$$

and characterize completely the range for \vec{y} .

5 The $E_{7(-5)}$ construction

The $E_{7(-5)}$ construction can be easily obtained from the $E_{7(-25)}$ one by following the analysis of Yokota in [13]. The maximal compact subgroup of $E_{7(-5)}$ is $U_5 = (\text{Spin}(12) \times \text{SU}(2))/\mathbb{Z}_2$. Let us consider the map

$$\sigma : \mathfrak{J} \longrightarrow \mathfrak{J}, \quad \begin{pmatrix} a & o_1 & o_2 \\ \bar{o}_1 & b & o_3 \\ \bar{o}_2 & \bar{o}_3 & c \end{pmatrix} \longmapsto \begin{pmatrix} a & -o_1 & -o_2 \\ -\bar{o}_1 & b & o_3 \\ -\bar{o}_2 & \bar{o}_3 & c \end{pmatrix}. \tag{5.1}$$

Thus σ lies in the group $F_4 \subset E_6 \subset E_7$. In [13], it is shown that U_5 is the subgroup of elements $g \in E_7$ such that $\sigma g = g\sigma$.

Let us go back to the Lie algebra. In Section 3.1, we have realized the Lie algebra E_7 as $E_6 + i\mathbb{R} + \mathfrak{J}_{\mathbb{C}}$, where \mathbb{R} is generated by the derivation D_i associated to the imaginary unit i , and $\mathfrak{J}_{\mathbb{C}} \simeq \mathfrak{J} \oplus i\mathfrak{J} \simeq (\mathbb{R}D_j \oplus j \otimes \mathfrak{J}') \oplus (\mathbb{R}D_k \oplus k \otimes \mathfrak{J}')$ is the complexification of the exceptional Jordan algebra \mathfrak{J} . Then, it follows that the Lie algebra of U_5 is generated by the subgroup $E_6^\sigma = \{h \in E_6 | \sigma h = h\sigma\} \simeq \text{spin}(10) \times i\mathbb{R}$ plus the elements $\mathfrak{J}_{\mathbb{C}}^\sigma$ which are invariant

under σ . E_6^σ is the subset of elements of the Lie algebra E_6 that leave the element

$$J_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

invariant. It is composed by the $A \in E_6$ such that $AJ_1 = 0$, which generate a $\text{spin}(10)$ subalgebra, plus the $B \in E_6$ with $BJ_1 = ibJ_1$ for some real b , generating the $U(1)$ factor $i\mathbb{R}$. With respect to the basis we have chosen for \mathfrak{J} , the element J_1 is

$$J_1 = \frac{1}{6} \left(3j_1 + \sqrt{3}j_{18} + \sqrt{6}j_{27} \right). \tag{5.2}$$

Thus, the matrices of E_6 generating $\text{spin}(10)$ are the ones having

$$\vec{v} = \frac{1}{6} \left(3\vec{e}_1 + \sqrt{3}\vec{e}_{18} + \sqrt{6}\vec{e}_{27} \right) \tag{5.3}$$

in the kernel (e_i are the canonical vectors of \mathbb{R}^{27}). These are⁴

$$\begin{aligned} L_b = Y_a, L_{45} = \frac{1}{2} \left(\sqrt{3}Y_{73} - Y_{56} \right), \quad a = 4, \dots, 24, 33, \dots, 39, \\ 48, \dots, 55, 74, \dots, 81, \quad b = 1, \dots, 44. \end{aligned} \tag{5.4}$$

For the $U(1)$ factor in E_6^σ from the above condition one finds the generator $(\sqrt{3}Y_{56} + Y_{73})/2$. However, we can obtain a $U(1)$ factor also by adding any multiple of Y_1 . In order to get an orthogonal basis (in particular orthogonal to the next generators) it is convenient to take the generator

$$L_{46} = \frac{1}{6} (3Y_{56} + \sqrt{3}Y_{73} + 2\sqrt{6}Y_1). \tag{5.5}$$

In order to construct the $SU(2)$ factor, note that J_1 is embedded in $\mathfrak{J}_\mathbb{C}$ and generates the real spaces $\mathbb{R}j \otimes J_1$ and $\mathbb{R}k \otimes J_1$, which algebraically generate the $SU(2)$ factor in U_5 [13]. As J_1 correspond to the vector \vec{v} above in our

⁴We reorder the numbering of the matrices by calling them L_b following the natural order. So, for example, $L_1 = Y_4, L_2 = Y_5, L_{22} = Y_{33}$ and so on. Moreover, when needed we will add multiplicative factors chosen so that the L_a have the same normalization as the Y_i .

basis it follows that the corresponding generators of $\mathfrak{su}(2)$ are

$$\begin{aligned} L_{67} &= \frac{1}{3\sqrt{2}}(3Y_{82} + \sqrt{3}Y_{99} + \sqrt{6}Y_2), & L_{68} &= \frac{1}{3\sqrt{2}}(3Y_{108} + \sqrt{3}Y_{125} + \sqrt{6}Y_3), \\ L_{69} &= \frac{1}{3\sqrt{2}}(3Y_{56} + \sqrt{3}Y_{73} - \sqrt{6}Y_1). \end{aligned} \tag{5.6}$$

Finally, we are left with the 20 generators in $\mathfrak{J}_{\mathbb{C}}^{\sigma}$ which are complementary to J_1 . These are

$$\begin{aligned} L_{47} &= \frac{1}{3\sqrt{2}}(-3Y_{82} + \sqrt{3}Y_{99} + \sqrt{6}Y_2), \\ L_{48} &= \frac{1}{3\sqrt{2}}(-3Y_{108} + \sqrt{3}Y_{125} + \sqrt{6}Y_3), \\ L_{49} &= \frac{\sqrt{2}}{3} \left(-\sqrt{3}Y_{99} + \sqrt{\frac{3}{2}}Y_2 \right), & L_{50} &= \frac{\sqrt{2}}{3} \left(-\sqrt{3}Y_{125} + \sqrt{\frac{3}{2}}Y_3 \right), \\ L_d &= Y_b, \quad b = 100, \dots, 107, 126, \dots, 133, \quad d = 51, \dots, 66. \end{aligned} \tag{5.7}$$

In this way, we have selected the $\text{Spin}(12) \times \text{SU}(2)$ subalgebra and we can obtain the $E_{7(-5)}$ real form by applying the unitary Weyl trick to the complementary generators

$$\begin{aligned} L_e &= Y_c, \quad c = 25, \dots, 32, 40, \dots, 47, 57, \dots, 72, 83, \dots, 98, 109, \dots, 124, \\ e &= 70, \dots, 133. \end{aligned} \tag{5.8}$$

5.1 Construction of the group

The symmetric manifold $E_{7(-5)}/U_5$ has rank 4, which means that we can find a Cartan subalgebra of E_7 with four generators in the complement of $\text{Lie}(U_5)$. A possible choice for such a complement is

$$H_4 := \langle L_{70}, L_{86}, L_{103}, L_{120} \rangle. \tag{5.9}$$

Let us indicate with \mathfrak{k} the maximal Lie subalgebra of $\mathfrak{u}_5 := \text{Lie}(U_5)$ that commute with H_4 . It results that the corresponding Lie group is $K \simeq \text{Spin}(4) \times \text{SU}(2) \times \mathbb{Z}_2^2 \subset U_5$, see appendix I. This means that we can write

the group elements in the form

$$\begin{aligned}
 &E_7[x_1, \dots, x_{60}; y_1, \dots, y_4; z_1, \dots, z_{69}] \\
 &= (U_5/K)[x_1, \dots, x_{60}]e^{H_4[y_1, \dots, y_4]}U_5[z_1, \dots, z_{69}], \tag{5.10}
 \end{aligned}$$

where

$$H_4[y_1, \dots, y_4] := y_1L_{70} + y_2L_{86} + y_3L_{103} + y_4L_{104}, \tag{5.11}$$

and $U_5[z_1, \dots, z_{69}]$ and $(U_5/K)[x_1, \dots, x_{60}]$ are parameterizations of U_5 and U_5/K respectively. Again we will now determine the range for the parameters y_i , the ranges for x_a and z_b being determinable as usual.

Let \mathfrak{p} the orthogonal complement of \mathfrak{u}_5 in $\text{Lie}E_7$, \mathfrak{t} the orthogonal complement of H_4 in \mathfrak{p} , and \mathfrak{s} the orthogonal complement of \mathfrak{k} in \mathfrak{u}_5 . The invariant measure over E_7 in this construction is

$$d\mu_{E_7} = d\mu_{U_5/K}[\underline{x}] d\mu_{U_5}[\underline{z}] |h(\vec{y})| d\vec{y}^4, \tag{5.12}$$

$$h(\vec{y}) := \det[\Pi \circ \text{Ad}_{e^{-H_4}} : \mathfrak{s} \rightarrow \mathfrak{t}], \tag{5.13}$$

where Π is the orthogonal projection on \mathfrak{t} . Again, we assume to work with orthonormal bases, so that the determinant function is well defined. Using a method similar to the one used in appendix H, one gets

$$|h(\vec{y})| = \prod_{\beta \in \text{Rad}'^+} \sin^{m_\beta}(|\beta(H_4(y_1, \dots, y_4))|), \tag{5.14}$$

where Rad'^+ is the set of positive restricted roots of E_7/U_5 w.r.t. H_4 , and m_β is the multiplicity of β . The restricted root lattice of E_7/U_5 is an F_4 lattice whose short roots have multiplicity 4. Thus, the equations for the range of \vec{y} are

$$0 < |\beta(H_4(y_1, \dots, y_4))| < \pi, \quad \beta \in \text{Rad}^+. \tag{5.15}$$

This is a set of 24 double inequalities, which, however, can be quickly reduced to a set of five equations as follows. Indeed, all positive roots can be obtained as non negative integer linear combinations of the simple roots. The simple

roots are

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(1, -1, -1, 1), & \alpha_2 &= (0, 0, 1, -1), & \alpha_3 &= (-1, 0, 0, 0), \\ \alpha_4 &= (0, 1, -1, 0). \end{aligned} \tag{5.16}$$

In particular, there exists a unique longest root

$$\beta_{\max} = 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4$$

whose coefficients are the highest ones. From this it follows that all inequalities are a consequence of the ones corresponding to the simple roots plus the one associated to the longest one. These are

$$0 < \frac{1}{2}(y^1 - y^2 - y^3 + y^4) < \pi, \tag{5.17}$$

$$0 < y^3 - y^4 < \pi, \tag{5.18}$$

$$0 < y^1 < \pi, \tag{5.19}$$

$$0 < y^2 - y^3 < \pi, \tag{5.20}$$

$$0 < y^3 + y^4 < \pi, \tag{5.21}$$

and characterize completely the range for \vec{y} .

