

Flopping and slicing: $\mathrm{SO}(4)$ and $\mathrm{Spin}(4)$ -models

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We study the geometric engineering of gauge theories with gauge group $\mathrm{Spin}(4)$ and $\mathrm{SO}(4)$ using crepant resolutions of Weierstrass models. The corresponding elliptic fibrations realize a collision of singularities corresponding to two fibers with dual graphs \tilde{A}_1 . There are eight different ways to engineer such collisions using decorated Kodaira fibers. The Mordell–Weil group of the elliptic fibration is required to be trivial for $\mathrm{Spin}(4)$ and $\mathbb{Z}/2\mathbb{Z}$ for $\mathrm{SO}(4)$.

Each of these models has two possible crepant resolutions connected by a flop. We also compute a generating function for the Euler characteristic of such elliptic fibrations over a base of arbitrary dimensions. In the case of a threefold, we also compute the triple intersection numbers of the fibral divisors. In the case of Calabi-Yau threefolds, we also compute their Hodge numbers and check the cancellations of anomalies in a six-dimensional supergravity theory.

1	Introduction	1004
2	Summary of results	1011
3	Preliminaries	1027
4	$\mathrm{SO}(4)$-model	1043
5	$\mathrm{Spin}(4)$-model	1049
	References	1061

1. Introduction

Crepant resolutions of Weierstrass models and their network of flops is a topic of interest for both mathematicians and physicists [4, 53, 57, 64]. Such crepant resolutions form a rich setup to study elliptic fibrations since any elliptic fibration over a smooth base is birational to a Weierstrass model, and in many important cases, the birational transformation to the Weierstrass model preserves the canonical class of the elliptic fibration (see also [38]). A crepant resolution of a Weierstrass model is a minimal model in the sense of Mori's theory, and minimal models are connected by flops [57, 64].

In F-theory and M-theory, elliptic fibrations are used as compactifying spaces to geometrically engineer gauge theories in space-times of dimension less than or equal to eight [11, 61, 62]. The F-theory's algorithm attaches to a given elliptic fibration a Lie algebra \mathfrak{g} , a Lie group G , and a representation \mathbf{R} of G . The Lie algebra \mathfrak{g} is determined by the singular fibers over generic points of the discriminant locus. As the fundamental group of G is isomorphic to the Mordell–Weil group of the elliptic fibration, the global structure of the Lie group G depends on not only the Lie algebra but also the Mordell–Weil group of the fibration [7, 58, 72]. The representation \mathbf{R} is derived from the singular fibers over codimension-two points of the base. In particular, the weights of \mathbf{R} are geometrically computed by intersection of rational curves composing the singular fibers and fibral divisors that project to irreducible components of the discriminant (see Section 3.2). An elliptic fibration for which the F-theory algorithm returns a Lie group G is called a G -model. The geometry of G -models are well understood mostly when G is a simple group. The semi-simple case is more subtle because of the appearance of colliding singularities.

1.1. Colliding singularities

The theory of elliptic surfaces is well understood since the seminal work of Kodaira and Néron [54, 65]. For an elliptic surface, the discriminant locus (i.e. the locus of singular fibers) is composed of isolated points and the singular fibers are classified by Kodaira symbols (see Section 3.1).

When the elliptic fibration is of dimension three or higher, the discriminant locus can have intersecting components. Kodaira fibers now classify the type of the geometric fiber over the generic point of a normal irreducible component of the discriminant locus. Singularities of the discriminant locus are called *collisions of singularities* [60]. A typical example of collisions

of singularities are intersection points of irreducible components of the discriminant locus. Let S_1 and S_2 be two smooth irreducible components of the discriminant locus with generic fibers of Kodaira type T_1 and T_2 . The generic fiber at the intersection of S_1 and S_2 is denoted by $T_1 + T_2$. The type of the generic fiber at the collision does not have to be one of Kodaira's types [42, 60]. But it is usually a contraction of a Kodaira type or obtained by letting some of the nodes of a Kodaira fiber to coincide. We will assume throughout this paper that we work over the complex numbers.

