

# The fermionic signature operator and space-time symmetries

FELIX FINSTER AND MORITZ REINTJES

We show that and specify how space-time symmetries give rise to corresponding symmetries of the fermionic signature operator and generalized fermionic projector states.

|          |   |             |
|----------|---|-------------|
| <b>1</b> | <b>Introduction</b>   | <b>1907</b> |
| <b>2</b> | <b>Preliminaries</b>  | <b>1908</b> |
| <b>3</b> | <b>Symmetries of the fermionic signature operator</b>       | <b>1914</b> |
| <b>4</b> | <b>Symmetries of generalized fermionic projector states</b> | <b>1925</b> |
| <b>5</b> | <b>Applications</b>   | <b>1927</b> |
| <b>6</b> | <b>Examples</b>   | <b>1931</b> |
|          | <b>References</b>   | <b>1935</b> |

## 1. Introduction

The fermionic signature operator introduced in [11, 12] provides a setting of spectral geometry in Lorentzian signature [7] and has been proven useful for constructing quasi-free Dirac states in globally hyperbolic space-times [6, 8]. In all known examples, the resulting so-called fermionic projector state respects the symmetries of space-time. The present paper is devoted to a systematic study of the general relationship between space-time symmetries and symmetries of the fermionic signature operator. We also study the symmetry properties of the resulting generalized fermionic projector states.

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We describe space-time symmetries by a Lie group  $\mathcal{G}$  which acts locally as isomorphisms  $\Phi$  of the spinor bundle  $S\mathcal{M}$  (see Section 3.1). Considering such local actions has the advantage that they can be obtained from Killing symmetries in a straightforward manner (see Section 5.1). The only condition needed is that the Killing fields are complete (see Definition 5.2). We construct a strongly continuous action of  $\mathcal{G}$  represented by unitary operators on the Hilbert space of Dirac solutions (Theorem 3.8). The corresponding Lie symmetries are represented by essentially self-adjoint operators acting on the smooth and spatially compact solutions (Theorem 3.9). Both the local representation of the group and the representation of the Lie algebra on the solution space commute with the fermionic signature operator (see Theorems 3.6, 3.7 and 3.9), except for a minus sign which appears if the time orientation is reversed. Moreover, the resulting generalized fermionic projector states are invariant under the symmetry transformations, again up to signs (Theorem 4.1 and Corollary 4.3). As applications we consider Killing symmetries and discrete symmetries (Section 5). The paper concludes with a discussion of several examples (Section 6).

## 2. Preliminaries

### 2.1. Lorentzian spin geometry

Let  $(\mathcal{M}, g)$  be a smooth, globally hyperbolic, time-oriented Lorentzian spin manifold of dimension  $k \geq 2$ . For the signature of the metric we use the convention  $(+, -, \dots, -)$ . We denote the corresponding spinor bundle by  $S\mathcal{M}$ . Its fibers  $S_p\mathcal{M}$  are endowed with an inner product  $\langle \cdot, \cdot \rangle_p$  of signature  $(n, n)$  with  $n = 2^{\lfloor k/2 \rfloor - 1}$  (where  $\lfloor \cdot \rfloor$  is the Gauß bracket; for details see [2, 16]), which we refer to as the spin scalar product. Clifford multiplication is described by a mapping  $\gamma$  which satisfies the anti-commutation relations,

$$\gamma : T_p\mathcal{M} \rightarrow L(S_p\mathcal{M}) \quad \text{with} \quad \gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u, v) \mathbf{1}_{S_p(\mathcal{M})}.$$

We also write Clifford multiplication in components with the Dirac matrices  $\gamma^j$ . The metric connections on the tangent bundle and the spinor bundle are denoted by  $\nabla$ . The sections of the spinor bundle are also referred to as wave functions.

In order to include the situation when an external potential is present, we introduce a multiplication operator  $\mathcal{B}(p) \in L(S_p\mathcal{M})$ , which we assume to

be smooth and symmetric with respect to the spin scalar product,

$$(2.1) \quad \mathcal{B} \in C^\infty(\mathcal{M}, L(S\mathcal{M})) \quad \text{with} \quad \langle \mathcal{B}\phi | \psi \rangle_p = \langle \phi | \mathcal{B}\psi \rangle_p \quad \forall \phi, \psi \in S_p\mathcal{M}.$$

### 2.2. The Dirac operator and inner products on wave functions

We denote the smooth sections of the spinor bundle by  $C^\infty(\mathcal{M}, S\mathcal{M})$ . Similarly,  $C_0^\infty(\mathcal{M}, S\mathcal{M})$  are the smooth sections with compact support. On the wave functions, one has the Lorentz invariant inner product

$$\begin{aligned} \langle \cdot | \cdot \rangle &: C^\infty(\mathcal{M}, S\mathcal{M}) \times C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow \mathbb{C}, \\ \langle \psi | \phi \rangle &= \int_{\mathcal{M}} \langle \psi | \phi \rangle_p d\mu_{\mathcal{M}}. \end{aligned}$$

The Dirac operator  $\mathcal{D}$  is defined by

$$(2.2) \quad \mathcal{D} := i\gamma^j \nabla_j + \mathcal{B} : C^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow C^\infty(\mathcal{M}, S\mathcal{M}).$$

For a given real parameter  $m \in \mathbb{R}$  (the “mass”), the Dirac equation reads

$$(\mathcal{D} - m)\psi_m = 0.$$

For clarity, we always denote solutions of the Dirac equation by a subscript  $m$ . We mainly consider solutions in the class  $C_{sc}^\infty(\mathcal{M}, S\mathcal{M})$  of smooth sections with spatially compact support. On such solutions, one has the scalar product

$$(2.3) \quad (\psi_m | \phi_m)_m = 2\pi \int_{\mathcal{N}} \langle \psi_m | \gamma(\nu) \phi_m \rangle_p d\mu_{\mathcal{N}}(p),$$

where  $\mathcal{N}$  denotes any Cauchy surface and  $\nu$  its future-directed normal (due to current conservation, the scalar product is in fact independent of the choice of  $\mathcal{N}$ ; for details see [11, Section 2]). Forming the completion gives the Hilbert space  $(\mathcal{H}_m, (\cdot | \cdot)_m)$ . It will be convenient to use the short notation

$$\mathcal{H}_m^\infty := \mathcal{H}_m \cap C_{sc}^\infty(\mathcal{M}, S\mathcal{M}).$$

### 2.3. The fermionic signature operator in finite lifetime

We now recall the construction of the fermionic signature operator in [11], which applies in particular to space-times of finite life-time. We here consider

a slightly more general setting which also applies to certain space-times involving horizons like the Rindler space-time [9].

**Definition 2.1.** The manifold  $(\mathcal{M}, g)$  is said to be **weakly  $m$ -finite** if for every  $\phi_m \in \mathcal{H}_m^\infty$ , there is a constant  $c(\phi_m) > 0$  such that for all  $\psi_m \in \mathcal{H}_m^\infty$ , the function  $\langle \psi_m | \phi_m \rangle_p$  is integrable on  $\mathcal{M}$  and

$$(2.4) \quad |\langle \psi_m | \phi_m \rangle| \leq c \|\psi_m\|_m.$$

Under this assumption, the Fréchet-Riesz theorem gives rise to a unique densely defined operator

$$\mathcal{S}_m : \mathcal{H}_m^\infty \rightarrow \mathcal{H}_m$$

with the property

$$\langle \psi_m | \phi_m \rangle = (\psi_m | \mathcal{S}_m \phi_m)_m \quad \text{for all } \psi_m \in \mathcal{H}_m,$$

referred to as the **fermionic signature operator**. We remark that the notion of  **$m$ -finiteness** in [11] instead of (2.4) imposes the stronger assumption

$$(2.5) \quad |\langle \psi_m | \phi_m \rangle| \leq c \|\phi_m\|_m \|\psi_m\|_m.$$

This inequality holds in every space-time of finite lifetime. It implies that the fermionic signature operator is bounded and can thus be extended to all of  $\mathcal{H}_m$ .

#### 2.4. The fermionic signature operator in infinite lifetime

In a space-time of infinite life time [12], one studies families of solutions. More precisely, we consider the mass parameter in a bounded open interval,  $m \in I := (m_L, m_R)$  with  $0 \notin I$ . By  $C_{\text{sc},0}^\infty(\mathcal{M} \times I, S\mathcal{M})$  we denote the smooth wave functions with spatially compact support which are also compactly supported in  $I$ . We often denote the dependence on  $m$  by a subscript,  $\psi_m(p) := \psi(p, m)$ . On families of solutions in  $C_{\text{sc},0}^\infty(\mathcal{M} \times I, S\mathcal{M})$ , for any fixed  $m$  we can take the scalar product (2.3). We introduce a scalar product on families of solutions by integrating over the mass parameter,

$$(\psi | \phi) := \int_I (\psi_m | \phi_m)_m dm$$

(where  $dm$  is the Lebesgue measure). Forming the completion gives the Hilbert space  $(\mathcal{H}, (\cdot|\cdot))$ . We denote the norm on  $\mathcal{H}$  by  $\|\cdot\|$ . Moreover, we set

$$\mathcal{H}^\infty := \mathcal{H} \cap C_{sc,0}^\infty(\mathcal{M} \times I, S\mathcal{M}).$$

We introduce  $T$  as the operator of multiplication by the mass parameter,

$$T : \mathcal{H} \rightarrow \mathcal{H}, \quad (T\psi)_m = m \psi_m.$$

Integrating over  $m$  gives the operation

$$\mathfrak{p} : \mathcal{H}^\infty \rightarrow C_{sc}^\infty(\mathcal{M}, S\mathcal{M}), \quad \mathfrak{p}\psi = \int_I \psi_m dm.$$