6 Conclusions

In this paper, we have solved the problem of giving an explicit construction of the exceptional E_7 simple Lie group and in particular of its compact form. We have solved the problem in three different ways. In the first one we have realized the generalized Euler construction w.r.t. the maximal compact subgroup $U = (E_6 \times U(1))/\mathbb{Z}_3$ of highest dimension. To this end we have first obtained the adjoint representation of the E_7 Lie algebra by using the Tits realization of the Magic Squares. This has allowed us to easily understand the structure of the commutators and then the main properties of the algebra in relation to the subalgebra $\text{Lie}(U)$. However, since the Adjoint representation of the group has a nontrivial kernel given by the center \mathbb{Z}_2 of E_7 , we have built also the fundamental representation **56**, which provides a faithful representation of the group. For this case we have reported a very careful analysis. Note that we can obtain a realization of E_7/\mathbb{Z}_2 , simply by replacing the matrices in the representation **56** with those in the **133**. In this case, the center is mapped into $\mathbb{1}_{133}$, so that we need to restrict the range of the $U(1)$ parameter x_{55} to the interval $[0, \sqrt{\frac{2}{3}} \pi]$.

Since there exists a generalized Euler parametrization for each maximal compact subgroup, we have worked out all such parametrization. Indeed, the other two correspond to the maximal compact subgroups $\tilde{U} := \text{SU}(8)/\mathbb{Z}_2$ and $U_5 = (\text{Spin}(12) \times \text{SU}(2))/\mathbb{Z}_2$. This is not merely an exercise. Indeed, it can be relevant to be able to recognize a specific subgroup for a given application. Moreover, from each Euler construction one can obtain the corresponding real form simply by means of the unitary Weyl trick. For these constructions we have omitted several details, in part to avoid annoying repetitions and in part because certain specific steps revealed to have a deeper meaning which can be understood in a more general context, which will be presented in a devoted paper [20].

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Appendix A The tits product

We will follow ([10]), and, for generality, we will allow for \mathbb{H} to be the non associative octonionic algebra.

Let $k \otimes j, k' \otimes j', k'' \otimes j'' \in \mathbb{H}' \otimes \mathfrak{J}'$ and define

$$[h \otimes j, h' \otimes j'] := \frac{\alpha}{3} \langle j, j' \rangle D_{h,h'} - \beta \langle h, h' \rangle [L_j, L_{j'}] + \gamma [h, h'] \otimes (j \star j') \quad (\text{A.1})$$

for some constants α, β, γ . Then, the only nontrivial Jacobi identities to be checked are

$$\begin{aligned} 0 &= [[h \otimes j, h' \otimes j'], h'' \otimes j''] + [[h'' \otimes j'', h \otimes j], h' \otimes j'] \\ &\quad + [[h' \otimes j', h'' \otimes j''], h \otimes j] \\ &=: \text{cyc}\{[[h \otimes j, h' \otimes j'], h'' \otimes j'']\}. \end{aligned} \quad (\text{A.2})$$

Now

$$\begin{aligned} [[h \otimes j, h' \otimes j'], h'' \otimes j''] &= \frac{\alpha}{3} \langle j, j' \rangle D_{k,k'}(k'') \otimes j'' - \beta \langle k, k' \rangle k'' \otimes [L_j, L_{j'}] j'' \\ &\quad + \frac{\alpha\gamma}{3} \langle j \star j', j'' \rangle D_{[k,k'],k''} - \gamma\beta \langle [k, k'], k'' \rangle \\ &\quad \otimes [L_{j \star j'}, L_{j''}] + \gamma^2 [[k, k'], k''] \otimes (j \star j') \star j''. \end{aligned} \quad (\text{A.3})$$

After cyclic summation the terms with coefficients $\alpha\gamma$ and $\beta\gamma$ disappear. Defining the associator $[k, k', k''] = (kk')k'' - k(k'k'')$ (which is totally anti-symmetric) we have the identities, which can be easily obtained adapting the relations in [10] to our notation,

$$D_{k,k'}(k'') = [[k, k'], k''] - 3[k, k', k''], \tag{A.4}$$

$$\langle c, a \rangle b - \langle c, b \rangle a = -\frac{1}{4}[[b, a], c] + \frac{1}{2}[c, b, a], \tag{A.5}$$

$$X \star (X \circ X) = \frac{1}{2}\langle X, X \rangle X, \quad X \in \mathfrak{J}'. \tag{A.6}$$

From the last one we also get

$$j \star (j' \circ j'') - \frac{1}{2}\langle j, j' \rangle j'' + \text{cyclic} = 0. \tag{A.7}$$

Using all this identities we finally obtain

$$\begin{aligned} & \text{cyc}\{[[h \otimes j, h' \otimes j'], h'' \otimes j'']\} \\ &= \left\{ [[k, k'], k''] \otimes \left[\frac{\alpha - \gamma^2}{3}\langle j, j' \rangle j'' + \frac{4\gamma^2 - \beta}{4}(j \circ j') \star j'' \right] \right\} \\ &+ \frac{\beta - 4\alpha}{4}[k, k', k''] \otimes \text{cyc}[\langle j, j' \rangle j''], \end{aligned} \tag{A.8}$$

which vanishes for all choices of k, k', k'' and j, j', j'' if and only if $\alpha = \gamma^2 = \frac{\beta}{4}$, which gives equation (3.6).

Appendix B The adjoint representation of E_7 in the $E_{7(-25)}$ construction

We can now realize the adjoint representation of E_7 . To this end, let us consider the basis $\{\Psi_A\}_{A=1}^{133}$ of \mathfrak{g} , with

$$\Psi_A = \begin{cases} H_L & \text{if } A = 1, 2, 3, \quad L = 1, 2, 3 \\ J_I & \text{if } A = I + 3, \quad I = 1, \dots, 52, \\ h_1 \otimes j_a & \text{if } A = a + 55, \quad a = 1, \dots, 26, \\ h_2 \otimes j_a & \text{if } A = a + 81, \quad a = 1, \dots, 26, \\ h_3 \otimes j_a & \text{if } A = a + 107, \quad a = 1, \dots, 26. \end{cases} \tag{B.1}$$

The dual basis is $\hat{\Psi}^B$, with $\hat{\Psi}^B(\Psi_A) = \delta_A^B$. Thus, the generic matrix of $\text{Lie}(E_7)$ is $(M_A)^C_B = \hat{\Psi}^C(ad_{\Psi_A}\Psi_B)$ where $ad_{\Psi_A}\Psi_B = [\Psi_A, \Psi_B]$ is the adjoint action of Ψ_A on Ψ_B . As usual, in $(M_A)^C_B$, the upper index denote the row, the lower the column and the index A labels the matrix we are considering. We will use the first capital Latin letters A, B, C , running from 1 to 133, to label the basis elements of $\text{Lie}(E_7)$, small Latin letters i, j, k , running from 1 to 3, to label the imaginary units of \mathbb{H} , capital Latin letters L, M, N , running from 1 to 3, for the derivations $D(\mathbb{H})$, capital Latin letters I, J, K , running from 1 to 52, for the derivation $D(\mathfrak{J})$, first small Latin letters a, b, c , running from 1 to 26, for the elements of \mathfrak{J}' and finally the Greek letters μ, ν, λ , running from 1 to 27, for the elements of \mathfrak{J} .

Let us proceed with the construction step by step.

B.1 The matrices M_A with $A = 1, 2, 3$

These matrices are determined by the adjoint action of the elements $\Psi_A = H_L$, with $L = 1, 2, 3$ on Ψ_B , $B = 1, \dots, 133$. We have to consider five cases for the index B .

- $B = 1, 2, 3$, i.e., $M = 1, 2, 3$: $[H_L, H_M] = 2\epsilon_{LM}^N H_N$.
- $B = 4, \dots, 55$, i.e., $J = 1, \dots, 52$: $[H_L, J_J] = 0$.
- $B = 56, \dots, 81$, i.e., $b = 1, \dots, 26$: $[H_L, h_1 \otimes j_b] = ad_{h_i} h_1 \otimes j_b = 2\epsilon_{i1}^k h_k \otimes j_b$, with the value of L equal to that of i .
- $B = 82, \dots, 107$, i.e., $b = 1, \dots, 26$: $[H_L, h_2 \otimes j_b] = ad_{h_i} h_2 \otimes j_b = 2\epsilon_{i2}^k h_k \otimes j_b$, with the value of L equal to that of i .
- $B = 108, \dots, 133$, i.e., $b = 1, \dots, 26$: $[H_L, h_3 \otimes j_b] = ad_{h_i} h_3 \otimes j_b = 2\epsilon_{i3}^k h_k \otimes j_b$, with the value of L equal to that of i .