In the early 1980s, Miranda introduced a systematic regularization procedure for elliptic threefolds defined by Weierstrass models [60]. He considers collisions of type $T_1 + T_2$ in which T_1 and T_2 are Kodaira fibers having the same j -invariant and such that the supporting divisors are smooth divisors intersecting transversally. Miranda's regularization produces elliptic fibrations that are flat fibrations and their j -invariant is a morphism. Miranda's work on threefolds was generalized to elliptic n -folds in Szydło's Ph.D. thesis [71]. Elliptic fibrations resulting from Miranda's regularization are called *Miranda models* [21]. Miranda models were used by Dolgachev and Gross to study the Tate-Shafarevich group of elliptic threefolds [21]. Using tools from the Minimal Model Program, Hodge theory, and toric geometry, Nakayama studied in [64] the local fibration structure of elliptic threefolds defining a collision $I_n + I_m$ with normal transverse divisors in a nonsingular surface.

Miranda regularization is not usually a crepant resolution. But when it is, it provides interesting examples of elliptic Calabi-Yau threefolds yielding non-Abelian semi-simple gauge theories. Applications of Miranda model to F-theory was first studied by Bershadsky and Johansen [12] and have applications in the classification of six-dimensional Conformal Field Theories (CFTs).

1.2. $SO(4)$ and $Spin(4)$ -models

In F-theory and M-theory, collisions of singularities are crucial for geometric engineering of gauge theories with semi-simple Lie groups, and matters charged under some representation of the Lie group [62, 62, 72]. In recent years, we have improved our understanding of crepant resolutions of elliptic fibrations corresponding to simple Lie groups [27, 33, 39–42]. We have extended these methods to the study of elliptic fibrations corresponding to semi-simple groups [32], including the case in which the gauge group is not simply connected. Following a point of view that started in [2, 3, 32, 43], we work relatively to an arbitrary base and do not impose the Calabi-Yau

condition. This allows us to understand the geometry of these elliptic fibrations in a larger setting before specializing to the particular cases relevant to string theory. In the case of Calabi-Yau threefolds, such analyses are closely related to basic questions on five-dimensional and six-dimensional supergravity theories since they are obtained by compactifications of M-theory and F-theory.

In the case of the five-dimensional theory, we determine the matter content, the structure of the Coulomb branch, and the Chern-Simons levels. In its six-dimensional theory uplift, we determine the tensor branch, the matter content, and the fine detail of cancellations of gravitational, gauge, and mixed anomalies.

In this paper, we study the geometry of $\mathrm{SO}(4)$ and $\mathrm{Spin}(4)$ -models. The $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ -models for $n \neq 4, n \geq 3$ are associated with gauge theories with simple gauge groups and do not require a collision of singularities. The $\mathrm{SO}(4)$ and $\mathrm{Spin}(4)$ -models are associated with the semi-simple Lie algebra of type D_2 , which is the unique reducible semi-simple Lie algebra \mathfrak{g} of rank two, and the direct sum of two Lie algebras of type A_1 :

$$D_2 \cong A_1 \oplus A_1.$$

In this sense, the $\mathrm{SO}(4)$ and $\mathrm{Spin}(4)$ -models are the simplest G -models with G a semi-simple group.¹

Geometrically, $\mathrm{SO}(4)$ and $\mathrm{Spin}(4)$ -models are elliptic fibrations whose discriminant locus contains two smooth irreducible components (S and T) such that the dual graphs of the fibers over the generic points of S and T is the affine Dynkin diagram \tilde{A}_1 , while the fiber over other generic points of the discriminant locus are irreducible singular fibers. Since $\mathrm{Spin}(4)$ has a trivial fundamental group and the fundamental group of $\mathrm{SO}(4)$ is $\mathbb{Z}/2\mathbb{Z}$, the Mordell–Weil group of a $\mathrm{Spin}(4)$ -model is trivial while the one of an $\mathrm{SO}(4)$ -model is $\mathbb{Z}/2\mathbb{Z}$.