**Definition 2.2.** The Dirac operator  $\mathcal{D}$  has the **weak mass oscillation property** in the interval  $I \subset \mathbb{R}$  with domain  $\mathcal{H}^\infty$  if the following conditions hold:

- (a) For every  $\psi, \phi \in \mathcal{H}^\infty$ , the function  $\langle \mathfrak{p}\phi | \mathfrak{p}\psi \rangle$  is integrable on  $\mathcal{M}$ . Moreover, for any  $\psi \in \mathcal{H}^\infty$  there is a constant  $c(\psi)$  such that

$$|\langle \mathfrak{p}\psi | \mathfrak{p}\phi \rangle| \leq c \|\phi\| \quad \forall \phi \in \mathcal{H}^\infty.$$

- (b) For all  $\psi, \phi \in \mathcal{H}^\infty$ ,

$$\langle \mathfrak{p}T\psi | \mathfrak{p}\phi \rangle = \langle \mathfrak{p}\psi | \mathfrak{p}T\phi \rangle.$$

This definition specifies the minimal requirements needed for the construction of the fermionic projector. More precisely, under these assumptions, the Fréchet-Riesz theorem gives rise to a densely defined symmetric operator  $\mathcal{S}$  acting on families of solutions defined by

$$(2.6) \quad \mathcal{S} : \mathcal{H}^\infty \rightarrow \mathcal{H}, \quad (\mathcal{S}\psi|\phi) = \langle \mathfrak{p}\psi | \mathfrak{p}\phi \rangle \quad \forall \phi \in \mathcal{H}.$$

This operator is shown to commute with  $T$ . After constructing the Friedrichs extension of  $\mathcal{S}^2$ , the spectral theorem for commuting operators gives rise to a joint spectral measure  $dE_{\rho,m}$  of the commuting operators  $\mathcal{S}^2$  and  $T$ . For the technical details of these constructions we refer to [12, Section 3]. This spectral measure makes it possible to define the fermionic signature

operator  $d\mathcal{S}_m$  as an operator-valued measure by

$$(2.7) \quad \int_I \eta(m) d\mathcal{S}_m := \int_{\sigma(\mathcal{S}^2) \times I} \eta(m) \mathcal{S} dE_{\rho,m}$$

(for any test function  $\eta \in C^0(I)$ ). This formula also makes it possible to introduce a functional calculus for  $\mathcal{S}_m$  in a straightforward way by

$$\begin{aligned} & \int_I \eta(m) dW(\mathcal{S}_m) \\ & := \int_{\sigma(\mathcal{S}^2) \times I} \eta(m) \left[ W(\sqrt{\rho}) (\mathcal{S} + \sqrt{\rho}) + W(-\sqrt{\rho}) (-\mathcal{S} + \sqrt{\rho}) \right] \frac{dE_{\rho,m}}{2\sqrt{\rho}}, \end{aligned}$$

where  $W$  is a bounded Borel function.

In order to construct the fermionic signature operator  $\mathcal{S}_m$  pointwise for any  $m \in I$ , one needs a stronger assumption:

**Definition 2.3.** The Dirac operator  $\mathcal{D}$  has the **strong mass oscillation property** in the interval  $I = (m_L, m_R)$  with domain  $\mathcal{H}^\infty$  if there is a constant  $c > 0$  such that

$$|\langle \mathbf{p}\psi | \mathbf{p}\phi \rangle| \leq c \int_I \|\phi_m\|_m \|\psi_m\|_m dm \quad \text{for all } \psi, \phi \in \mathcal{H}^\infty.$$

The following theorem is proved in [12, Theorem 4.2, Proposition 4.3 and Theorem 4.7]:

**Theorem 2.4.** *Assume that the Dirac operator  $\mathcal{D}$  has the strong mass oscillation property in the interval  $I = (m_L, m_R)$ . Then there exists a family of self-adjoint linear operators  $(\mathcal{S}_m)_{m \in I}$  with  $\mathcal{S}_m \in L(\mathcal{H}_m)$  which are uniformly bounded,*

$$\sup_{m \in I} \|\mathcal{S}_m\| < \infty,$$

such that

$$(2.8) \quad \langle \mathbf{p}\psi | \mathbf{p}\phi \rangle = \int_I (\psi_m | \mathcal{S}_m \phi_m)_m dm \quad \text{for all } \psi, \phi \in \mathcal{H}^\infty.$$

The operator  $\mathcal{S}_m$  is uniquely determined for every  $m \in I$  by demanding that for all  $\psi, \phi \in \mathcal{H}^\infty$ , the functions  $(\psi_m | \mathcal{S}_m \phi_m)_m$  are continuous in  $m$ . Moreover, the operator  $\mathcal{S}_m$  is the same for all choices of  $I$  containing  $m$ .

### 2.5. Quasi-free Dirac fields and generalized fermionic projector states

We now explain the connection to quantum field theory as worked out in [8]. Assume that the fermionic signature operator is bounded (as is the case if space-time is strongly  $m$ -finite or if the strong mass oscillation property holds). Then the *fermionic projector*  $P$  is introduced as the operator (for details see [11, Section 3] and [12, Section 4.2])

$$(2.9) \quad P = -\chi_{(-\infty,0)}(\mathcal{S}_m) k_m : C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow \mathcal{H}_m,$$

where  $k_m$  is the *causal fundamental solution* defined as the difference of the advanced and retarded Green's operators,

$$k_m := \frac{1}{2\pi i} (s_m^\vee - s_m^\wedge) : C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow \mathcal{H}_m^\infty.$$

The fermionic projector  $P$  can be written as an integral operator involving a uniquely determined distributional *kernel*  $\mathcal{P} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ , i.e. (for details see [11, Section 3.5])

$$(2.10) \quad \langle \phi | P \psi \rangle = \mathcal{P}(\bar{\phi} \otimes \psi) \quad \text{for all } \phi, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M}).$$

A main application of our constructions is that the projection operator  $\chi_{(-\infty,0)}(\mathcal{S}_m)$  gives rise to a distinguished quasi-free ground state of the second-quantized Dirac field with the property that the two-point distribution coincides with the kernel of the fermionic projector. Indeed, applying Araki's construction in [1] gives the following result (see [8, Theorem 1.4]):

**Theorem 2.5.** *There is an algebra of smeared fields generated by  $\Psi(g)$ ,  $\Psi^*(f)$  together with a pure quasi-free state  $\omega$  with the following properties:*

(a) *The canonical anti-commutation relations hold:*

$$\{\Psi(g), \Psi^*(f)\} = \langle g^* | k_m f \rangle, \quad \{\Psi(g), \Psi(g')\} = 0 = \{\Psi^*(f), \Psi^*(f')\}.$$

(b) *The two-point distribution of the state is given by*

$$\omega(\Psi(g) \Psi^*(f)) = - \iint_{\mathcal{M} \times \mathcal{M}} g(p) \mathcal{P}(p, q) f(q) d\mu_{\mathcal{M}}(q) d\mu_{\mathcal{M}}(y).$$

The state  $\omega$  is referred to as the *fermionic projector state* (or FP state) [6].

We finally note that, using the functional calculus, for any non-negative bounded Borel function  $W$  we obtain in generalization of (2.9) the operator

$$(2.11) \quad P_W := -W(\mathcal{S}_m) k_m : C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow \mathcal{H}_m,$$

which can again be represented according to (2.10) as an integral operator with a kernel  $\mathcal{P}_W \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ . Again using Araki's construction (this time for the positive operator  $W(\mathcal{S}_m)$ ) gives a corresponding quasi-free state. In this way, the fermionic signature operator gives rise to a whole class of distinguished quasi-free states, which we refer to as *generalized fermionic projector states*. As shown in [9, Section 11] in the example of two-dimensional Rindler space-time, this makes it possible to obtain thermal states from the fermionic signature operator.

### 3. Symmetries of the fermionic signature operator

#### 3.1. Symmetries of space-time

Let  $\mathcal{G}$  be a Lie group (possibly non-compact, of finite dimension  $d \geq 0$ , where the case  $d = 0$  is a discrete group). In view of our applications, we want to allow for the possibility that  $\mathcal{G}$  acts only *locally* as a symmetry group. To this end, let  $\mathcal{U} \subset \mathcal{G}$  be an open neighborhood of the neutral element  $e \in \mathcal{G}$  (by choosing  $\mathcal{U} = \mathcal{G}$ , one recovers standard group actions). To every group element  $h \in \mathcal{U}$  we want to associate an isomorphism of the spinor bundle  $S\mathcal{M}$ . Moreover, these isomorphisms should be compatible with the group operations, whenever the group multiplication stays in  $\mathcal{U}$ . In order to have the inverse element to our disposal, we assume that the implication

$$(3.1) \quad g \in \mathcal{U} \implies g^{-1} \in \mathcal{U}$$

holds. This property can always be arranged by intersecting  $\mathcal{U}$  with the set  $\{g^{-1} \mid g \in \mathcal{U}\}$ . This leads us to the following definition:

**Definition 3.1.** Let  $\mathcal{U} \subset \mathcal{G}$  be an open neighborhood of  $e$  with the property (3.1). Moreover, let  $\Phi$  be a smooth mapping

$$\Phi \in C^\infty(S\mathcal{M} \times \mathcal{U}, S\mathcal{M})$$

with the following properties:

(i)  $\Phi$  is compatible with the local group operations, i.e.

$$(3.2) \quad \Phi_g \circ \Phi_h = \Phi_{gh} \quad \text{for all } g, h \in \mathcal{U} \text{ with } gh \in \mathcal{U},$$

where  $\Phi_g := \Phi(\cdot, g)$ . In view of (3.1), this implies that  $\Phi_g$  is a diffeomorphism on  $S\mathcal{M}$  and that  $\Phi_e = \text{id}_{S\mathcal{M}}$ .

