ADV. THEOR. MATH. PHYS.
Volume 22, Number 3, 509–564, 2018

SYZ transformation for coisotropic A-branes

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Kapustin and Orlov observed that natural boundary conditions in A-model are coisotropic A-branes, and also they need to be included for mirror symmetry.

In the SYZ conjecture, the transformation which takes a holomorphic bundle E in $\check{X}$ to a Lagrangian A-brane in its mirror manifold X uses the property that the restriction of E to any Lagrangian torus fiber in $\check{X}$ is topologically trivial.

In the semiflat setting, without assuming that E is fiberwise topologically trivial, we construct a SYZ transformation which takes holomorphic bundles in $\check{X}$ to coisotropic A-branes in X and vice versa. The construction uses fiberwise Nahm transformations for twisted Dirac operators on tori.

1. introduction

Inspired by string theory, it is conjectured that the symplectic geometry (A-model) of a Calabi-Yau manifold $\check{X}$ is equivalent to the complex geometry (B-model) of a mirror Calabi-Yau manifold X and vice versa, which is known as the mirror symmetry phenomenon. One mathematical formulation of this proposed by Kontsevich in [16] is the homological mirror symmetry (HMS) conjecture, which interprets mirror symmetry as the equivalence of the derived Fukaya category $D\mathcal{F}uk(\check{X})$ of $\check{X}$ and the derived category of coherent sheaves $D^b(X)$ of the mirror X. Objects of $D\mathcal{F}uk(\check{X})$ are roughly speaking Lagrangian submanifolds carrying flat unitary bundles (Lagrangian A-branes) while objects of $D^b(X)$ are bounded complexes of coherent sheaves (B-branes). From physical perspectives, branes are boundary conditions for strings. Kapustin and Orlov pointed out in [15] that, in general, extra objects called coisotropic A-branes should be added to the Fukaya category for the HMS conjecture to be true. A coisotropic A-brane is roughly a coisotropic submanifold whose leaf space admits a holomorphic symplectic structure. In particular, coisotropic A-branes on a four-torus was studied in [1].

Strominger-Yau-Zaslow [26] proposed a more geometric explanation to mirror symmetry which is known as the SYZ conjecture. It asserts that, for a pair of mirror Calabi-Yau manifolds X and $\check{X}$, there exist special Lagrangian torus fibrations $p : X \to B$ and $\check{p} : \check{X} \to B$ over the same base manifold B which are fiberwise dual to each other, at least in the large complex structure (volume) limit. It was studied extensively, such as [8, 9, 11, 17–19, 22].

Moveover, it is conjectured that HMS can be revealed by the SYZ approach in the sense that $D\mathcal{F}uk(\check{X})$ and $D^b(X)$ are exchanged by a fibrewise Fourier-type transformation. The construction of the Fourier transformation is based on the natural identification between the dual torus $\check{\mathbf{T}}$ and the moduli space of flat $U(1)$-bundles over $\mathbf{T}$. Suppose we are given a holomorphic line bundle $\mathcal{E} \to X$ which is flat along fibers of $p : X \to B$. Then its restriction to each fiber is a flat $U(1)$-bundle over $\mathbf{T}$ and it becomes a point in $\check{\mathbf{T}}$ under the Fourier transformation. Therefore, family Fourier transformation takes $\mathcal{E}$ to a section of $\check{p} : \check{X} \to B$ which can be shown to be a Lagrangian submanifold of $\check{X}$. This program has been carried out successfully in the semi-flat case, namely no singular fiber appears in those torus fibrations [2, 6, 21, 23, 24]. In fact, this approach can be further extended beyond the semi-flat case [3, 7, 10, 12, 13].

However, a coisotropic A-brane appears as mirror when $\mathcal{E}$ is no longer fiberwise flat. Therefore, we need to define the SYZ transformation which is a generalization of the fiberwise Fourier transformation. It is motivated by the Nahm transformation which was used to transform anti-self-dual bundles on flat four-tori in [5] and [25]. The basic idea is to use spinor bundles and a family version of kernels of Dirac operators to incorporate coisotropic A-branes into the SYZ picture.

Furthermore, at the large complex structure (volume) limit, A- and B-branes are conjectured to be families of Yang-Mills bundles over semi-flat submanifolds. It leads us to consider the class of semi-affine branes (see Definition 10). Our main theorem says that in the semi-flat case, if we relax the fiberwise flatness condition to fiberwise Yang-Mills, then SYZ transformation still works precisely if we include coisotropic A-branes.

Theorem 1 (Main Theorem). *The SYZ transformation exchanges semi-affine coisotropic A-branes and B-branes between a semi-flat Calabi-Yau manifold* $\check{X}$ *and its mirror manifold* X.

Acknowledgement. The authors thank Kwokwai Chan, Siu-Cheong Lau and Ziming Nikolas Ma for useful discussions. The work of the second author was substantially supported by a grant from the Research Grants Council of the Hong Kong Council of the Hong Kong Special Administrative Region, China (Project No. CUHK402012).

2. Construction of transformation

2.1. SYZ mirror symmetry for semi-flat Calabi-Yau manifolds

In this section, we will briefly review the SYZ mirror symmetry for the semi-flat Calabi-Yau manifolds and the construction of the Fourier transformation on tori which appears in [21, 23]. Details can be also found in [6].

2.1.1. Semi-flat Calabi-Yau manifolds. Let $M = \mathbb{Z}^n$ be a lattice and $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual lattice. Define $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Let B be an affine manifold such that all transition maps are in $M_\mathbb{R} \rtimes \mathrm{SL}(n, \mathbb{Z})$. We can construct a pair $(X, \check{X})$ of mirror Calabi-Yau manifolds from the tangent and cotangent bundles of B as follows:

Construction of X

Let $(x_1, \ldots, x_n)$ be local affine coordinates of B which induces fiber coordinates $(y_1, \ldots, y_n)$ of the tangent bundle TB via the base $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Since the transition maps of the affine manifold B lie in $M_\mathbb{R} \rtimes \mathrm{SL}(n, \mathbb{Z})$, we have a lattice bundle $\Lambda \subset TB$ which is generated by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. We define X to be TB/Λ and then

$$p : X \to B$$

is a torus fibration over B. Also $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ gives a set of local coordinates on X which is called a *semi-flat coordinate system*. Furthermore, X is a complex manifold with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$ and a holomorphic volume form $\Omega_X := dz_1 \wedge \ldots \wedge dz_n$.

If $\phi : B \to \mathbb{R}$ is a convex function, then

$$g_X = \sum_{1\le i,j\le n} \phi^{ij} \left(dx_i \otimes dx_j + dy_i \otimes dy_j\right)$$

defines a Kähler metric on X and the corresponding Kähler form is

$$\omega_X = \sum_{1\le i,j\le n} \phi^{ij} dx_i \wedge dy_j = \frac{\sqrt{-1}}{2} \sum_{1\le i,j\le n} \phi^{ij} dz_i \wedge d\bar{z}_j,$$

where $\phi^{ij} := \frac{\partial^2\phi}{\partial x_i \partial x_j}$.

Construction of $\check{X}$

To construct $\check{X}$ which is the mirror of X, we consider the cotangent bundle T^*B with coordinates $(x_1, \ldots, x_n; y^1, \ldots, y^n)$, where $(y^1, \ldots, y^n)$ are fiber coordinates induced by $dx_1, \ldots, dx_n$. Also, $dx_1, \ldots, dx_n$ gives rise to a lattice bundle $\Lambda^* \subset T^*B$. As a consequence, if we define $\check{X}$ to be T^*B/Λ^*, we get the dual torus fibration

$$\check{p} : \check{X} \to B$$

over B. Also $(x_1, \ldots, x_n; y^1, \ldots, y^n)$ gives a set of coordinates on $\check{X}$ which is called the mirror semi-flat coordinate system. Moreover, $\check{X}$ carries a symplectic structure $\omega_{\check{X}} = \sum_{j=1}^n dx_j \wedge dy^j$.

Furthermore, if $\phi : B \to \mathbb{R}$ is a convex function, then $\check{X}$ is a Kähler manifold with complex coordinates $z^i = \phi^{ij} x_j + \sqrt{-1}\, y^i$ and Kähler metric

$$g_{\check{X}} = \sum_{1\le i,j\le n} \phi^{ij} dx_i \otimes dx_j + \sum_{1\le i,j\le n} \phi_{ij} dy_i \otimes dy_j,$$

where $(\phi_{ij}) = (\phi^{ij})^{-1}$, such that the Kähler form is exactly $\omega_{\check{X}}$.

In summary, we get a pair of torus fibrations $p : X \to B$ and $\check{p} : \check{X} \to B$ which are dual to each other over the same affine manifold B. Furthermore, if ϕ satisfies the real *Monge-Ampére* equation:

$$\det\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right) = \text{constant},$$

then one can check easily that X and $\check{X}$ are Calabi-Yau manifolds and we call them a mirror pair of semi-flat Calabi-Yau manifolds.

2.1.2. Fourier transformation on tori. Let V be a real n-dimensional vector space and Λ be a lattice in V which gives rise to a torus $\mathbf{T} := V/\Lambda$. Let $\check{V}$ and $\check{\Lambda}$ be the dual vector space of V and the dual lattice of Λ respectively, then we can define the dual torus $\check{\mathbf{T}} := \check{V}/\check{\Lambda}$. Observe that the moduli space $\mathcal{M}_{\mathbf{T}}$ of flat $U(1)$-bundles over $\mathbf{T}$ can be identified with $\check{\mathbf{T}}$ as follows: Given any point $\check{y} \in \check{\mathbf{T}}$ with a representative $\check{\mathbf{y}} \in \check{V}$, then $\check{\mathbf{y}}$ can be regarded as a constant one-form on $\mathbf{T}$. No matter which representative is chosen,

$$d - 2\pi\sqrt{-1}\,\check{\mathbf{y}}$$

defines the same connection on the trivial complex line bundle over $\mathbf{T}$. We denote the connection and the bundle associated by $\nabla_{\check{y}}$ and $L_{\check{y}}$ respectively. Then, $L_{\check{y}}$ is a flat $U(1)$-bundle and the identification of $\check{\mathbf{T}}$ and $\mathcal{M}_{\mathbf{T}}$ is given by

$$\check{y} \longleftrightarrow L_{\check{y}}.$$

In fact, this identification can be encoded in the Poincaré line bundle $P \to \mathbf{T} \times \check{\mathbf{T}}$, which is a line bundle possessing an universal property that $P|_{\mathbf{T}\times\{\check{y}\}} \cong L_{\check{y}}$ for any point $\check{y} \in \check{\mathbf{T}}$. The bundle P can be constructed as the following: We first choose a linear coordinate system $(y_1, \ldots, y_n)$ of V and its dual coordinates $(y^1, \ldots, y^n)$ of $\check{V}$, which give local coordinates of $\mathbf{T}$ and $\check{\mathbf{T}}$. Consider the trivial bundle

$$L := \mathbf{T} \times \check{V} \times \mathbb{C} \to \mathbf{T} \times \check{V}$$

with connection

$$\nabla^L := d - 2\pi\sqrt{-1}\sum_{i=1}^{n} y^i dy_i,$$

then we define a $\check{\Lambda}$-action on L:

$$\check{\lambda}\cdot(y, \check{\mathbf{y}}, v) = (y, \check{\mathbf{y}} + \check{\lambda}, e^{-2\pi\sqrt{-1}\,\langle \check{n}, y\rangle} v),$$

where $\check{\lambda} \in \check{\Lambda}$, $y \in \mathbf{T}$, $\check{\mathbf{y}} \in \check{V}$ and $v \in \mathbb{C}$. It is easy to check that this action preserves the connection ∇^L on L. Therefore, the quotient bundle

$$P := L/\check{\Lambda} \to \mathbf{T} \times \check{\mathbf{T}} \tag{2.1}$$

is a well defined $U(1)$-bundle over the product $\mathbf{T} \times \check{\mathbf{T}}$ with the connection ∇^P descended from ∇^L. This bundle is called the Poincaré line bundle. The curvature of P is given by

$$2\pi\sqrt{-1}\,F^P = 2\pi\sum_{i=1}^{n}\sqrt{-1}\,dy_i\wedge dy^i. \tag{2.2}$$

Let π and $\breve{\pi}$ be the projections of $\mathbf{T}\times\check{\mathbf{T}}$ on $\mathbf{T}$ and $\check{\mathbf{T}}$ respectively. We can define the Fourier transformation of differential forms between dual tori by using F^P as follows:

Definition 2. The Fourier transformation for differential forms between dual tori

$$\mathbf{F}:\Omega^*(\mathbf{T})\to\Omega^{n-*}(\check{\mathbf{T}})$$

is defined by

$$\mathbf{F}(\alpha) := \breve{\pi}_*(\pi^*(\alpha)\wedge e^{-F^P}) = \int_{\mathbf{T}}\pi^*(\alpha)\wedge e^{-F^P}.$$

Remark 3. The Fourier transformation $\mathbf{F}$ in fact descends to be an isomorphism

$$\mathbf{F}:H^*(\mathbf{T})\to H^{n-*}(\check{\mathbf{T}}).$$

Furthermore, we can use the Poincaré line bundle P to construct the Fourier transformation for flat branes on tori.

Definition 4. A flat brane (C,E) on a torus $\mathbf{T}$ is a pair of affine subtorus C of $\mathbf{T}$ and a flat unitary bundle E over C. We denote the set of all flat branes on $\mathbf{T}$ by $\mathcal{B}_0(\mathbf{T})$.

Let (C,E) be a flat brane in $\mathbf{T}$ with a flat unitary connection ∇^E. For each point $\check{y}$ in $\check{\mathbf{T}}$, we define $\check{E}_{\check{y}}$ to be the vector space of flat sections of the bundle $\pi^*E\otimes P|_{C\times\{\check{y}\}}$ with respect to the connection $\pi^*\nabla^E\otimes\nabla^P$. We also define

$$\check{C} := \{\check{y}\in\check{\mathbf{T}}|\ \check{E}_{\check{y}}\neq 0\}\subset\check{\mathbf{T}}.$$

It can be shown that $\check{C}$ is an affine subtorus in $\check{T}$ and $\check{E} := \bigsqcup_{\check{y}\in\check{C}}\check{E}_{\check{y}}$ defines a flat unitary bundle over $\check{C}$. Then, we have the following definition:

Definition 5. The Fourier transformation

$$\mathcal{F}:\mathcal{B}_0(\mathbf{T})\to\mathcal{B}_0(\check{\mathbf{T}})$$

is defined by $\mathcal{F}(C,E) = (\check{C},\check{E})$.

In fact, the flat unitary bundle E can be split orthogonally into a direct sum of flat unitary line bundles. In case that E is a flat unitary bundle, the Fourier transformation can be explicitly written down in local coordinates. If the lifting of an affine subtorus C in V is given by

$$\{\mathbf{y} \in V : y_{k+1} = b_{k+1}, \ldots, y_n = b_n\} \tag{2.3}$$

and

$$\nabla^E = d - 2\pi\sqrt{-1}\sum_{j=1}^{k} b^j dy_j \tag{2.4}$$

with respect to some trivialization of E, then $\check{C}$ can be described by the lifting

$$\{\check{\mathbf{y}} \in \check{V} : y^1 = b^1, \ldots, y^k = b^k\} \tag{2.5}$$

and the flat $U(1)$-connection $\nabla^{\check{E}}$ on $\check{E}$ is

$$\nabla^{\check{E}} = d - 2\pi\sqrt{-1}\sum_{j=n-k+1}^{n} b_j dy^j \tag{2.6}$$

with respect to some suitable trivialization of $\check{E}$.

Remark 6. From (2.3) and (2.5), it can be seen easily that C and $\check{C}$ are of complementary dimensions.

This transformation for flat branes in fact descends to the Fourier transformation $\mathbf{F}$ under the Chern character map ch:

$$\begin{array}{ccc} \mathcal{B}_0(\mathbf{T}) & \xrightarrow{\mathcal{F}} & \mathcal{B}_0(\check{\mathbf{T}}) \\ \big\downarrow ch & & \big\downarrow ch \\ H^*(\mathbf{T}) & \xrightarrow{\mathbf{F}} & H^{n-*}(\check{\mathbf{T}}) \end{array}$$

Conversely, one can also define the map

$$\check{\mathcal{F}} : \mathcal{B}_0(\check{\mathbf{T}}) \longrightarrow \mathcal{B}_0(\mathbf{T})$$

by regarding $\check{\check{\mathbf{T}}} = \mathbf{T}$. Then, $\mathcal{F} \circ \check{\mathcal{F}}$ and $\check{\mathcal{F}} \circ \mathcal{F}$ are identity maps.

2.1.3. Fiberwise Fourier transformation for semi-flat Calabi-Yau manifolds. Let X and $\check{X}$ be a pair of mirror semi-flat Calabi-Yau manifolds. Then the fiber product $X \times_B \check{X}$ can be regarded as a family of product of $\mathbf{T} \times \check{\mathbf{T}}$ parametrized over the base B. In particular, branes that are families of flat branes on tori are called semi-flat branes. Under fiberwise Fourier transformation, semi-flat A- and B-branes in X are transformed to semi-flat B- and A-branes in $\check{X}$ respectively.

For instance, if $\mathcal{E}$ is a fiberwise flat unitary bundle over X, then $(X, \mathcal{E})$ is a semi-flat B-brane in X. From Remark 6, since the restriction of X on any fiber torus is the whole fiber, the corresponding A-brane obtained must be a Lagrangian section with a fiberwise flat unitary bundle over it. However, Kapustin and Orlov in [15] constructed an example of B-brane which is a four-torus with a non-fiberwise flat $U(1)$-bundle, but the corresponding A-brane is the mirror four-torus which is coisotropic (see example 20). Therefore, in order to include this case, we have to generalize the Fourier transformation in Definition 2.1.2.

2.2. SYZ transformation on tori

In this section, we are going to construct the SYZ transformation which is motivated by the Nahm transformation (see [5] and [25]) Simply speaking, besides twisting the Poincaré bundle, a spinor bundle is also twisted to give extra information.

Definition 7. A constant curvature brane (or simply brane) on a torus $\mathbf{T}$ is a pair (C, E), where C is an affine subtorus of $\mathbf{T}$ and E is a projectively flat unitary bundle over C such that the curvature is $2\pi\sqrt{-1}\, F^E \cdot I_E$, where F^E is a constant real two form on C. The set of all branes on a torus $\mathbf{T}$ is denoted by $\mathcal{B}(\mathbf{T})$.

