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Geometric kernel formula relating prime forms

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We use geometric representation for the Szegő kernel on genus $g+1$ and genus g Riemann surfaces in order to derive formulas relating corresponding prime forms. The result will be useful for computation of fermionic vertex algebra cohomology of smooth manifolds foliations.

1. Differentials and kernels on a Riemann surface

1.1. The self-sewing formalism of a Riemann surface

In this subsection we recall the construction of the Szegő kernel on a Riemann surface $\Sigma^{(g+1)}$ constructed by self-sewing a Riemann surface $\Sigma^{(g)}$ of genus g. It is based on the formalism of the paper [Y]. Consider a Riemann surface $\Sigma^{(g)}$ of genus g. Let z_1, z_2 be local coordinates in the vicinity of two separated points p_1, p_2. Consider two disks $|z_a| \leq r_a$, for $r_a > 0$ and $a = 1, 2$. We require that r_1, r_2 must be sufficiently small to ensure that the disks do not intersect. Let us introduce a complex parameter ρ such that $|\rho| \leq r_1 r_2$ and restrict the disks $\{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\} \subset \Sigma^{(g)}$, to form a twice-punctured surface $\widehat{\Sigma}^{(g)} = \Sigma^{(g)} \backslash \bigcup_{a=1,2} \{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\}$. Here we use the notation $\bar{1} = 2$, $\bar{2} = 1$. Next let us define annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(g)}$ such that $\mathcal{A}_a = \{z_a : |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$ and identify them as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ subject to the sewing relation $z_1 z_2 = \rho$, to construct a compact Riemann surface $\Sigma^{(g+1)} = \widehat{\Sigma}^{(g)} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$ of genus $g+1$. The sewing relation may be viewed as a parameterization of a cylinder connecting the punctured Riemann surface to itself.

We define a standard homology basis of cycles $\{a_1, b_1, \ldots, a_{g+1}, b_{g+1}\}$ on $\Sigma^{(g+1)}$ where $\{a_1, b_1, \ldots, a_g, b_g\}$ is the initial basis on $\Sigma^{(g)}$. Let $\mathcal{C}_a(z_a) \subset \mathcal{A}_a$ denote a closed anti-clockwise contour parameterized by z_a surrounding the puncture at $z_a = 0$. Then by applying the sewing relation one infers that $\mathcal{C}_2(z_2) \sim -\mathcal{C}_1(z_1)$. We then define the cycle a_{g+1} to be $\mathcal{C}_2(z_2)$, and the cycle b_{g+1} to be a path chosen in $\widehat{\Sigma}^{(g)}$ between identified points $z_1 = z_0$ and $z_2 =$

ρ/z_0 on the sewn surface. The the holomorphic one-forms $\nu_i^{(g+1)}$ and the period matrix $\Omega^{(g+1)}$ can be computed in terms of data coming from $\Sigma^{(g)}$ [Y].

1.2. Differentials on a Riemann surface

Consider a compact Riemann surface $\Sigma^{(g)}$ of genus g endowed with canonical homology cycle basis $a_1, \ldots, a_g, b_1, \ldots, b_g$. In general there exists g holomorphic one-forms $\nu_i^{(g)}$, $i = 1, \ldots, g$ which may be normalized [FK, Sp] by

$$\oint_{a_i} \nu_j^{(g)} = 2\pi i \delta_{ij}. \tag{1.1}$$

The genus g period matrix $\Omega^{(g)}$ is defined by

$$\Omega_{ij}^{(g)} = \frac{1}{2\pi i} \oint_{b_i} \nu_j^{(g)}, \tag{1.2}$$

for $i, j = 1, \ldots, g$. $\Omega^{(g)}$ is symmetric with positive imaginary part, i.e., $\Omega^{(g)} \in \mathbb{H}_g$, the Siegel upper half plane. Here We recall the definition of the theta function with real characteristics [Mu, FK]