(ii)  $\Phi$  is compatible with the bundle projection  $\pi$ , meaning that the following diagram commutes:

$$(3.3) \quad \begin{array}{ccc} \Phi : S\mathcal{M} \times \mathcal{U} & \longrightarrow & S\mathcal{M} \\ & \pi \downarrow & \pi \downarrow \\ & \mathcal{M} \times \mathcal{U} & \longrightarrow & \mathcal{M} \end{array}$$

(iii) The mapping  $f : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$  defined by the lowest line of this commutative diagram is a family of isometries, i.e.

$$(3.4) \quad (f_h)^* g = g \quad \text{for all } h \in \mathcal{U}.$$

(iv)  $\Phi$  is compatible with Clifford multiplication and preserves the spin scalar product up to a sign, i.e.

$$\gamma((f_h)_* u) = \Phi_h \gamma(u) \Phi_h^{-1} \quad \text{and} \quad \langle \Phi_h \psi | \Phi_h \phi \rangle_{f(p)} = \epsilon(h) \langle \psi | \phi \rangle_p,$$

valid for all  $h \in \mathcal{U}$ ,  $u \in T_p \mathcal{M}$  and  $\psi, \phi \in S_p \mathcal{M}$ . Here  $\epsilon(h)$  is defined by

$$\epsilon(h) = \begin{cases} 1 \\ -1 \end{cases} \quad \text{if } f_h \begin{cases} \text{preserves} \\ \text{reverses} \end{cases} \text{ the time orientation.}$$

(v)  $\Phi$  describes a symmetry of the external potential  $\mathcal{B}$ , i.e.

$$\mathcal{B}(f_h(p)) = \Phi_h \mathcal{B}(p) \Phi_h^{-1}.$$

We refer to  $\Phi$  as a **local group of isomorphisms of the spinor bundle**  $S\mathcal{M}$ .

For clarity, we here explain our notation: Suppose that  $f_h$  maps the space-time point  $p$  to  $q$ . Then the lower star is the usual derivative, i.e.

$$(f_h)_*|_p = Df_h|_p : T_p \mathcal{M} \rightarrow T_q \mathcal{M}.$$

The upper star denotes the pull-back defined by the identity

$$((f_h)^* g)_p(u, v) = g_q((f_h)_* u, (f_h)_* v).$$

We also note for clarity that, as a consequence of the commutativity of the diagram (3.3), also  $f$  is compatible with the group operations, i.e.

$$(3.5) \quad f_g \circ f_h = f_{gh} \quad \text{for all } g, h \in \mathcal{U} \text{ with } gh \in \mathcal{U}.$$

Again using (3.1), this implies that  $f_g$  is a diffeomorphism on  $\mathcal{M}$  and that  $f_e = \text{id}_{\mathcal{M}}$ .

### 3.2. Unitary symmetries on Hilbert spaces of Dirac solutions

Throughout this section, we fix a group element  $h \in \mathcal{U}$ .

**Lemma 3.2.** *The diffeomorphism  $f_h$  maps Cauchy surfaces to Cauchy surfaces.*

*Proof.* Recall that a Cauchy surface in a globally hyperbolic space-time is defined to be a subset with the property that any non-extendible causal curve intersects the set exactly once (see for example [3, p. 62]). Being an isometry of  $\mathcal{M}$ , the mapping  $f_h$  clearly maps inextendible causal curves to inextendible causal curves. Since  $f_h$  is invertible, a curve  $\gamma$  intersects a subset  $\mathcal{N} \subset \mathcal{M}$  if and only if  $f_h(\gamma)$  intersects  $f_h(\mathcal{N})$ . This concludes the proof.  $\square$

**Lemma 3.3.** *Let  $\psi_m \in C^\infty(\mathcal{M}, S\mathcal{M})$  be a solution of the Dirac equation. Then the push-forward  $(\Phi_h)_*\psi \in C^\infty(\mathcal{M}, S\mathcal{M})$  defined by*

$$(3.6) \quad ((\Phi_h)_*\psi)(f_h(p)) := \Phi_h(\psi(p))$$

*is again a solution of the Dirac equation.*

*Proof.* It follows immediately from Definition 3.1 (iii)–(v) and (2.2) that the Dirac operator is invariant under the symmetry transformations. Therefore, solutions are mapped to solutions.  $\square$

**Lemma 3.4.** *The push-forward  $(\Phi_h)_*$  maps the following solution spaces to each other,*

$$(\Phi_h)_* : \mathcal{H}_m^\infty \rightarrow \mathcal{H}_m^\infty.$$

*Moreover, this mapping is bijective and preserves the scalar product  $(\cdot|\cdot)_m$ .*

*Proof.* Let  $\psi_m \in \mathcal{H}_m^\infty$ . In view of the unique solvability of the Cauchy problem,  $\psi_m$  is determined by its initial data  $\psi_m|_{\mathcal{N}} \in C_0^\infty(\mathcal{N}, S\mathcal{M})$  on a Cauchy

surface  $\mathcal{N}$ . By Lemma 3.2, the transformed set  $f_h(\mathcal{N})$  is again a Cauchy surface. Moreover, as the mapping (3.6) maps compact sets to compact sets, the restriction of the transformed wave function  $(\Phi_h)_*\psi_m$  to the transformed Cauchy surface is again compact. Moreover, since  $(\Phi_h)_*\psi_m$  is again a solution of the Dirac equation (see Lemma 3.3), it follows from the unique solvability of the Cauchy problem for initial data on the transformed Cauchy surface that  $(\Phi_h)_*\psi_m$  again has spatially compact support.

In order to show that the scalar product is preserved, we make use of the fact that it can be computed on any Cauchy surface. Hence

$$\begin{aligned} (\psi_m|\phi_m)_m &= \int_{\mathcal{N}} \langle \psi_m | \gamma(\nu) \phi_m \rangle_p d\mu_{\mathcal{N}}(p) \\ &\stackrel{(*)}{=} \int_{f_h(\mathcal{N})} \langle (\Phi_h)_*\psi_m | \gamma(\nu) (\Phi_h)_*\phi_m \rangle_q d\mu_{f_h(\mathcal{N})}(q) \\ &= ((\Phi_h)_*\psi_m | (\Phi_h)_*\phi_m)_m, \end{aligned}$$

where in (\*) we used that  $\Phi_h$  preserves the Lorentzian metric and the spin scalar product (see Definition 3.1 (iii) and (iv)). If  $f_h$  reverses the time orientation, then both the spin scalar product and the future-directed normal  $\nu$  change their signs, so that (\*) again holds. This completes the proof.  $\square$

In view of this result, we can uniquely extend the operator  $(\Phi_h)_*$  by continuity to a unitary mapping on  $\mathcal{H}_m$ . For notational clarity, we denote this operator by  $U_m^h$ ,

$$(3.7) \quad U_m^h := \overline{(\Phi_h)_*} : \mathcal{H}_m \rightarrow \mathcal{H}_m \quad \text{unitary.}$$

The above construction can immediately be extended to families of solutions, simply by carrying it out pointwise for each  $m \in I$ . This gives the following result:

**Lemma 3.5.** *The push-forward  $(\Phi_h)_*$  maps the following spaces of families of solutions to each other,*

$$(\Phi_h)_* : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty.$$

*Moreover, this mapping is bijective and preserves the scalar product  $(\cdot|\cdot)$ . It uniquely extends by continuity to a unitary operator on families of solutions,*

$$U^h := \overline{(\Phi_h)_*} : \mathcal{H} \rightarrow \mathcal{H} \quad \text{unitary.}$$

The operator  $U^h$  acts pointwise in  $m$  and commutes with  $T$ ,

$$(3.8) \quad (U^h \psi)_m = U_m^h \psi_m \quad \text{and} \quad TU^h = U^h T.$$

### 3.3. Symmetries of the fermionic signature operator

**Theorem 3.6.** *Assume that  $(\mathcal{M}, g)$  is weakly  $m$ -finite (see Definition 2.1). Then, up to a sign, the fermionic signature operator is invariant under the symmetry transformation, meaning that for every  $h \in \mathcal{U}$ ,*

$$(3.9) \quad (U_m^h)^* \mathcal{S}_m U_m^h = \epsilon(h) \mathcal{S}_m.$$

*Proof.* For any  $\psi_m, \phi_m \in \mathcal{H}_m$ ,

$$\begin{aligned} (\psi_m | \mathcal{S}_m \phi_m)_m &= \langle \psi_m | \phi_m \rangle \\ &\stackrel{(*)}{=} \epsilon(h) \langle (\Phi_h)_* \psi_m | (\Phi_h)_* \phi_m \rangle = \epsilon(h) \langle (\Phi_h)_* \psi_m | \mathcal{S}_m (\Phi_h)_* \phi_m \rangle_m \\ &\stackrel{(3.7)}{=} \epsilon(h) (U_m^h \psi_m | \mathcal{S}_m U_m^h \phi_m)_m = \epsilon(h) (\psi_m | (U_m^h)^* \mathcal{S}_m U_m^h \phi_m)_m, \end{aligned}$$

where in  $(*)$  we used that  $\Phi_h$  keeps the integration measure unchanged and preserves the spin scalar product up to a sign.  $\square$