A flat brane on $\mathbf{T}$ may be regarded as a brane on $\mathbf{T}$ with $F^E = 0$ and hence $\mathcal{B}_0(\mathbf{T}) \subset \mathcal{B}(\mathbf{T})$. Note that

$$V_0 := \{X \in TC : \iota_X F^E = 0 \in T^*C\}$$

defines a subbundle of TC. The metric on $\mathbf{T}$ induces a metric on C and so TC can be decomposed as $TC = V_0 \oplus V_0^{\perp}$, where $V_0^{\perp}$ is the orthogonal complement of V_0. Then $V_0^{\perp}$ associates a spinor bundle $\mathcal{S}$ over C. Furthermore, the restriction of the Levi-Civita connection of TC on $V_0^{\perp}$ induces a connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$. As a result, $\nabla^{\mathcal{S}\otimes E} := \nabla^{\mathcal{S}} \otimes \nabla^E$ defines a connection on

$\mathcal{S} \otimes E$ over C, where ∇^E is the unitary connection of E. We define $\Gamma(\mathcal{S} \otimes E)$ to be the space of sections of $\mathcal{S} \otimes E$ and define

$$\Gamma(\mathcal{S} \otimes E)^{V_0} := \{s \in \Gamma(\mathcal{S} \otimes E) : \nabla_X^{\mathcal{S}\otimes E} s = 0 \text{ for all } X \in \Gamma(V_0)\}$$

to be the space of smooth invariant sections along V_0, where $\Gamma(V_0)$ is the space of sections of V_0. Furthermore, by using $\nabla^{\mathcal{S}\otimes E}$, we associate the Dirac operator

$$\mathcal{D} : \Gamma(\mathcal{S} \otimes E)^{V_0} \to \Gamma(\mathcal{S} \otimes E)^{V_0}.$$

We claim that $\ker \mathcal{D}$ is finite dimensional, see Proposition 21. Following the idea of the construction of the Fourier transformation, if $\ker \mathcal{D}$ is nontrivial, we twist the pullback of $\mathcal{S} \otimes E$ over $C \times \check{\mathbf{T}} \subset \mathbf{T} \times \check{\mathbf{T}}$ with the Poincaré bundle defined by (2.1), then for any $\check{y} \in \check{\mathbf{T}}$, we construct the induced Dirac operator

$$\mathcal{D}_{\check{y}} : \Gamma(\pi^*(\mathcal{S} \otimes E) \otimes P|_{C\times\{\check{y}\}})^{V_0} \to \Gamma(\pi^*(\mathcal{S} \otimes E) \otimes P|_{C\times\{\check{y}\}})^{V_0}, \tag{2.7}$$

where π and $\check{\pi}$ are projections of $\mathbf{T} \times \check{\mathbf{T}}$ on $\mathbf{T}$ and $\check{\mathbf{T}}$ respectively. We claim that the $\ker \mathcal{D}_{\check{y}}$ is finite dimensional. Furthermore,

$$\check{C} := \{\check{y} \in \check{\mathbf{T}} : \ker \mathcal{D}_{\check{y}} \text{ is nontrival}\}$$

defines an affine subtorus of $\check{\mathbf{T}}$ and

$$\check{E} := \bigsqcup_{\check{y}\in\check{C}} \ker \mathcal{D}_{\check{y}}$$

defines an projectively flat unitary bundle over $\check{C}$. Hence, $(\check{C}, \check{E})$ is a brane in $\check{\mathbf{T}}$ and we can define

Definition 8. The SYZ-transformation $\mathcal{F}^{SYZ} : \mathcal{B}(\mathbf{T}) \to \mathcal{B}(\check{\mathbf{T}})$ is defined by

$$\mathcal{F}^{SYZ}(C, E) = (\check{C}, \check{E}).$$

Remark 9. In particular, if (C, E) is a flat brane on torus, then $F^E = 0$. In this case, $V_0 = TC$ and so $\mathcal{S} \otimes E$ is just E. Then, we have

$$\mathcal{F}^{SYZ}(C, E) = \mathcal{F}(C, E),$$

and hence the SYZ transformation in Definition 8 can be regarded as a generalization of the Fourier transformation in Definition 5.

2.3. Fiberwise SYZ transformation for semi-flat Calabi-Yau manifolds

As the SYZ transformation is a generalization of the Fourier transformation, it is expected that the family version of the SYZ transformation is able to transform a larger class of branes, which are called semi-affine branes.

Let X be a semi-flat Calabi-Yau manifold and $p : X \to B$ is a torus fibration over an affine manifold B.

Definition 10. A semi-affine brane is a pair $(\mathcal{C}, \mathcal{E})$, where

1) $\mathcal{C}$ is a submanifold in X such that $p|_{\mathcal{C}} : \mathcal{C} \to p(\mathcal{C})$ is a torus bundle with each fiber over $x \in p(\mathcal{C})$ being an affine subtorus of $p^{-1}(x)$.
2) $\mathcal{E}$ is a projectively flat unitary bundle over $\mathcal{C}$ such that the curvature of $\mathcal{E}$ is constant along any fiber of the fibration $p|_{\mathcal{C}} : \mathcal{C} \to p(\mathcal{C})$.

The set of all semi-affine branes in X is denoted by $\mathcal{B}(X)$.

Basically, a semi-affine brane is a family of branes in Definition 7. Therefore, the family version of the transformation in Definition 2.3 associates the SYZ transformation

$$\mathcal{F}^{SYZ} : \mathcal{B}(X) \to \mathcal{B}(\check{X}),$$

see Theorem 26. Furthermore, we can show that $(\mathcal{F}^{SYZ})^2$ is the identity map, see Theorem 38. Comparing to the Fourier transformation, the SYZ transformation incorporates the spinor bundle. It is worth to note that even the spinor bundle is trivial along torus fibers, it is a nontrivial bundle over the semi-affine branes which provides extra information to transform a larger class of objects.

Recall that if (X, ω) is a symplectic manifold and $\mathcal{C}$ is a submanifold of X, we define $T\mathcal{C}^\omega$ to be the orthogonal complement of $T\mathcal{C}$ in TX with respect to the symplectic structure ω. The submanifold $\mathcal{C}$ is said to be Lagrangian if $T\mathcal{C}^\omega = T\mathcal{C}$ and coisotropic if $T\mathcal{C}^\omega \leq T\mathcal{C}$. For a coisotropic submanifold $\mathcal{C}$, the subbundle $T\mathcal{C}^\omega \leq T\mathcal{C}$ is an integrable distribution and hence it induces a foliation for $\mathcal{C}$ by the Frobenius theorem. We call $T\mathcal{C}^\omega$ the tangent bundle of this foliation and

$$\mathcal{N}\mathcal{C} := T\mathcal{C}/T\mathcal{C}^\omega$$

the normal bundle of this foliation. Note that the symplectic structure ω induces an invertible bundle map $\omega : \mathcal{N}\mathcal{C} \to \mathcal{N}\mathcal{C}^*$ with inverse ω^{-1}.

Definition 11. A coisotropic A-brane on a symplectic manifold (X, ω) is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a coisotropic submanifold of X and $\mathcal{E}$ is a unitary line bundle over it, such that

1) The curvature two-form F of $\mathcal{E}$, regarded as a bundle map $F : T\mathcal{C} \to T\mathcal{C}^*$, annihilates $T\mathcal{C}^\omega$. This induces a bundle map $F : \mathcal{N}\mathcal{C} \to \mathcal{N}\mathcal{C}^*$.
2) The composition $J := \omega^{-1} \circ F$ gives a complex vector bundle structure on $\mathcal{N}\mathcal{C}$, i.e., $J^2 = -I$.

Remark 12. In general, if we consider the mirror brane in $\check{X}$ of a brane in X which is not semi-flat, then the ranks of the attached bundles may be different. Therefore, we slightly generalize the definition given by Kapustin and Orlov in [15] and allow $\mathcal{E}$ to be a projectively flat unitary bundle.

Definition 13. A B-brane on a complex manifold X is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a submanifold of X and $\mathcal{E}$ is a unitary bundle over $\mathcal{C}$ satisfying that

1) $\mathcal{C}$ is a complex submanifold, of X;
2) $\mathcal{E}$ is a projectively flat unitary bundle satisfying that the $(0,1)$-part of its connection $\nabla^{\mathcal{E}}$ gives a holomorphic structure on $\mathcal{E}$.

With the above definitions, we are ready to state the main result of this paper as follows

Theorem 14 (Main Theorem). *The fiberwise SYZ transformation $\mathcal{F}^{SYZ}$ transforms a semi-affine coisotropic A-brane to a semi-affine B-brane and vice versa.*

Remark 15. Following from Remark 9, the SYZ transformation is a generalization of the Fourier transformation which also transforms a semi-flat Lagrangian A-brane to a semi-flat B-brane and vice versa.

3. Explicit computations

In fact, the SYZ transformation on a pair of dual tori or mirror semi-flat Calabi-Yau manifolds can be written down explicitly by choosing normalized coordinates as we explain below.

3.1. SYZ transformation on tori

Let V be a n-dimensional real vector space and Λ be a lattice in V. Then the quotient V/Λ gives a torus $\mathbf{T}$. Let (C, E) be a brane in the torus $\mathbf{T}$. Note that E is a projectively flat unitary bundle over the affine subtorus C such that the curvature is $2\pi\sqrt{-1}\,F^E \cdot I_E$, where $F^E \in \Omega^2(C, \mathbb{R})$. Since F^E can be regarded as a two form on the universal cover $\tilde{C}$ of C,

$$\tilde{C}_0 := \{w \in \tilde{C} : \iota_w F^E = 0\} \tag{3.1}$$

defines an affine subspace of V and $\tilde{C}_0/(\tilde{C}_0 \cap \Lambda)$ defines a subtorus C_0 of C. Note that the codimension of C_0 in C must be even, we can choose coordinates of $\mathbf{T} \supset C \supset C_0$ as

$$(\mathbf{u}, \mathbf{y}, \mathbf{v}) = (u_1, \ldots, u_s, y_1, \ldots, y_{2r}, v_1, \ldots, v_k) \tag{3.2}$$

such that

$$C = \{(\mathbf{u}, \mathbf{y}, \mathbf{v}) : \mathbf{u} = \mathbf{b}\} \tag{3.3}$$

for some $\mathbf{b} = (b_1, \ldots, b_s) \in \mathbb{R}^s$ and

$$\nabla^E = d + 2\pi\sqrt{-1}\left(-\check{\mathbf{b}}\, d\mathbf{v}^T + \frac{1}{2}\mathbf{y}Ad\mathbf{y}^T\right) \cdot I_E, \tag{3.4}$$

where $\check{\mathbf{b}} = (b^1, \ldots, b^k) \in \mathbb{R}^k$, $d\mathbf{v} = (dv_1, \ldots, dv_k)$, $d\mathbf{y} = (dy_1, \ldots, dy_{2r})$ and

$$A = \operatorname{diag}\left\{\begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & a_r \\ -a_r & 0 \end{bmatrix}\right\} \tag{3.5}$$

which is a $(2r) \times (2r)$-matrix with the listed block matrices on the diagonal and $a_1, \ldots, a_r$ are nonzero real numbers. Then we have

$$F^E = \frac{1}{2} d\mathbf{y} \wedge A \wedge d\mathbf{y}^T. \tag{3.6}$$

Similarly, we can write down the dual brane $(\check{C}, \check{E})$ by choosing a set of normalized coordinates explicitly:

Proposition 16. *The dimensions of $\check{C}$ and $\check{C}_0$ are given by*

$$\dim \check{C}_0 = \operatorname{codim}(C) \qquad \textit{and} \qquad \dim C_0 = \operatorname{codim}(\check{C}).$$

Furthermore, there exists a set of normalized coordinates of $\check{\mathbf{T}} \supset \check{C} \supset \check{C}_0$ *as*

$$(\check{\mathbf{u}}, \check{\mathbf{y}}, \check{\mathbf{v}}) = (u^1, \ldots, u^k, y^1, \ldots, y^{2r}, v^1, \ldots, v^s) \tag{3.7}$$

such that

$$\check{C} = \{(\check{\mathbf{u}}, \check{\mathbf{y}}, \check{\mathbf{v}}) : \check{\mathbf{u}} = \check{\mathbf{b}}\} \tag{3.8}$$

$$\nabla^{\check{E}} = d + 2\pi\sqrt{-1}\left(-\mathbf{b}\, d\check{\mathbf{v}} + \frac{1}{2}\check{\mathbf{y}} A^{-1} d\check{\mathbf{y}}^T\right) \cdot I_{\check{E}} \tag{3.9}$$

and so

$$F^{\check{E}} = \frac{1}{2} d\check{\mathbf{y}} \wedge A^{-1} \wedge d\check{\mathbf{y}}^T. \tag{3.10}$$

Moreover, by Proposition 21, the rank of $\check{E}$ equal to

$$\int_C ch(E).$$

Example 17. Let $\mathbf{T}$ be a two dimensional torus and let (C, E) be a brane in $\mathbf{T}$ such that $C = \mathbf{T}$ and E is a $U(1)$-bundle over C with connection

$$\nabla^E = d + \pi\sqrt{-1}\,(y_1 dy_2 - y_2 dy_1).$$

Then, we have

$$F^E = dy_1 \wedge dy_2,$$

In this case, $C_0 = \{*\}$ and $\check{C} = \check{\mathbf{T}}$. Also $\check{E}$ is a $U(1)$-bundle over $\check{C}$ with

$$\nabla^{\check{E}} = d + \pi\sqrt{-1}\,(-y^1 dy^2 + y^2 dy^1)$$

and

$$F^{\check{E}} = -dy^1 \wedge dy^2.$$

Example 18. If the connection ∇^E in example 17 is changed such that

$$F^E = 2dy_1 \wedge dy_2,$$

then $\check{C}$ is still the dual torus $\check{\mathbf{T}}$. However, $\check{E}$ is a projectively flat unitary bundle with rank 2 and

$$F^{\check{E}} = -\frac{1}{2}dy^1 \wedge dy^2.$$

3.2. SYZ transformation on semi-flat Calabi-Yau manifolds

Let X and $\check{X}$ be a pair of mirror semi-flat Calabi-Yau manifolds, where $p : X \to B$ and $\check{p} : \check{X} \to B$ are torus and dual torus fibrations over an affine manifold B, see the construction in Section 2.1.1.

Let $(\mathcal{C}, \mathcal{E})$ be a semi-affine brane in a semi-flat Calabi Yau manifold X. In order to describe $\mathcal{C}$ locally, we choose an open set $U \cong \mathbb{R}^m$ such that $U \subset p(\mathcal{C}) \subset B$ and we define $\mathbf{T}_x := p^{-1}(x)$ to be the fiber torus over $x \in U$. From the Definition 10, $\mathcal{C}_x := \mathbf{T}_x \cap \mathcal{C}$ is an affine subtorus in $\mathbf{T}_x$ and

$$\mathcal{C}_U = \bigsqcup_{x \in U} \mathcal{C}_x \cong U \times C,$$

where C is a torus.

Furthermore, the restriction of $F^{\mathcal{E}}$ on $\mathcal{C}_x$ is a constant two form, so it defines a subtorus $\mathcal{C}_{0,x}$ in $\mathcal{C}_x$ as (3.1) and

$$\mathcal{C}_{0,U} := \bigsqcup_{x \in U} \mathcal{C}_{0,x} \cong U \times C_0,$$

where C_0 is a subtorus of C with even codimension.

We choose normalized coordinates, as in (3.2), of $p^{-1}(U) \cong U \times \mathbf{T}$ as

$$(\mathbf{x}; \mathbf{u}, \mathbf{y}, \mathbf{v}) = (x_1, \ldots, x_n; u_1, \ldots, u_s, y_1, \ldots, y_{2r}, v_1, \ldots, v_k),$$

with $(\mathbf{u}, \mathbf{y}, \mathbf{v})$ as normalized coordinates on $\mathbf{T} \supset C \supset C_0$. Under this set of coordinates,

$$\mathcal{C}_U = \{(\mathbf{x}; \mathbf{u}, \mathbf{y}, \mathbf{v}) : \mathbf{u} = \mathbf{g}(\mathbf{x})\} \tag{3.11}$$

for some function $\mathbf{g}(\mathbf{x})$ with valued in $\mathbb{R}^s$, spanned by u_i's coordinates. The connection of $\mathcal{E}$ is

$$\nabla^{\mathcal{E}} = d + 2\pi\sqrt{-1}\left(\alpha - \breve{\mathbf{g}}d\mathbf{v}^T + \frac{1}{2}\mathbf{y}Ad\mathbf{y}^T - \mathbf{f}d\mathbf{y}^T\right)\cdot I_{\mathcal{E}}, \tag{3.12}$$

for some $\alpha = \alpha(\mathbf{x}) \in \Omega^1(U,\mathbb{R})$, $\mathbf{f} = \mathbf{f}(\mathbf{x})$ and $\breve{\mathbf{g}} = \breve{\mathbf{g}}(\mathbf{x})$ are some functions and

$$A = \operatorname{diag}\left\{\begin{bmatrix} 0 & a_1 \\ -a_1 & 0\end{bmatrix}, \dots, \begin{bmatrix} 0 & a_r \\ -a_r & 0\end{bmatrix}\right\}$$

for some nonzero real numbers $a_1, \dots, a_r$. Furthermore,

$$F^{\mathcal{E}} = d\alpha - d\breve{\mathbf{g}}^T \wedge d\mathbf{v} + \frac{1}{2}d\mathbf{y} \wedge A \wedge d\mathbf{y}^T - d\mathbf{f} \wedge d\mathbf{y}^T. \tag{3.13}$$

Then, the dual semi-affine brane in $\check{X}$ $(\check{\mathcal{C}}, \check{\mathcal{E}})$ can be written down explicitly:

Proposition 19. *There exists a set of fiberwise dual normalized coordinates*

$$(\mathbf{x}; \check{\mathbf{u}}, \check{\mathbf{y}}, \check{\mathbf{v}}) = (x_1, \dots, x_n; u^1, \dots, u^k, y^1, \dots, y^{2r}, v^1, \dots, v^s) \tag{3.14}$$

on $\check{p}^{-1}(U) \cong U \times \check{\mathbf{T}}$ *such that*

$$\check{\mathcal{C}}_U := \{(\mathbf{x}; \check{\mathbf{u}}, \check{\mathbf{y}}, \check{\mathbf{v}}) : \check{\mathbf{u}} = \breve{\mathbf{g}}(\mathbf{x})\}, \tag{3.15}$$

and

$$\begin{aligned}\nabla^{\check{\mathcal{E}}} &= d + 2\pi\sqrt{-1}\left(\alpha - \mathbf{g}d\check{\mathbf{v}}^T + \frac{1}{2}(\check{\mathbf{y}} + \mathbf{f})Ad(\check{\mathbf{y}} + \mathbf{f})^T\right)\cdot I_{\check{\mathcal{E}}} \\ &= d + 2\pi\sqrt{-1}\left(\check{\alpha} - \breve{\mathbf{g}}d\check{\mathbf{v}}^T + \frac{1}{2}\check{\mathbf{y}}A^{-1}d\check{\mathbf{y}}^T - \check{\mathbf{f}}d\check{\mathbf{y}}^T\right)\cdot I_{\check{\mathcal{E}}}\end{aligned} \tag{3.16}$$

with

$$\begin{aligned}F^{\check{\mathcal{E}}} &= d\alpha - d\mathbf{g} \wedge d\check{\mathbf{v}}^T + \frac{1}{2}d(\check{\mathbf{y}} + \mathbf{f}) \wedge A \wedge d(\check{\mathbf{y}} + \mathbf{f})^T \\ &= d\check{\alpha} - d\breve{\mathbf{g}} \wedge d\check{\mathbf{v}}^T + \frac{1}{2}d\check{\mathbf{y}} \wedge A^{-1} \wedge d\check{\mathbf{y}}^T - d\check{\mathbf{f}} \wedge d\check{\mathbf{y}}^T.\end{aligned} \tag{3.17}$$

where $\check{\alpha} = \alpha + \frac{1}{2}(\check{\mathbf{y}} + \mathbf{f})A^{-1}d\mathbf{f}^T \in \Omega^1(U,\mathbb{R})$ *and* $\check{\mathbf{f}}(\mathbf{x}) = -\frac{1}{2}\mathbf{f}A^{-1}$.