Applying the dual basis $\hat{\Psi}^C$, and organizing the result in block matrices (with block structure $C = \{N, K, a, b, c\}$) we obtain the first three matrices:

$$(M_1)^C_B = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & & & & \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & & & & \\ \hline & 0 & & 0 & 0 & 0 & 0 \\ & 0 & & 0 & 0 & 0 & 0 \\ & 0 & & 0 & 0 & 0 & -2I_{26} \\ \hline & 0 & & 0 & 0 & 2I_{26} & 0 \end{array} \right),$$

$$\begin{aligned}
 (M_2)^C_B &= \left(\begin{array}{ccc|c|c|c|c}
 0 & 0 & 2 & & & & \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -2 & 0 & 0 & & & & \\
 \hline
 0 & & & 0 & 0 & 0 & 0 \\
 0 & & & 0 & 0 & 0 & 2I_{26} \\
 0 & & & 0 & 0 & 0 & 0 \\
 \hline
 0 & & & 0 & -2I_{26} & 0 & 0
 \end{array} \right), \\
 (M_3)^C_B &= \left(\begin{array}{ccc|c|c|c|c}
 0 & -2 & 0 & & & & \\
 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & & & & \\
 \hline
 0 & & & 0 & 0 & 0 & 0 \\
 0 & & & 0 & 0 & -2I_{26} & 0 \\
 0 & & & 0 & 2I_{26} & 0 & 0 \\
 \hline
 0 & & & 0 & 0 & 0 & 0
 \end{array} \right), \tag{B.2}
 \end{aligned}$$

where the five diagonal blocks have dimensions 3×3 , 52×52 , 26×26 , 26×26 and 26×26 respectively.

B.2 The matrices M_A with $A = 4, \dots, 55$

These matrices are given by the adjoint action of the elements $\Psi_{I+3} = J_I$, with $I = 1, \dots, 52$ on Ψ_B , $B = 1, \dots, 133$.

- $B = 1, 2, 3$, i.e., $M = 1, 2, 3$: $[J_I, H_M] = 0$.
- $B = 4, \dots, 55$, i.e., $J = 1, \dots, 52$: $[J_I, J_J] = f_{IJ}^K J_K$, where f_{IJ}^K are the structure constants of $\text{Lie}(F_4)$.
- $B = 56, \dots, 81$, i.e., $b = 1, \dots, 26$: $[J_I, h_1 \otimes j_b] = h_1 \otimes J_I j_b = h_1 \otimes (C_I)^c_b j_c$. The $(C_I)^c_b$ are the matrices of $\text{Lie}(F_4)$, the algebra of the derivation of the octonionic Jordan matrices. Note that j_b are traceless, actually we choose as basis for \mathfrak{J} 26 traceless matrices and the identity. As derivations vanish on the identity, $\text{Lie}(F_4)$ can be obtained as the derivations on \mathfrak{J}' .
- $B = 82, \dots, 107$, i.e., $b = 1, \dots, 26$: $[J_I, h_2 \otimes j_b] = h_2 \otimes J_I j_b = h_2 \otimes (C_I)^c_b j_c$.
- $B = 108, \dots, 133$, i.e., $b = 1, \dots, 26$: $[J_I, h_3 \otimes j_b] = h_3 \otimes J_I j_b = h_3 \otimes (C_I)^c_b j_c$.

We obtain:

$$(M_{I+3})^C_B = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & f_{IJ}^K & 0 & 0 & 0 \\ \hline 0 & 0 & (C_I)^c_b & 0 & 0 \\ \hline 0 & 0 & 0 & (C_I)^c_b & 0 \\ \hline 0 & 0 & 0 & 0 & (C_I)^c_b \end{array} \right), \quad (B.3)$$

$I = 1, \dots, 52$.

B.3 The matrices M_A with $A = 56, \dots, 81$

These matrices are given by the adjoint action of the elements $\Psi_A = h_1 \otimes j_a$, with $a = 1, \dots, 26$ on Ψ_B , $B = 1, \dots, 133$.

- $B = 1, 2, 3$, i.e., $M = 1, 2, 3$: $[h_1 \otimes j_a, H_M] = -[H_M, h_1 \otimes j_a] = -ad_{h_j} h_1 \otimes j_a = -2\epsilon_{j1}^k h_k \otimes j_a$, with the value of M equal to that of j .
- $B = 4, \dots, 55$, i.e., $J = 1, \dots, 52$: $[h_1 \otimes j_a, J_J] = -[J_J, h_1 \otimes j_a] = -h_1 \otimes J_J j_a = -h_1 \otimes (C_J)^c_a j_c$.
- $B = 56, \dots, 81$, i.e., $b = 1, \dots, 26$: $[h_1 \otimes j_a, h_1 \otimes j_b] = \frac{1}{12} H_{[h_1, h_1]} \langle j_a, j_b \rangle - \langle h_1, h_1 \rangle [L_{j_a}, L_{j_b}] + \frac{1}{2} [h_1, h_1] \otimes (j_a \star j_b) = -\langle h_1, h_1 \rangle [L_{j_a}, L_{j_b}] = -[L_{j_a}, L_{j_b}] = -\alpha_{ab}^K J_K$. The last equality follows from $[L_{j_a}, L_{j_b}] j_\mu = L_{j_a}(L_{j_b}(j_\mu)) - L_{j_b}(L_{j_a}(j_\mu))$. The Jordan algebra is commutative so that the left action and the right action on j_μ coincide. On the other hand, the right action of \mathfrak{J}' on \mathfrak{J} gives the 26 new elements of the extension the Lie algebra of F_4 to $\text{Lie}(E_6)$, so that $[L_{j_a}, L_{j_b}]$ can be expressed in terms of the 26 $\text{Lie}(E_6)$ matrices that do not belong to $\text{Lie}(F_4)$: $[L_{j_a}, L_{j_b}] j_\mu = j_\lambda [\tilde{C}_a^\lambda \nu \tilde{C}_b^\nu \mu - \tilde{C}_b^\lambda \nu \tilde{C}_a^\nu \mu] = j_\lambda [\tilde{C}_a, \tilde{C}_b]^\lambda_\mu = j_\lambda \alpha_{ab}^K C_K^\lambda_\mu = \alpha_{ab}^K J_K j_\mu = \alpha_{ab}^K J_K j_c + \alpha_{ab}^K J_K j_{27} = \alpha_{ab}^K J_K j_c$, because a derivation on the identity vanishes. The third equality follows from the fact that the commutator between two of the 26 matrices of $\text{Lie}(E_6)/\text{Lie}(F_4)$, lies in $\text{Lie}(F_4)$.
- $B = 82, \dots, 107$, i.e., $b = 1, \dots, 26$: $[h_1 \otimes j_a, h_2 \otimes j_b] = [h_1 \otimes j_a, h_2 \otimes j_b] = \frac{1}{12} H_{[h_1, h_2]} \langle j_a, j_b \rangle - \langle h_1, h_2 \rangle [L_{j_a}, L_{j_b}] + \frac{1}{2} [h_1, h_2] \otimes (j_a \star j_b) = \frac{1}{12} H_{[h_1, h_2]} \tau \delta_{ab} + \frac{1}{2} [h_1, h_2] \otimes (j_a \star j_b) = \frac{1}{6} \tau \delta_{ab} H_3 + h_3 \otimes j_c (\tilde{C}_a)^c_b$. The last equality follows from $(j_a \star j_b) = j_a \circ j_b - \frac{1}{3} \langle j_a, j_b \rangle I_3 = R_{j_a}(j_b) - \frac{1}{3} \langle j_a, j_b \rangle I_3 = (\tilde{C}_a)^c_b j_\mu - \frac{1}{3} \tau \delta_{ab} I_3 = (\tilde{C}_a)^c_b j_c + (\tilde{C}_a)^{27}_b j_{27} - \frac{1}{3} \tau \delta_{ab} I_3 = (\tilde{C}_a)^c_b j_c$. The last two terms cancel because the \star product gives traceless elements by definition. This fact can also be verified using the explicit expression of the matrices \tilde{C}_a .
- $B = 108, \dots, 133$, i.e., $b = 1, l, 26$: $[h_1 \otimes j_a, h_3 \otimes j_b] = -\frac{1}{6} \tau \delta_{ab} H_2 - h_2 \otimes j_c (\tilde{C}_a)^c_b$.

Collecting all these results, we get

$$(M_{a+55})^C_B = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & \frac{1}{6}\tau\delta_3^N\delta_{ab} & -\frac{1}{6}\tau\delta_2^N\delta_{ab} \\ \hline 0 & 0 & -\alpha_{ab}^K & 0 & 0 \\ \hline 0 & -(C_J)^c_a & 0 & 0 & 0 \\ \hline -2\delta_a^c\delta_{3M} & 0 & 0 & 0 & -(\tilde{C}_a)^c_b \\ \hline 2\delta_a^c\delta_{2M} & 0 & 0 & (\tilde{C}_a)^c_b & 0 \end{array} \right), \tag{B.4}$$

$a = 1, \dots, 26$.