**Theorem 3.7.** *Assume that  $(\mathcal{M}, g)$  satisfies the weak mass oscillation property (see Definition 2.2). Then, up to a sign, the operator  $\mathcal{S}$  defined by (2.6) is invariant under the symmetry transformation, i.e.*

$$(3.10) \quad (U^h)^* \mathcal{S} U^h = \epsilon(h) \mathcal{S}.$$

Moreover, the operator-valued measure  $d\mathcal{S}_m$  in (2.7) is invariant up to a sign,

$$(3.11) \quad (U^h)^* d\mathcal{S}_m U^h = \epsilon(h) d\mathcal{S}_m.$$

If  $(\mathcal{M}, g)$  satisfies the strong mass oscillation property (see Definition 2.3), then the fermionic signature operators are all invariant up to a sign, i.e.

$$(3.12) \quad (U_m^h)^* \mathcal{S}_m U_m^h = \epsilon(h) \mathcal{S}_m \quad \text{for all } m \in I.$$

*Proof.* For any  $\psi, \phi \in \mathcal{H}^\infty$ ,

$$\begin{aligned} (\psi | \mathcal{S} \phi) &= \langle \mathbf{p}\psi | \mathbf{p}\phi \rangle \\ &= \epsilon(\mathfrak{h}) \langle (\Phi_{\mathfrak{h}})_* \mathbf{p}\psi | (\Phi_{\mathfrak{h}})_* \mathbf{p}\phi \rangle \stackrel{(*)}{=} \epsilon(\mathfrak{h}) \langle \mathbf{p} (\Phi_{\mathfrak{h}})_* \psi | \mathbf{p} (\Phi_{\mathfrak{h}})_* \phi \rangle \\ &= \epsilon(\mathfrak{h}) ((\Phi_{\mathfrak{h}})_* \psi | \mathcal{S} (\Phi_{\mathfrak{h}})_* \phi) = \epsilon(\mathfrak{h}) (U^{\mathfrak{h}} \psi | \mathcal{S} U^{\mathfrak{h}} \phi) = \epsilon(\mathfrak{h}) (\psi | (U^{\mathfrak{h}})^* \mathcal{S} U^{\mathfrak{h}} \phi), \end{aligned}$$

giving (3.10). Here the only major difference to the proof of Theorem 3.6 is that in (\*) we used that the transformation  $\Phi_{\mathfrak{h}}$  is independent of  $m$  and therefore trivially commutes with  $\mathbf{p}$ . The relation (3.11) follows from the definition (2.7) and the fact that  $U^{\mathfrak{h}}$  commutes with  $T$ .

If the strong mass oscillation property holds, (3.12) follows from the computation

$$\mathcal{S}_m \psi_m = (\mathcal{S} \psi)_m = \epsilon(\mathfrak{h}) ((U^{\mathfrak{h}})^* \mathcal{S} U^{\mathfrak{h}} \psi)_m = \epsilon(\mathfrak{h}) (U_m^{\mathfrak{h}})^* \mathcal{S}_m U_m^{\mathfrak{h}} \psi_m,$$

where we used that both  $U^{\mathfrak{h}}$  and  $\mathcal{S}$  act on  $\mathcal{H}$  pointwise for every  $m \in I$  (see (3.8) and (2.8)). □

### 3.4. Strongly continuous unitary representations of the symmetry group

Varying the group element  $\mathfrak{h}$ , we obtain a mapping

$$(3.13) \quad U_m : \mathcal{U} \rightarrow L(\mathcal{H}_m), \quad \mathfrak{h} \mapsto U_m^{\mathfrak{h}}$$

(with  $U_m^{\mathfrak{h}}$  as defined in (3.7)).

**Theorem 3.8.** *The mapping (3.13) is a local unitary representation of  $\mathcal{G}$  which is strongly continuous, i.e.*

$$\lim_{\mathfrak{h} \rightarrow \mathfrak{g}} \|U_m^{\mathfrak{h}} \psi_m - U_m^{\mathfrak{g}} \psi_m\|_m = 0 \quad \text{for all } \psi_m \in \mathcal{H}_m.$$

*Proof.* From the compatibility with the group operations (3.2) and (3.5) it is straightforward to verify that  $U_m$  is a local group representation. Moreover, in view of (3.7) this representation is obviously unitary. Therefore, it remains to show strong continuity. To this end, let  $\mathfrak{h}(\tau)$  for  $\tau \in (-\delta, \delta)$  and  $\delta > 0$  be a smooth curve in  $\mathcal{U}$  with  $\mathfrak{h}(0) = \mathfrak{e}$ . Using the group properties, it suffices to show strong continuity of  $U_m^{\mathfrak{h}(\tau)}$  at  $\tau = 0$ .

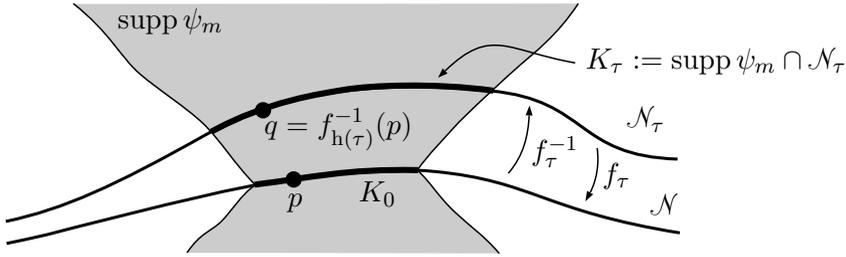


Figure 1: Strong continuity for wave functions with spacelike compact support.

Our first step is to prove this strong continuity for smooth and spatially compact solutions, i.e.

$$(3.14) \quad \lim_{\tau \rightarrow 0} \|U_m^{h(\tau)} \psi_m - \psi_m\|_m = 0 \quad \text{for all } \psi_m \in \mathcal{H}_m^\infty.$$

To this end, given  $\psi_m \in \mathcal{H}_m^\infty$  and a Cauchy surface  $\mathcal{N}$ , we set

$$\mathcal{N}_\tau = f_{h(\tau)}^{-1} \mathcal{N}.$$

Moreover, we define the compact sets

$$K_\tau = \text{supp } \psi_m \cap \mathcal{N}_\tau.$$

For a point  $p \in \mathcal{N}$  we denote the corresponding point on  $\mathcal{N}_\tau$  by  $q = f_{h(\tau)}^{-1} p$ . These notions are illustrated in Figure 1. According to (3.7) and (3.6),

$$(3.15) \quad \begin{aligned} (U_m^{h(\tau)} \psi_m - \psi_m)(p) &= ((\Phi_{h(\tau)})_* \psi_m - \psi_m)(p) \\ &= \Phi_{h(\tau)}(\psi_m(q)) - \psi_m(p). \end{aligned}$$

Integrating over the Cauchy surface  $\mathcal{N}$ , we obtain for the norm corresponding to the scalar product (2.3)

$$\frac{1}{4\pi^2} \|U_m^{h(\tau)} \psi_m - \psi_m\|_m^2 = \int_{\mathcal{N}} \left\| \Phi_{h(\tau)}(\psi_m(f_\tau^{-1}(p))) - \psi_m(p) \right\|_p^2 d\mu_{\mathcal{N}}(p),$$

where the norm in the integrand denotes the pointwise norm on spinors

$$\|\psi\|_p = (\prec \psi | \gamma(\nu) \psi \succ_p)^{\frac{1}{2}}.$$

Since all mappings as well as the wave function  $\psi_m$  are smooth, the integrand obviously tends to zero pointwise as  $\tau \rightarrow 0$ . Moreover, the integrand

is supported in the set

$$K_0 \cup f_\tau(K_\tau) \subset \mathcal{N},$$

and it is bounded pointwise by

$$\sup_{p \in \mathcal{N}, q \in \mathcal{N}_\tau} \left( \|\psi_m(p)\|_p + \|\psi_m(q)\|_q \right)^2.$$

Using that all transformations are smooth and that the speed of propagation is finite, the integrand is bounded and has compact support, both uniformly for  $\tau \in [-\delta/2, \delta/2]$ . Therefore, we may take the limit  $\tau \rightarrow 0$  with the help of Lebesgue's dominated convergence theorem to obtain (3.14).

In order to extend the strong continuity to all of  $\mathcal{H}_m$ , we use a standard  $3\varepsilon$ -argument: Given  $\varepsilon > 0$  and  $\phi_m \in \mathcal{H}_m$ , we choose  $\psi_m \in \mathcal{H}_m^\infty$  with  $\|\psi_m - \phi_m\|_m < \varepsilon$ . Then

$$\begin{aligned} \|U_m^{h(\tau)}\phi_m - \phi_m\|_m &\leq \|U_m^{h(\tau)}(\phi_m - \psi_m)\|_m + \|U_m^{h(\tau)}\psi_m - \psi_m\|_m \\ &\quad + \|\psi_m - \phi_m\|_m \\ &< 2\varepsilon + \|U_m^{h(\tau)}\psi_m - \psi_m\|_m \rightarrow 2\varepsilon, \end{aligned}$$

where we used that  $U_m^{h(\tau)}$  is unitary and took the limit  $\tau \rightarrow 0$  using (3.14). Since  $\varepsilon$  is arbitrary, the result follows.  $\square$

### 3.5. Lie algebra representations and commutators

We denote the Lie algebra corresponding to  $\mathcal{G}$  by  $\mathfrak{g} = T_e\mathcal{G}$ . Given an element  $x \in \mathfrak{g}$  the exponential map gives a one-parameter subgroup of  $\mathcal{G}$ , which we denote by

$$(3.16) \quad h_x(\tau) := \exp_e(\tau x).$$

**Theorem 3.9.** *The following statements hold:*

- (i) *For any  $x \in \mathfrak{g}$  and  $\psi_m \in \mathcal{H}_m^\infty$ , the derivative*

$$(3.17) \quad X\psi_m := -i \frac{d}{d\tau} \left( U_m^{h_x(\tau)}\psi_m \right) \Big|_{\tau=0}$$

*exists in  $\mathcal{H}_m$  and defines a linear operator*

$$(3.18) \quad X : \mathcal{H}_m^\infty \rightarrow \mathcal{H}_m^\infty.$$

Considered as a densely defined operator on  $\mathcal{H}_m$  with domain  $\mathcal{H}_m^\infty$ , this operator is essentially self-adjoint.