Example 20. Let $B = \mathbf{T}^2$ be a 2-torus with affine coordinates (x_1, x_2). Then $X = B \times \mathbf{T}^2$ and $\check{X} = B \times \check{\mathbf{T}}^2$ is a mirror pair with semi-flat coordinates (x_1, x_2, y_1, y_2) and (x_1, x_2, y^1, y^2) respectively. The following examples can be regarded as the family version of the example 17 which transform semi-affine B-branes in X to semi-affine coisotropic A-branes in $\check{X}$ under fiberwise SYZ transformation:

1) If $(\mathcal{C}_1, \mathcal{E}_1)$ is a semi-affine B-brane in X, where $\mathcal{C}_1 = X$ and $\mathcal{E}_1$ is a $U(1)$-bundle over $\mathcal{C}_1$ with curvature $2\pi\sqrt{-1}\, F^{\mathcal{E}}$ and

$$F^{\mathcal{E}} = dx_1 \wedge dx_2 + dy_1 \wedge dy_2.$$

Then, its mirror $(\check{\mathcal{C}}_1, \check{\mathcal{E}}_1)$ is a semi-affine coisotropic A-brane in $\check{X}$, where $\check{\mathcal{C}}_1 = \check{X}$ and $\check{\mathcal{E}}_1$ is a line bundle with

$$F^{\check{\mathcal{E}}_1} = dx_1 \wedge dx_2 - dy_1 \wedge dy_2.$$

2) If $(\mathcal{C}_2, \mathcal{E}_2)$ is a semi-affine B-brane in X, where $\mathcal{C}_2 = X$ and $\mathcal{E}_2$ is the trivial line bundle with

$$\nabla^{\mathcal{E}_2} = d + 2\pi\sqrt{-1}\left(\frac{1}{2}(x_1 dx_2 - x_2 dx_1) + \frac{1}{2}(y_1 dy_2 - y_2 dy_1) - f_1 dy_1 - f_2 dy_2\right),$$

where f_1 and f_2 are functions on B. Moreover,

$$F^{\mathcal{E}_2} = (dx_1 \wedge dx_2) - (df_1 \wedge dy_1 - df_2 \wedge dy_2) + dy_1 \wedge dy_2.$$

Then, its mirror $(\check{\mathcal{C}}_2, \check{\mathcal{E}}_2)$ is a semi-affine coisotropic A-brane in $\check{X}$, where $\check{\mathcal{C}}_2 = \check{X}$ and $\check{\mathcal{E}}_2$ is the $U(1)$-bundle with

$$F^{\check{\mathcal{E}}_2} = (dx_1 \wedge dx_2 - df_1 \wedge df_2) + (df_1 \wedge dy^2 - df_2 \wedge dy^1) - dy^1 \wedge dy^2.$$

4. Proof

4.1. Kernel of Dirac operator

In this section, we will study the Dirac operator defined on torus.

Proposition 21. *Let (C, E) be a brane on a torus $\mathbf{T}$ and let $\mathcal{D} : \Gamma(\mathcal{S} \otimes E)^{V_0} \to \Gamma(\mathcal{S} \otimes E)^{V_0}$ be the Dirac operator defined on $\mathbf{T}$. Then $\ker \mathcal{D} \neq 0$ if and only if $E|_{C_0}$ is trivial. When this happens, one has*

$$\dim(\ker \mathcal{D}) = \int_C ch(E).$$

Proof. We will first compute $\dim(\ker \mathcal{D})$ when F^E is non-degenerate and then show the general situation can be reduced to this case.

Firstly, if F^E is a non-degenerate two-form, $\dim C$ must be even, say $2r$. Therefore, the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ and the Dirac operator has the form

$$\mathcal{D} = \begin{bmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{bmatrix}.$$

We will show that $\ker \mathcal{D}^-$ vanishes and this proposition follows from the Atiyah-Singer Index Theorem.

We choose an orthonormal frame $\{e_1, \dots, e_n\}$ of the tangent bundle $T(\mathbf{T})$ with dual frame $\{\omega_1, \dots, \omega_n\}$, such that

$$F^{\mathcal{E}} = \lambda_1 \omega_1 \wedge \omega_2 + \cdots + \lambda_r \omega_{2r-1} \wedge \omega_{2r}$$

for some positive real numbers $\lambda_1, \dots, \lambda_r$. Then the *Lichnerowicz Formula* gives:

$$\mathcal{D}^2 = \nabla^* \nabla + \sum_{i=1}^{r} \frac{2\pi}{\sqrt{-1}} \lambda_i e_{2i-1} e_{2i} \cdot \ , \tag{4.1}$$

where ∇ denotes the connection of the spinor bundle $\mathcal{S} \otimes E$ and '$\cdot$' denotes the Clifford multiplication.

Remark 22. Since C is a flat torus, the spinor bundle $\mathcal{S} \cong C \times \mathbb{C}^{2^r}$ is trivial with trivial connection. To simplify our notations, we use ∇ to denote both connections of the bundles ∇^E and $\nabla^{\mathcal{S} \otimes E}$ when no confusion occurs. Readers may refer to Appendix 5.1 for the details of spinor bundles and Clifford multiplications.

Lemma 23. *Let λ be the first eigenvalue of the operator*

$$\nabla^* \nabla : \Gamma(E) \to \Gamma(E),$$

then $\lambda \geq 2\pi \sum_{i=1}^{r} \lambda_i$.

Proof. Suppose $\lambda < \sum_{i=1}^{r} \lambda_i$. Then there exists an non-zero section $\alpha \in \Gamma(E)$ such that

$$\nabla^* \nabla \alpha = 2\pi \lambda \alpha.$$

For each $I = (\iota_1, \dots, \iota_r) \in \Upsilon := \{(1, -1)\}^r$, we define

$$\chi_I := \chi_{\iota_1} \otimes \cdots \otimes \chi_{\iota_r} \in \bigotimes^r \mathbb{C}^2,$$

where $\chi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\chi_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose that $I_0 = (1, 1, \ldots, 1) \in \Upsilon$, then $\eta = \chi_{I_0} \otimes \alpha$ is a nonzero section of $\mathcal{S} \otimes E$ which satisfies

$$\nabla^* \nabla \eta = 2\pi \lambda \eta$$

since $\mathcal{S}$ is a trivial bundle with fiber $\overset{r}{\otimes} \mathbb{C}^2$. Using (4.1),

$$\mathcal{D}^2 \eta = 2\pi \left(\lambda - \sum_{i=1}^{r} \lambda_i \right) \eta.$$

This is a contradiction as $\lambda < \sum_{i=1}^{r}$ and $\mathcal{D}^2$ is an non-negative operator. □

As F^E is non-degenerate, we have $V_0^{\perp} = TC$ and so $\Gamma(\mathcal{S} \otimes E)^{V_0}$ is simply $\Gamma(\mathcal{S} \otimes E)$.

Lemma 24. *For any $s \in \ker \mathcal{D} \subset \Gamma(\mathcal{S} \otimes E)$, s must be in the form of*

$$s = \chi_{I_0} \otimes \alpha,$$

where α is a section of E satisfying that $\nabla^* \nabla \alpha = 2\pi (\sum_{i=1}^{r} \lambda_i) \alpha$.

Proof. Suppose that $s \in \ker \mathcal{D} \subset \Gamma(\mathcal{S} \otimes E)$ and

$$s = \sum_{I \in \Upsilon} \chi_I \otimes \alpha_I$$

for some sections $\alpha_I \in \Gamma(E)$. By plugging it into (4.1), we obtain

$$\sum_{I \in \Upsilon} \chi_I \otimes (\nabla^* \nabla \alpha_I - 2\pi \mu_I \alpha_I) = 0,$$

where $\mu_I = \sum_{i=1}^{r} \iota_i \lambda_i$. Therefore,

$$\nabla^* \nabla \alpha_I = 2\pi \mu_I \alpha_I \tag{4.2}$$

for all $I \in \Upsilon$. However,

$$\mu_I < \mu_{I_0} = 2\pi \sum_{i=1}^{r} \lambda_i$$

for any $I \neq I_0$ since $\lambda_i > 0$ for all i. By Lemma 23, $\alpha_I = 0$ for $I \neq I_0$ and so

$$s = \chi_{I_0} \otimes \alpha_{I_0}$$

with

$$\nabla^* \nabla \alpha_{I_0} = 2\pi \sum_{i=1}^{r} \lambda_i \alpha_{I_0}$$

by (4.2). □

An immediate consequence is $\ker \mathcal{D}^- = 0$ because $\chi_{I_0} \otimes \alpha_{I_0} \in \mathcal{S}^+$. By the Atiyah-Singer Index Theorem, we have

$$\dim(\ker \mathcal{D}) = \int_C ch(E)$$

In the general case, the subbundle

$$V_0 = \{X \in TC : \iota_X F^E = 0 \in T^*C\} \leq TC$$

is integrable since F^E is a constant two-form.

Since C_0 is an affine subtorus of C, we can choose a complementary affine subtorus C_1 to get a decomposition

$$C = C_1 \times C_0.$$

Suppose that $\phi_1 : C \to C_1$ and $\phi_0 : C \to C_0$ are projection maps. Note that the bundles E and $\phi_1^*(E|_{C_1})$ have the same rank and curvature.

Lemma 25. *Let E and E' be two projectively flat unitary bundles over a torus C with the same rank and curvature. Suppose that E is irreducible, then E and E' differ by tensoring with a flat $U(1)$-bundle.*

Proof. For simplicity, we assume that $\operatorname{rank} \mathcal{E} = 2$. Clearly, $\mathcal{E}' \otimes \mathcal{E}^*$ is a flat unitary bundle. Therefore,

$$\mathcal{E}' \otimes \mathcal{E}^* \cong \bigoplus_{i=1}^{4} \mathcal{L}_i$$

for four flat $U(1)$-bundles $\mathcal{L}_i$. Similarly, we also have

$$\mathcal{E}^* \otimes \mathcal{E} = \bigoplus_{i=1}^{4} \mathcal{L}'_i$$

for another four flat $U(1)$-bundles $\mathcal{L}'_i$. Therefore,

$$\bigoplus_{i=1}^{4} (\mathcal{E} \otimes \mathcal{L}_i) = \mathcal{E}' \otimes \mathcal{E}^* \otimes \mathcal{E} = \bigoplus_{i=1}^{4} (\mathcal{E}' \otimes \mathcal{L}'_i). \tag{4.3}$$

We obtain two decompositions for the bundle $\mathcal{E}' \otimes \mathcal{E}^* \otimes \mathcal{E}$. For this bundle, there must be at least one non-zero map among the orthogonal projections from $\mathcal{E} \otimes \mathcal{L}_i$ to $\mathcal{E}' \otimes \mathcal{L}'_1$ for $1 \leq i \leq 4$. Without loss of generality, assume that the map

$$\phi : \mathcal{E} \otimes \mathcal{L}_1 \to \mathcal{E}' \otimes \mathcal{L}'_1$$

is non-zero. We claim that ϕ is actually a unitary bundle isomorphism up to a constant.

Let p be an arbitrary point in $\mathbf{T}$ and V be the fiber of the bundle $\mathcal{E}' \otimes \mathcal{E}^* \otimes \mathcal{E}$ with the holonomy group G acting on it. According to 4.3, the representation of G on V has two decompositions $\oplus_{i=1}^4 V_i$ and $\oplus_{i=1}^4 V'_i$. Note that the decomposition $V = \oplus_{i=1}^4 V_i$ is irreducible since $\mathcal{E}$ is irreducible. Note that the linear map

$$\phi : V_1 \to V'_1$$

induced by the projection ϕ is a homomorphisms of representations. Moreover, ϕ is a non-zero map and V_1 and V'_1 have the same dimension, so ϕ is actually an isomorphism of representations by Schurs lemma. Hence, by Schurs lemma again, one can prove that there exists a positive number $c(p)$ such that $c(p) \cdot \phi$ preserves the Hermitian metrics on the fiber over the point p.

On the other hand, it is easy to see that ϕ preserves the connections, by letting $c(p)$ to be a constant c independent of p, then $c \cdot \phi$ gives a unitary bundle isomorphism which completes the proof. □

By the above lemma and the fact that any flat $U(1)$-bundle on $C = C_1 \times C_0$ is of the form

$$\phi_1^* L_1 \otimes \phi_0^* L_0$$

where L_1 and L_0 are some flat U(1)-bundles over C_1 and C_0. As a result,

$$\begin{aligned} E &= \phi_1^*(E|_{C_1}) \otimes (\phi_1^* L_1 \otimes \phi_0^* L_0) \\ &= \phi_1^*(E|_{C_1} \otimes L_1) \otimes \phi_0^* L_0 \end{aligned}$$

Hence if we let $E_1 := E|_{C_1} \otimes L_1$ and $E_0 := L_0$, then we can decompose E as

$$E = \phi_1^* E_1 \otimes \phi_0^* E_0$$

such that the curvature of E_1 is non-degenerate and E_0 is flat.

We claim that E_0 is trivial. Otherwise, $\Gamma(\mathcal{S} \otimes E)^{V_0}$ is nontrivial and there exists a non-zero section

$$s = \sum_{I \in \Upsilon} \chi_I \otimes \alpha_I \in \Gamma(\mathcal{S} \otimes E)^{V_0}$$

for some sections $\alpha_I \in \Gamma(E)$. Since s is nonzero, there exists nonzero α_I for some I, which contradicts to the assumption that E_0 is nontrivial.

As a result, $E = \phi_1^* E_1$. Let $\mathcal{S}_1$ be the spinor bundle of C_1 associated by the tangent bundle TC_1 and let

$$\mathcal{D}_1 : \Gamma(\mathcal{S}_1 \otimes E_1) \to \Gamma(\mathcal{S}_1 \otimes E_1)$$

be the corresponding Dirac operator. Note that $\mathcal{S} \cong \phi_1^* \mathcal{S}_1$, so we have the bundle isomorphism

$$\mathcal{S}_1 \otimes \mathcal{E}_1 \cong \phi_1^*(\mathcal{S}_1 \otimes \mathcal{E}_1)$$

and it induces an isomorphism $\Gamma(\mathcal{S}_1 \otimes \mathcal{E}_1) \cong \Gamma(\mathcal{S} \otimes \phi_1^* \mathcal{E}_1)^{V_0}$ which is defined by

$$s \mapsto \phi_1^* s.$$

This map gives an isomorphism between $\ker \mathcal{D}_1$ and $\ker \mathcal{D}$. Hence, $\dim \mathcal{D} = \dim \mathcal{D}_1$ and the general situation reduces to the non-degenerate case which finishes the proof Proposition 21. □

4.2. Transformation of semi-affine branes

The main goal of this section is proving the following:

Theorem 26. *The fiberwise SYZ transformation $\mathcal{F}^{SYZ}$ transforms a semi-affine brane in X to be a semi-affine brane in $\check{X}$.*

Since $\mathcal{F}^{SYZ}$ is defined fiberwisely, we can prove this theorem locally on base. Without loss of generality, we assume that $X \cong B \times \mathbf{T}$ where B is a convex subset of $\mathbb{R}^n$, then we have:

Proposition 27. *Let $(\mathcal{C}, \mathcal{E})$ be a semi-affine brane on X. Then there exists a decomposition of the torus $\mathbf{T}$ as a product of subtori $C_2 \times C_1 \times C_0$ and a semi-flat coordinate system $(\mathbf{x}; \mathbf{u}, \mathbf{y}, \mathbf{v})$ of $X \cong B \times C_2 \times C_1 \times C_0$ which satisfies:*

1)

$$\mathcal{C} = \{(\mathbf{x}; \mathbf{u}, \mathbf{y}, \mathbf{v}) | \ \mathbf{x} \in p(\mathcal{C}), \ \mathbf{u} = \mathbf{g}(\mathbf{x})\}$$

for some vector valued function $\mathbf{g}(\mathbf{x})$ *on* $p(\mathcal{C})$.

2)

$$\mathcal{E} \cong \mathcal{E}_b \otimes \mathcal{E}_m \otimes \mathcal{E}_f$$

for three bundles $\mathcal{E}_b$, $\mathcal{E}_m$ *and* $\mathcal{E}_f$ *over* $\mathcal{C}$ *such that:*

(i) $\mathcal{E}_b$ *is the pull back of a* $U(1)$*-bundle over* $p(\mathcal{C})$ *with curvature* $2\pi\sqrt{-1}\,F_b$.

(ii) $\mathcal{E}_m = \mathbb{C} \times \mathcal{C}$ *with connection form*

$$2\pi\sqrt{-1}\,\left(\mathbf{f} d\mathbf{y}^T + \check{\mathbf{g}} d\mathbf{v}^T\right)$$

for two vector valued functions $\mathbf{f}$ *and* $\check{\mathbf{g}}$ *on* $p(\mathcal{C})$. *We denote the corresponding curvature by* $2\pi\sqrt{-1}\,F_m$.

(iii) $\mathcal{E}_f$ *is the pull back of a projectively flat unitary bundle over* C_1 *with curvature to be* $2\pi\sqrt{-1}\,F_f \cdot I_{\mathcal{E}_f}$ *where*

$$F_f = \frac{1}{2} d\mathbf{y} \wedge A \wedge d\mathbf{y}^T$$

and $A = \operatorname{diag}\left\{\begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & a_r \\ -a_r & 0 \end{bmatrix}\right\}$ *is a constant matrix with* $a_i \in \mathbb{R}\backslash\{0\}$.