The $\mathrm{SO}(4)$ -model as a collision $I_2^{\mathrm{ns}} + I_2^{\mathrm{ns}}$ in an elliptic fibration with Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$ has been discussed in [63] in the case of a Calabi-Yau threefold elliptically fibered over a \mathbb{P}^2 base. The $\mathrm{SO}(4)$ -model was studied in more detail using toric methods in [58]. The first appearing of elliptic fibrations with a $\mathbb{Z}/2\mathbb{Z}$ Mordell–Weil group in F-theory is in [13] in relation

¹As discussed in Section 1.3, there are also other compact groups with Lie algebra D_2 , namely the half-spin groups $\mathrm{HSpin}^\pm(4)$ which are isomorphic to $\mathrm{SO}(3) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2) \times \mathrm{SO}(3)$. However, these groups are not compatible with the typical representation $(\mathbf{2}, \mathbf{2})$ observed at the collisions of two fibers with dual graphs \tilde{A}_1 .

to the CHL model (see also [70]) and shortly after in [7] in the study of non-simply connected Lie groups in F-theory.

In this paper, we add to previous work on SO(4) and Spin(4)-models in many ways. We consider other collisions than $I_2^{\text{ns}} + I_2^{\text{ns}}$ and before to specialize to the case of Calabi-Yau threefolds, we study these geometries over a base arbitrary dimension. Geometrically, SO(4) and Spin(4)-models are characterized by the collision of two Kodaira fibers whose dual graphs are of type \tilde{A}_1 . While the Kodaira fiber of type I_2 has dual graph \tilde{A}_1 , it is not the only one. That means there are many more ways to realize geometrically a model of type SO(4) or Spin(4) than a collision of type $I_2 + I_2$. In fact, the dual graph \tilde{A}_1 is the most versatile one as it can be realized geometrically by five distinct types of singular fibers, namely

$$(1.1) \quad \text{Fibers with dual graph } \tilde{A}_1 : I_2^{\text{ns}}, I_2^{\text{s}}, \text{ III}, I_3^{\text{ns}}, \text{ IV}^{\text{ns}}.$$

We avoid realizing A_1 with fibers of type I_3^{ns} to stay away from terminal singularities. The elliptic fibrations that we consider are constructed by crepant resolutions of Weierstrass models corresponding to the following collisions of singularities

$$(1.2) \quad \begin{aligned} &I_2^{\text{ns}} + I_2^{\text{ns}}, \quad I_2^{\text{s}} + I_2^{\text{ns}}, \quad I_2^{\text{s}} + I_2^{\text{s}}, \quad \text{III} + I_2^{\text{ns}}, \quad \text{III} + I_2^{\text{s}}, \\ &\text{III} + \text{III}, \quad \text{IV}^{\text{ns}} + I_2^{\text{s}}, \quad \text{IV}^{\text{ns}} + I_2^{\text{ns}}. \end{aligned}$$

These collisions define a gauge theory with Lie algebra $A_1 \oplus A_1$. There are many gauge groups with this Lie algebra, but as we will explain, we only get SO(4) or Spin(4) in this case respectively when the Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z}$ or trivial. It is important to distinguish between I_2^{s} and I_2^{ns} and to fix the Mordell–Weil group as they lead to completely different fiber structures as seen by comparing Figures 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.

The models considered in this paper are given by Weierstrass equations listed in Table 1. The fiber at the collisions are listed in Table 4. For the SO(4)-model, the Mordell–Weil condition forces the class of the two divisors S and T to satisfy the linear relation

$$S + T = 4L,$$

where $L = c_1(\mathcal{L})$ is the first Chern class of the fundamental line bundle \mathcal{L} of the Weierstrass model (see Definition 3.10). In the Calabi-Yau case, $L = -K$ where K is the canonical class of the base B of the elliptic fibration.