- (ii) The mapping  $x \mapsto X$  is a representation of the Lie algebra  $\mathfrak{g}$  on  $\mathcal{H}_m^\infty$ .
- (iii) The following weak commutation relations hold,

$$(3.19) \quad (X\psi_m | \mathcal{S}_m\phi_m)_m = (\mathcal{S}_m\psi_m | X\phi_m)_m \quad \text{for all } \psi_m, \phi_m \in \mathcal{H}_m^\infty.$$

Moreover, if  $\mathcal{S}_m$  has a self-adjoint extension, then the operators  $\overline{X}$  and  $\mathcal{S}_m$  commute in the sense that their spectral measures commute (see [18, p. 271]).

- (iv) In the setting of the weak mass oscillation property, the spectral measure  $dE_{\rho,m}$  in (2.7) commutes with the spectral measure of the operator  $\overline{X}$  (where  $X$  acts on  $\mathcal{H}^\infty$  pointwise in the mass, i.e.  $(X\psi)_m := X\psi_m$ ). Moreover, the following weak commutation relations hold,

$$(X\psi | \mathcal{S}\phi) = (\mathcal{S}\psi | X\phi) \quad \text{for all } \psi, \phi \in \mathcal{H}^\infty.$$

Finally, if  $\mathcal{S}$  has a self-adjoint extension, then the operators  $\overline{X}$  and  $\mathcal{S}$  commute in the sense that their spectral measures commute.

For clarity, we note that the assumption in (iii) that  $\mathcal{S}_m$  must have a self-adjoint extension is obviously satisfied if  $\mathcal{S}_m$  is a bounded operator. This is the case if either space-time is  $m$ -finite (see (2.5)) or if the strong mass oscillation property holds (see Definition 2.3). If space-time is only weakly  $m$ -finite (see Definition 2.1), the resulting fermionic signature operator  $\mathcal{S}_m$  is only symmetric. In this case, in order to obtain strong commutation relations, one must first construct a self-adjoint extension of  $\mathcal{S}_m$  (as is done in Rindler space-time in [9, Section 8]). Similarly, if only the weak mass oscillation property holds, the operator  $\mathcal{S}$  defined by (2.6) is only symmetric.

Before entering the proof, we remark that, knowing from Theorem 3.8 that  $U_{h_x(\tau)}$  is a strongly continuous one-parameter group, Stone’s theorem (see for example [18, Theorem VIII.8]) implies that there is a self-adjoint generator, i.e.

$$U_{h_x(\tau)} = e^{i\tau X} \quad \text{with } X : \mathcal{D}(\mathcal{H}_m) \subset \mathcal{H}_m \rightarrow \mathcal{H}_m \text{ self-adjoint.}$$

However, this abstract result does give explicit information on the domain. In particular, Stone’s theorem does not yield that the domain contains the subset  $\mathcal{H}_m^\infty \subset \mathcal{D}(\mathcal{H}_m)$ , nor that this subspace is mapped to itself. For this reason, we here prefer to use Chernoff’s method [5].

**Lemma 3.10.** *For any  $x \in \mathfrak{g}$ , the derivative (3.17) exists and is in  $\mathcal{H}_m^\infty$ .*

*Proof.* We fix  $x \in \mathfrak{g}$  and denote the one-parameter subgroup (3.16) for simplicity by  $h(\tau)$ . Given  $\psi_m \in \mathcal{H}_m^\infty$ , we know from (3.15) that on a Cauchy surface  $\mathcal{N}$ ,

$$(U_m^{\mathfrak{h}(\tau)}\psi_m - \psi_m)(p) = \Phi_{h(\tau)}\left(\psi_m(f_\tau^{-1}(p))\right) - \psi_m(p).$$

Since all transformations as well as the wave function  $\psi_m$  is smooth, for any  $p \in \mathcal{N}$  the  $\tau$ -derivative exists,

$$(3.20) \quad \frac{d}{d\tau}(U_m^{\mathfrak{h}(\tau)}\psi_m)(p) = \frac{d}{d\tau}\Phi_{h(\tau)}\left(\psi_m(f_\tau^{-1}(p))\right).$$

Moreover, these derivatives are bounded uniformly in  $\tau \in [-\delta/2, \delta/2]$ , locally uniformly in  $p \in \mathcal{N}$ . Exactly as in the proof of Theorem 3.8, one sees that the support of the function in (3.20) is compact, again uniformly in  $\tau \in [-\delta/2, \delta/2]$ . Therefore, we may apply Lebesgue’s dominated convergence theorem to the difference quotient to conclude that the  $\tau$ -derivative exists in  $\mathcal{H}_m$ . Differentiating the Dirac equation

$$(\mathcal{D} - m)(U_m^{\mathfrak{h}(\tau)}\psi_m) = 0$$

with respect to  $\tau$ , we know furthermore that the derivative is again a solution. Since the resulting derivative (3.20) is obviously smooth and has compact support on  $\mathcal{N}$ , this solution is in  $\mathcal{H}_m^\infty$ . This concludes the proof.  $\square$

**Lemma 3.11.** *For any  $x \in \mathfrak{g}$ , the operator  $X$  in (3.18) is essentially self-adjoint on  $\mathcal{H}_m$ .*

*Proof.* From Lemma 3.4 we know that  $U_{h(\tau)}$  maps  $\mathcal{H}_m^\infty$  to itself. Moreover, using the group property,

$$U_m^{\mathfrak{h}(s)}U_m^{\mathfrak{h}(\tau)}\psi_m = U_m^{\mathfrak{h}(s+\tau)}\psi_m = U_m^{\mathfrak{h}(\tau)}U_m^{\mathfrak{h}(s)}\psi_m,$$

and differentiating the left and right with respect to  $\tau$  at  $\tau = 0$  gives

$$U_m^{\mathfrak{h}(s)}X\psi_m = XU_m^{\mathfrak{h}(s)}\psi_m,$$

showing that the operators  $U_m^{\mathfrak{h}(s)}$  and  $X$  commute on  $\mathcal{H}_m^\infty$ . Now we can apply the result by Chernoff [5, Lemma 2.1] to conclude the proof.  $\square$

*Proof of Theorem 3.9.* The previous two lemmas prove part (i) of Theorem 3.9. In order to prove (ii), given  $x, y \in \mathfrak{g}$ , the group properties imply that for every  $\psi_m \in \mathcal{H}_m^\infty$  and all  $\tau \in \mathbb{R}$  with sufficiently small  $|\tau|$ ,

$$(U_m^{\mathfrak{h}_x(\tau)})^{-1} (U_m^{\mathfrak{h}_y(\tau)})^{-1} U_m^{\mathfrak{h}_x(\tau)} U_m^{\mathfrak{h}_y(\tau)} \psi_m = U_m^{\mathfrak{h}_x(\tau)^{-1} \mathfrak{h}_y(\tau)^{-1} \mathfrak{h}_x(\tau) \mathfrak{h}_y(\tau)} \psi_m.$$

In view of (3.17) we may differentiate twice with respect to  $\tau$  at  $\tau = 0$ . On the left side, this gives the commutator of operators  $-2[X, Y]$ . On the right side, on the other hand, we may use the Lie algebra relation

$$\mathfrak{h}_x(\tau)^{-1} \mathfrak{h}_y(\tau)^{-1} \mathfrak{h}_x(\tau) \mathfrak{h}_y(\tau) = \mathfrak{h}_{[x,y]}(\tau^2) + \mathcal{O}(\tau^3)$$

(which can be verified for example by using the Baker-Campbell-Hausdorff formula) to obtain  $2iZ$ , where  $Z$  is the operator corresponding to the commutator  $z = [x, y] \in \mathfrak{g}$ . We thus obtain the relation  $Z = i[X, Y]$ , proving (ii).

For the proof of the weak commutation relations in (iii), we use that the operator  $\mathcal{S}_m$  with domain  $\mathcal{H}_m^\infty$  is symmetric and that the operators  $U_m$  are unitary and commute with  $\mathcal{S}_m$ . This gives

$$(U_m^{\mathfrak{h}_x(\tau)} \psi_m \mid \mathcal{S}_m \phi_m)_m = (\mathcal{S}_m \psi_m \mid (U_m^{\mathfrak{h}_x(\tau)})^{-1} \phi_m)_m$$

According to (3.17) we may differentiate with respect to  $\tau$  at  $\tau = 0$ , giving (3.19).

Now assume that  $\mathcal{S}_m$  has a self-adjoint extension (which we again denote by  $\mathcal{S}_m$ ). According to (3.12), we know that for all  $t \in \mathbb{R}$ ,

$$(U_m^{\mathfrak{h}_x(t)})^* \mathcal{S}_m U_m^{\mathfrak{h}_x(t)} = \mathcal{S}_m.$$

Using the spectral calculus, this equation also holds if  $\mathcal{S}_m$  is replaced by powers of  $\mathcal{S}_m$  or by  $W(\mathcal{S}_m)$ , where  $W$  is any bounded Borel function. In particular, it follows that for all  $s, t \in \mathbb{R}$ ,

$$e^{is\mathcal{S}_m} U_m^{\mathfrak{h}_x(t)} = U_m^{\mathfrak{h}_x(t)} e^{is\mathcal{S}_m}.$$

Noting that  $U_m^{\mathfrak{h}_x(t)} = e^{it\bar{X}}$ , we can apply [18, Theorem VIII.13 (c)] to conclude that  $\bar{X}$  and  $\mathcal{S}_m$  commute. This concludes the proof of (iii).