Proof. Recall the definition of a semi-flat submanifold that, if $p : X \cong B \times \mathbf{T} \to B$ is the projection map, then the restriction map $p|_{\mathcal{C}} : \mathcal{C} \to p(\mathcal{C})$ gives a torus fibration. Hence $\mathcal{C}$ can be regarded as a family of affine subtori of $\mathbf{T}$ which gives a continuous family of homology classes in $H_*(\mathbf{T}, \mathbb{Q})$. Since $H_*(\mathbf{T}, \mathbb{Q})$ is totally disconnected, this family actually gives only one homology class in $H_*(\mathbf{T}, \mathbb{Q})$, which means $\mathcal{C}$ is homotopic to C for some subtorus C in $\mathbf{T}$. Then we can choose a subtorus C_2 which is complementary to C in $\mathbf{T}$ so that $X \cap p^{-1}(p(\mathcal{C})) = p(\mathcal{C}) \times C_2 \times C$ with a semi-flat coordinate system $(\mathbf{x}; \mathbf{u}, \mathbf{w})$ and $\mathcal{C}$ is given by

$$\mathcal{C} = \{(\mathbf{x}; \mathbf{u}, \mathbf{w}) : \mathbf{u} = \mathbf{g}(\mathbf{x})\}$$

for some vector valued function $\mathbf{g}(\mathbf{x})$. Furthermore, the cohomology class of $F^{\mathcal{E}}$ can be represented by a constant two-form F_f on C. After carrying out a suitable coordinate change, we may assume that

$$F_f = \frac{1}{2} d\mathbf{w} \wedge \tilde{A} \wedge d\mathbf{w}^T$$

where

$$\tilde{A} = \mathrm{diag}\{A, \mathbf{0}\} \qquad \text{and} \qquad A = \mathrm{diag}\left\{ \begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & a_r \\ -a_r & 0 \end{bmatrix} \right\}$$

where $a_1, \ldots, a_r$ are nonzero real numbers. So we can further decompose coordinates $\mathbf{w} = (\mathbf{y}, \mathbf{v})$ such that

$$F_f = \frac{1}{2} d\mathbf{w} \wedge \tilde{A} \wedge d\mathbf{w}^T = \frac{1}{2} d\mathbf{y} \wedge A \wedge d\mathbf{y}^T.$$

Since $[F^{\mathcal{E}}] = [F_f]$, one has $F^{\mathcal{E}} - F_f = d\alpha$ for some one form α on $\mathcal{C}$. Moreover, both $F^{\mathcal{E}}$ and F_f are semi-affine forms, so α can be chosen to be a semi-affine form which can be expressed as

$$\alpha = \mathbf{f} d\mathbf{y}^T + \check{\mathbf{g}} d\mathbf{v} + \alpha_b$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x})$ and $\check{\mathbf{g}} = \check{\mathbf{g}}(\mathbf{x})$ are vector valued functions on $p(\mathcal{C})$ and α_b is the pull back of some one-form on $p(\mathcal{C})$. It then follows that

$$\begin{aligned} F^{\mathcal{E}} &= d\alpha_b + (d\mathbf{f} \wedge d\mathbf{y}^T + d\check{\mathbf{g}} \wedge d\mathbf{v}^T) + \frac{1}{2} d\mathbf{y} \wedge A \wedge d\mathbf{y}^T \\ &=: F_b + F_m + F_f. \end{aligned}$$

Since $\mathcal{C}$ is homotopic to $\mathbf{T}$ by our assumption, we can construct three bundles $\mathcal{E}_b$, $\mathcal{E}_m$, $\mathcal{E}_f$ with curvatures as stated in the proposition and the curvature of the bundle $\mathcal{E}_b \otimes \mathcal{E}_m \otimes \mathcal{E}_f$ over $\mathcal{C}$ equals to a multiple of the two-form $2\pi\sqrt{-1}\, F^{\mathcal{E}}$. Then the result just follows from the family version of Lemma 25. □

The Proposition 27 gives a decomposition of $F^{\mathcal{E}}$ into the base part F_b, the fiber part F_f and the mixed part F_m. With the above, we are ready to come back:

Proof of Theorem 26. Note that $p|_{\mathcal{C}} : \mathcal{C} \to p(\mathcal{C})$ is in fact a Riemannian fiber bundle (see Section 5.2). In other words, $\mathcal{C} = \bigsqcup_{x \in p(\mathcal{C})} \mathcal{C}_x$ is a family of flat tori. Let

$$\mathcal{V}_{\mathcal{C}} = \bigsqcup_{x \in p(\mathcal{C})} T(\mathcal{C}_x) \to \mathcal{C}$$

be the vertical tangent bundle of this Riemannian fiber bundle and let

$$\mathcal{V}_0 := \{X \in \mathcal{V}_{\mathcal{C}} : \iota_X F^{\mathcal{E}} = 0 \in \mathcal{V}_{\mathcal{C}}^*\}$$

be a subbundle of $\mathcal{V}_{\mathcal{C}}$. Then $\mathcal{V}_0^{\perp}$ associates a spinor bundle $\$ \to \mathcal{C}$ which can be regarded as a family of spinor bundles over the family of tori $\{\mathcal{C}_x : x \in p(\mathcal{C})\}$ (see Section 5.2). Furthermore, let $\pi : X \times_B \check{X} \to X$ and $\check{\pi} : X \times_B \check{X} \to \check{X}$ be projection maps, we can define the Dirac operator $\not{D}$ which is parametrized by points $(x, \check{y}) \in \check{X} = B \times \check{\mathbf{T}}$

$$\not{D}_{(x,\check{y})} : \Gamma(\pi^*(\$ \otimes \mathcal{E}) \otimes \mathcal{P}|_{\mathcal{C}_x \times \{\check{y}\}})^{\mathcal{V}_0|_{\mathcal{C}_x}} \to \Gamma(\pi^*(\$ \otimes \mathcal{E}) \otimes \mathcal{P}|_{\mathcal{C}_x \times \{\check{y}\}})^{\mathcal{V}_0|_{\mathcal{C}_x}},$$

which is a family version of the one in (2.7). Then

$$\check{\mathcal{C}} := \{(x, \check{y}) \in \check{X} : \ker \not{D}_{(x,\check{y})} \neq 0\},$$

Note that $\mathcal{C}_x$ is nonempty if and only if $x \in p(\mathcal{C})$. Therefore, by Propositions 21 and 27,

$$\check{\mathcal{C}} = \{(\mathbf{x}; \check{\mathbf{u}}, \check{\mathbf{v}}, \check{\mathbf{w}}) : \mathbf{x} \in p(\mathcal{C}), \check{\mathbf{u}} = \check{\mathbf{g}}(\mathbf{x})\}$$

defines a semi-affine submanifold in $\check{X}$ and $\check{\mathcal{E}}$ is a bundle over $\check{\mathcal{C}}$, where $\check{\mathcal{E}}_x = \ker \not{D}_{(x,\check{y})}$.

What remains to show is that the curvature of $\check{\mathcal{E}}$ is constant along any fiber of the fibration $p|_{\mathcal{C}} : \mathcal{C} \to p(\mathcal{C})$. The computation is divided into two cases, the special case $\mathcal{C} = X$ and the general case:

(i) special case $\mathcal{C} = X$

We begin from the case that $\mathcal{C} = X$ and the curvature of $\mathcal{E}$ is non-degenerate along each fiber. In this case, $\dim_{\mathbb{C}} X = n = 2r$ is even and

$$\mathcal{V} := \mathcal{V}_0^{\perp} = \mathcal{V}_{\mathcal{C}} = \bigsqcup_{x \in B} T(\mathbf{T}_x),$$

so the fiber coordinates of X in Proposition 27 only consist of $\mathbf{y}$ and we have $\mathcal{E} \cong \mathcal{E}_b \otimes \mathcal{E}_m \otimes \mathcal{E}_f$, where

(i) $\mathcal{E}_b$ is the pull back of a $U(1)$-bundle over $p(\mathcal{C})$ with curvature $2\pi\sqrt{-1}\, F_b$;

(ii) $\mathcal{E}_m = \mathbb{C} \times \mathcal{C}$ is a $U(1)$-bundle with connection $2\pi\sqrt{-1}\,\mathbf{f}d\mathbf{y}^T$ where $\mathbf{f}(\mathbf{x})$ is a vector valued function on B;

(iii) $\mathcal{E}_f$ is a pull back of a projectively flat bundle over the fiber $\mathbf{T}$ with curvature $2\pi\sqrt{-1}\,F_f \cdot I_{\mathcal{E}_f}$, where

$$F_f = \frac{1}{2}d\mathbf{y} \wedge A \wedge d\mathbf{y}^T$$

for a constant invertible matrix

$$A = \operatorname{diag}\left\{ \begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & a_r \\ -a_r & 0 \end{bmatrix} \right\}.$$

Suppose that $(\mathbf{x}; \check{\mathbf{y}})$ is the semi-flat coordinates in $\check{X}$ mirror to $(\mathbf{x}; \mathbf{y})$. For our convenience, we perform a change of coordinates which is replacing $\check{\mathbf{y}} + \mathbf{f}$ by $\check{\mathbf{y}}$. Then

$$(\mathbf{x}; \mathbf{y}, \check{\mathbf{y}}) = (x_1, \dots, x_n; y_1, \dots, y_n, y^1, \dots, y^n) \tag{4.4}$$

gives a set of coordinates of $X \times_B \check{X}$.

Remark 28. The reason of performing this change of coordinates can be seen from the equation (3.17).

We first investigate the connection of $\check{\mathcal{E}}$. Note that $\check{\mathcal{E}}$ is a subbundle of the bundle

$$\Gamma(\pi^*(\$ \otimes \mathcal{E}) \otimes \mathcal{P}|_{\mathcal{C}_x \times \{\check{y}\}})^{\mathcal{V}_0|_{\mathcal{C}_x}} = \Gamma(\pi^*(\$ \otimes \mathcal{E}) \otimes \mathcal{P}|_{\mathbf{T} \times \{\check{y}\}}) \to \mathcal{H} \to \check{X}.$$

A section s of $\mathcal{H}$ can also be regarded as a section of $\pi^*(\$ \times \mathcal{E}) \otimes \mathcal{P}$. Therefore, $\mathcal{H}$ equips a connection which is defined by

$$\nabla^{\mathcal{H}}_v s := \nabla_{\hat{v}} s,$$

where v is a vector field on $\check{X}$ with a lift $\hat{v}$ via the connection of the Riemannian fiber bundle $\pi : X \times_B \check{X} \to \check{X}$.

Remark 29. A connection for a Riemannian fiber bundle $\pi : M \to B$ with a vertical tangent bundle V in TM is a choice of splitting $TM = V \oplus H$. In our case, the connection is induced by the Riemannian metric of $X \times_B \check{X}$. See Section 5.2 for more details.

Since $\check{\mathcal{E}}$ is a subbundle of $\mathcal{H}$, it has connection $P\nabla^{\mathcal{H}}$ where $P : \mathcal{H} \to \check{\mathcal{E}}$ is the orthogonal projection map. To calculate the curvature of $\check{\mathcal{E}}$, we let

$$\{\varphi^1_{(x,\check{y})}(y), \ldots, \varphi^m_{(x,\check{y})}(y)\}$$

be an orthonormal framing of $\check{\mathcal{E}}$, where m is the rank of $\check{\mathcal{E}}$. For simplicity, we denote $\varphi^p_{(x,\check{y})}(y)$ as φ^p. Then the connection matrix of $\ker \not{D}$ is

$$\check{A}_{pq} := \langle \varphi^p, P\nabla^{\mathcal{H}}\varphi^q \rangle_{L^2} = \langle \varphi^p, \nabla^{\mathcal{H}}\varphi^q \rangle_{L^2},$$

where $\langle \cdot, \cdot \rangle_{L^2}$ to denote the L^2 metric of the infinity dimensional vector bundle $\mathcal{H}$. Then the calculation in [5] shows that the curvature

$$\begin{aligned} 2\pi\sqrt{-1}\, F^{\check{\mathcal{E}}}_{pq} &= \check{d}\check{A}_{pq} + \sum_k \check{A}_{pk} \wedge \check{A}_{kq} \\ &= \langle \varphi^p, \nabla^{\mathcal{H}}\nabla^{\mathcal{H}}\varphi^q \rangle_{L^2} + \langle (Id - P)\nabla^{\mathcal{H}}\varphi^p, \nabla^{\mathcal{H}}\varphi^q \rangle_{L^2} \end{aligned}$$

where $\check{d}$ is the exterior derivative of $\check{X}$. The Dirac operator $\not{D}_{(x,\check{y})}$ is simply denoted by $\not{D}$ when no confusion occurs and we let G be the corresponding Green's operator, which is a self-adjoint operator satisfying

$$\mathrm{Id} = P + \not{D}G\not{D}.$$

Therefore,

$$\begin{aligned} 2\pi\sqrt{-1}\, F^{\check{\mathcal{E}}}_{pq} &= \langle \varphi^p, \nabla^{\mathcal{H}}\nabla^{\mathcal{H}}\varphi^q \rangle_{L^2} + \langle G\not{D}\nabla^{\mathcal{H}}\varphi^p, \not{D}\nabla^{\mathcal{H}}\varphi^q \rangle_{L^2} \\ &=: I + J \end{aligned}$$

Note that for any two vector fields v, w of $\check{X}$ with lifts $\hat{v}, \hat{w}$ to $X \times_B \check{X}$, we have

$$\begin{aligned} I(v, w) &= \langle \varphi^p, \nabla^{\mathcal{H}}\nabla^{\mathcal{H}}(v, w)\varphi^q \rangle_{L^2} \\ &= 2\pi\sqrt{-1}\, F^{\pi^*(\$\otimes\mathcal{E})\otimes\mathcal{P}}(\hat{v}, \hat{w}) \\ &= I_{\$} \otimes 2\pi\sqrt{-1}\, F^{\pi^*\mathcal{E}\otimes\mathcal{P}}(\hat{v}, \hat{w}) + 2\pi\sqrt{-1}\, F^{\$}(\hat{v}, \hat{w}) \otimes I_{\pi^*\mathcal{E}\otimes\mathcal{P}} \\ &=: I_1(v, w) + I_2(v, w). \end{aligned}$$

We decompose $J = J_b + J_m + J_f$ to be a sum of the base part, the mix part and the fiber part, where

$$\begin{aligned} J_b :&= \sum_{1\le i,j\le n} \langle G\not{D}\nabla_{\frac{\partial}{\partial x_i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial x_j}}\varphi^q\rangle_{L^2} dx_i \wedge dx_j \\ J_m :&= \sum_{1\le i,j\le n} \langle G\not{D}\nabla_{\frac{\partial}{\partial x_i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial y^j}}\varphi^q\rangle_{L^2} dx_i \wedge dy^j \\ &\quad + \sum_{1\le i,j\le n} \langle G\not{D}\nabla_{\frac{\partial}{\partial y^i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial x_j}}\varphi^q\rangle_{L^2} dy^i \wedge dx_j \\ J_f :&= \sum_{1\le i,j\le n} \langle G\not{D}\nabla_{\frac{\partial}{\partial y^i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial y^j}}\varphi^q\rangle_{L^2} dy^i \wedge dy^j. \end{aligned}$$

Hence,

$$2\pi\sqrt{-1}\, F^{\check{\mathcal{E}}}_{pq} = I_1 + I_2 + J_b + J_m + J_f \tag{4.5}$$

while we have the following lemma:

Lemma 30. *If $2\pi\sqrt{-1}\, F^{\check{\mathcal{E}}}_{pq}$ is decomposed as a sum of terms as in (4.5), then*

1) $I_1 = 2\pi\sqrt{-1}\,\delta_{pq} F_b$;
2) $I_2 + J_b = 0$;
3) $J_f = 2\pi\sqrt{-1}\,\delta_{pq}\left(\frac{1}{2} d\check{\mathbf{y}} \wedge A^{-1} \wedge d\check{\mathbf{y}}^T\right)$;
4) $J_m = 0$.

An immediate result from this lemma that

$$F^{\check{\mathcal{E}}} = F_b + \frac{1}{2} d\check{\mathbf{y}} \wedge A^{-1} \wedge d\check{\mathbf{y}}^T$$

which is constant along any fiber of the fibration $p|_{\mathcal{C}} : \mathcal{C} \to p(\mathcal{C})$. The proof of Lemma 30 will be deferred to the next section.

(ii) general case

For the general situation, we recall that the identification

$$\Gamma(S_1 \otimes \mathcal{E}_1) \cong \Gamma(S \otimes \pi_1^* \mathcal{E}_1)^{V_0}$$

which is also true for the family case. Hence we can apply the same method and obtain a decomposition

$$2\pi\sqrt{-1}\,\check{F}_{pq} = I_1 + I_2 + J_b + J_m + J_f$$

as before. The only difference is the term I_1 which is now given by

$$I_1 = 2\pi\sqrt{-1}\,\delta_{pq}(F_b + d\mathbf{g} \wedge \check{\mathbf{v}}).$$

Furthermore, note that

$$F^{\mathcal{P}}|_{\mathcal{C}\times_{p(\mathcal{C})}\check{X}} = d\mathbf{g} \wedge d\check{\mathbf{v}}^T + d\mathbf{y} \wedge d\check{\mathbf{y}}^T + d\mathbf{v} \wedge d\check{\mathbf{u}}^T,$$

so there is an extra term

$$\begin{aligned} I_1(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{v}}) &= \left\langle \varphi^p, \left(I_{\mathcal{S}} \otimes 2\pi\sqrt{-1}\,F^{\pi^*\mathcal{E}\otimes\mathcal{P}} \left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{v}} \right) \right) \varphi^q \right\rangle_{L^2} \\ &= 2\pi\sqrt{-1}\,\delta_{pq}\, d\mathbf{g} \wedge d\check{\mathbf{v}} \left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{v}} \right). \end{aligned}$$

As a result, we have

$$F^{\check{\mathcal{E}}} = F_b + d\mathbf{g} \wedge \check{\mathbf{v}} + \frac{1}{2} d\mathbf{y} \wedge A^{-1} \wedge d\mathbf{y}^T.$$

Finally, we recall that we have used a change of coordinates which is replacing $\check{\mathbf{y}} + \mathbf{f}$ by $\check{\mathbf{y}}$. If we express $F^{\check{\mathcal{E}}}$ in the original semi-flat coordinates, then we have

$$\begin{aligned} F^{\check{\mathcal{E}}} = &\left(F_b + \frac{1}{2} d\mathbf{f} \wedge A^{-1} \wedge d\mathbf{f} \right) + (d\mathbf{g} \wedge \check{\mathbf{v}} + d\mathbf{f} \wedge A^{-1} \wedge d\check{\mathbf{y}}^T) \\ &+ \frac{1}{2} d\check{\mathbf{y}} \wedge A^{-1} \wedge d\check{\mathbf{y}}^T. \end{aligned}$$

Therefore, the restriction of $F^{\check{\mathcal{E}}}$ is constant along any torus fiber of the fibration $\check{p} : \check{\mathcal{C}} \to \check{p}(\check{\mathcal{C}})$ and so $(\check{\mathcal{C}}, \check{\mathcal{E}})$ is semi-affine brane on $\check{X}$. This completes the proof of theorem. □

4.3. Proof of lemmas

We will give the proof of the Lemma 30 in this section.