$$\begin{aligned} &\vartheta^{(g)} \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(z | \Omega^{(g)} \right) \\ &= \sum_{m \in \mathbb{Z}^g} \exp \left(i\pi (m + \alpha^{(g)}).\Omega^{(g)}. \left(m + \alpha^{(g)} \right) + (m + \alpha^{(g)}). \left(z + 2\pi i \beta^{(g)} \right) \right), \end{aligned}$$

for $\alpha^{(g)} = (\alpha_i)$, $\beta^{(g)} = (\beta_i) \in \mathbb{R}^g$, $z = (z_i) \in \mathbb{C}^g$, and $i = 1, \ldots, g$. There exists [Mu, Fay] a non-singular odd character $\begin{bmatrix} \gamma^{(g)} \\ \delta^{(g)} \end{bmatrix}$ such that

$$\vartheta^{(g)} \begin{bmatrix} \gamma^{(g)} \\ \delta^{(g)} \end{bmatrix} (0 | \Omega^{(g)}) = 0, \qquad \partial_{z_i} \vartheta^{(g)} \begin{bmatrix} \gamma^{(g)} \\ \delta^{(g)} \end{bmatrix} \left(0 | \Omega^{(g)} \right) \neq 0.$$

Then let us introduce

$$\zeta^{(g)}(x) = \sum_{i=1}^{g} \partial_{z_i} \vartheta^{(g)} \begin{bmatrix} \gamma^{(g)} \\ \delta^{(g)} \end{bmatrix} \left(0 | \Omega^{(g)} \right) \nu_i^{(g)}(x), \tag{1.3}$$

a holomorphic one-form, and let $\left(\zeta^{(g)}(x) \right)^{\frac{1}{2}}$ denote the form of weight $\frac{1}{2}$ on the double cover $\widetilde{\Sigma^{(g)}}$ of $\Sigma^{(g)}$. We call $\left(\zeta^{(g)}(x) \right)^{\frac{1}{2}}$ a double-valued $\frac{1}{2}$-form on

$\Sigma^{(g)}$. One defines the prime form $\mathcal{E}^{(g)}(x,y)$ by (here the sign in the definition differs with [Mu, Fay])

$$\mathcal{E}^{(g)}(x,y) = \vartheta^{(g)} \begin{bmatrix} \gamma^{(g)} \\ \delta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} | \Omega^{(g)} \right) \left(\zeta^{(g)}(x) \right)^{-\frac{1}{2}} \left(\zeta^{(g)}(y) \right)^{-\frac{1}{2}}, \tag{1.4}$$

where $\int_y^x \nu^{(g)} = \left(\int_y^x \nu_i^{(g)} \right) \in \mathbb{C}^g$. The prime form $\mathcal{E}^{(g)}(x,y) = -\mathcal{E}^{(g)}(y,x)$ is a holomorphic differential form of weight $(-\frac{1}{2}, -\frac{1}{2})$ on $\widetilde{\Sigma^{(g)}} \times \widetilde{\Sigma^{(g)}}$.

We define the Szegö kernel [Sc, HS, Fay] for $\vartheta^{(g)} \left[\begin{smallmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{smallmatrix}\right] (0|\Omega^{(g)}) \neq 0$ by

$$\mathcal{S}^{(g)} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (x, y|\Omega^{(g)}) = \frac{\vartheta^{(g)} \left[\begin{smallmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{smallmatrix}\right] \left(\int_y^x \nu^{(g)} | \Omega^{(g)} \right)}{\vartheta^{(g)} \left[\begin{smallmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{smallmatrix}\right] (0|\Omega^{(g)}) \, \mathcal{E}^{(g)}(x,y)}, \tag{1.5}$$

where $\theta^{(g)} = (\theta_i)$, $\phi^{(g)} = (\phi_i) \in U(1)^n$ for $\theta_j = -e^{-2\pi i \beta_j}$, $\phi_j = -e^{2\pi i \alpha_j}$, $j = 1, \ldots, g$. This can be written as

$$\mathcal{S}^{(g)} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (x, y|\Omega^{(g)}) = \Theta^{(g)} \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(x, y; \nu^{(g)}, 0 | \Omega^{(g)} \right) \left(\mathcal{E}^{(g)}(x,y) \right)^{-1},$$

with the functional

$$\Theta^{(g)} \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(x, y; f_1, f_2 | \Omega^{(g)} \right) = \frac{\vartheta^{(g)} \left[\begin{smallmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{smallmatrix}\right] \left(\int_y^x f_1 | \Omega^{(g)} \right)}{\vartheta^{(g)} \left[\begin{smallmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{smallmatrix}\right] \left(f_2 | \Omega^{(g)} \right)}.$$