For the proof of (iv), we make use of the fact that the group  $\mathcal{G}$  as well as its Lie algebra act pointwise in  $m$ . Hence their representations on  $\mathcal{H}$  commute with  $T$ . Therefore, one can adapt the proof of (iii) in a straightforward way by inserting integrals over  $m$  to obtain the result. □

#### 4. Symmetries of generalized fermionic projector states

We now make precise in which sense the generalized fermionic projector state preserves symmetries:

**Theorem 4.1.** *For any non-negative bounded Borel function  $W$ , the operator  $P_W$  defined in (2.11) has the symmetry property*

$$(4.1) \quad (U_m^h)^* P_W((\Phi_h)_* \psi) = \epsilon(h) P_{W^h}(\psi) \quad \text{for all } \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M}),$$

where  $W^h$  is defined by

$$W^h(\lambda) = W(\epsilon(h) \lambda).$$

Likewise, the kernel  $\mathcal{P}_W \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$  defined by (2.10) has the symmetry property

$$(4.2) \quad \langle (\Phi_h)_* \phi | P_W(\Phi_h)_* \psi \rangle = \langle \phi | P_{W^h} \psi \rangle \quad \text{for all } \phi, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M}).$$

Before giving the proof of this theorem, we emphasize a special case relevant for the applications:

**Corollary 4.2.** *Assume that  $f$  preserves the time orientation. Then  $P_W$  is invariant under the symmetries in the sense that for all  $\phi, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$ ,*

$$(4.3) \quad (U_m^h)^* P_W((\Phi_h)_* \psi) = P_W(\psi)$$

$$(4.4) \quad \langle (\Phi_h)_* \phi | P_W(\Phi_h)_* \psi \rangle = \langle \phi | P_W \psi \rangle.$$

*Proof of Theorem 4.1.* We first derive the symmetries of the causal Green's operators  $s_m^\vee$  and  $s_m^\wedge$  (for basic definitions see for example [11, Section 2]). Since  $\Phi_h$  preserves the Dirac equation, from the defining equation of the Green's operator we have for any  $\psi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$ ,

$$\begin{aligned} \Phi_h^{-1} \psi &= \Phi_h^{-1} (\mathcal{D} - m) s_m \psi = (\mathcal{D} - m) \Phi_h^{-1} s_m \psi \\ &= (\mathcal{D} - m) (\Phi_h^{-1} s_m \Phi_h) \Phi_h^{-1} \psi. \end{aligned}$$

Using that  $\Phi_h$  preserves the Lorentzian structure up to the time orientation, we obtain

$$\Phi_h^{-1} s_m^\wedge \Phi_h = \begin{cases} s_m^\wedge & \text{if } f_h \text{ preserves the time orientation} \\ s_m^\vee & \text{if } f_h \text{ reverses the time orientation.} \end{cases}$$

Being defined as the difference of the causal Green’s operators (see again [11, Section 2]), the causal fundamental solution  $k_m$  transforms according to

$$\Phi_h^{-1} k_m \Phi_h = \epsilon(\mathfrak{h}) k_m.$$

Since  $k_m$  maps to solutions, we can also write this identity as

$$(4.5) \quad (U_m^h)^* k_m ((\Phi_h)_* \psi) = \epsilon(\mathfrak{h}) k_m(\psi).$$

Next, applying the spectral calculus to the symmetry statement in (3.9) and (3.12), we obtain

$$W \left( (U_m^h)^* \mathcal{S}_m U_m^h \right) = (U_m^h)^* W \left( \epsilon(\mathfrak{h}) \mathcal{S}_m \right) U_m^h = (U_m^h)^* W^h(\mathcal{S}_m) U_m^h.$$

Multiplying by (4.5) gives (4.1).

In order to prove (4.2), we use the identity  $\langle \phi | \psi_m \rangle = (k_m \phi | \psi_m)_m$ , valid for all  $\psi_m \in \mathcal{H}_m$  and  $\phi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$  (see [11, Proposition 3.1]). We thus obtain

$$\begin{aligned} \langle \phi | P_{W^h} \psi \rangle &= (k_m \phi | P_{W^h} \psi)_m \\ &\stackrel{(4.1)}{=} \epsilon(\mathfrak{h}) (k_m \phi | (U_m^h)^* P_W(\Phi_h)_* \psi)_m = \epsilon(\mathfrak{h}) (U_m^h k_m \phi | P_W(\Phi_h)_* \psi)_m \\ &\stackrel{(4.5)}{=} (k_m (\Phi_h)_* \phi | P_W(\Phi_h)_* \psi)_m = \langle (\Phi_h)_* \phi | P_W(\Phi_h)_* \psi \rangle, \end{aligned}$$

giving the result. □

We finally state an infinitesimal version of the above theorem. To this end, for any  $x \in \mathfrak{g}$  we consider the curve  $h_x(\tau)$  in (3.16) and introduce the Lie-type derivative

$$\begin{aligned} L_x : C_0^\infty(\mathcal{M}, S\mathcal{M}) &\rightarrow C_0^\infty(\mathcal{M}, S\mathcal{M}), \\ (L_x \psi)(p) &:= \frac{d}{d\tau} \left( (\Phi_{h_x(\tau)})_* \psi \right)(p) \Big|_{\tau=0} \stackrel{(3.6)}{=} \frac{d}{d\tau} \Phi_{h_x(\tau)} \left( \psi(f_{h_x(\tau)}^{-1}(p)) \right) \Big|_{\tau=0}. \end{aligned}$$

**Corollary 4.3.** *For any  $x \in \mathfrak{g}$ , the generalized fermionic projector state has the infinitesimal symmetries*

$$\begin{aligned} (iX \phi_m | P_W \psi)_m + (\phi_m | P_W L_x \psi)_m &= 0 \\ \langle L_x \eta | P_W \psi \rangle + \langle \eta | P_W L_x \psi \rangle &= 0, \end{aligned}$$

valid for all  $\eta, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$  and  $\phi_m \in \mathcal{H}_m^\infty$ .

*Proof.* Follows immediately by evaluating the identities of Corollary 4.2 for  $h = h_x(\tau)$  and differentiating with respect to  $\tau$  at  $\tau = 0$ . Before differentiating (4.3), one must take the inner product with  $\psi_m$ , making it possible to apply (3.17).  $\square$

## 5. Applications

We now consider two typical applications: infinitesimal symmetries as described by Killing fields and discrete symmetries.

### 5.1. Killing symmetries

In many applications, the symmetries of space-time are expressed in terms of Killing fields. Since the commutator of two Killing fields is again Killing, we may assume without loss of generality that the Killing fields form a Lie algebra  $\mathfrak{g}$  of dimension  $d$ ,

$$\mathfrak{g} \subset C^\infty(\mathcal{M}, T\mathcal{M}) \quad \text{with} \quad \dim \mathfrak{g} = d \geq 1.$$

In general, a Killing field does *not* give rise to a corresponding symmetry of the fermionic signature operator, as we now illustrate.

**Example 5.1. (The Minkowski drum)** Let  $\mathcal{M} \subset \mathbb{R}^{1,1}$  be a globally hyperbolic subset of two-dimensional Minkowski space as considered in [7, Section 1.1]. Then the restriction of the Killing fields of Minkowski space (the three generators of the Poincaré group in two dimensions) are clearly Killing fields in  $\mathcal{M}$ . However, these Killing fields do not correspond to global symmetries of  $\mathcal{M}$ . Accordingly, the fermionic signature operator does *not* reflect the Killing symmetries. This can be seen explicitly in the following counter example:

For the triangular domain

$$\mathcal{M} = \{(t, x) \in \mathbb{R}^{1,1} \mid 0 < t < \pi - |x|\},$$

the fermionic signature operator is computed explicitly in [7, Example 3.6]. Choosing the representation of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the fermionic signature operator in the massless case  $m = 0$  maps the plane wave solutions

$$\psi_L^n(t, x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{in(x+t)}, \quad \psi_R^n(t, x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{in(x-t)}$$

with  $n \geq 1$  to each other. However, these plane waves are eigenfunctions of the Hamiltonian

$$H = -i\gamma^0\gamma^1\partial_x = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x$$

with two different eigenvalues. Therefore,  $H$  and  $\mathcal{S}_0$  do not commute. Since  $H$  is the generator of time translations on  $\mathcal{H}_0$ , we conclude that the Killing field  $\partial_t$  does not correspond to a symmetry of  $\mathcal{S}_0$ .

This example also explains why we need global symmetries as described by local group actions on  $\mathcal{M}$ . This leads us to impose an additional condition on the Killing fields:

**Definition 5.2.** The Killing field  $K$  is **complete** if there is an  $\varepsilon > 0$  such that for every  $p \in \mathcal{M}$  the integral curve  $\gamma$  of  $K$  with  $\gamma(0) = p$  exists on  $(-\varepsilon, \varepsilon)$ .

By patching the solutions, this definition immediately implies that the integral curve exists on all of  $\mathbb{R}$ . We remark that in the special case when the Killing field  $K$  describes the time translation symmetry of a static space-time, the completeness of  $K$  follows from geodesic completeness (see [19, Theorem 2.1 (i)]). We also remark that the Killing field is complete if it can be written as  $K = \frac{\partial}{\partial t}$  with a global time function  $t \in \mathbb{R}$ .