Each of the Weierstrass models listed in Table 1 has two crepant resolutions connected by an Atiyah flop. Using the main theorem of [29], we

determine a generating function for the Euler characteristic of $\mathrm{SO}(4)$ and $\mathrm{Spin}(4)$ -models (see Theorem 2.3 and Theorem 2.1). We study in detail the fiber degeneration of each of these elliptic fibrations and identify new non-Kodaira fibers (see Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12).

When the elliptic fibration is a threefold, we compute the triple intersection numbers of the fibral divisors (see Theorem 5.3). Assuming the Calabi-Yau condition, we can also compute the Hodge numbers of the elliptic fibration. The five-dimensional theory has two distinct Coulomb phases separated by a wall defined by a weight of the vector representation corresponding to the difference of the fundamental weights of each A_1 forming the Lie algebra of type D_2 . When the elliptic fibration is a Calabi-Yau threefold, the triple intersection numbers of the fibral divisors give the Chern-Simons levels of the theory in a Calabi-Yau compactification of M-theory to a five-dimensional supergravity theory with eight supercharges. In such a five-dimensional theory, we can constrain the number of charged hypermultiplets by comparing the triple intersection numbers of fibral divisors of the elliptic fibration with the cubic $5d$ prepotential. This point of view was presented in [49] and explicitly implemented in [27, 28, 30, 31, 33, 35–37, 39]. For an $\mathrm{SO}(4)$ -model this is enough to completely fix the number of charged hypermultiplets. But for a $\mathrm{Spin}(4)$ -model, this only gives two unresolved linear relations. The number of charged multiplets are fixed by using intersecting-brane techniques or anomaly cancellations from an uplifted chiral six-dimensional gauged supergravity theory, coming from Calabi-Yau compactifications of F-theory. In all cases, we check that the matter content we obtained is compatible with an anomaly free six-dimensional supergravity theory.

1.3. Spin groups, orthogonal groups, and half-spin groups

Assuming that a Lie group G is complex and connected, we only need to know its fundamental group $\pi_1(G)$ and the type of its Lie algebra \mathfrak{g} to determine G up to isomorphism. The Lie algebra only determines the local structure of the group G . In F-theory, the Mordell–Weil group of the elliptic fibration is isomorphic to the fundamental group of the Lie group. Assuming that the Mordell–Weil group has a trivial rank, and is therefore purely a torsion group T , we can then retrieve the group G as the quotient $G = \tilde{G}/\tilde{T}$, where $\tilde{G} = \exp(\mathfrak{g})$ is the simply connected and compact connected group with Lie algebra \mathfrak{g} and \tilde{T} is a normal subgroup of the center $Z(\tilde{G})$ of \tilde{G} isomorphic to T . Note that different isomorphic subgroups \tilde{T}_1 and \tilde{T}_2 of the center $Z(\tilde{G})$ can give non-isomorphic quotient \tilde{G}/\tilde{T}_1 and \tilde{G}/\tilde{T}_2 . The fundamental group $\pi_1(G)$ and the center $Z(G)$ of G are isomorphic to T

and the quotient $Z(\tilde{G})/\tilde{T}$ respectively:

$$(1.3) \quad \begin{aligned} G &= \tilde{G}/\tilde{T}, & \tilde{G} &:= \exp(\mathfrak{g}), & T &\cong \tilde{T} \subset Z(\tilde{G}), \\ \pi_1(G) &\cong T, & Z(G) &= Z(\tilde{G})/\tilde{T}. \end{aligned}$$

In our case of interest, we recall that the universal covering of a compact gauge group with Lie algebra of type $D_2 \cong A_1 \oplus A_1$ is $\mathrm{Spin}(4)$

$$(1.4) \quad \tilde{G} = \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2).$$