1.3. Szegő Kernel in the ρ-Formalism

Now let us determine the Szegő kernel

$$\mathcal{S}^{(g+1)}(x,y) = \mathcal{S}^{(g+1)} \begin{bmatrix} \theta^{(g+1)} \\ \phi^{(g+1)} \end{bmatrix} (x,y),$$

on the self-sewn Riemann surface $\Sigma^{(g+1)}$ in terms of genus g Szegő kernel and the multiplier parameters associated with the handle cycles. The $\mathcal{S}^{(g+1)}$ multipliers on the cycles a_i, b_i for $i = 1, \ldots, g$ are determined by the multipliers of $\mathcal{S}^{(g)}$ with $\phi_i^{(g+1)} = \phi_i^{(g)}$ and $\theta_i^{(g+1)} = \theta_i^{(g)}$, i.e., $\alpha_i^{(g+1)} = \alpha_i^{(g)}$ and $\beta_i^{(g+1)} = \beta_i^{(g)}$. The remaining two multipliers associated with the cycles a_{g+1} and b_{g+1} $\phi_{g+1} = \phi_{g+1}^{(g+1)} = -e^{2\pi i \alpha_{g+1}^{(g+1)}}$, $\theta_{g+1} = \theta_{g+1}^{(g+1)} = -e^{-2\pi i \beta_{g+1}^{(g+1)}}$, must be specified so that $\mathcal{S}^{(g+1)}(e^{2\pi i} x_a, y) = -\phi_{g+1}^{a-\bar{a}}$, $\mathcal{S}^{(g+1)}(x_a, y)$, $\mathcal{S}^{(g+1)}(x_a, y) = -\theta_{g+1}^{a-\bar{a}}\, \mathcal{S}^{(g+1)}(x_{\bar{a}}, y)$, for $x_a \in \mathcal{A}_a$ and $x_{\bar{a}} \in \mathcal{A}_{\bar{a}}$.

Let $k_a = k + (-1)^{\bar{a}}\kappa$, for $a = 1, 2$ and integer $k \geq 1$. For a kernel $S(x, y)$ on a genus g Riemann surface we next define the infinite matrix $G_S = \left(G_{S;ab} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (k, l) \right)$ of moments

$$G_{S;ab} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (k, l) = \frac{\rho^{\frac{1}{2}(k_a + l_b - 1)}}{(2\pi i)^2} \times \oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})} \oint_{\mathcal{C}_b(y_b)} (x_{\bar{a}})^{-k_a} (y_b)^{-l_b} S(x_{\bar{a}}, y_b)\, dx_{\bar{a}}^{\frac{1}{2}}\, dy_b^{\frac{1}{2}}.$$

Finally we introduce infinite row vectors (indexed by a, k)

$$h_S(x) = \left(h_a \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (S; k, x) \right),$$
$$\bar{h}_S(y) = \left(\bar{h}_a \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (S; k, y) \right)$$

of half-order differentials

$$h_a \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (S; k, x) = \frac{\rho^{\frac{1}{2}(k_a - \frac{1}{2})}}{2\pi i} \oint_{\mathcal{C}_a(y_a)} y_a^{-k_a} S^{(g)}(x, y_a)\, dy_a^{\frac{1}{2}}, \tag{1.6}$$

$$\bar{h}_a \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (S; k, y) = \frac{\rho^{\frac{1}{2}(k_a - \frac{1}{2})}}{2\pi i} \oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})} x_{\bar{a}}^{-k_a} S^{(g)}(x_{\bar{a}}, y)\, dx_{\bar{a}}^{\frac{1}{2}}. \tag{1.7}$$