B.4 The matrices M_A with $A = 82, \dots, 133$

With analogous computations as the previous case we get the last 52 matrices:

$$(M_{a+81})^C_B = \left(\begin{array}{c|c|c|c|c} 0 & 0 & -\frac{1}{6}\tau\delta_3^N\delta_{ab} & 0 & \frac{1}{6}\tau\delta_1^N\delta_{ab} \\ \hline 0 & 0 & 0 & -\alpha_{ab}^K & 0 \\ \hline 2\delta_a^c\delta_{3M} & 0 & 0 & 0 & (\tilde{C}_a)^c_b \\ \hline 0 & -(C_J)^c_a & 0 & 0 & 0 \\ \hline -2\delta_a^c\delta_{1M} & 0 & -(\tilde{C}_a)^c_b & 0 & 0 \end{array} \right), \tag{B.5}$$

$$(M_{a+107})^C_B = \left(\begin{array}{c|c|c|c|c} 0 & 0 & \frac{1}{6}\tau\delta_2^N\delta_{ab} & -\frac{1}{6}\tau\delta_1^N\delta_{ab} & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha_{ab}^K \\ \hline -2\delta_a^c\delta_{2M} & 0 & 0 & -(\tilde{C}_a)^c_b & 0 \\ \hline 2\delta_a^c\delta_{1M} & 0 & (\tilde{C}_a)^c_b & 0 & 0 \\ \hline 0 & -(C_J)^c_a & 0 & 0 & 0 \end{array} \right), \tag{B.6}$$

$a = 1, \dots, 26$. This completes the construction of the representation **133**.

Appendix C The representation 56 of E_7

By applying the Yokota method we obtain the following 133 56×56 matrices:

$$Y_1 = \left(\begin{array}{c|c|c|c} \frac{i}{\sqrt{6}}I_{27} & \vec{0}_{27} & 0_{27} & \vec{0}_{27} \\ \hline {}^t\vec{0}_{27} & -i\sqrt{\frac{3}{2}} & \vec{0}_{27}^t & 0 \\ \hline 0_{27} & \vec{0}_{27} & -\frac{i}{\sqrt{6}}I_{27} & \vec{0}_{27} \\ \hline {}^t\vec{0}_{27} & 0 & \vec{0}_{27}^t & i\sqrt{\frac{3}{2}} \end{array} \right), \tag{C.1}$$

$$Y_2 = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \vec{0}_{27} & -i\sqrt{\frac{2}{3}}\tilde{I} & i\sqrt{2}\vec{e}_{27} \\ \hline \vec{0}_{27}^t & 0 & i\sqrt{2}\vec{e}_{27}^t & 0 \\ \hline -i\sqrt{\frac{2}{3}}\tilde{I} & i\sqrt{2}\vec{e}_{27} & 0_{27} & \vec{0}_{27} \\ \hline i\sqrt{2}\vec{e}_{27}^t & 0 & \vec{0}_{27}^t & 0 \end{array} \right), \tag{C.2}$$

$$Y_3 = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \vec{0}_{27} & \sqrt{\frac{2}{3}}\tilde{I} & \sqrt{2}\vec{e}_{27} \\ \hline \vec{0}_{27}^t & 0 & \sqrt{2}\vec{e}_{27}^t & 0 \\ \hline -\sqrt{\frac{2}{3}}\tilde{I} & -\sqrt{2}\vec{e}_{27} & 0_{27} & \vec{0}_{27} \\ \hline -\sqrt{2}\vec{e}_{27}^t & 0 & \vec{0}_{27}^t & 0 \end{array} \right), \tag{C.3}$$

$$Y_{I+3} = \left(\begin{array}{c|c|c|c} \phi_I & \vec{0}_{27} & 0_{27} & \vec{0}_{27} \\ \hline \vec{0}_{27}^t & 0 & \vec{0}_{27}^t & 0 \\ \hline 0_{27} & \vec{0}_{27} & -\phi_I^t & \vec{0}_{27} \\ \hline \vec{0}_{27}^t & 0 & \vec{0}_{27}^t & 0 \end{array} \right), \quad I = 1, \dots, 78, \tag{C.4}$$

$$Y_{a+81} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \vec{0}_{27} & 2iA_a & i\sqrt{2}\vec{e}_a \\ \hline \vec{0}_{27}^t & 0 & i\sqrt{2}\vec{e}_a^t & 0 \\ \hline 2iA_a & i\sqrt{2}\vec{e}_a & 0_{27} & \vec{0}_{27} \\ \hline i\sqrt{2}\vec{e}_a^t & 0 & \vec{0}_{27}^t & 0 \end{array} \right), \quad a = 1, \dots, 26, \tag{C.5}$$

$$[3pt]Y_{a+107} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \vec{0}_{27} & -2A_a & \sqrt{2}\vec{e}_a \\ \hline \vec{0}_{27}^t & 0 & \sqrt{2}\vec{e}_a^t & 0 \\ \hline 2A_a & -\sqrt{2}\vec{e}_a & 0_{27} & \vec{0}_{27} \\ \hline -\sqrt{2}\vec{e}_a^t & 0 & \vec{0}_{27}^t & 0 \end{array} \right), \quad a = 1, \dots, 26, \tag{C.6}$$

where I_n is the $n \times n$ identity matrix, 0_{27} is the 27×27 null matrix, $\vec{0}_n$ is the zero vector in \mathbb{R}^n , \vec{e}_μ , $\mu = 1, \dots, 27$, is the canonical basis of \mathbb{R}^{27} ,

$$\tilde{I} = \left(\begin{array}{c|c} I_{26} & \vec{0}_{26} \\ \hline \vec{0}_{26}^t & -2 \end{array} \right), \tag{C.7}$$

$$\phi_I = \begin{cases} C_I & I = 1, \dots, 52 \\ \tilde{C}_{I-52} & I = 53, \dots, 78, \end{cases} \tag{C.8}$$

where C_I and \tilde{C}_a are defined in (3.8) and (3.9) respectively. Finally, A_a , $a = 1, \dots, 26$ are the 27×27 symmetric matrices representing, by means of

the linear isomorphism $\mathfrak{J} \simeq \mathbb{R}^{27}, j_\mu \mapsto \vec{e}_\mu$, the action of \mathfrak{J}' on \mathfrak{J} defined by

$$j'(j) := j' \circ j - \frac{1}{2} \text{Tr}(j)j' - \frac{1}{2} \langle j, j' \rangle I_3.$$

In particular j_μ is the basis of \mathfrak{J} defined in the previous section, and A_a is the matrix associated to $j_a, a = 1, \dots, 26$.

The matrices $Y_A, A = 1, \dots, 133$ are orthonormalized w.r.t. to the product

$$\langle Y, Y' \rangle_{\mathbf{56}} = -\frac{1}{12} \text{Tr}(YY').$$

It is easy to check, for example by means of a computer, that the Y_A satisfy exactly the same commutation relations of the M_A .

Appendix D The range of the parameters for the $E_{7(-25)}$

To determine the range of the parameters, we can proceed as shown in [9]. In particular, we will be able to make use of the results already obtained in [7, 8], thus simplifying most of the computations.

Setting $U := (E_6 \times U(1))/\mathbb{Z}_3$, from our previous construction we can write the generic element of E_7 as

$$E_7[x_1, \dots, x_{133}] = B[x_1, \dots, x_{51}] \exp V[x_{52}, x_{53}, x_{54}] U[x_{55}, \dots, x_{133}], \quad (\text{D.1})$$

which is (3.36) realized with the matrices Y_A . Then, the range of the parameters x_{55}, \dots, x_{133} can be chosen in such the way that x_{55} covers the whole $U(1)$ and x_{56}, \dots, x_{133} cover the whole E_6 . The last ones have been determined in [8] and will be reported in the conclusions for convenience. The $U(1)$ is covered if x_{55} runs over a period T_g . However, because of the action of \mathbb{Z}_3 , we must reduce its range to $T_g/3 = 2\sqrt{\frac{2}{3}}\pi$.

For the remaining parameters, we have to construct the invariant measure over the quotient E_7/U . This is given by

$$d\mu_{B_{E_7}} = |\det J_{\mathfrak{p}}^\perp|, \quad (\text{D.2})$$

where

$$J_{\mathfrak{p}} = e^{-V} B^{-1} d(Be^V) = dV + e^{-V} B^{-1} dB e^V$$

and \perp means the part orthogonal to $\text{Lie}(U)$. Concretely, this means that we have to project J_p on $Y_2, Y_3, Y_{82}, \dots, Y_{133}$. Since

$$dV = dx_{52}Y_2 + dx_{53}Y_{82} + dx_{54}Y_{99}, \tag{D.3}$$

we only need to concentrate on the term $e^{-V}B^{-1}dB e^V$, which must be projected on $Y_3, Y_{83}, \dots, Y_{98}, Y_{100}, \dots, Y_{133}$. The details of the computations are given in Appendix E. However, to express the result of the computation we need to look better at the structure of $B = U/U_o$.