Proof of Lemma 30. The proof is divided into four parts:

Proof of (1). First note that

$$I_1(v,w) = \langle \varphi^p, (I_{\$} \otimes 2\pi\sqrt{-1}\, F^{\pi^*\mathcal{E}\otimes\mathcal{P}}(\hat{v},\hat{w}))\varphi^q\rangle_{L^2}$$

for any two vector fields v, w with lifts $\hat{v}, \hat{w}$. Since we are able to write down the curvatures of both bundles $\mathcal{E}$ and $\mathcal{P}$, by direct computation, we have

$$I_1 = -2\pi\sqrt{-1}\,\delta_{pq}F_b.$$

Proof of (2). Let $(x, \check{y})$ be a fixed point in $\check{X}$. We start from the computation of I_2. Suppose that $\$$ is the spinor bundle associated to the vertical tangent bundle $\mathcal{V}$ of the Riemannian fiber bundle $\pi : X \times_B \check{X} \to \check{X}$, then by [20], we know that $F^{\$}$ and $F^{\mathcal{V}}$ are related by

$$F^{\$} = \frac{1}{2}\sum_{i<j}\langle F^{\mathcal{V}}e_i, e_j\rangle e_i e_j,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of $\mathcal{V}$. As a result, we have

$$I_2(v,w) = \frac{1}{2}\sum_{i<j}\langle \varphi^p, \langle F^{\mathcal{V}}(\hat{v},\hat{w})e_i, e_j\rangle e_i e_j \varphi^q\rangle_{L^2} \tag{4.6}$$

for any two vector fields v, w of $\check{X}$ with lifts $\hat{v}, \hat{w}$ to $X \times_B \check{X}$. In order to compute I_2, we have to study $F^{\mathcal{V}}$.

For our convenience, we first choose an orthonormal frame of $\mathcal{V}$ as follows: Since each fiber of the Riemannian fiber bundle $\pi : X \times_B \check{X} \to \check{X}$ is a flat torus, we can find an $n \times n$ matrix valued function $H(x)$ on B such that

$$\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} := H(x) \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} \tag{4.7}$$

is an orthonormal frame of V with dual $\{\omega_1, ..., \omega_n\}$ and the restriction of the curvature of $\mathcal{E}$ to the fiber torus over the point $(x, \check{y})$ equals to a multiple of

$$-2\pi\sqrt{-1}\,(\lambda_1\omega_1 \wedge \omega_2 + \cdots + \lambda_r\omega_{2r-1} \wedge \omega_{2r})$$

for some positive real numbers $\lambda_1, \ldots, \lambda_r$. We then calculate the connection and curvature of $\mathcal{V}$.

Lemma 31. *With respect to the orthonormal frame $\{e_1, \dots, e_n\}$, the connection of $\mathcal{V}$ equals to*

$$\nabla^{\mathcal{V}} = \frac{1}{2}\sum_l ((B^l)^T - B^l)dx_l$$

and

$$F^{\mathcal{V}} = \frac{1}{4}\sum_{l<m}\left[B^m + (B^m)^T, B^l + (B^l)^T\right] dx_l \wedge dx_m$$

where

$$B^l := \frac{\partial H}{\partial x_l} H^{-1}.$$

Proof. Recall that the bundle

$$\mathcal{V} \leq T^*(X \times_B \check{X})$$

inherits a connection from the Levi-Civita connection of $X \times_B \check{X}$ with the metric which is a sum of the fiber metric and an arbitrary metric on the base $\check{X}$. Moreover, this metric on $\mathcal{V}$ is independent of the choice of the base metric. Therefore, we simply take a flat metric on the base $\check{X}$ such that

$$\{dx_1, \dots, dx_n; \omega_1, \dots, \omega_n, dy^1, \dots, dy^n\}$$

forms an orthonormal frame of $T^*(X \times_B \check{X})$.

If we let $d\mathbf{z} = (dx_1, \dots, dx_n; dy^1, \dots, dy^n)$ and $\omega = (\omega_1, \dots, \omega_n)$ and let Ω be the Levi-Civita connection of $T(X \times_B \check{X})$, then

$$d\begin{bmatrix} d\mathbf{z}^T \\ \omega^T \end{bmatrix} = -\Omega \wedge \begin{bmatrix} d\mathbf{z}^T \\ \omega^T \end{bmatrix}. \tag{4.8}$$

Furthermore, we can partition Ω into four blocks

$$\Omega = \begin{bmatrix} \mathbf{0} & Q \\ -Q^T & \Gamma \end{bmatrix},$$

where the block Γ is the connection of $\mathcal{V}$. Note that $d\omega^T = -B^T \wedge \omega^T$ by (4.7), where

$$B := (\mathrm{d}H)H^{-1} = \sum_{l=1}^{n} B^l dx_l \tag{4.9}$$

Then equality (4.8) becomes

$$\begin{bmatrix} 0 \\ -B^T \wedge \omega^T \end{bmatrix} = - \begin{bmatrix} Q \wedge \omega^T \\ -Q^T \wedge d\mathbf{z}^T + \Gamma \wedge \omega^T \end{bmatrix}.$$

and so

$$Q \wedge \omega^T = 0 \qquad \text{and} \qquad B^T \wedge \omega^T = -Q^T \wedge d\mathbf{z}^T + \Gamma \wedge \omega^T.$$

Multiplying the second equation by ω, together with the first equation, we have

$$\omega \wedge B^T \wedge \omega^T = \omega \wedge \Gamma \wedge \omega^T.$$

Moreover, since Γ is antisymmetric,

$$\Gamma = \frac{1}{2}(B^T - B) = \frac{1}{2}\sum_{l=1}^{n}((B^l)^T - B^l)dx_l.$$

Therefore,

$$\begin{aligned} F^{\mathcal{V}} &= d\Gamma + \Gamma \wedge \Gamma \\ &= \sum_{1\le l<m\le n} \left\{ \frac{1}{2}\left(\frac{\partial B^m}{\partial x_l} - \frac{\partial B^l}{\partial x_m}\right)^T - \frac{1}{2}\left(\frac{\partial B^m}{\partial x_l} - \frac{\partial B^l}{\partial x_m}\right) \right. \\ &\qquad \left. + \frac{1}{4}[(B^l)^T - B^l, (B^m)^T - B^m] \right\} \mathrm{d}x_l \wedge \mathrm{d}x_m. \end{aligned} \tag{4.10}$$

By direct computations, we obtain that

$$\frac{\partial B^m}{\partial x_l} = \frac{\partial^2 H}{\partial x_l \partial x_m} H^{-1} - B^m B^l \tag{4.11}$$

Therefore, by plugging (4.11) into (4.10), we obtain

$$F^{\mathcal{V}} = \frac{1}{4}\sum_{1\le l<m\le n}\left[B^m + (B^m)^T, B^l + (B^l)^T\right] dx_l \wedge dx_m. \tag{4.12}$$

□

From the above lemma,

$$I_2\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial y_m}\right) = I_2\left(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_m}\right) = 0 \tag{4.13}$$

and

$$I_2\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) = \frac{1}{2}\sum_{i<j}\left\langle F^{\mathcal{V}}\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) e_i, e_j\right\rangle \langle \varphi^p, e_i e_j \varphi^q\rangle_{L^2}, \tag{4.14}$$

where

$$F^{\mathcal{V}}(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}) = \frac{1}{4}\left[B^m + (B^m)^T, B^l + (B^l)^T\right].$$

Then we compute (4.14) at $(x, \check{y})$. Note that the fiber torus over $(x, \check{y})$ of $\check{\pi} : X \times_B \check{X} \to \check{X}$ is just $\mathbf{T}_x$. Recall that

$$\not{S}|_{\mathbf{T}_x} \cong \mathbb{C}^{2^r} \times \mathbf{T}_x$$

and the Clifford multiplication of each e_i is the matrix multiplication by E_i. Under this trivialization, by Lemma 24, $\varphi^p = \chi_{I_0} \otimes \alpha^p$ for some section α^p of the bundle $(\mathcal{E} \otimes \mathcal{P})|_{\mathbf{T}_x}$. From the construction of matrices E_i's, every non-zero term among $\langle \varphi^p, e_i e_j \varphi^q\rangle_{L^2}$'s comes from

$$\langle \varphi^p, e_{2i-1} e_{2i} \varphi^p\rangle_{L^2} = -\sqrt{-1}$$

for $i = 1, \ldots, r$ and $p = 1, \ldots, m$. Thus equation (4.14) becomes

$$I_2\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) = -\delta_{pq}\frac{\sqrt{-1}}{2}\sum_{i=1}^{r}\left\langle F^{\mathcal{V}}\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) e_{2i-1}, e_{2i}\right\rangle. \tag{4.15}$$

If we express

$$B^l = [B^l_{st}]_{1\le s,t\le r}$$

and each B^l_{st} is a 2×2 block matrix. We can further obtain

$$I_2\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) = -\frac{\sqrt{-1}}{8}\delta_{pq}\sum_{1\le s,t\le r}\begin{bmatrix}0 & 1\end{bmatrix} Q^{lm}_{st}\begin{bmatrix}1\\0\end{bmatrix} \tag{4.16}$$

where

$$Q^{lm}_{st} = \left(B^m_{ik} + (B^m_{ki})^T\right)\left(B^l_{ik} + (B^l_{ki})^T\right)^T - \left(B^m_{ik} + (B^m_{ki})^T\right)^T\left(B^l_{ik} + (B^l_{ki})^T\right).$$

Next, we are going to compute J_b. Recall that

$$J_b = \sum_{1\le i,j\le 2r} \left\langle G\not{D}\nabla_{\frac{\partial}{\partial x_i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial x_j}}\varphi^q \right\rangle_{L^2} dx_i \wedge dx_j \tag{4.17}$$

at the point $(x, \check{y})$. We first express each $\not{D}\nabla_{\frac{\partial}{\partial x_i}}\varphi^p$ as a linear combination of $e_i\nabla_{e_j}\varphi^p$'s.

Lemma 32.

$$\not{D}\nabla_{\frac{\partial}{\partial x_l}}\varphi^p = -\sum_{1\le i,j\le 2r} \frac{1}{2}\Big((B^l)^T + B^l\Big)_{ij}(e_i\nabla_{e_j}\varphi^p) \tag{4.18}$$

where $(\ )_{ij}$ denotes the entry of a matrix at the i-th row and the j-th column.

Proof. Since $\not{D}\varphi^p = 0$, one has

$$\begin{aligned} 0 = \nabla_{\frac{\partial}{\partial x_l}}(\not{D}_{(x,\check{y})}\varphi^p) &= \nabla_{\frac{\partial}{\partial x_l}}\left(\sum_{i=1}^n e_i\nabla_{e_i}\varphi^p\right) \\ &= \sum_{i=1}^n \left(\nabla^{\mathcal{V}}_{\frac{\partial}{\partial x_l}} e_i\right)\nabla_{e_i}\varphi^p + \sum_{i=1}^n e_i\left(\nabla_{\frac{\partial}{\partial x_l}}\nabla_{e_i}\varphi^p\right). \end{aligned} \tag{4.19}$$

In addition,

$$0 = F^{\pi^*(\$\otimes\mathcal{E})\otimes\mathcal{P}}\left(\frac{\partial}{\partial x_l}, e_i\right) = \nabla_{\frac{\partial}{\partial x_l}}\nabla_{e_i} - \nabla_{e_i}\nabla_{\frac{\partial}{\partial x_l}} - \nabla_{[\frac{\partial}{\partial x_l}, e_i]}.$$

The first equality holds because the $F^{\pi^*(\$\otimes\mathcal{E})\otimes\mathcal{P}}$ does not contain any $dx_i \wedge dy_j$. Therefore,

$$\begin{aligned} \not{D}\nabla_{\frac{\partial}{\partial x_l}}\varphi^p &= \sum_{i=1}^n e_i\nabla_{e_i}\left(\nabla_{\frac{\partial}{\partial x_l}}\varphi^p\right) \\ &= -\sum_{i=1}^n \left(\nabla^{\mathcal{V}}_{\frac{\partial}{\partial x_l}} e_i\right)\nabla_{e_i}\varphi^p - \sum_{i=1}^n e_i\nabla_{[\frac{\partial}{\partial x_l}, e_i]}\varphi^p. \end{aligned} \tag{4.20}$$

From Lemma 31, we have

$$\nabla^{\mathcal{V}}_{\frac{\partial}{\partial x_l}} e_i = \frac{1}{2}\sum_{j=1}^n ((B^l)^T - B^l)_{ji} e_j \tag{4.21}$$

On the other hand, we recall that

$$\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = H(x) \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} \qquad \text{and} \qquad B^l = (\frac{\partial H}{\partial x_l})H^{-1},$$

so we have

$$\left[\frac{\partial}{\partial x_l}, e_i\right] = (B^l)_{ij} e_j. \tag{4.22}$$

Finally, by putting (4.21) and (4.22) into (4.20), the result follows. □

From Lemma 34, for $s, t = 1, 2, \ldots, r$, we have

$$\begin{aligned} &e_{2s-1}\nabla_{e_{2t-1}}\varphi^p = -e_{2s}\nabla_{e_{2t}}\varphi^p \qquad \text{and} \\ &e_{2s}\nabla_{e_{2t-1}}\varphi^p = e_{2s-1}\nabla_{e_{2t}}\varphi^p = \sqrt{-1}\, e_{2s}\nabla_{e_{2t}}. \end{aligned}$$

Therefore, the formula 4.18 can be further expressed as

$$\not{D}\nabla_{\frac{\partial}{\partial x_l}}\varphi^p = -\frac{1}{2}\sum_{1\le s,t\le r}\left(\begin{bmatrix}\sqrt{-1} & 1\end{bmatrix}\left(B^l_{st} + (B^l_{ts})^T\right)\begin{bmatrix}\sqrt{-1} \\ 1\end{bmatrix}\right)(e_{2s}\nabla_{e_{2t}}\varphi^p). \tag{4.23}$$

By Lemma 36, for $s, t = 1, 2, \ldots, r$,

$$G(e_{2s}\nabla_{e_{2t}}\varphi^p) = \frac{\lambda_t}{4\pi(\lambda_s + \lambda_t)} e_{2s}\nabla_{e_{2t}}\varphi^p,$$

and by Lemma 37, for $k, l, s, t = 1, 2, \ldots, r$,

$$\langle e_{2s}\nabla_{e_{2t}}\varphi^p, e_{2k}\nabla_{e_{2l}}\varphi^q\rangle_{L^2} = \lambda_t \pi \delta_{sk}\delta_{tl}(-1)^{s+t+k+l}\langle\varphi^p, \varphi^q\rangle_{L^2}.$$

As a result, we obtain

$$
\begin{aligned}
& J_b\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) \\
&= \frac{1}{16}\delta_{pq} \sum_{1\le s,t\le r} \frac{\lambda_t}{\lambda_s+\lambda_t} \begin{bmatrix} \sqrt{-1} & 1 \end{bmatrix} \\
&\quad \times \left(\left(B^l_{st} + (B^l_{ts})^T\right) \begin{bmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{bmatrix} \left(B^m_{st} + (B^m_{ts})^T\right) \right. \\
&\qquad \left. - \left(B^m_{st} + (B^m_{ts})^T\right) \begin{bmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{bmatrix} \left(B^l_{st} + (B^l_{ts})^T\right) \right) \begin{bmatrix} -\sqrt{-1} \\ 1 \end{bmatrix} \\
&= \frac{1}{16}\delta_{pq} \sum_{1\le s,t\le r} \frac{\lambda_t}{\lambda_s+\lambda_t} (2\sqrt{-1}\,) \begin{bmatrix} 0 & 1 \end{bmatrix} (Q^{lm}_{st} + Q^{lm}_{ts}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{\sqrt{-1}}{8}\delta_{pq} \sum_{1\le s,t\le r} \begin{bmatrix} 0 & 1 \end{bmatrix} Q^{lm}_{st} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= -I_2\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right)
\end{aligned}
$$

which means $I_2 + J_b = 0$.

Proof of (3). Without loss of generality, we assume in this proof that $\frac{\partial}{\partial y_i} = e_i$ on the fiber torus $\mathbf{T}_x$. We first express $\not{D}\nabla_{\frac{\partial}{\partial y^i}}\varphi^p$ as a linear combination of $e_i\varphi^p$'s:

Lemma 33. *For $i = 1, 2, \ldots, 2r$,*

$$\not{D}\nabla_{\frac{\partial}{\partial y^i}}\varphi^p = 2\pi\sqrt{-1}\, e_i\varphi^p.$$

The proof is similar to that of Lemma 32.

Then, by Lemma 36, we have

$$G\not{D}\nabla_{\frac{\partial}{\partial y^i}}\varphi^p = \frac{\sqrt{-1}}{2\lambda_{[i/2]}} e_i\varphi^p$$

and by Lemma 37,

$$\langle e_i\varphi^p, e_j\varphi^q\rangle_{L^2} = \delta_{[\frac{i}{2}][\frac{j}{2}]}(\sqrt{-1}\,)^i(-\sqrt{-1}\,)^j\langle\varphi^p, \varphi^q\rangle_{L^2}.$$

As a result, we obtain

$$\begin{aligned} J_f &= \sum_{1\le i,j\le n} -\frac{\pi}{\lambda_{[i/2]}}\langle e_i\varphi^p, e_j\varphi^q\rangle_{L^2} dy^i\wedge dy^j \\ &= 2\pi\delta_{pq}\sqrt{-1}\left(\frac{1}{\lambda_1}dy^1\wedge dy^2+\cdots+\frac{1}{\lambda_r}dy^{2r-1}\wedge dy^{2r}\right) \\ &= 2\pi\delta_{pq}\sqrt{-1}\left(\frac{1}{2}d\check{\mathbf{y}}\wedge A^{-1}\wedge d\check{\mathbf{y}}^T\right). \end{aligned}$$

The last equality is true because $\lambda_i = a_i$ by the assumption that $\frac{\partial}{\partial y_i} = e_i$ on the torus $\mathbf{T}_x$.