Under the above assumption, the Killing symmetry can indeed be lifted to a local symmetry of the spinor bundle:

**Proposition 5.3.** *Assume that  $\mathfrak{g}$  consists of complete Killing fields. Then there is a Lie group  $\mathcal{G}$  with  $T_e\mathcal{G} = \mathfrak{g}$  as well as a local group of isomorphism  $\Phi$  of  $S\mathcal{M}$  having all properties (i)–(v) in Definition 3.1. Moreover, the corresponding space-time symmetry  $f$  in (3.4) is generated by the Killing fields in the sense that*

$$(5.1) \quad K(p) = \frac{d}{d\tau} f_{\exp_e(\tau K)}(p) \Big|_{\tau=0} \quad \text{for all } K \in \mathfrak{g}.$$

For clarity, we remind the reader that the Lie algebra  $\mathfrak{g}$  consists of vector fields on  $\mathcal{M}$  (see (5.3)), so that both sides of (5.1) give a vector in  $T_p\mathcal{M}$ .

*Proof of Proposition 5.3.* Let  $\mathcal{G}$  be a Lie group with  $T_e\mathcal{G} = \mathfrak{g}$  (for example, one can choose the unique simply-connected Lie group with this property; see [15, Theorem 3.15] or the “converse of Lie’s third theorem” in [14, p. 108]). Since the exponential map is locally invertible, there is a neighborhood  $V$  of  $0 \in \mathfrak{g}$  such that the restriction

$$\exp_e : V \rightarrow \exp_e(V) =: \mathcal{U} \subset \mathcal{G}$$

is a diffeomorphism, and  $\mathcal{U}$  is an open neighborhood of  $e$ . We denote its inverse by  $\log : \mathcal{U} \rightarrow V$ .

Evaluating the integral curves of the Killing fields at  $\tau = 1$ , we obtain a smooth mapping

$$E : \mathcal{M} \times \mathfrak{g} \rightarrow \mathcal{M},$$

which can be thought of as a realization of the exponential map on  $\mathcal{M}$ . Decomposing this mapping by the logarithm gives the desired local group action on space-time

$$(5.2) \quad f : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}, \quad f(p, h) := E(p, \log h).$$

From the unique local characterization of Lie groups from their Lie algebras (following from the convergence of the Baker-Campbell-Hausdorff formula, see [20, Theorems 2.15.4 and 2.16.6]; see also the “converse of Lie’s second theorem” in [14, p. 107]), this mapping is indeed compatible with the local group operations (3.5) if the neighborhood  $\mathcal{U}$  is chosen sufficiently small. Moreover, (5.1) follows immediately from (5.2).

In order to construct the local group action on the spinor bundle, we define  $\Phi_h\psi(p)$  as the spinor obtained from  $\psi(p)$  by parallel transport with respect to the spin connection along the integral curve of the Killing field  $\log h$  through the point  $p$ , evaluated at  $f_h(p)$ . The compatibility with the group operation is verified as follows: Let  $\psi, \phi \in S_p\mathcal{M}$  be two spinors. Then the corresponding Dirac current is in the complexified tangent space,

$$\langle \psi | \gamma^j \phi \rangle_p \partial_j \Big|_p =: v \in T_p^{\mathbb{C}}\mathcal{M},$$

and since the spin connection is the lift of the Levi-Civita connection, it follows that

$$\langle \Phi_g \Phi_h \psi | \gamma^j \Phi_g \Phi_h \phi \rangle_{f_{gh}(p)} = \langle \Phi_{gh} \psi | \gamma^j \Phi_{gh} \phi \rangle_{f_{gh}(p)} = ((f_{gh})_* v)^j.$$

Moreover, knowing that the spinorial parallel transport is unitary with respect to the spin scalar product and that the last equation holds for

all  $\psi, \phi \in S_p \mathcal{M}$ , we conclude that

$$\Phi_h^{-1} \Phi_g^{-1} \gamma^j \Phi_g \Phi_h = \Phi_{gh}^{-1} \gamma^j \Phi_{gh}$$

or, equivalently, that the following operators commute,

$$[A, \gamma^j] = 0 \quad \text{with} \quad A := \Phi_{gh} \Phi_h^{-1} \Phi_g^{-1}.$$

As a consequence,  $A$  commutes with all elements of the Clifford algebra. Since spinor representations are irreducible (see [16]), it follows by Schur's lemma that  $A$  is plus or minus the identity. We conclude that

$$(5.3) \quad \Phi_g \Phi_h \Big|_{S_p \mathcal{M}} = \pm \Phi_{gh} \Big|_{S_p \mathcal{M}}.$$

A continuity argument shows that, choosing the neighborhood  $\mathcal{U}$  sufficiently small, only the plus sign appears. This concludes the proof.  $\square$

We finally remark that by [20, Theorem 2.16.13], it is even possible to define the action  $f$  on  $M$  on the whole group  $G$ , if  $G$  is chosen to be simply connected. However, in view of the sign ambiguity (5.3), it is unclear to us how to construct the lift to  $S\mathcal{M}$  globally. This is the reason why we restrict attention to local group actions.

## 5.2. Discrete symmetries

As another typical application assume that  $\mathcal{U} = \mathcal{G}$  is a discrete group acting as a group of isometries  $f : \mathcal{M} \times \mathcal{G} \rightarrow \mathcal{M}$  on space-time. In order to get into the setting of Section 3, the group of isometries on  $\mathcal{M}$  must be lifted to a group of isomorphisms of the spinor bundle  $S\mathcal{M}$ . More precisely, we need to construct a mapping  $\Phi$  which has all the properties in Definition 3.1. This is a non-trivial task which involves a detailed knowledge of the group action  $f$  and the spin structure of  $S\mathcal{M}$ . Therefore, in order not to distract from the main topic of this paper, we shall not enter this construction, but instead assume that  $\Phi$  is given (however, in the simple example of Minkowski space and the group of parity and time reversal transformations, the detailed construction is carried out in Section 6.1 below). Then all the results of Sections 3.2, 3.3 and 4 apply. In particular, one obtains a unitary representation on  $\mathcal{H}_m$  (see (3.7) and Lemma 3.5). Moreover, the fermionic signature operator is invariant up to signs (see Theorems 3.6 and 3.7), and the generalized fermionic projector state has the symmetry properties stated in Theorem 4.1.

## 6. Examples

We now illustrate our constructions and results in a few examples in which the fermionic signature operator has been studied previously.

### 6.1. Minkowski space

Let  $\mathcal{M} = \mathbb{R}^{1,3}$  be Minkowski space. We work with the Dirac equation in the Dirac representation, i.e.  $S\mathcal{M} = \mathcal{M} \times \mathbb{C}^4$  and

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix},$$

where  $\sigma^\alpha$  are the three Pauli matrices. The Dirac equation reads

$$(i\gamma^j \partial_j - m)\psi(x) = 0.$$

The symmetries of Minkowski space are described by the Poincaré group  $\mathcal{P}$ , being the semi-direct product of translations with the Lorentz group,

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3).$$

The Lorentz group, in turn, is the semi-direct product of the proper, orthochronous Lorentz group and a discrete group,

$$\mathrm{O}(1,3) = \mathrm{SO}^+(1,3) \rtimes \{1, P, T, PT\},$$

where  $P$  is the parity transformation and  $T$  denotes time reflections, i.e.

$$f_P(x) = (t, -\vec{x}) \quad \text{and} \quad f_T(x) = (-t, \vec{x}),$$

where  $x$  has the components  $(t, \vec{x})$ .

We begin with the Killing symmetries. The Lie algebra of the Poincaré group gives rise to ten Killing fields (4 translations, 3 rotations and 3 boosts). The rotations and boosts are lifted infinitesimally by the well-known transformations (see for example [4, Chapter 2])

$$(\Phi_{\mathfrak{h}}\psi)(\Lambda x) = S\psi(x)$$

with

$$(6.1) \quad dS = -\frac{i}{4} d\Lambda_{jk} \sigma^{jk} S$$

(where  $\Lambda \in \text{SO}^+(1, 3)$  and  $\sigma^{jk} = i[\gamma^j, \gamma^k]/2$  are the bilinear covariants). Integrating this equation globally gives rise to the usual representation of the spin group on the spinor bundle. With our concept of working with local group actions, one can avoid topological issues and work with the proper, orthochronous Lorentz group instead of the spin group. Thus we choose

$$\mathcal{U} \subsetneq \mathcal{G} := \mathbb{R}^{1,3} \rtimes \text{SO}^+(1, 3).$$

Restricting the group actions to a small subset of the identity (necessarily with all rotation angles smaller than  $2\pi$ ), and taking  $\Phi$  as the lift obtained by integrating (6.1), we obtain a local group of isomorphism of the spinor bundle (see Definition 3.1). Thus all results of Sections 3 and 4 apply: The unitary mappings  $\mathcal{U}_m^h$  describe spatial translations and time evolutions of Dirac solutions as well as rotations and boosts. All these mappings are strongly continuous according to Theorem 3.8. Furthermore, Theorem 3.9 shows that the corresponding generators  $X$  (being the momentum and angular momentum operators, the Hamiltonian and infinitesimal boost operators) are all essentially self-adjoint. Both the local representation of the group and the representation of the Lie algebra on  $\mathcal{H}_m$  commute with the fermionic signature operator (see Theorems 3.7 and 3.9). Moreover, the resulting generalized fermionic projector states are Lorentz invariant as specified in Corollaries 4.2 and 4.3.