We can write it as $B = U(1)/\mathbb{Z}_3 \times E_6/SO(8)$, where $SO(8)$ is the subgroup $SO(8) \subset F_4 \subset E_6$ which commutes with $Y_{56} := \psi(c_{53})$ and $Y_{73} := \psi(c_{70})$, used in [8] to construct E_6 . By using the results in [7, 8], we then see that

$$B[x_1, \dots, x_{51}] = e^{x_1 Y_1} \psi_*(B_{E_6})[x_2, \dots, x_{27}] \psi_*(B_{F_4})[x_{28}, \dots, x_{43}] \times \psi_*(B_{SO(9)})[x_{44}, \dots, x_{51}], \tag{D.4}$$

and B_{E_6}, B_{F_4} and $B_{SO(9)}$ are the basis in the construction of the groups

$$E_6 = B_{E_6}F_4, \quad F_4 = B_{F_4}SO(9), \quad SO(9) = B_{SO(9)}SO(8), \tag{D.5}$$

defined in [7, 8] respectively. Also, ψ indicates the map $\psi(c_I) = Y_{I+3}$, $I = 1, \dots, 78$, c_I being the generators of E_6 . From now on we will omit the map ψ , for simplicity. Starting from this structure, as shown in Appendix E, we get

$$d\mu_{B_{E_7}}(x_1, \dots, x_{55}) = dx_1 d\mu_{B_{E_6}}(x_2, \dots, x_{27}) d\mu_{B_{F_4}}(x_{28}, \dots, x_{43}) d\mu_{B_{SO(9)}}(x_{44}, \dots, x_{51}) W(x_{52}, x_{53}, x_{54}) dx_{52} dx_{53} dx_{54}, \tag{D.6}$$

$$W(x_1, x_2, x_3) := \sin\left(\frac{\sqrt{3}x_2 + \sqrt{2}x_1 + x_3}{\sqrt{3}}\right) \sin\left(\frac{\sqrt{3}x_2 - \sqrt{2}x_1 - x_3}{\sqrt{3}}\right) \times \sin\left(\frac{\sqrt{6}x_1 - 2\sqrt{3}x_3}{3}\right) := \sin\left(\frac{\sqrt{3}x_2 + 2\sqrt{2}x_1 - x_3}{2\sqrt{3}}\right)^8 \sin\left(\frac{\sqrt{3}x_2 - 2\sqrt{2}x_1 + x_3}{2\sqrt{3}}\right)^8$$

$$\begin{aligned} & \times \sin\left(\frac{x_2 - \sqrt{3}x_3}{2}\right)^8 \\ & := \sin\left(\frac{x_2 + \sqrt{3}x_3}{2}\right)^8 \sin\left(\frac{\sqrt{2}x_1 + x_3}{\sqrt{3}}\right)^8 \sin(x_2)^8. \end{aligned} \tag{D.7}$$

At this point, we have to fix the ranges so that $d\mu$ is positive definite. This fixes only the ranges of the parameters, which explicitly appear in the measure. The other ones can be chosen over a period. In this way, one covers the whole space an integer number N of times. This can be checked by comparing the volume obtained by integrating $d\mu$ over the obtained ranges, with the volume of E_7/U obtained by means of the Macdonald's formula (see [14]). Finally, one has to identify the finite symmetry group responsible of the N -covering to reduce the ranges over the periods to get the $N = 1$ case. Fortunately, in our case most of the work has been done in [7,8]. Indeed, the terms $d\mu_{BE_6}$, $d\mu_{BF_4}$ $d\mu_{BSO(9)}$ are the same appearing there and determine the same corresponding ranges of parameters given in that papers. Moreover, x_1 does not appears in $d\mu$, so that its range is the period of $U(1)/\mathbb{Z}_3$ which is $T_g/3 = 2\sqrt{\frac{2}{3}}\pi$. Thus, we are left with the problem to determine the ranges for the parameters x_{52}, x_{53}, x_{54} . To this end, it is convenient to introduce the change of variables

$$\begin{aligned} x &= \frac{\sqrt{6}x_{52} - 2\sqrt{3}x_{54}}{3}, & y &= \frac{\sqrt{3}x_{53} + \sqrt{2}x_{52} + x_{54}}{\sqrt{3}}, \\ z &= \frac{\sqrt{3}x_{53} - \sqrt{2}x_{52} - x_{54}}{\sqrt{3}}, \end{aligned} \tag{D.8}$$

with inverse:

$$x_{52} = \frac{1}{\sqrt{6}}x - \frac{1}{\sqrt{3}}z, \quad x_{53} = \frac{1}{2}(y + z), \quad x_{54} = -\frac{1}{3}x + \frac{1}{2\sqrt{3}}(y - z). \tag{D.9}$$

Then,

$$\begin{aligned} W(x_{52}, x_{53}, x_{54}) &= \sin(x) \sin(y) \sin(z) \sin^8\left(\frac{x-y}{2}\right) \sin^8\left(\frac{x+y}{2}\right) \\ & \sin^8\left(\frac{x-z}{2}\right) \sin^8\left(\frac{x+z}{2}\right) \sin^8\left(\frac{y-z}{2}\right) \sin^8\left(\frac{y+z}{2}\right), \end{aligned} \tag{D.10}$$

which is positive for

$$x \in [0, \pi], \quad y \in [0, x], \quad z \in [0, y]. \tag{D.11}$$

To check if the range of parameters is the right one, let us compute the integral

$$\mathcal{I} = \int_R d\mu_{B_{E_7}},$$

with the determined ranges, to be compared with the volume $\text{Vol}(E_7/U)$ computed in Appendix F by means of the formula of Macdonald. As

$$dx_{52} dx_{53} dx_{54} = \frac{1}{2\sqrt{2}} dx dy dz,$$

we obtain

$$\mathcal{I} = \frac{T_g}{3} \frac{\text{Vol}(E_6)}{\text{Vol}(SO(8))} \frac{1}{2\sqrt{2}} I, \tag{D.12}$$

$$I = \int_0^\pi dx \int_0^x dy \int_0^y dz \sin(x) \sin(y) \sin(z) \sin^8\left(\frac{x-y}{2}\right) \sin^8\left(\frac{x+y}{2}\right) \sin^8\left(\frac{x-z}{2}\right) \sin^8\left(\frac{x+z}{2}\right) \sin^8\left(\frac{y-z}{2}\right) \sin^8\left(\frac{y+z}{2}\right). \tag{D.13}$$

Using the change of variables

$$\chi = \frac{1}{2}(1 - \cos x), \quad \xi = \frac{1}{2}(1 - \cos y), \quad \zeta = \frac{1}{2}(1 - \cos z), \tag{D.14}$$

the last integral becomes, see Appendix G:

$$I = \frac{2^3 \int_0^1 d\chi \int_0^\chi d\xi \int_0^\xi d\zeta (\chi - \xi)^8 (\xi - \zeta)^8 (\zeta - \chi)^8}{2} = \frac{1}{3^5 \times 5 \times 11 \times 13^2 \times 17}. \tag{D.15}$$

Since the volumes of E_6 and $SO(8)$ can be computed by means of the Macdonald formula giving

$$\text{Vol}(E_6) = \frac{\sqrt{3} 2^{17} \pi^{42}}{3^{10} \times 5^5 \times 7^3 \times 11}, \quad \text{Vol}(SO(8)) = \frac{2^{12} \pi^{16}}{3^3 \times 5}, \tag{D.16}$$

we get

$$\mathcal{I} = \frac{2^6 \pi^{27}}{3^{12} \times 5^5 \times 7^3 \times 11^2 \times 13^2 \times 17}, \tag{D.17}$$

which is just twice the volume of E_7/U computed in Appendix F. However, we can easily see the origin of this double covering: consider the element

$$\tau := -I_{56}\omega^2 = g_1[T_g/2]g_1[2T_g/3] = U(1)[T_g/6], \quad \tau^6 = I_{56}. \tag{D.18}$$

One can check that τ commutes with E_6 and with V . This means that we can write

$$Be^V U = B\tau^{-1}\tau e^V U = B\tau^{-1}e^V \tau U. \tag{D.19}$$

Since the right τ can be reabsorbed in the parametrization of U , and, on the left, it acts as \mathbb{Z}_2 on the factor $U(1)/\mathbb{Z}_3$ in B , we see that it identifies the points in the range by means of the relation

$$x_1 \sim x_1 + T_g/6.$$

Then, to have an injective parametrization, apart from a subset of vanishing measure, we must further restrict the range of x_1 to the interval $x_1 \in [0, \sqrt{\frac{2}{3}}\pi]$.

As a result, we find the following ranges for the E_7 Euler parameters:

$$\begin{array}{llll} x_1 \in \left[0, \sqrt{\frac{2}{3}}\pi\right] & x_2 \in [0, 2\pi] & x_3 \in [0, 2\pi] & x_4 \in [0, 2\pi] \\ x_5 \in [0, \pi] & x_6 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_7 \in \left[0, \frac{\pi}{2}\right] & x_8 \in \left[0, \frac{\pi}{2}\right] \\ x_9 \in [0, \pi] & x_{10} \in [0, 2\pi] & x_{11} \in [0, 2\pi] & x_{12} \in [0, 2\pi] \\ x_{13} \in [0, \pi] & x_{14} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{15} \in \left[0, \frac{\pi}{2}\right] & x_{16} \in \left[0, \frac{\pi}{2}\right] \\ x_{17} \in [0, \pi] & x_{18} \in [0, 2\pi] & x_{19} \in [0, 2\pi] & x_{20} \in [0, 2\pi] \\ x_{21} \in [0, \pi] & x_{22} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{23} \in \left[0, \frac{\pi}{2}\right] & x_{24} \in \left[0, \frac{\pi}{2}\right] \\ x_{25} \in [0, \pi] & x_{26} \in [0, \pi] & -\frac{x_{26}}{\sqrt{3}} \leq x_{27} \leq \frac{x_{26}}{\sqrt{3}} & x_{28} \in [0, 2\pi] \end{array}$$