Proof of (4). It suffices to show that for $i, j, p, q = 1, 2, \ldots, 2r$,

$$\left\langle G\not{D}\nabla_{\frac{\partial}{\partial x_i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial y^j}}\varphi^q\right\rangle_{L^2} = \left\langle G\not{D}\nabla_{\frac{\partial}{\partial y^i}}\varphi^p, \not{D}\nabla_{\frac{\partial}{\partial x_j}}\varphi^q\right\rangle_{L^2} = 0$$

From Lemma 32 and Lemma 33, we know that $G\not{D}\nabla_{\frac{\partial}{\partial x_i}}\varphi^p$ can be expressed as a linear combination of $e_i\nabla_{e_j}\varphi^p$'s while $\not{D}\nabla_{\frac{\partial}{\partial y^j}}\varphi^q$ is a linear combination of $e_i\varphi^q$'s. The result follows from the fact that the inner product $\langle e_i\nabla_{e_j}\varphi^p, e_l\varphi^q\rangle_{L^2} = 0$, see Lemma 37.

This completes the proof of Lemma 30. □

In the rest of the section, we will prove the computational lemmas used in the proof of Lemma 30.

Lemma 34. *For $s = 1, 2, \ldots r$ and for any $\varphi \in \ker \not{D}_{(x,\check{y})}$, we have*

1) $e_{2s}\varphi = \sqrt{-1}\, e_{2s-1}\varphi$;
2) $e_{2s}\nabla_{e_j}\varphi = \sqrt{-1}\, e_{2s-1}\nabla_{e_j}\varphi$ *for* $j = 1, 2, \ldots, 2r$;
3) $\nabla_{e_{2s}}\varphi = \sqrt{-1}\,\nabla_{e_{2s-1}}\varphi$.

Proof. By Lemma 24, if $\varphi \in \ker \not{D}_{(x,\check{y})}$, then $\varphi = \chi_{I_0}\otimes\alpha$. Therefore, in order to show (1) and (2), it suffices to show that $e_{2s}\alpha = \sqrt{-1}\, e_{2s-1}\alpha$ and $e_{2s}\nabla_{e_j}\alpha = \sqrt{-1}\, e_{2s-1}\nabla_{e_j}\alpha$ for $j = 1, 2, \ldots, 2r$ which are true by the properties of Clifford multiplication. Furthermore, first note that

$$\begin{aligned} 0 = \not{D}\varphi &= \sum_{s=1}^{r}(e_{2s-1}\nabla_{e_{2s-1}}\varphi + e_{2s}\nabla_{e_{2s}}\varphi) \\ &= \sum_{s=1}^{r}\chi_{I_s}\otimes(\sqrt{-1}\,\nabla_{e_{2s-1}}\alpha + \nabla_{e_{2s}}\alpha), \end{aligned}$$

where

$$I_s := (1, \dots, 1, \underbrace{-1}_{s-\text{th}}, 1, \dots, 1)$$

and χ_{I_s} is defined in Lemma 24. Note that $\{\chi_{I_1}, ..., \chi_{I_r}\}$ is a linear independent set, so

$$\sqrt{-1}\,\nabla_{e_{2s-1}}\alpha + \nabla_{e_{2s}}\alpha = 0,$$

and equivalently,

$$\sqrt{-1}\,\nabla_{e_{2s-1}}\varphi + \nabla_{e_{2s}}\varphi = 0$$

for $s = 1, \dots, r$. □

To prove Lemma 36 and 37, we need following relations. For simplicity, write $f_{2i-1} = e_{2i}$ and $f_{2i} = -e_{2i-1}$.

Lemma 35. *Let $[\cdot,\cdot]$ denotes the superbracket. Then we have*

1) $[\not{D}, e_i] = -2\nabla_{e_i}$;
2) $[\not{D}, \nabla_{e_i}] = 2\pi\sqrt{-1}\,\lambda_{[i/2]} f_i$;
3) $[\not{D}^2, e_i] = -4\pi\sqrt{-1}\,\lambda_{[i/2]} f_i$;
4) $[\not{D}^2, \nabla_{e_i}] = -4\pi\sqrt{-1}\,\lambda_{[i/2]}\nabla_{f_i}$.

Proof. (1) just follows directly from $e_ie_j + e_je_i = -2\delta_{ij}$. Note that $\nabla_{e_i} \circ (e_l) = (e_l) \circ \nabla_{e_i}$ as $\nabla_{e_i} e_j = 0$. Therefore,

$$\not{D}\nabla_{e_i} - \nabla_{e_i}\not{D} = \sum_{l=1}^{n} e_l \nabla_{e_l}\nabla_{e_i} - \sum_{l=1}^{n} e_l \nabla_{e_i}\nabla_{e_l} = \sum_{l=1}^{n} e_l \otimes F^{S\otimes\mathcal{E}}(e_l, e_i).$$

Then, (2) follows from the fact that

$$F^{\mathcal{E}} = \frac{2\pi}{\sqrt{-1}}(\lambda_1 \omega_1 \wedge \omega_2 + \dots + \lambda_r \omega_{2r-1} \wedge \omega_{2r}).$$

Finally, (3) and (4) follows from (1) and (2) directly. □

Lemma 36. *For any $\varphi \in \ker \not{D}_{(x,\check{y})}$, we have*

1) $G(e_i\nabla_{e_j}\varphi) = \frac{1}{4\pi(\lambda_{[i/2]} + \lambda_{[\frac{j}{2}]})}(e_i\nabla_{e_j}\varphi)$;
2) $G(e_i\varphi) = \frac{1}{4\pi\lambda_{[i/2]}}(e_i\varphi)$.

Proof. By (3) and (4) in Lemma 35 and noting that $\not{D}\varphi = 0$, we have

$$\not{D}^2(e_i\nabla_{e_j}\varphi) = \frac{4\pi\lambda_{[j/2]}}{\sqrt{-1}} e_i\nabla_{f_j}\varphi + \frac{4\pi\lambda_{[i/2]}}{\sqrt{-1}} f_i\nabla_{e_j}\varphi.$$

Also by Lemma 34, $\sqrt{-1}\, e_i\nabla_{e_j}\varphi = e_i\nabla_{f_j}\varphi = f_i\nabla_{e_j}\varphi$. Therefore,

$$\not{D}^2(e_i\nabla_{e_j}\varphi) = 4\pi(\lambda_{[i/2]} + \lambda_{[j/2]})e_i\nabla_{e_j}\varphi. \tag{4.24}$$

Note that $e_i\nabla_{e_j}\varphi \perp \ker\not{D}$ as $e_i\nabla_{e_j} \in \Gamma(S^- \otimes \mathcal{E})$ and $\ker\not{D}$ is a subset of $\Gamma(S^+ \otimes \mathcal{E})$, so the restriction of the Green's operator G on $\ker\not{D}^\perp$ is the inverse of the operator $\not{D}^2$ and (1) is proved. The proof of (2) is similar. □

Lemma 37. *For any $\varphi, \psi \in \ker\not{D}_{(x,\check{y})}$, we have*

1) $\langle e_i\varphi, e_j\psi\rangle_{L^2} = \delta_{[\frac{i}{2}][\frac{j}{2}]}(\sqrt{-1}\,)^i(-\sqrt{-1}\,)^j\langle\varphi,\psi\rangle_{L^2}$;
2) $\langle e_i\nabla_{e_k}\varphi, e_j\nabla_{e_l}\psi\rangle_{L^2} = \pi\delta_{[\frac{i}{2}][\frac{j}{2}]}\delta_{[\frac{k}{2}][\frac{l}{2}]}(\sqrt{-1}\,)^{i+k}(-\sqrt{-1}\,)^{j+l}\lambda_t\langle\varphi,\psi\rangle_{L^2}$;
3) $\langle e_i\nabla_{e_j}\varphi, e_k\psi\rangle_{L^2} = \langle\nabla_{e_j}\varphi,\psi\rangle_{L^2} = 0$.

Proof. First note that

$$\langle e_i\varphi, e_j\varphi\rangle_{L^2} = -\langle\varphi, e_ie_j\varphi\rangle_{L^2}.$$

Then (1) just follows from Lemma 24 and the properties of Clifford multiplication.

To prove (2), similar to the method of proving (1), it suffices to show that

$$\langle\nabla_{e_{2t-1}}\varphi, \nabla_{e_{2v-1}}\varphi\rangle_{L^2} = \pi\delta_{tv}\lambda_t\langle\varphi,\psi\rangle_{L^2}.$$

By (1) in Lemma 35 and $\not{D}\varphi = 0$, we have

$$\nabla_{e_i}\varphi = -\frac{1}{2}\not{D}e_i\varphi. \tag{4.25}$$

As a result,

$$\langle\nabla_{e_{2t-1}}\varphi, \nabla_{e_{2v-1}}\varphi\rangle_{L^2} = \frac{1}{4}\langle\not{D}e_{2t-1}\varphi, \not{D}e_{2v-1}\varphi\rangle_{L^2} = \frac{1}{4}\langle\not{D}^2e_{2t-1}\varphi, e_{2v-1}\varphi\rangle_{L^2}.$$

On the other hand, by the proof of Lemma 36, $\not{D}^2e_{2t-1}\varphi = 4\pi\lambda_te_{2t-1}\varphi$. Therefore,

$$\langle\nabla_{e_{2t-1}}\varphi, \nabla_{e_{2v-1}}\varphi\rangle_{L^2} = \pi\lambda_t\langle e_{2t-1}\varphi, e_{2v-1}\varphi\rangle_{L^2}$$

and hence (2) follows from (1).

Again, to prove (3), it is enough to show that $\langle \nabla_{e_j}\varphi, \varphi\rangle_{L^2} = 0$. By equation (4.25),

$$\langle \nabla_{e_j}\varphi, \varphi\rangle_{L^2} = -\frac{1}{2}\langle \not{D} e_i\varphi, \varphi\rangle_{L^2} = -\frac{1}{2}\langle e_i\varphi, \not{D}\varphi\rangle_{L^2} = 0.$$

□

4.4. Invertibility of the SYZ transformation

Theorem 38. *$(\mathcal{F}^{SYZ})^2$ is the identity map.*

Proof. Suppose that $(\check{\mathcal{C}}, \check{\mathcal{E}}) = \mathcal{F}^{SYZ}(\mathcal{C}, \mathcal{E})$ and $(\check{\check{\mathcal{C}}}, \check{\check{\mathcal{E}}}) = \mathcal{F}^{SYZ}(\check{\mathcal{C}}, \check{\mathcal{E}})$. We need to show that $\mathcal{C} = \check{\check{\mathcal{C}}}$ and $\mathcal{E} \cong \check{\check{\mathcal{E}}}$.

We first consider the case that $\mathcal{C} = X$ and the restriction of $F^{\mathcal{E}}$ on each fiber is non-degenerate, and come back to the general case after. In this particular case, it is easy to see that $\mathcal{C} = \check{\check{\mathcal{C}}} = X$ by Theorem 26 and we just need to prove that $\mathcal{E}$ is isomorphic to $\check{\check{\mathcal{E}}}$. We prove this by the following steps:

Step 1. The construction of the map $\mathcal{I} : \mathcal{E} \to \check{\check{\mathcal{E}}}$

Recall that $\check{\mathcal{E}}_{(x,\check{y})}$ is defined to be $\ker \not{D}_{(x,\check{y})}$ whose elements are sections of the bundle

$$\pi^*(\not{S} \otimes \mathcal{E}) \otimes \mathcal{P}|_{\check{\pi}^{-1}(x,\check{y})} \to X \times_B \check{X}.$$

In other words, for each vector in $\check{\mathcal{E}}_{(x,\check{y})}$, we can associate a section of the bundle $\pi^*(\not{S} \otimes \mathcal{E}) \otimes \mathcal{P}|_{\check{\pi}^{-1}(x,\check{y})}$. This gives rise to a section Ψ of the bundle

$$(\check{\pi}^*\check{\mathcal{E}})^* \otimes \pi^*(\not{S} \otimes \mathcal{E}) \otimes \mathcal{P}$$

The authors of [5] and [25] also introduced a section to show that the square of the Nahm transformation for a four-torus is an identity map, whose family version is exactly the section Ψ we have just defined. As each point (x, y) in X, the section Ψ induces a map

$$\Psi : \mathcal{E}^*_{(x,y)} \longrightarrow \Gamma((\check{\pi}^*\check{\mathcal{E}})^* \otimes \pi^*\not{S} \otimes \mathcal{P}|_{\pi^{-1}(x,y)}).$$

Also, we use δ to denote the map from a unitary bundle to its dual by its Hermitian metric. Then the composition of the maps

$$\begin{aligned}
\text{(4.26)} \qquad \mathcal{E}_{(x,y)} &\xrightarrow{\delta} \mathcal{E}^*_{(x,y)} \xrightarrow{\Psi} \Gamma((\check{\pi}^*\check{\mathcal{E}})^* \otimes \pi^*\not{S} \otimes \mathcal{P}|_{\pi^{-1}(x,y)}) \\
&\xrightarrow{\delta} \Gamma(((\check{\pi}^*\check{\mathcal{E}})^* \otimes \pi^*\not{S} \otimes \mathcal{P}|_{\pi^{-1}(x,y)})^*) \\
&= \Gamma((\check{\pi}^*\check{\mathcal{E}}) \otimes (\pi^*\not{S})^* \otimes \mathcal{P}^*|_{\pi^{-1}(x,y)})
\end{aligned}$$

gives a map

$$\mathcal{I} := \frac{\mathrm{vol}(\mathbf{T}_x)}{\sqrt{\mathrm{rk}(\check{\mathcal{E}})}}\delta \circ \Psi \circ \delta \,:\, \mathcal{E}_{(x,y)} \longrightarrow \Gamma((\check{\pi}^*\check{\mathcal{E}}) \otimes (\pi^*\$)^* \otimes \mathcal{P}^*|_{\pi^{-1}(x,y)}),$$

where $\mathrm{vol}(\mathbf{T}_x)$ is the volume of the fiber torus $\mathbf{T}_x$ and $\mathrm{rk}(\check{\mathcal{E}})$ is the rank of the bundle $\check{\mathcal{E}}$.

Since $T(\mathbf{T}_x)$ is naturally identified with $T^*(\check{\mathbf{T}}_x)$, the bundle $(\pi^*\$)^* \to X \times_B \check{X}$ is in fact the pullback bundle $\check{\pi}^*\check{\$}$, where $\check{\$}$ is the family spinor bundle of the Riemannian fiber bundle $\check{\pi} : X \times_B \check{X} \to \check{X}$. Therefore, we can define the Dirac operator

$$\check{\not{D}}_{(x,y)} : \Gamma((\check{\pi}^*(\check{\mathcal{E}} \otimes \check{\mathcal{S}}) \otimes \mathcal{P}^*|_{\pi^{-1}(x,y)}) \to \Gamma((\check{\pi}^*(\check{\mathcal{E}} \otimes \check{\mathcal{S}}) \otimes \mathcal{P}^*|_{\pi^{-1}(x,y)})$$

and carry out the SYZ transformation. Hence we have

$$\check{\check{\mathcal{E}}}_{(x,y)} = \ker \check{\not{D}}_{(x,y)}.$$

The following proposition says that $\mathcal{I}$ is in fact a map from $\mathcal{E}$ to $\check{\check{\mathcal{E}}}$.

Proposition 39. $\check{\not{D}}_{(x,y)}(\mathcal{I}(f)) = 0$ *for any point* (x, y) *in* X *and any vector* f *in the fiber* $\mathcal{E}_{(x,y)}$.

Proof. By choosing a local orthonormal frame $\{\varphi^i\}$ of $\ker \not{D}$ with corresponding frame $\{f^i\}$ of $\check{\mathcal{E}}$, for $i = 1, 2, \ldots, \mathrm{rk}(\check{\mathcal{E}})$, we are able to write down the map $\mathcal{I}$ explicitly:

$$\mathcal{I}(f) = \frac{\mathrm{vol}(\mathbf{T}_x)}{\sqrt{\mathrm{rk}(\check{\mathcal{E}})}} \sum_{i=1}^{n} \delta(\langle f, \varphi^i\rangle_{\mathcal{E}}) \otimes f^i, \tag{4.27}$$

where $\langle\ ,\ \rangle_{\mathcal{E}}$ denotes the metric of the bundle $\mathcal{E}$. Also, we choose a frame $\{e_1, \ldots, e_n\}$ of $V_{\check{\mathcal{C}}}$ with the same property on the torus over the point $(x, \check{y})$ as in the proof of Proposition 21. Let $\{\check{e}_1, \ldots, \check{e}_n\}$ be the frame of $V_{\mathcal{C}}$ that are identified with $\{e_1, \ldots, e_n\}$ under the identification between the bundles $V_{\check{\mathcal{C}}}$ and $V_{\mathcal{C}}$.

Then we have

$$\check{\not{D}}_{(x,y)}(\mathcal{I}(f)) \tag{4.28}$$

$$\begin{aligned}
&= \frac{\mathrm{vol}(\mathbf{T}_x)}{\sqrt{\mathrm{rk}(\check{\mathcal{E}})}} \sum_{1\le i,j\le n} \check{e}_j \check{\nabla}_{\check{e}_j}(\delta(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}})\otimes f^i) \qquad (4.29)\\
&= \frac{\mathrm{vol}(\mathbf{T}_x)}{\sqrt{\mathrm{rk}(\check{\mathcal{E}})}} \sum_{1\le i,j\le n} \check{e}_j \Bigg(\check{\nabla}^{\pi^*\$^*\otimes\mathcal{P}^*}_{\check{e}_j} \Bigg(\delta(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}})\otimes f^i \\
&\qquad + \mathrm{vol}(\mathbf{T}_x) \sum_{1\le i,j\le n} \check{e}_j \delta(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}}) \Bigg) \otimes \check{\nabla}^{\check{\mathcal{E}}}_{\check{e}_j} f^i \Bigg),
\end{aligned}$$

where $\check{\nabla}$ is the connection of the bundle $\check{\pi}^*(\check{\mathcal{E}}\otimes\check{\$})\otimes\mathcal{P}^*$. We further note that

$$\begin{aligned}
\check{\nabla}^{\pi^*\$^*\otimes\mathcal{P}^*}_{\check{e}_j}(\delta(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}})) &= \delta(\nabla^{\pi^*\$\otimes\mathcal{P}}_{\check{e}_j}(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}}))\\
&= \delta(\langle f,\nabla_{\check{e}_j}\varphi^i\rangle_{\mathcal{E}_{(x,y)}}),
\end{aligned} \tag{4.30}$$

where ∇ denotes the connection of the bundle $\pi^*(\mathcal{E}\otimes\$)\otimes\mathcal{P}$.