From the sewing relation we have $dz_a^{\frac{1}{2}} = (-1)^{\bar{a}} \xi \rho^{\frac{1}{2}} z_{\bar{a}}^{-1} dz_{\bar{a}}^{\frac{1}{2}}$, for $\xi \in \{\pm\sqrt{-1}\}$. We also define the matrix $T_S^{(g)} = \xi G_S D^\theta$, for infinite diagonal matrix

$$D^\theta(k, l) = \begin{bmatrix} \theta_{g+1}^{-1} & 0 \\ 0 & -\theta_{g+1} \end{bmatrix} \delta(k, l).$$

Let us also introduce $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ by $\phi_{g+1} = -e^{2\pi i \kappa}$, i.e., $\kappa = \alpha_{g+1}^{(g+1)} \mod 1$.

2. Geometric formulas relating prime forms

In [TZ1] we have proved that $\mathcal{S}^{(g+1)}$ is holomorphic in ρ for $|\rho| < r_1 r_2$ with $\mathcal{S}^{(g+1)}(x,y) = S^{(g)}_\kappa(x,y) + O(\rho)$, for some kernel $S^{(g)}_\kappa(x,y)$. For the genus $g+1$ prime form $\mathcal{E}^{(g+1)}$ and genus g prime form $\mathcal{E}^{(g)}$ we obtain here the following result. Let us define

$$\mathcal{U}^{(g)}(x,y) = \frac{\mathcal{E}^{(g)}(x,p_2)\ \mathcal{E}^{(g)}(y,p_1)}{\mathcal{E}^{(g)}(x,p_1)\ \mathcal{E}^{(g)}(y,p_2)},$$

and $z_{p_1,p_2} = \int\limits_{p_1}^{p_2} \nu^{(g)}$, for holomorphic one-forms $\nu^{(g)}$.

Proposition 1. *One can relate the genus $g+1$ and g prime forms $\mathcal{E}^{(g+1)}$ and $\mathcal{E}^{(g)}$ by means of the following formulas. For $\kappa \neq -1/2$,*

$$(2.1)\quad \mathcal{E}^{(g+1)}(x,y) = \Theta^{(g+1)}\begin{bmatrix}\alpha^{(g+1)}\\ \beta^{(g+1)}\end{bmatrix}\left(x,y;\nu^{(g+1)},0|\Omega^{(g+1)}\right) \times \Bigg[\Theta^{(g)}\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(x,y;\nu^{(g)}+\kappa z_{p_1,p_2},\kappa z_{p_1,p_2}|\Omega^{(g)}\right) \times \left(\mathcal{U}^{(g)}(x,y)\right)^{\kappa}\left(\mathcal{E}^{(g)}(x,y)\right)^{-1} + \xi h_{S^{(g)}_\kappa}(x) D^\theta \left(I - T^{(g)}_{S^{(g)}_\kappa}\right)^{-1} \bar{h}^T_{S^{(g)}_\kappa}(y)\Bigg]^{-1}.$$

For $\kappa = -1/2$ one obtains:

(2.2)

$$\mathcal{E}^{(g+1)}(x,y) = \Theta^{(g+1)}\begin{bmatrix}\alpha^{(g+1)}\\ \beta^{(g+1)}\end{bmatrix}\left(x,y;\nu^{(g+1)},0|\Omega^{(g+1)}\right)$$
$$\times \Bigg[\Bigg(\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(\int_y^x \nu^{(g)} + \frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right)\left(\mathcal{U}^{(g)}(x,y)\right)^{\frac{1}{2}}$$
$$- \theta_{g+1}\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(\int_y^x \nu^{(g)} - \frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right)\left(\mathcal{U}^{(g)}(x,y)\right)^{-\frac{1}{2}}\Bigg)\mathcal{E}^{(g)}(x,y)^{-1}$$
$$\times \left(\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(\frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right) - \theta_{g+1}\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(-\frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right)\right)^{-1}$$
$$+ \xi h_{S^{(g)}_\kappa}(x) D^\theta (I - T^{(g)}_{S^{(g)}_\kappa})^{-1} \bar{h}^T_{S^{(g)}_\kappa}(y)\Bigg]^{-1}.$$