$$\begin{array}{llll}
x_{29} \in [0, 2\pi] & x_{30} \in [0, 2\pi] & x_{31} \in [0, \pi] & x_{32} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
x_{33} \in \left[0, \frac{\pi}{2}\right] & x_{34} \in \left[0, \frac{\pi}{2}\right] & x_{35} \in [0, \pi] & x_{36} \in [0, 2\pi] \\
x_{37} \in [0, 2\pi] & x_{38} \in [0, 2\pi] & x_{39} \in [0, \pi] & x_{40} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
x_{41} \in \left[0, \frac{\pi}{2}\right] & x_{42} \in \left[0, \frac{\pi}{2}\right] & x_{43} \in [0, \pi] & x_{44} \in [0, 2\pi] \\
x_{45} \in [0, 2\pi] & x_{46} \in [0, 2\pi] & x_{47} \in [0, \pi] & x_{48} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
x_{49} \in \left[0, \frac{\pi}{2}\right] & x_{50} \in \left[0, \frac{\pi}{2}\right] & x_{51} \in [0, \pi] & x_{55} \in \left[0, 2\sqrt{\frac{2}{3}}\pi\right] \\
x_{56} \in [0, 2\pi] & x_{57} \in [0, 2\pi] & x_{58} \in [0, 2\pi] & x_{59} \in [0, \pi] \\
x_{60} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{61} \in \left[0, \frac{\pi}{2}\right] & x_{62} \in \left[0, \frac{\pi}{2}\right] & x_{63} \in [0, \pi] \\
x_{64} \in [0, 2\pi] & x_{65} \in [0, 2\pi] & x_{66} \in [0, 2\pi] & x_{67} \in [0, \pi] \\
x_{68} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{69} \in \left[0, \frac{\pi}{2}\right] & x_{70} \in \left[0, \frac{\pi}{2}\right] & x_{71} \in [0, \pi] \\
x_{72} \in [0, 2\pi] & x_{73} \in [0, 2\pi] & x_{74} \in [0, 2\pi] & x_{75} \in [0, \pi] \\
x_{76} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{77} \in \left[0, \frac{\pi}{2}\right] & x_{78} \in \left[0, \frac{\pi}{2}\right] & x_{79} \in [0, \pi] \\
x_{80} \in [0, \pi] & -\frac{x_{80}}{\sqrt{3}} \leq x_{81} \leq \frac{x_{80}}{\sqrt{3}} & x_{82} \in [0, 2\pi] & x_{83} \in [0, 2\pi] \\
x_{84} \in [0, 2\pi] & x_{85} \in [0, \pi] & x_{86} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{87} \in \left[0, \frac{\pi}{2}\right] \\
x_{88} \in \left[0, \frac{\pi}{2}\right] & x_{89} \in [0, \pi] & x_{90} \in [0, 2\pi] & x_{91} \in [0, 2\pi] \\
x_{92} \in [0, 2\pi] & x_{93} \in [0, \pi] & x_{94} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{95} \in \left[0, \frac{\pi}{2}\right] \\
x_{96} \in \left[0, \frac{\pi}{2}\right] & x_{97} \in [0, \pi] & x_{98} \in [0, 2\pi] & x_{99} \in [0, 2\pi] \\
x_{100} \in [0, 2\pi] & x_{101} \in [0, \pi] & x_{102} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{103} \in \left[0, \frac{\pi}{2}\right] \\
x_{104} \in \left[0, \frac{\pi}{2}\right] & x_{105} \in [0, \pi] & x_{106} \in [0, 2\pi] & x_{107} \in [0, 2\pi] \\
x_{108} \in [0, 2\pi] & x_{109} \in [0, \pi] & x_{110} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{111} \in \left[0, \frac{\pi}{2}\right] \\
x_{112} \in \left[0, \frac{\pi}{2}\right] & x_{113} \in [0, 2\pi] & x_{114} \in [0, \pi] & x_{115} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
\end{array}$$

$$\begin{aligned}
 x_{116} &\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{117} &\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{118} &\in [0, \pi] & x_{119} &\in [0, 2\pi] \\
 x_{120} &\in [0, \pi] & x_{121} &\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{122} &\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{123} &\in [0, \pi] \\
 x_{124} &\in [0, 2\pi] & x_{125} &\in [0, \pi] & x_{126} &\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & x_{127} &\in [0, \pi] \\
 x_{128} &\in [0, 2\pi] & x_{129} &\in [0, \pi] & x_{130} &\in [0, \pi] & x_{131} &\in [0, 2\pi] \\
 x_{132} &\in [0, \pi] & x_{133} &\in [0, 4\pi] & & & &
 \end{aligned}$$

and

$$x_{52} = \frac{1}{\sqrt{6}}x - \frac{1}{\sqrt{3}}z, \quad x_{53} = \frac{1}{2}(y + z), \quad x_{54} = -\frac{1}{3}x + \frac{1}{2\sqrt{3}}(y - z),$$

with $x \in [0, \pi]$, $y \in [0, x]$ and $z \in [0, y]$.

Appendix E Deduction of the measure

The invariant measure on E_7/U is

$$d\mu_{B_{E_7}} = |\det J_{\mathfrak{p}}^\perp|, \tag{E.1}$$

where $J_{\mathfrak{p}}^\perp$ is the projection of

$$J_{\mathfrak{p}} = e^{-V} B^{-1} d(B e^V) = dV + e^{-V} B^{-1} dB e^V$$

on the real linear subspace of $\text{Lie}(E_7)$ spanned by $Y_2, Y_3, Y_{82}, \dots, Y_{133}$. More specifically we have

$$\begin{aligned}
 J_{\mathfrak{p}} &= dV + e^{-V} U_1^{-1} dU_1 e^V + e^{-V} B_{SO(9)}^{-1} dB_{SO(9)} e^V \\
 &\quad + e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} dB_{F_4} B_{SO(9)} e^V \\
 &\quad + e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} B_{E_6}^{-1} dB_{E_6} B_{F_4} B_{SO(9)} e^V,
 \end{aligned}$$

where we used the fact that the $U(1)$ factor commutes with E_6 . To compute $\det J_{\mathfrak{p}}^\perp$ we can then proceed by analyzing the summands term by term (here we will set $H := \text{Lie}(U)$):

- $dV = Y_2 dx_2 + Y_{82} dx_{82} + Y_{99} dx_{99}$, which has obviously non zero projection just on the matrices Y_2, Y_{82} and Y_{99} . Thus, the remaining terms have to be projected on $Y_3, Y_{83}, \dots, Y_{98}, Y_{100}, \dots, Y_{133}$.

- The $SO(9)$ current $J_{B_{SO(9)}} := B_{SO(9)}^{-1} dB_{SO(9)}$ can be split in two orthogonal pieces $J_{B_{SO(9)}} = J_{SO(8)} \oplus J_{SO(9)\setminus SO(8)}$, where $J_{SO(8)}$ is the projection of $J_{B_{SO(9)}}$ into $\text{Lie}(SO(8))$. The space V commute with this $SO(8)$, thus $e^{-V} J_{SO(8)} e^V = J_{SO(8)}$ and as $SO(8) \subset E_6$ it has vanishing projection over H^\perp . Then $e^{-V} B_{SO(9)}^{-1} dB_{SO(9)} e^V|_{H^\perp} = e^{-V} J_{SO(9)\setminus SO(8)} e^V|_{H^\perp}$. Thus, this term will contribute to the determinant with the term

$$A = \det e^{-V} J_{SO(9)\setminus SO(8)} e^V|_{H^\perp}.$$

To compute it we first note that $J_{SO(9)\setminus SO(8)} = \sum_{a=1}^8 J_{B_{SO(9)}}^a Y_{a+47}$ and that $Ad_{e^{-V}} Y_{a+47}|_{H^\perp}$, $a = 1, \dots, 8$ have non vanishing components only on the subspace generated by Y_{a+99} , $a = 1, \dots, 8$. This means that if we define the eight dimensional spaces

$$W_{47} := \sum_{a=1}^8 \mathbb{R} Y_{a+47}, \quad W_{99} := \sum_{a=1}^8 \mathbb{R} Y_{a+99}, \tag{E.2}$$

define the linear map

$$\rho : W_{47} \longrightarrow W_{99}, \quad Y \longmapsto e^{-V} Y e^V|_{H^\perp} \tag{E.3}$$

and define the matrix R associated to ρ by the bases $\{Y_{a+n}\}_{a=1}^8$, $n = 47, 99$ of W_n , respectively, then we can write $A = \det R \det J_{SO(9)\setminus SO(8)}$. Now, $\det R$ can be easily computed by means of a computer, whereas $\det J_{SO(9)\setminus SO(8)} = d\mu_{B_{SO(9)}}$ by construction and has yet been computed in [7]. The result is then

$$A = \sin^8 \left(\frac{x_2 - \sqrt{3}x_3}{2} \right) d\mu_{B_{SO(9)}}. \tag{E.4}$$

Moreover, the remaining terms have to be projected on $H_1^\perp := \text{Span}\{Y_3, Y_{83}, \dots, Y_{98}, Y_{108}, \dots, Y_{133}\}_{\mathbb{R}}$.

- Let us now consider the term $e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} dB_{F_4} B_{SO(9)} e^V$. The F_4 current $J_{B_{F_4}} := B_{F_4}^{-1} dB_{F_4}$ can be split in two orthogonal parts $J_{B_{F_4}} = J_{SO(9)} \oplus J_{F_4\setminus SO(9)}$, where $J_{SO(9)}$ is the projection of $J_{B_{F_4}}$ over $\text{Lie}(SO(9))$. The adjoint action of $SO(9)$ on $J_{SO(9)}$ has value in $\text{Lie}(SO(9))$ which under $Ad_{e^{-V}}$ has vanishing projection over H_1^\perp , as seen before.