On the other hand, recall that the connection of $\check{\mathcal{E}}$ is given by

$$\check{A}^j_{ik} = \langle\varphi^k, \nabla_{\check{e}_k}\varphi^i\rangle_{L^2},$$

so

$$\sum_{i=1}^n(\delta(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}})\otimes\check{\nabla}^{\check{\mathcal{E}}}_{\check{e}_j}f^i) = \sum_{i=1}^n\left(\delta\left(\left\langle f,\sum_{k=1}^n\overline{\check{A}^j_{ki}}\varphi^k\right\rangle_{\mathcal{E}_{(x,y)}}\right)\otimes f^i\right).$$

Also, we note that

$$\sum_{k=1}^n\overline{\check{A}^j_{ki}}\varphi^k = -\sum_{k=1}^n\langle\varphi^k,\nabla_{\check{e}_j}\varphi^i\rangle_{L^2}\varphi^k = -P\nabla_{\check{e}_j}\varphi^i,$$

so we have

$$\sum_{i=1}^n(\delta(\langle f,\varphi^i\rangle_{\mathcal{E}_{(x,y)}})\otimes\check{\nabla}^{\check{\mathcal{E}}}_{\check{e}_j}f^i) = \sum_{i=1}^n(\delta(\langle f,-P\nabla_{\check{e}_j}\varphi^i\rangle_{\mathcal{E}_{(x,y)}})\otimes f^i). \tag{4.31}$$

By plugging equations (4.30) and (4.31) into (4.28), we get

$$\begin{aligned}
\check{\not{D}}_{(x,y)}(\mathcal{I}(f)) &= \frac{\operatorname{vol}(\mathbf{T}_x)}{\sqrt{\operatorname{rk}(\check{\mathcal{E}})}} \sum_{1\le i,j\le n} \check{e}_j(\delta(\langle f, (I-P)\nabla_{\check{e}_j}\varphi^i\rangle_{\mathcal{E}_{(x,y)}}) \otimes f^i) \\
&= \frac{\operatorname{vol}(\mathbf{T}_x)}{\sqrt{\operatorname{rk}(\check{\mathcal{E}})}} \sum_{1\le i,j\le n} \check{e}_j(\delta(\langle f, \not{D}G\not{D}\nabla_{\check{e}_j}\varphi^i\rangle_{\mathcal{E}_{(x,y)}}) \otimes f^i) \\
&= \frac{\operatorname{vol}(\mathbf{T}_x)}{\sqrt{\operatorname{rk}(\check{\mathcal{E}})}} \sum_{i=1}^{n} (\delta(\langle f, \sum_{j=1}^{n} e_j\not{D}G\not{D}\nabla_{\check{e}_j}\varphi^i\rangle_{\mathcal{E}_{(x,y)}}) \otimes f^i)
\end{aligned}$$

Then by Lemma 36, 35 , 33 and noting that $\not{D}\varphi^i = 0$, we have

$$\begin{aligned}
\sum_{j=1}^{n} e_j\not{D}G\not{D}\nabla_{\check{e}_j}\varphi^i &= -\sqrt{-1}\sum_{j=1}^{n} \frac{1}{\lambda_{[j/2]}} e_j\nabla_{e_j}\varphi^i \\
&= -\sqrt{-1}\sum_{s=1}^{n} \frac{1}{\lambda_s}(e_{2s-1}\nabla_{e_{2s-1}}\varphi^i + e_{2s}\nabla_{e_{2s}}\varphi^i) \\
&= 0.
\end{aligned}$$

The last equality follows from Lemma 34 that $e_{2s-1}\nabla_{e_{2s-1}}\varphi^i + e_{2s}\nabla_{e_{2s}}\varphi^i = 0$. As a result,

$$\check{\not{D}}_{(x,y)}(\mathcal{I}(f)) = 0.$$

□

Step 2. Show that $\langle f, g\rangle_{\mathcal{E}} = \langle \mathcal{I}f, \mathcal{I}g\rangle_{\check{\mathcal{E}}}$ for any $f, g \in \mathcal{E}$

Without loss of generality, we prove this on the fiber of $\mathcal{E}$ over a point $(x, 0)$ in X. We start with a more careful choice of local orthonormal frame $\{\varphi^i\}$, for $i = 1, 2, \ldots, \operatorname{rk}(\mathcal{E})$, of $\ker\not{D}$ as below: Let E be a projectively flat bundle over $\mathbf{T}$ with curvature to be $2\pi\sqrt{-1}\,F^E \cdot I_E$ for an non-degenerate constant two-form F^E on $\mathbf{T}$. As a non-degenerate invariant two-form, F^E induces an invertible linear map

$$\varepsilon : V \to \check{V}.$$

Note that V acts on the torus $\mathbf{T}$ by translation, so the inverse map ε^{-1} gives an action of the dual vector space $\check{V}$ on the torus $\mathbf{T}$. Moreover, this action can be lift to the bundle E by parallel translation of E. By direct calculations, one can prove the following proposition, due to the fact that F^E is a constant two-form.

Proposition 40. *Let $\check{y}$ be a point in $\check{V}$ which induces a map $\check{y}: E \to E$. Then*

$$\check{y}_* \nabla^E = \nabla^E - 2\pi\sqrt{-1}\,\check{y},$$

where $\check{y}$ on the right hand side is regarded as an one-form on the torus $\mathbf{T}$.

The action of $\check{V}$ on $\mathbf{T}$ discussed above can be generalized to the family case so that $\check{V}$ acts on X with a lifting to the bundle $\mathcal{E}$. Similarly, it has the property that

$$\check{y}_* \nabla^{\mathcal{E}} = \nabla^{\mathcal{E}} - 2\pi\sqrt{-1}\,\check{y}$$

for any point $\check{y}$ in $\check{V}$.

Therefore, if a section φ lies in $\ker \not{D}_{(x,\check{0})}$, then $\check{y}_*\varphi$ gives a section in $\ker \not{D}_{(x,\check{y})}$. This motivates us to construct an orthonormal frame $\{\varphi^i\}$, of $\ker \not{D}$ as below: We first choose an orthonormal frame $\{\varphi^i_x(y)\}$ of $\ker \not{D}|_{B\times\{0\}}$ and define $\varphi^i_{(x,\check{y})}(y)$ to be $\check{y}_*\varphi^i_x(y)$, then we get an orthonormal frame $\{\varphi^i_{(x,\check{y})}(y)\}$ of $\ker \not{D}_{(x,\check{y})}$.

We have

(4.32)
$$\langle \mathcal{I}f, \mathcal{I}g\rangle_{\check{\mathcal{E}}} = \frac{(\mathrm{vol}(\mathbf{T}_x))^2}{\mathrm{rk}(\check{\mathcal{E}})}\langle f, g\rangle_{\mathcal{E}} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\check{\mathbf{T}}_x} \langle \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\check{\mathbf{T}}_x) d\check{y}$$

by (4.27), where $\mathrm{vol}(\check{\mathbf{T}}_x)$ equals to $1/\mathrm{vol}(\mathbf{T}_x)$. Therefore, it remains to show that

$$\frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\check{\mathbf{T}}_x} \langle \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} = 1.$$

To prove this, first note that we can choose a region $\mathcal{A}$ of V with image $\check{\mathcal{A}} = \varepsilon(\mathcal{A})$ such that $\mathcal{A}$ and $\check{\mathcal{A}}$ cover the torus $\mathbf{T}$ and $\check{\mathbf{T}}$ for $\mathrm{rk}(\mathcal{E})^2$ and $\mathrm{rk}(\check{\mathcal{E}})^2$ times respectively. By Proposition 40, we have

$$\begin{aligned}\int_{\check{\mathcal{A}}} \langle \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} &= \int_{\check{\mathcal{A}}} \langle \varphi^i_x(\varepsilon^{-1}(\check{y})), \varphi^i_x(\varepsilon^{-1}(\check{y}))\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} \\ &= \int_{\mathcal{A}} \langle \varphi^i_x(y), \varphi^i_x(y)\rangle \mathrm{vol}(\mathbf{T}_x) \det(\varepsilon) dy.\end{aligned}$$

Since $\det(\varepsilon) = \frac{\mathrm{rk}(\check{\mathcal{E}})^2}{\mathrm{rk}(\mathcal{E})^2}$ and the choices of $\mathcal{A}$ and $\check{\mathcal{A}}$, we then obtain

$$\begin{aligned}\int_{\check{\mathbf{T}}_x} \langle \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} &= \int_{\mathbf{T}_x} \langle \varphi^i_x(y), \varphi^i_x(y)\rangle \mathrm{vol}(\mathbf{T}_x) dy \\ &= \langle \varphi^i, \varphi^i\rangle_{L^2} = 1.\end{aligned} \tag{4.33}$$

Hence,

$$\frac{1}{\mathrm{rk}(\check{\mathcal{E}})}\sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})}\int_{\check{\mathbf{T}}}\langle\varphi^i_{(x,\check{y})}(0),\varphi^i_{(x,\check{y})}(0)\rangle\mathrm{vol}(\mathbf{T}_x)d\check{y}=1 \tag{4.34}$$

which completes the step 2.

Step 3. Prove that $\mathcal{I}$ preserves the connections

Without loss of generality, we show that

$$\langle\nabla^{\mathcal{E}}_Z f,g\rangle_{\mathcal{E}}=\langle\nabla^{\check{\check{\mathcal{E}}}}_Z\mathcal{I}(f),\mathcal{I}(g)\rangle_{\check{\check{\mathcal{E}}}}$$

at a point $(x,0)$ for any sections f,g of the bundle $\mathcal{E}$ and any vector Z in $T_{(x,0)}X$.

Recall that ∇ and $\check{\nabla}$ are the connections of $\pi^*(\mathcal{E}\otimes\$)\otimes\mathcal{P}$ and $\check{\pi}^*(\check{\mathcal{E}}\otimes\check{\$})\otimes\mathcal{P}^*$ respectively. Let $\check{P}$ be the projection to $\ker\check{\not{D}}$ and we have $\nabla^{\check{\check{\mathcal{E}}}}_Z=\check{P}\check{\nabla}_{\hat{Z}}$, where $\hat{Z}$ is the lift of Z to $X\times_B\check{X}$. Then,

$$\langle\nabla^{\check{\check{\mathcal{E}}}}_Z\mathcal{I}(f),\mathcal{I}(g)\rangle_{\check{\check{\mathcal{E}}}}=\langle\check{P}\check{\nabla}_{\hat{Z}}\mathcal{I}(f),\mathcal{I}(g)\rangle_{L^2}=\langle\check{\nabla}_{\hat{Z}}\mathcal{I}(f),\mathcal{I}(g)\rangle_{L^2}.$$

Note that

$$\begin{aligned}\check{\nabla}(\mathcal{I}(f))=\frac{\mathrm{vol}(\mathbf{T}_x)}{\sqrt{\mathrm{rk}(\check{\mathcal{E}})}}\sum_{i=1}^{n}\Big(&\delta(\langle f,\nabla\varphi^i\rangle_{\mathcal{E}})\otimes f^i+\delta(\langle\nabla^{\mathcal{E}}f,\varphi^i\rangle_{\mathcal{E}})\otimes f^i\\&+\delta(\langle f,\varphi^i\rangle_{\mathcal{E}})\otimes\nabla^{\check{\pi}^*\check{\mathcal{E}}}f^i\Big),\end{aligned}$$

so

$$\begin{aligned}\langle\nabla^{\check{\check{\mathcal{E}}}}_Z\mathcal{I}(f),\mathcal{I}(g)\rangle_{\check{\check{\mathcal{E}}}}&=\frac{1}{\mathrm{rk}(\check{\mathcal{E}})}\sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})}\langle f,g\rangle_{\mathcal{E}}\int_{\check{\mathbf{T}}_x}\overline{\langle\nabla_{\hat{Z}}\varphi^i,\varphi^i\rangle}\mathrm{vol}(\mathbf{T}_x)d\check{y}\\&\quad+\frac{1}{\mathrm{rk}(\check{\mathcal{E}})}\sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})}\langle\nabla^{\mathcal{E}}_Z f,g\rangle_{\mathcal{E}}\int_{\check{\mathbf{T}}_x}\overline{\langle\varphi^i,\varphi^i\rangle}\mathrm{vol}(\mathbf{T}_x)d\check{y}\\&\quad+\frac{1}{\mathrm{rk}(\check{\mathcal{E}})}\sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})}\langle f,g\rangle_{\mathcal{E}}\int_{\check{\mathbf{T}}_x}\overline{\langle\varphi^i,\varphi^i\rangle}\langle\nabla^{\check{\pi}^*\check{\mathcal{E}}}_Z f^i,f^i\rangle_{\check{\mathcal{E}}}\mathrm{vol}(\mathbf{T}_x)d\check{y}\\&=:N_1+N_2+N_3.\end{aligned}$$

Next we compute N_1, N_2 and N_3.

Proposition 41. *If a vector Z in $T_{(x,0)}X$ is decomposed as the sum of a horizontal vector W and a vertical vector T with respect to the Riemannian fiber bundle $\check{p}: \check{X} \to B$, then*

$$N_1 = -\langle f, g\rangle_{\mathcal{E}} \frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle \mathrm{vol}(\mathbf{T}_x) dy; \tag{4.35}$$

$$N_2 = \langle \nabla_Z^{\mathcal{E}} f, g\rangle_{\mathcal{E}}; \tag{4.36}$$

$$N_3 = \langle f, g\rangle_{\mathcal{E}} \frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle \mathrm{vol}(\mathbf{T}_x) dy. \tag{4.37}$$

Proof. First note that $\overline{\langle \nabla_{\hat{Z}} \varphi^i, \varphi^i\rangle} = \langle \varphi^i, \nabla_{\hat{Z}} \varphi^i\rangle = -\langle \nabla_{\hat{Z}} \varphi^i, \varphi^i\rangle$. Therefore,

$$N_1 = -\frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \langle f, g\rangle_{\mathcal{E}} \int_{\check{\mathbf{T}}_x} \langle \nabla_{\hat{Z}} \varphi^i, \varphi^i\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y}.$$

By Proposition 40,

$$\begin{aligned}
&\int_{\check{\mathbf{T}}_x} \langle \nabla_{\hat{Z}} \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} \\
&= \int_{\check{\mathbf{T}}_x} \langle (\nabla^{\mathcal{E}\otimes\$} - 2\pi\sqrt{-1}\,\check{y})_Z \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} \\
&= \int_{\check{\mathbf{T}}_x} \langle \nabla_Z^{\mathcal{E}\otimes\$} \varphi_x^i(\varepsilon^{-1}(\check{y})), \varphi_x^i(\varepsilon^{-1}(\check{y}))\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y}.
\end{aligned}$$

Then, by changing the integral variables to be y, we get

$$\begin{aligned}
&\int_{\check{\mathbf{T}}_x} \langle \nabla_{\hat{Z}} \varphi^i_{(x,\check{y})}(0), \varphi^i_{(x,\check{y})}(0)\rangle \mathrm{vol}(\mathbf{T}_x) d\check{y} \\
&= \int_{\mathbf{T}_x} \langle \nabla_Z^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle \mathrm{vol}(\mathbf{T}_x) dy \\
&= \langle \nabla_W^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle_{L^2} + \langle \nabla_T^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle_{L^2} \\
&= \langle \nabla_W^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle_{L^2}.
\end{aligned}$$

The last equality holds because of Lemma 37. Then,

$$N_1 = -\langle f, g\rangle_{\mathcal{E}} \frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E}\otimes\$} \varphi_x^i(y), \varphi_x^i(y)\rangle \mathrm{vol}(\mathbf{T}_x) dy.$$

Note that $\overline{\langle \varphi^i, \varphi^i \rangle} = \langle \varphi^i, \varphi^i \rangle$, so

$$N_2 = \langle \nabla_Z^{\mathcal{E}} f, g \rangle_{\mathcal{E}} \left(\frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\check{\mathbf{T}}_x} \overline{\langle \varphi^i, \varphi^i \rangle} \mathrm{vol}(\mathbf{T}_x) d\check{y} \right) = \langle \nabla_Z^{\mathcal{E}} f, g \rangle_{\mathcal{E}}$$

by equation 4.33. Because the choice of the frame $\{\varphi^i\}$ and the term $\langle \nabla_Z^{\check{\pi}^* \check{\mathcal{E}}} f^i, f^i \rangle_{\check{\mathcal{E}}}$ are independent from $\check{y}$,

$$\langle \nabla_Z^{\check{\pi}^* \check{\mathcal{E}}} f^i, f^i \rangle_{\check{\mathcal{E}}} = \langle \nabla_W^{\check{\pi}^* \check{\mathcal{E}}} f^i, f^i \rangle_{\check{\mathcal{E}}} = \int_{\mathbf{T}_x} \langle \nabla_{\hat{W}} \varphi^i_{(x,\check{y})}(y), \varphi^i_{(x,\check{y})}(y) \rangle \mathrm{vol}(\mathbf{T}_x) dy$$

because $\nabla^{\check{\pi}^* \check{\mathcal{E}}}$ is the pull back connection such that $\nabla_T^{\check{\pi}^* \check{\mathcal{E}}} f^i = 0$. Then, we have

$$\begin{aligned}
&\int_{\mathbf{T}_x} \langle \nabla_{\hat{W}} \varphi^i_{(x,\check{y})}(y), \varphi^i_{(x,\check{y})}(y) \rangle \mathrm{vol}(\mathbf{T}_x) dy \\
&= \int_{\mathbf{T}_x} \langle (\nabla^{\mathcal{E} \otimes \$} - 2\pi\sqrt{-1}\, \check{y})_W \varphi^i_{(x,\check{y})}(y), \varphi^i_{(x,\check{y})}(y) \rangle \mathrm{vol}(\mathbf{T}_x) dy \\
&= \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E} \otimes \$} \varphi^i_x(\varepsilon^{-1}(\check{y}) + y), \varphi^i_x(\varepsilon^{-1}(\check{y}) + y) \rangle \mathrm{vol}(\mathbf{T}_x) dy \\
&= \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E} \otimes \$} \varphi^i_x(y), \varphi^i_x(y) \rangle \mathrm{vol}(\mathbf{T}_x) dy
\end{aligned}$$

which is independent of $\check{y}$. Therefore,

$$\begin{aligned}
N_3 &= \left(\langle f, g \rangle_{\mathcal{E}} \frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E} \otimes \$} \varphi^i_x(y), \varphi^i_x(y) \rangle \mathrm{vol}(\mathbf{T}_x) dy \right) \\
&\quad \times \left(\int_{\check{\mathbf{T}}_x} \overline{\langle \varphi^i, \varphi^i \rangle} \mathrm{vol}(\mathbf{T}_x) d\check{y} \right) \\
&= \langle f, g \rangle_{\mathcal{E}} \frac{1}{\mathrm{rk}(\check{\mathcal{E}})} \sum_{i=1}^{\mathrm{rk}(\check{\mathcal{E}})} \int_{\mathbf{T}_x} \langle \nabla_W^{\mathcal{E} \otimes \$} \varphi^i_x(y), \varphi^i_x(y) \rangle \mathrm{vol}(\mathbf{T}_x) dy,
\end{aligned}$$

where the second equality follows from equality 4.33. □

By this proposition,

$$\langle \nabla_Z^{\mathcal{E}} f, g \rangle_{\mathcal{E}} = \langle \nabla_Z^{\check{\check{\mathcal{E}}}} \mathcal{I}(f), \mathcal{I}(g) \rangle_{\check{\check{\mathcal{E}}}}$$

which finishes step 3. As a result, the bundle $\mathcal{E}$ is isomorphic to the bundle $\check{\check{\mathcal{E}}}$ for the case that $\mathcal{C} = X$ and the restriction of $F^{\mathcal{E}}$ on each fiber is non-degenerate.