Proof. For $x,\, y \in \widehat{\Sigma}^{(g)}$, where $S^{(g)}_\kappa(x,y)$ is defined [TZ1] as follows: for $\kappa \neq -\frac{1}{2}$

$$S^{(g)}_\kappa(x,y) = \frac{\vartheta\left[\begin{smallmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{smallmatrix}\right]\left(\int_y^x \nu^{(g)} + \kappa z_{p_1,p_2}|\Omega^{(g)}\right)\ \left(\mathcal{U}^{(g)}(x,y)\right)^\kappa}{\vartheta\left[\begin{smallmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{smallmatrix}\right]\left(\kappa z_{p_1,p_2}|\Omega^{(g)}\right)\ \mathcal{E}^{(g)}(x,y)}, \tag{2.3}$$

For $\kappa = -\frac{1}{2}$, $S^{(g)}_{-\frac{1}{2}}(x,y)$ is given by

(2.4)
$$\begin{aligned} S^{(g)}_{-\frac{1}{2}}(x,y) = &\left(\frac{\left(\mathcal{U}^{(g)}(x,y)\right)^{\frac{1}{2}}}{\mathcal{E}^{(g)}(x,y)}\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(\int_y^x \nu^{(g)} + \frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right)\right.\\ &\left. -\theta_{g+1}\frac{\left(\mathcal{U}^{(g)}(x,y)\right)^{-\frac{1}{2}}}{\mathcal{E}^{(g)}(x,y)}\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(\int_y^x \nu^{(g)} - \frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right)\right)\\ &\times\left(\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(\frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right) - \theta_{g+1}\vartheta\begin{bmatrix}\alpha^{(g)}\\ \beta^{(g)}\end{bmatrix}\left(-\frac{1}{2}z_{p_1,p_2}|\Omega^{(g)}\right)\right)^{-1}. \end{aligned}$$

Then we obtain in [TZ1] that $\mathcal{S}^{(g+1)}(x,y)$ is given by

$$\mathcal{S}^{(g+1)}(x,y) = \mathcal{S}^{(g)}_\kappa(x,y) + \xi h_{S^{(g)}_\kappa}(x) D^\theta \left(I - T^{(g)}_{S^{(g)}_\kappa}\right)^{-1} \bar{h}^T_{S^{(g)}_\kappa}(y).$$

Note that $\det\left(I - T^{(g)}\right)$ is non-vanishing and holomorphic in ρ for $|\rho| < r_1 r_2$. Using the definition (1.5) and expressing the genus $g+1$ prime form we find:

$$\begin{aligned} \mathcal{E}^{(g+1)}(x,y) = &\frac{\vartheta^{(g+1)}\left[\begin{smallmatrix}\alpha^{(g+1)}\\ \beta^{(g+1)}\end{smallmatrix}\right]\left(\int_y^x \nu^{(g+1)}|\Omega^{(g+1)}\right)}{\vartheta^{(g+1)}\left[\begin{smallmatrix}\alpha^{(g+1)}\\ \beta^{(g+1)}\end{smallmatrix}\right](0|\Omega^{(g+1)})}\\ &\times\left[\mathcal{S}^{(g)}_\kappa(x,y) + \xi h_{S^{(g)}_\kappa}(x) D^\theta\left(I - T^{(g)}_{S^{(g)}_\kappa}\right)^{-1}\bar{h}^T_{S^{(g)}_\kappa}(y)\right]^{-1}. \end{aligned}$$

Then substituting either (2.3) or (2.4) for $\kappa \neq -1/2$ or $\kappa = -1/2$ correspondingly, we obtain (2.1) and (2.2). □

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