On the other hand,

$$\Phi := Ad_{SO(9)} : \text{Lie}(F_4/SO(9)) \longrightarrow \text{Lie}(F_4/SO(9))$$

acts as an orthogonal map so that we have

$$\begin{aligned} B &:= \det(e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} dB_{F_4} B_{SO(9)} e^V)|_{H_1^\perp} = \det(e^{-V} \Phi(J_{F_4 \setminus SO(9)}) e^V)|_{H_1^\perp} \\ &= \det \Phi \det(e^{-V} J_{F_4 \setminus SO(9)} e^V)|_{H_1^\perp} = \det(e^{-V} J_{F_4 \setminus SO(9)} e^V)|_{H_1^\perp}. \end{aligned}$$

Now we can proceed as in the previous case: first, one can check that

$$J_{F_4 \setminus SO(9)} \in W_{24,39} = \text{Span}\{Y_{24+a}, Y_{39+a}, a = 1, \dots, 8\}_{\mathbb{R}}. \tag{E.5}$$

Then, one sees that

$$Ad_{e^{-V}}(W_{24,39})|_{H_1^\perp} \subseteq W_{82} := \text{Span}\{Y_{82+a}, a = 1, \dots, 16\}_{\mathbb{R}},$$

so that if we define the linear map

$$\rho' W_{24,39} \longrightarrow W_{82}, Y \longmapsto e^{-V} Y e^V|_{H_1^\perp}, \tag{E.6}$$

and M' is the associated matrix, then we get

$$B = \det M' \det J_{F_4 \setminus SO(9)} = \sin^8 x_2 \sin^8 \left(\frac{x_2 + \sqrt{3}x_3}{2} \right) d\mu_{B_{F_4}}, \tag{E.7}$$

where $d\mu_{B_{F_4}}$ has been computed in [7]. The remaining terms in J_p must be projected on $H_2^\perp := \text{Span}\{Y_3, Y_{108}, \dots, Y_{133}\}_{\mathbb{R}}$.

- The computation we have to consider is the contribution of the terms

$$e^{-V} U_1^{-1} dU_1 e^V + e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} B_{E_6}^{-1} dB_{E_6} B_{F_4} B_{SO(9)} e^V,$$

which we rewrite conveniently in the form

$$\begin{aligned} &e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} (J_{U(1)} \oplus J_{B_{E_6}}) B_{F_4} B_{SO(9)} e^V \\ &:= e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} (U_1^{-1} dU_1 + B_{E_6}^{-1} dB_{E_6}) B_{F_4} B_{SO(9)} e^V. \end{aligned}$$

As before, $J_{B_{E_6}}$ can be split in two orthogonal parts as $J_{B_{E_6}} = J_{F_4} \oplus J_{E_6 \setminus F_4}$, where J_{F_4} is the projection of $J_{B_{E_6}}$ over $\text{Lie}(F_4)$, and by construction

$$e^{-V} B_{SO(9)}^{-1} B_{F_4}^{-1} J_{F_4} B_{F_4} B_{SO(9)} e^V|_{H_2^\perp} = 0.$$

Moreover, Ad_{F_4} (and then $Ad_{SO(9)}$) acts as an orthogonal map on the space

$$W_{1,55} := \text{Span}\{Y_1, Y_{55+a}, a = 1, \dots, 26\}_{\mathbb{R}}$$

so that, with the same arguments as before, we can restrict to compute the term

$$C := \det(e^{-V}(U_1^{-1}dU_1 \oplus J_{E_6 \setminus F_4})e^V|_{H_2^\perp}). \tag{E.8}$$

If we construct the map

$$\rho'' : W_{1,55} \longrightarrow H_2^\perp, Y \longmapsto e^{-V}Ye^V|_{H_2^\perp} \tag{E.9}$$

and the natural associated matrix M'' , then we get

$$\begin{aligned} C &= \det M'' dx_1 \det J_{E_6 \setminus F_4} \\ &= \frac{1}{2^9} \left(\cos(2x_2) - \cos \frac{2\sqrt{2}x_1 + 2x_3}{\sqrt{3}} \right) \sin \frac{\sqrt{6}x_1 - 2\sqrt{3}x_3}{3} \\ &\quad \cdot \left(\cos x_2 - \cos \frac{2\sqrt{6}x_1 - \sqrt{3}x_3}{3} \right) dx_1 d\mu_{BE_6}, \end{aligned} \tag{E.10}$$

where $d\mu_{BE_6}$ is the measure computed in [8].

Collecting all the terms, we finally get $d\mu_{BE_7} = ABC$, which after application of the formulas of prosthapheresis give the result (D.6).

Appendix F Macdonald formulas

We can compute the volume of the group by means of the Macdonald formula, see [14]. We are using the invariant measure induced by an invariant scalar product on the algebra. The Cartan subalgebra \mathcal{C} is generated by the matrices $Y_1, Y_4, Y_9, Y_{18}, Y_{39}, Y_{56}, Y_{73}$. These matrices are orthonormal, so that the same holds true for the dual basis. In this normalization the roots have length $\sqrt{2}$. In particular, by diagonalizing ad_{Y_i} , $i = 1, 4, 9, 18, 39, 56, 73$ simultaneously, one sees that any choice of simple roots can be written in

the form

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 + \sqrt{2}\lambda_7), \\ \alpha_2 &= \lambda_1 + \lambda_2, \\ \alpha_3 &= \lambda_2 - \lambda_1, \\ \alpha_4 &= \lambda_3 - \lambda_2, \\ \alpha_5 &= \lambda_4 - \lambda_3, \\ \alpha_6 &= \lambda_5 - \lambda_4, \\ \alpha_7 &= \lambda_6 - \lambda_5, \end{aligned}$$

where λ_j is an orthonormal basis (w.r.t. the product induced by the one on \mathcal{C}). The coroots α_i^\vee then coincide with the roots, and the fundamental region, that is the fundamental torus generated on \mathcal{C} by the root lattice, has then volume

$$V_T = |\alpha_1 \wedge \cdots \wedge \alpha_7| = \sqrt{2}.$$

Moreover, the rational cohomology of E_7 is same of the product of seven spheres [17, 18]:

$$H(E_7; \mathbb{Q}) = H(S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}; \mathbb{Q}).$$

Applying the formula of Macdonald, we obtain

$$\text{Vol}(E_7) = V_T \prod_{i=1}^7 \text{Vol}(S^{d_i}) \prod_{\alpha \neq 0} |\alpha^\vee| = \frac{\sqrt{2} \times 2^{23} \pi^{70}}{3^{22} \times 5^{10} \times 7^6 \times 11^3 \times 13^2 \times 17}, \tag{F.1}$$

where S^{d_i} are the spheres appearing in the cohomology, and the second product is over all non vanishing roots.

In a similar way, we can compute the volume of the subgroups needed in the computations. In particular:

$$\text{Vol}(E_6) = \frac{\sqrt{3} 2^{17} \pi^{42}}{3^{10} \times 5^5 \times 7^3 \times 11}, \tag{F.2}$$

$$\text{Vol}(U) = \text{Vol}(E_6)\text{Vol}(U(1))/3 = \frac{\sqrt{2} 2^{18} \pi^{43}}{3^{10} \times 5^5 \times 7^3 \times 11}, \tag{F.3}$$

and then

$$\text{Vol}(E_7/U) = \frac{2^5 \pi^{27}}{3^{12} \times 5^5 \times 7^3 \times 11^2 \times 13^2 \times 17}. \tag{F.4}$$

Appendix G Computation of the integral

To compute the integral I in (D.15), we here compute the more general integral

$$I(a, b, c) = \int_0^1 dx \int_0^x dy \int_0^y dz (x - y)^{a-1} (y - z)^{b-1} (x - z)^{c-1}, \quad (G.1)$$

and we will get I from the the relation $I = 2^3 I(9, 9, 9)$. To this hand we will use the following useful representation of the hypergeometric function:

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)}{\Gamma(d)\Gamma(\gamma - d)} \int_0^1 dt t^{d-1} (1 - t)^{\gamma-d-1} \\ &\times F(\alpha, \beta; d; zt), \end{aligned} \quad (G.2)$$

which for $d = \alpha$ takes the form

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 dt t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} (1 - zt)^{-\beta}. \quad (G.3)$$

First, let us change the variables by introducing the new coordinates (x, s, t) such that $(x, y, z) = (x, xs, xst)$. This gives

$$I(a, b, c) = \frac{1}{a + b + c} \int_0^1 ds \int_0^1 dt s^b (1 - s)^{a-1} (1 - t)^{b-1} (1 - st)^{c-1}. \quad (G.4)$$

Next we can first integrate over t . By using (G.3), we obtain

$$\begin{aligned} I(a, b, c) &= \frac{1}{b} \frac{1}{a + b + c} \int_0^1 ds s^b (1 - s)^{a-1} F(1, 1 - c; b + 1; s) \\ &= \frac{1}{b} \frac{1}{a + b + c} \frac{\Gamma(a)\Gamma(b + 1)}{\Gamma(a + b + 1)} \left[\frac{\Gamma(a + b + 1)}{\Gamma(a)\Gamma(b + 1)} \int_0^1 ds s^b (1 - s)^{a-1} \right. \\ &\quad \left. \times F(1, 1 - c; b + 1; s) \right]. \end{aligned} \quad (G.5)$$

The therm in the square brackets has exactly the form (G.2), with $\alpha = 1, \beta = 1 - c, d = b + 1, \gamma = a + b + 1, z = 1$. Then we can write

$$I(a, b, c) = \frac{1}{b} \frac{1}{a + b + c} \frac{\Gamma(a)\Gamma(b + 1)}{\Gamma(a + b + 1)} F(1, 1 - c; a + b + 1; 1). \quad (G.6)$$

Finally, by using

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \tag{G.7}$$

and the property $z\Gamma(z) = \Gamma(z + 1)$, we find

$$I(a, b, c) = \frac{1}{a + b + c} \frac{1}{a + b + c - 1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}. \tag{G.8}$$

In particular:

$$I = 2^3 I(9, 9, 9) = \frac{2^2}{3^3 \times 13} \frac{(8!)^2}{17!} = \frac{2}{3^5 \times 5 \times 11 \times 13^2 \times 17}. \tag{G.9}$$

Appendix H Roots and range of parameters for the $E_{7(7)}$ construction

To our aim we need to perform a choice of positive roots w.r.t. the Cartan subalgebra $H = \langle D_\alpha \rangle_{\mathbb{R}}$. First, we can write

$$\begin{aligned} L = H \oplus \langle J^+ \rangle_{\mathbb{R}} \oplus \langle J^- \rangle_{\mathbb{R}} &:= H \oplus \left\langle \left\{ J_{kl}^+ = \frac{1}{\sqrt{2}}(-iS_{kl} + A_{kl}) \mid k < l \right\} \right\rangle_{\mathbb{R}} \\ &\oplus \left\langle \left\{ J_{kl}^- = \frac{1}{\sqrt{2}}(-iS_{kl} - A_{kl}) \mid k < l \right\} \right\rangle_{\mathbb{R}}, \\ \lambda^4 &:= \langle \mathcal{J} \rangle_{\mathbb{R}} := \langle \{ \mathcal{J}_I = \lambda_I \mid I \in \mathcal{I} \} \rangle_{\mathbb{R}}. \end{aligned}$$

where \mathcal{I} is the set of 4/indices.