For the general case, we consider $\mathcal{V}_0$-invariant sections and recall that the vector space $\check{\mathcal{E}}_{(x,\check{y})}$ is identified with a finite dimensional subspace of

$$\Gamma(\pi^*(\mathcal{E} \otimes \$) \otimes \mathcal{P}|_{\pi^{-1}(x,\check{y})})^{\mathcal{V}_0}.$$

Similar to the previous case, this identification gives us a section $\Psi \in \Gamma((\check{\pi}^*\check{\mathcal{E}})^* \otimes \pi^*\$ \otimes \pi^*\mathcal{E} \otimes \mathcal{P})$ which induces a map

$$\Psi : \mathcal{E}^*_{(x,y)} \longrightarrow \Gamma((\check{\pi}^*\check{\mathcal{E}})^* \otimes \pi^*\$ \otimes \mathcal{P}|_{\pi^{-1}(x,\check{y})}).$$

Then we can construct a map $\mathcal{I} : \mathcal{E}_{(x,y)} \longrightarrow \Gamma((\check{\pi}^*\check{\mathcal{E}}) \otimes (\pi^*\$)^* \otimes \mathcal{P}^*|_{\pi^{-1}(x,\check{y})})^{\check{\mathcal{V}}_0}$ defined by

$$\mathcal{I} := \frac{\mathrm{vol}(\mathbf{T}_x)}{\sqrt{\mathrm{rk}(\check{\mathcal{E}})}}\, \delta \circ \Psi \circ \delta,$$

where $(x, y) \in \mathcal{C}$ and $(x, \check{y}) \in \check{\mathcal{C}}$. By the definition of $\mathcal{F}^{SYZ}$, the fiber $\check{\check{\mathcal{E}}}_{(x,\check{y})}$ is the kernel space of

$$\check{\not{D}}_{(x,y)} : \Gamma(\check{\pi}^*(\check{\mathcal{E}} \otimes \check{\$}) \otimes \mathcal{P}^*|_{\pi^{-1}(x,y)})^{\check{\mathcal{V}}_0} \longrightarrow \Gamma(\check{\pi}^*(\check{\mathcal{E}} \otimes \check{\$}) \otimes \mathcal{P}^*|_{\pi^{-1}(x,y)})^{\check{\mathcal{V}}_0}.$$

Now by the same arguments in the proof for the non-degenerate case, we can show that $\mathcal{I}$ is in fact an isomorphism of vector space between $\mathcal{E}_{(x,y)}$ and $\ker \check{\not{D}}_{(x,y)}$ for $(x, y) \in \mathcal{C}$. Hence by the construction of $\mathcal{F}^{SYZ}$, we have

$$\check{\check{\mathcal{C}}} = \{(x, y) \in X : \ker \check{\not{D}}_{(x,y)} \neq 0\} = \mathcal{C}.$$

Moreover, similar arguments can show that

$$\mathcal{I} : \mathcal{E} \to \check{\check{\mathcal{E}}}$$

preserves both Hermitian metrics and connections. Therefore, we obtain a bundle isomorphism $\mathcal{I}$ between $\mathcal{E}$ and $\check{\check{\mathcal{E}}}$ and we finish the proof of Theorem 38. $\square$

4.5. Transformation between semi-affine coisotropic A-branes and semi-affine B-branes

The mechanism of transformation between semi-affine coisotropic A-branes and semi-affine B-branes can be reformulated in a more concise way in terms of generalized complex geometry. We are going to have a quick review and details can be found in [14]. For any real vector space V, we define its generalized space

$$\mathfrak{V} := V \oplus V^*$$

with a natural non-degenerate inner product of signature (n, n) defined by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))$$

for $X, Y \in V$ and $\xi, \eta \in V^*$. A generalized complex structure on $\mathfrak{V}$ is a complex structure $\mathcal{J}$ on $\mathfrak{V}$ which preserves the above inner product. There are two examples of generalized complex structures that we are particularly interested:

Example 42 (Complex case). If V is a vector space equipped with a complex structure J, then J induces a generalized complex structure

$$\mathcal{J}_J := \begin{bmatrix} J & 0 \\ 0 & -J^* \end{bmatrix}.$$

Example 43 (Symplectic case). If V is a vector space equipped with a symplectic structure ω, then ω induces a generalized complex structure

$$\mathcal{J}_\omega := \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}.$$

A generalized subspace of V is defined to be a pair (W, F), where W is a subspace of V and F is a two-form on W. Also, we define the generalized tangent space of (W, F) to be

$$\mathcal{T}_{(W,F)} := \{X + \xi \in W \oplus V^* : \xi|_W = \iota_X F\} < \mathfrak{V}.$$

A generalized complex subspace of V with respect to a generalized complex structure $\mathcal{J}$ is a generalized subspace (W, F) such that the associated subspace V_W^F is stable under $\mathcal{J}$.

Let $(\mathcal{C}, \mathcal{E})$ be a semi-affine brane on a Calabi-Yau manifold X. The curvature of $\mathcal{E}$ is the multiple of a two-form $2\pi\sqrt{-1}\, F^{\mathcal{E}}$ on $\mathcal{C}$. For each point p in $\mathcal{C}$, we can define a generalized space $\mathcal{T}_p X = T_p X \oplus T_p^* X$ and its generalized subspace $(T_p\mathcal{C}, F^{\mathcal{E}})$. Furthermore, since X is Calabi-Yau, we can construct two generalized structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$ on $T_p X$ which come from the complex and symplectic structures of X respectively.

Proposition 44. *$(\mathcal{C},\mathcal{E})$ is a coisotropic A-brane (B-brane) on X if and only if $(T_p\mathcal{C}, F^{\mathcal{E}})$ is a generalized complex subspace of T_pX with respect to the generalized structure $\mathcal{J}_\omega$ ($\mathcal{J}_J$) for any point p in $\mathcal{C}$.*

With the above preparation, we can now state and prove the main theorem:

Theorem 45 (Main Theorem). *The fiberwise SYZ transformation $\mathcal{F}^{SYZ}$ transforms a semi-affine coisotropic A-brane to a semi-affine B-brane and vice versa.*

Proof. Let $(\mathcal{C},\mathcal{E})$ be a semi-affine brane on X. By Proposition 26, $(\check{\mathcal{C}},\check{\mathcal{E}}) = \mathcal{F}^{SYZ}(\mathcal{C},\mathcal{E})$ is a semi-affine brane in $\check{X}$. Without loss of generality, assume $X = B\times\mathbf{T}$ and $\check{X} = B\times\check{\mathbf{T}}$, where B is a convex subset of a real vector space V. Hence for any point $(x,y)\in X$ and $(x,\check{y})\in\check{X}$, we have

$$T_{(x,y)}X \cong T_xB\times T_y\mathbf{T}\cong V\oplus V$$

and

$$T_{(x,\check{y})}\check{X} \cong T_xB\times T_{\check{y}}\check{\mathbf{T}}\cong V\oplus V^*.$$

Therefore, the corresponding generalized space

$$\mathcal{T}_{(x,y)}X\cong V\oplus V\oplus V^*\oplus V^*$$

and

$$\mathcal{T}_{(x,\check{y})}\check{X}\cong V\oplus V^*\oplus V^*\oplus V$$

Note that there is an obvious identification

$$\sigma : \mathcal{T}_{(x,y)}X\to\mathcal{T}_{(x,\check{y})}\check{X} \tag{4.38}$$

defined by swapping the second and fourth summands above. Moreover, under this map, the generalized complex structure $\mathcal{J}_J$ coming from the complex structure J of X is mapped to the generalized complex structure $\mathcal{J}_{\check{\omega}}$ coming from the mirror symplectic structure $\check{\omega}$ of $\check{X}$ and vice versa. Therefore, by Proposition 44, it remains to prove that the generalized tangent spaces $\mathcal{T}_{(T_{(x,y)}\mathcal{C},F)}$ and $\mathcal{T}_{(T_{(x,\check{y})}\check{\mathcal{C}},\check{F})}$ are exchanged by σ for each $(x,y)\in\mathcal{C}$ and $(x,\check{y})\in\check{\mathcal{C}}$.

If we follow the notations in Proposition 26, then $\frac{\partial}{\partial\mathbf{u}}$, $\frac{\partial}{\partial\mathbf{y}}$, $\frac{\partial}{\partial\mathbf{v}}$, $d\mathbf{u}$, $d\mathbf{y}$ and $d\mathbf{v}$ are mapped to $d\check{\mathbf{v}}$, $d\check{\mathbf{y}}$, $d\check{\mathbf{u}}$, $\frac{\partial}{\partial\check{\mathbf{v}}}$, $\frac{\partial}{\partial\check{\mathbf{y}}}$ and $\frac{\partial}{\partial\check{\mathbf{u}}}$ respectively under the identification σ. Then, it is straightforward to show that $\mathcal{T}_{(T_{(x,y)}\mathcal{C},F)}$ and $\mathcal{T}_{(T_{(x,\check{y})}\check{\mathcal{C}},\check{F})}$ are exchanged by σ. □

5. Appendix

We are going to review those results of spin geometry which are used in this article, readers may refer to [4] for more details.

5.1. Clifford algebra, spinor and Dirac operator

Let V be a n dimensional real vector space with the Euclidean metric g. The Clifford algebra $Cl(V,g)$ is defined to be the quotient algebra

$$\sum_{k=0}^{\infty} V^{\otimes k}/\langle v\cdot w + w\cdot v - 2g(v,w)\rangle.$$

Let

$$\mathbb{C}l(n) := \mathbb{C}\otimes_{\mathbb{R}} Cl(V,g)$$

be the complex Clifford algebra. It has a natural complex representation

$$\rho : \mathbb{C}l(n) \to End(\mathcal{S})$$

on the complex vector space $S := \mathbb{C}^{2^{[\frac{n}{2}]}}$ constructed as below:

Case 1. When $n = 2r$ is even. Let $\{e_1, e_2, ..., e_{2r-1}, e_{2r}\}$ be an orthonormal basis of V. We define $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}$ and for $i = 1, \dots, r$, define

$$E_{2i-1} := \underbrace{I_{2\times 2}\otimes\cdots\otimes I_{2\times 2}}_{r-i\ \ times}\otimes A\otimes \underbrace{\sqrt{-1}\,AB\otimes\cdots\otimes\sqrt{-1}\,AB}_{i-1\ \ times}$$

and

$$E_{2i} := \underbrace{I_{2\times 2}\otimes\cdots\otimes I_{2\times 2}}_{r-i\ \ times}\otimes B\otimes \underbrace{\sqrt{-1}\,AB\otimes\cdots\otimes\sqrt{-1}\,AB}_{i-1\ \ times}$$

Note that these are matrices acting on the vector space

$$\mathcal{S} = \bigotimes_{k=1}^{r}\mathbb{C}^2.$$

Then the representation ρ is determined by $\rho(e_k) = E_k$ for $k = 1, \dots, 2r$ since $\{e_1, \dots, e_n\}$ is a generating set of $\mathbb{C}l(n)$.

Case 2. When $n = 2r + 1$ is odd. Let $\{e_1, e_2, \ldots, e_{2r-1}, e_{2r}, e_{2r+1}\}$ be an orthonormal basis of V. For $i = 1, \ldots, r$, we define

$$\begin{aligned} E'_{2i-1} &: = [1] \otimes E_{2i-1} \\ E'_{2i} &: = [1] \otimes E_{2i} \\ E'_{2r+1} &: = \left[-\sqrt{-1}\,\right] \otimes \sqrt{-1}\,AB \otimes \cdots \otimes \sqrt{-1}\,AB \end{aligned}$$

Note that these are matrices acting on the vector space

$$\mathcal{S} := \mathbb{C}^1 \otimes \left(\bigotimes_{k=0}^{r} \mathbb{C}^2 \right) = \mathbb{C}^{2^{[\frac{n}{2}]}}.$$

Also, the representation ρ is determined by $\rho(e_k) := E_k$ for $k = 1, \ldots, 2r + 1$.

With the above definition, we can define the spin group

$$\mathrm{Spin}(n) := \bigcup_{k=1}^{\infty} \{v_1 \cdots v_{2k} : v_i \in V, \|v_i\| = 1, i = 1, \ldots, 2k\} \subset \mathbb{C}l(n)$$

which is the universal cover of the orthogonal group $\mathrm{SO}(n)$. As a subset of the Clifford algebra $\mathbb{C}l(n)$, $\mathrm{Spin}(n)$ has a representation induced by the representation ρ of $\mathbb{C}l(n)$:

$$\rho : \mathrm{Spin}(n) \hookrightarrow \mathbb{C}l(n) \to \mathrm{End}(\mathcal{S}).$$

When n is odd, this is an irreducible representation. However, if n is even, this representation has two irreducible components:

$$\rho^{\pm} : Spin(n) \to End(\mathcal{S}^{\pm})$$

for $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$.

Let E be an n dimensional oriented Riemannian vector bundle over a manifold M and denote P to be the orthonormal frame bundle of it, which is a principle $\mathrm{SO}(n)$ bundle. A spin structure is a lift $\mathrm{Spin}(n)$-bundle $\tilde{P}$ of P. The spinor bundle associated to this spin structure is

$$\mathcal{S} = \tilde{P} \times_\rho \mathcal{S}.$$

Note that each fiber of E is an Euclidean space and so we can associate the Clifford bundle $\mathbb{C}l(E)$. The spinor bundle $\mathcal{S}$ is in fact a bundle of modules over the bundle of algebras $\mathbb{C}l(E)$:

$$\mathbb{C}l(E) \times \mathcal{S} \to \mathcal{S}.$$

This is called the Clifford multiplication and we denote it by "$\cdot$".

In particular, if E is a rank n subbundle of the tangent bundle TM of a Riemannian manifold M, we can define the Dirac operator $\mathcal{D} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ which is given by

$$\mathcal{D} := \sum_{i=1}^{n} e_i \nabla_{e_i},$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of E. When n is even, we have a further decomposition of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ with Dirac operator

$$\mathcal{D} = \begin{bmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{bmatrix}.$$

5.2. Family spinor bundle

A Riemannian fiber bundle is a fiber bundle $\pi : M \longrightarrow B$ such that $M_b := \pi^{-1}(b)$ is a Riemannian manifold for each point b in B. Then,

$$V := \bigsqcup_{b \in B} T(M_b)$$

forms a subbundle of TM, which is called the vertical tangent bundle of $\pi : M \to B$. A connection of $M \to B$ is defined to be a choice of splitting $TM = V \oplus H$ of the following exact sequence of vector bundles

$$0 \to V \to TM \to \pi^* TB \to 0$$

and H is called a horizontal bundle.

For a Riemannian fiber bundle $\pi : M \to B$ with a connection, one can associate a natural connection to the bundle V as below: We first choose a Riemannian metric on the base manifold B and pull it up to get a metric on the horizontal bundle H. Also note that the Riemannian metric for each M_b gives a metric on the vertical tangent bundle V. Then by combining the horizontal and the vertical metrics, one obtains a Riemannian metric on M. Then the Levi-Civita connection on TM gives a connection ∇^V on the subbundle V of TM. By Proposition 10.2 in [4], ∇^V is independent of the choice of the metric on B:

Proposition 46. *Let $\pi : M \to B$ be a Riemannian fiber bundle equipped with a connection. Then its vertical tangent bundle V is naturally a Riemannian vector bundle with a compatible connection ∇^V.*

Assume that the bundle V is orientable and it is equipped with a spin structure, then we can associate it with a spinor bundle $\not{S}$ which the Clifford algebra bundle $\mathbb{C}l(V)$ of V acts on it. Moreover, the Riemannian connection ∇^V induces an Hermitian connection ∇ on $\not{S}$. Since $V|_{M_b} = TM_b$, the bundle $\not{S}$ can be viewed as a family of spinor bundles $\not{S}|_{M_b}$ with Dirac operators $\not{D}$ such that

$$\not{D}_b := \not{D}|_{M_b} = \sum_{i=1}^{n} e_i \nabla_{e_i}$$

parametrized by points b in B, where $e_1, \ldots, e_n$ is an orthonormal frame of the bundle $V|_{M_b} = TM_b$.

Now, we can construct an infinity dimensional vector bundle

$$\bigsqcup_{b \in B} L^2(S|_{M_b}) \to \mathcal{H} \to B.$$

In order to get a finite dimensional subbundle of $\mathcal{H}$ over B, we consider the kernels of the Dirac operators $\ker(\not{D}_b) \leq L^2(S|_{M_b})$, for $b \in B$, which forms an object

$$\bigsqcup_{b \in B} \ker(\not{D}_b) \to \ker(\not{D}) \to B$$

of $\mathcal{H}$. Although the dimension of $\ker(\not{D}_b)$ may jumps as b varies in B such that $\ker \not{D}$ is not a bundle over B, it sometimes happens that $B' := \{b \in B | \ker(\not{D}_b) \neq 0\}$ is a submanifold of B and $\ker(\not{D})$ is a bundle over B'. This turns out to be our case and it is such a submanifold on which a brane supports.

Remark 47. We can tensor S with a Hermitian bundle $E \to M$ and consider the family twisted Dirac operators

$$\not{D}_b^E : L^2\left((S \otimes E)|_{M_b}\right) \to L^2\left((S \otimes E)|_{M_b}\right)$$

to obtain

$$\ker(\not{D}_b^E) \to \ker(\not{D}^E) \to B$$

in B.

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