We next consider the discrete group by choosing

$$\mathcal{U} = \mathcal{G} = \{1, P, T, PT\}.$$

Setting<sup>1</sup>

$$(\Phi_P \psi)(f_P(x)) = \gamma^0 \psi(x) \quad \text{and} \quad (\Phi_T \psi)(f_T(x)) = \gamma^5 \gamma^0 \psi(x)$$

(where  $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ ), the relations

$$\begin{aligned} \langle \Phi_P \psi | \Phi_P \psi \rangle_{f_P(x)} &= \langle \psi | \psi \rangle_x \\ \langle \Phi_P \psi | \gamma^0 \Phi_P \psi \rangle_{f_P(x)} &= \langle \psi | \gamma^0 \psi \rangle_x \\ \langle \Phi_P \psi | \gamma^\alpha \Phi_P \psi \rangle_{f_P(x)} &= \langle \psi | (-\gamma^\alpha) \psi \rangle_x \end{aligned}$$

---

<sup>1</sup>We remark that our lift of  $T$  does not agree with the  $T$ -transformation in the physics literature (see [4, Section 5.4] or [17, Section 3.6]), where an *anti*-linear transformation is used. Using the common notions in physics, our transformation corresponds to  $CT$ , where  $C$  is charge conjugation.

$$\begin{aligned} \langle \Phi_T \psi | \Phi_T \psi \rangle_{f_T(x)} &= -\langle \psi | \psi \rangle_x \\ \langle \Phi_T \psi | \gamma^0 \Phi_T \psi \rangle_{f_T(x)} &= -\langle \psi | (-\gamma^0) \psi \rangle_x \\ \langle \Phi_T \psi | \gamma^\alpha \Phi_T \psi \rangle_{f_T(x)} &= -\langle \psi | \gamma^\alpha \psi \rangle_x \end{aligned}$$

(where  $\alpha = 1, 2, 3$  denotes the spatial index) show that the transformation indeed leaves Clifford multiplication invariant. Moreover, the  $T$ -transformation flips the sign of the spin scalar product, in agreement with Definition 3.1 (iv). Therefore,  $\Phi$  is indeed a group of isomorphisms of the spinor bundle (see Definition 3.1). Consequently, the results of Sections 3 and 4 apply and show that the fermionic signature operator is invariant under the group transformation, except that the  $T$ -transformation gives rise to a sign, i.e.

$$(6.2) \quad (U_m^T)^* \mathcal{S}_m U_m^T = -\mathcal{S}_m.$$

These symmetry properties carry over to the generalized fermionic projector states (see Theorem 4.1).

Taken together, these symmetry properties imply that all the positive-frequency solutions (and similarly all the negative-frequency solutions) form an eigenspace of  $\mathcal{S}_m$ . Moreover, the  $T$ -transformation (6.2) shows that the corresponding eigenvalues have the same absolute value but opposite signs. Hence these symmetry considerations determine the fermionic signature operator (as computed in [12, Theorem 5.1] to have eigenvalues  $\pm 1$ ) up to a real prefactor. The symmetry considerations also show that the generalized fermionic projector states do not give more than the usual frequency splitting.

### 6.2. Static and ultrastatic space-times

In *static* space-times, the symmetry under time translations is described by a timelike Killing field  $K = \partial_t$ . Integrating this Killing symmetry gives the group action  $f$  of  $\mathcal{G} = \mathbb{R}$  on  $\mathcal{M}$ . The lift  $\Phi$  to  $S\mathcal{M}$  is obtained simply by parallel transport of spinors along the Killing field. Thus all results of Sections 3 and 4 again apply. The resulting strongly continuous family of operators  $U_h$  are the time evolution operators, and its essentially self-adjoint generator  $X$  is the Dirac Hamiltonian. Moreover, we conclude that the spectral measures of  $\bar{X}$  and  $\mathcal{S}_m$  commute.

In static space-times one also has the symmetry  $T$  under time reversals. Similar as explained in Minkowski space, this again gives rise to the symmetry (6.2).

For *ultrastatic space-times* it was shown in [12, Theorem 5.1] that the fermionic signature operator has eigenvalues  $\pm 1$ , and that the corresponding eigenspaces coincide with the subspaces of positive and negative frequency, respectively. This is consistent with the above results obtained from the symmetry considerations, but of course the symmetry considerations give much less information.

### 6.3. Rindler space-time

In [9] the fermionic signature operator is computed in two- and four-dimensional Rindler space-time. We now discuss to which extent these results can be obtained from symmetry considerations.

In two-dimensional Rindler space-time, there is a timelike Killing field  $K$  describing Lorentz boosts. Therefore, space-time is static with respect to so-called Rindler time  $\tau$  (defined by the relation  $\partial_\tau = K$ ; for details see [9, Section 10]), and the symmetry considerations of the previous section imply that  $\mathcal{S}_m$  commutes with the Hamiltonian in Rindler time and is anti-symmetric under reversals of Rindler time (6.2). In [9, Theorem 10.1] it is shown that  $\mathcal{S}_m$  is indeed a multiple of the Hamiltonian. This is compatible with the symmetry considerations but clearly is a much stronger result.

In four-dimensional Rindler space-time, the two additional spatial coordinates  $y$  and  $z$  give rise to an additional symmetry group  $\mathbb{R}^2 \rtimes \mathrm{O}(2)$ . Thus the total symmetry group is

$$(6.3) \quad \mathcal{G} = \mathbb{R}^1 \times (\mathbb{R}^2 \rtimes \mathrm{O}(2)).$$

The additional symmetry means that, after separating the  $y$ - and  $z$ -dependence by plane waves with momenta  $k_y$  and  $k_z$ , the fermionic signature operator must depend only on  $k_y^2 + k_z^2$ . This is compatible with the result in [9, Theorem 13.2], but again the explicit computation of the fermionic signature operator gives more detailed information.

### 6.4. Closed Friedmann-Robertson-Walker space-times

In [11, Section 5] and [12, Section 6] the fermionic signature operator is computed in spatially symmetric space-times. The symmetry results in the present paper show that the fermionic signature operator is diagonal on the spatial modes (i.e. the eigenspinors of the spatial Dirac operator; for details see [10]). Thus our results give a more abstract explanation for the separation procedure used in the the above-cited papers.

### 6.5. A plane electromagnetic wave

In [13] the fermionic signature operator is computed in Minkowski space in the presence of an electromagnetic potential of the form of a plane wave, i.e.

$$A = A(t + x).$$

As shown in [9, Theorem 5.5], the fermionic signature operator has eigenvalues  $\pm 1$ , and the corresponding eigenspaces are the solutions of positive and negative momentum  $u$  in the separation ansatz in null direction

$$\psi_m(t, x, y, z) = e^{-iu(t-x)} \chi_m(t + x, y, z).$$

This result can again be partly understood from symmetry considerations: The symmetries are again described by (6.3), where the group  $\mathbb{R}^1$  now describes translations in the null direction  $(1, -1, 0, 0)$ , and the group  $\mathbb{R}^2 \times \text{O}(2)$  again acts on the spatial coordinates  $y$  and  $z$ . The momentum  $u$  is precisely the spectral parameter of the generator  $X$  of the translations in null directions.

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### References

- [1] H. Araki, *On quasifree states of CAR and Bogoliubov automorphisms*, Publ. Res. Inst. Math. Sci. **6** (1970/71), 385–442.
- [2] H. Baum, *Spinor structures and Dirac operators on pseudo-Riemannian manifolds*, Bull. Polish Acad. Sci. Math. **33** (1985), no. 3-4, 165–171.
- [3] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry*, second ed., Monographs and Textbooks in Pure and Applied Mathematics, Vol. 202, Marcel Dekker, Inc., New York, 1996.
- [4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill Book Co., New York, 1964.
- [5] P.R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis **12** (1973), 401–414.

- [6] C.J. Fewster and B. Lang, *Pure quasifree states of the Dirac field from the fermionic projector*, *Class. Quantum Grav.* **32** (2015), no. 9, 095001, 30.
- [7] F. Finster and O. Müller, *Lorentzian spectral geometry for globally hyperbolic surfaces*, *Adv. Theor. Math. Phys.* **20** (2016), no. 4, 751–820.
- [8] F. Finster, S. Murro, and C. Röken, *The fermionic projector in a time-dependent external potential: Mass oscillation property and Hadamard states*, *J. Math. Phys.* **57** (2016), 072303.
- [9] F. Finster, S. Murro, and C. Röken, *The fermionic signature operator and quantum states in Rindler space-time*, *J. Math. Anal. Appl.* **454** (2017), no. 1, 385–411.
- [10] F. Finster and M. Reintjes, *The Dirac equation and the normalization of its solutions in a closed Friedmann-Robertson-Walker universe*, *Class. Quantum Grav.* **26** (2009), no. 10, 105021.
- [11] F. Finster and M. Reintjes, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime*, *Adv. Theor. Math. Phys.* **19** (2015), no. 4, 761–803.
- [12] F. Finster and M. Reintjes, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II – Space-times of infinite lifetime*, *Adv. Theor. Math. Phys.* **20** (2016), no. 5, 1007–1048.
- [13] F. Finster and M. Reintjes, *The fermionic signature operator and Hadamard states in the presence of a plane electromagnetic wave*, *Ann. Henri Poincaré* **18** (2017), no. 5, 1671–1701.
- [14] R. Gilmore, *Lie groups, Lie Algebras, and Some of Their Applications*, John Wiley & Sons, New York, 1974.
- [15] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*, Graduate Texts in Mathematics, Vol. 222, Springer-Verlag, New York, 2003.
- [16] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series, Vol. 38, Princeton University Press, Princeton, NJ, 1989.
- [17] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1995.

- [18] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I, Functional analysis*, second ed., Academic Press Inc., New York, 1980.
- [19] M. Sánchez, *On the geometry of static spacetimes*, *Nonlinear Analysis* **63** (2005), e455–e463.
- [20] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, *Graduate Texts in Mathematics*, Vol. 102, Springer-Verlag, New York, 1984, Reprint of the 1974 edition.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT REGENSBURG  
D-93040 REGENSBURG, GERMANY  
*E-mail address:* `finster@ur.de`

DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO  
1049-001 LISBON, PORTUGAL  
*E-mail address:* `moritzreintjes@gmail.com`