Proposition 1. *The set $J^+ \cup J^- \cup \mathcal{J}$ diagonalizes simultaneously the adjoint action of H .*

Proof. By direct computation of the action of $[D_\alpha, J_{kl}^\pm]$ on e_{ij} and on ε^{ij} we get:

$$[D_\alpha, J_{kl}^\pm] = \pm i(D_\alpha^k - D_\alpha^l)J_{kl}^\pm, \tag{H.1}$$

where for $\mathcal{J}_I = \lambda_{i_1 i_2 i_3 i_4}$, using $\text{tr} D_\alpha = 0$, we obtain

$$[D_\alpha, \mathcal{J}_I] = i(D_\alpha^{i_1} + D_\alpha^{i_2} + D_\alpha^{i_3} + D_\alpha^{i_4})\mathcal{J}_I. \tag{H.2}$$

□

We now fix an explicit choice for a basis of H , suitable for our purposes:

$$\begin{aligned}
 D_1 &= \frac{1}{\sqrt{2}} \text{diag}\{1, -1, -1, 1, 0, 0, 0, 0\}, \\
 D_2 &= \frac{1}{\sqrt{2}} \text{diag}\{1, -1, 1, -1, 0, 0, 0, 0\}, \\
 D_3 &= \frac{1}{\sqrt{2}} \text{diag}\{1, 1, -1, -1, 0, 0, 0, 0\}, \\
 D_4 &= \frac{1}{\sqrt{2}} \text{diag}\{0, 0, 0, 0, 1, -1, -1, 1\}, \\
 D_5 &= \frac{1}{\sqrt{2}} \text{diag}\{0, 0, 0, 0, 1, -1, 1, -1\}, \\
 D_6 &= \frac{1}{\sqrt{2}} \text{diag}\{0, 0, 0, 0, 1, 1, -1, -1\}, \\
 D_7 &= \frac{1}{2} \text{diag}\{1, 1, 1, 1, -1, -1, -1, -1\}.
 \end{aligned}$$

Before continuing, let us recall the structure of the roots for the E_7 type algebras. Let R^+ the set of positive roots and L_i an orthonormal basis for $H_{\mathbb{R}}^*$ the real space spanned by all roots. Then, the positive roots system for an E_7 Lie algebra is (see [19], chapter 21, p. 333)

$$\begin{aligned}
 R^+ &= \{L_j + L_i\}_{i < j \leq 6} \cup \{L_j + L_i\}_{i < j \leq 6} \cup \{\sqrt{2}L_7\} \\
 &\cup \left\{ \frac{\pm L_1 \pm \dots \pm L_6 + \sqrt{2}L_7}{2} \right\}_{\text{odd number of sign-}}
 \end{aligned} \tag{H.3}$$

and in particular a choice of simple roots is

$$\begin{aligned}
 \alpha_1 &= \frac{L_1 - L_2 - L_3 - L_4 - L_5 - L_6 + \sqrt{2}L_7}{2}, & \alpha_2 &= L_1 + L_2, \\
 \alpha_3 &= L_2 - L_1, & \alpha_4 &= L_3 - L_2, & \alpha_5 &= L_4 - L_3, \\
 \alpha_6 &= L_5 - L_4, & \alpha_7 &= L_6 - L_5.
 \end{aligned} \tag{H.4}$$

We normalized the basis for H so that $D_\alpha \cdot D_\beta = 2\delta_{\alpha\beta}$. An orthonormal basis for $H_{\mathbb{R}}^*$ is thus provided by $L_\alpha(H_\beta) = \sqrt{2}\delta_{\alpha\beta}$. Let us then introduce the subset $\mathcal{I}_0 \subset \mathcal{I}$ such that $\mathcal{I} = \mathcal{I}_0 \cup \tilde{\mathcal{I}}_0$ defined as follows:

$$\mathcal{I}_0 = \{I \in \mathcal{I} : i_1, i_2, i_3 \in \{1, 2, 3, 4, 5\}\}. \tag{H.5}$$

In other words it is the set of ordered tetra-indices such that or all indices run from 1 to 5, or $i_7 \in 6, 7, 8$. Its cardinality is then $\binom{5}{4} + 3\binom{5}{3} = 35$. Set

$$\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^- := \{\lambda_I \in \mathcal{J} : I \in \mathcal{I}_0\} \cup \{\lambda_I \in \mathcal{J} : \tilde{I} \in \mathcal{I}_0, \epsilon_{I, \tilde{I}} = 1\}. \tag{H.6}$$

Proposition 2. *The set $\mathcal{J}^+ \cup \mathcal{J}^-$ consists of all eigenvectors associated to all positive roots of $\text{Lie}E_7$. The corresponding roots are*

$$\left\{ \beta_{kl} := \sum_{\alpha=1}^7 \frac{D_\alpha^k - D_\alpha^l}{\sqrt{2}} L_\alpha \right\}_{k < l} \cup \left\{ \beta_{i_1 i_2 i_3 i_4} := \sum_{\alpha=1}^7 \frac{D_\alpha^{i_1} + D_\alpha^{i_2} + D_\alpha^{i_3} + D_\alpha^{i_4}}{\sqrt{2}} L_\alpha \right\}_{i_1 i_2 i_3 i_4 \in \mathcal{I}_0}. \tag{H.7}$$

In particular the simple roots are

$$\begin{aligned} \alpha_1 &= \beta_{45}, & \alpha_2 &= \beta_{12}, & \alpha_3 &= \beta_{34}, & \alpha_4 &= \beta_{23}, & \alpha_5 &= \beta_{3458}, \\ \alpha_6 &= \beta_{78}, & \alpha_7 &= \beta_{67}. \end{aligned} \tag{H.8}$$

The proof is by direct inspection. Note that the order for the simple roots in the theorem is the same as in figure 1. This result is very helpful for computing the function $f = \det[\Pi \circ \text{Ad}_{e^{-V}} : \mathfrak{u} \rightarrow \mathfrak{t}]$. Indeed, from both Propositions we see that a basis for \mathfrak{u} is given by the matrices of the form

$$S = i \frac{J^+ + J^-}{\sqrt{2}},$$

whereas a basis for \mathfrak{t} is given by the elements of the form

$$S = \frac{J^+ - J^-}{\sqrt{2}}.$$

Using this and

$$\text{Ad}_{e^{\sum_a y^a D_a}} J_\beta^\pm = e^{\pm \sum_a y^a \beta(D_a)} J_\beta^\pm$$

for a given root β , one finally obtains

$$|f(\vec{y})| = \prod_{\beta \in \text{Rad}^+} \sin \left(\left| \sum_{a=1}^7 y^a \beta(D_a) \right| \right), \tag{H.9}$$

where Rad^+ is the set of positive roots w.r.t. V .

Appendix I Some details for the $E_{7(-5)}$ construction

Here we will specify the subgroup K of U_5 commuting with the torus e^{H_4} . To this end start we start by looking for the subalgebra \mathfrak{k} of \mathfrak{u}_5 commuting with H_4 . This can be done by means of Mathematica and gives

$$\mathfrak{k} = \langle M_1, \dots, M_9 \rangle_{\mathbb{R}}, \quad (\text{I.1})$$

with

$$M_1 = \frac{1}{2}(L_1 - L_9 + L_{21} - L_{25}), \quad (\text{I.2})$$

$$M_2 = \frac{1}{2}(L_2 + L_8 + L_{14} - L_{28}), \quad (\text{I.3})$$

$$M_3 = \frac{1}{2}(L_3 - L_7 + L_{19} + L_{27}), \quad (\text{I.4})$$

$$M_4 = \frac{1}{2}(L_4 - L_{13} + L_{20} + L_{23}), \quad (\text{I.5})$$

$$M_5 = \frac{1}{2}(L_5 - L_{15} + L_{18} - L_{22}), \quad (\text{I.6})$$

$$M_6 = \frac{1}{2}(L_{10} - L_{12} + L_{16} + L_{24}), \quad (\text{I.7})$$

$$M_7 = \frac{1}{\sqrt{2}}(L_{45} + L_{46}), \quad (\text{I.8})$$

$$M_8 = L_{49}, \quad (\text{I.9})$$

$$M_9 = L_{50}. \quad (\text{I.10})$$

These generate an algebra $\mathfrak{so}(4) \oplus \mathfrak{su}(2)$ whose exponentiation gives the group $K^0 = \text{Spin}(4) \times \text{SU}(2)$. A general analysis, which will be presented in [20], shows that K contains an extra \mathbb{Z}_2^2 factor so that

$$\text{Spin}(4) \times \text{SU}(2) \times \mathbb{Z}_2^2. \quad (\text{I.11})$$

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