The center of $\mathrm{Spin}(4 + 4n)$ is the Klein’s four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$(1.5) \quad Z = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \{\pm\mathbb{I}, \pm\Gamma_*\}.$$

where \mathbb{I} is the identity and Γ_* is the product of all gamma matrices and squares to the identity $\Gamma_*^2 = \mathbb{I}$. The matrix Γ_* is used to define Weyl spinors of $\mathrm{Spin}(4 + 4n)$. Each non-neutral element g of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generates a subgroup $\langle g \rangle$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the three possibilities account for all the possible embedding of $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Each of the corresponding quotient $Z/\langle g \rangle$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. But each quotient $\mathrm{Spin}(4 + n)/\langle g \rangle$ is a different group as expressed by the following exact sequences:

$$(1.6) \quad \begin{aligned} 1 &\longrightarrow \langle -\mathbb{I} \rangle \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(4 + 4n) \longrightarrow \mathrm{SO}(4 + 4n) \longrightarrow 1, \\ 1 &\longrightarrow \langle +\Gamma_* \rangle \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(4 + 4n) \longrightarrow \mathrm{HSpin}^+(4 + 4n) \longrightarrow 1, \\ 1 &\longrightarrow \langle -\Gamma_* \rangle \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(4 + 4n) \longrightarrow \mathrm{HSpin}^-(4 + 4n) \longrightarrow 1. \end{aligned}$$

The group $\mathrm{SO}(4 + 4n)$ is the $\mathbb{Z}/2\mathbb{Z}$ quotient of $\mathrm{Spin}(4 + 4n)$ when $\mathbb{Z}/2\mathbb{Z}$ is generated by minus the identity of $\mathrm{Spin}(4 + 4n)$. When $\mathbb{Z}/2\mathbb{Z}$ is generated by Γ_* or $-\Gamma_*$, we get a *half-spin group*. In the case of $\mathrm{SO}(4)$, we have

$$(1.7) \quad \mathrm{HSpin}^+(4) \cong \mathrm{SO}(3) \times \mathrm{SU}(2), \quad \mathrm{HSpin}^-(4) \cong \mathrm{SU}(2) \times \mathrm{SO}(3).$$

Half-spin groups are called *semi-spin groups* in [59], see also [16, Chap VIII, §13.4]. The quotient of $\mathrm{Spin}(4 + 4n)$ by its center is the adjoint group $\mathrm{PSO}(4 + 4n)$. Hence, there are four compact groups of type D_{2+2n} , namely the simply connected group $\mathrm{Spin}(4 + 4n)$, the adjoint group $\mathrm{PSO}(4 + 4n)$, the isomorphic half-spin groups $\mathrm{HSpin}^\pm(4 + 4n)$, and the orthogonal group $\mathrm{SO}(4 + 4n)$. We recall that the half-spin groups $\mathrm{HSpin}^\pm(4m)$ are always isomorphic to each other but are isomorphic to $\mathrm{SO}(4m)$ if and only if $m = 2$, namely $\mathrm{Spin}^\pm(8) \cong \mathrm{SO}(8)$. For more information, see Table 5 and [9, 59].

1.4. The simplest $\mathrm{SO}(4)$ -model as a Miranda model

The simplest $\mathrm{SO}(4)$ -model is realized by III+III, the collision of two fibers of type III, in an elliptic fibration with Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$. Its defining equation is

$$(1.8) \quad y^2 z = x(x^2 + stz^2).$$

As an illustration, we quickly derive this equation. First, the general elliptic fibration with Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$ is [7, 52]

$$(1.9) \quad y^2 z = x(x^2 + a_2 z^2 + a_4 z^3).$$

The generator of the Mordell–Weil group is the section $x = y = 0$ and the neutral element is $x = z = 0$. A fiber of type III over the generic point of $S = V(S)$ requires that the valuation of a_2 and a_4 be:

$$v_S(a_2) \geq 1, \quad v_S(a_4) = 1.$$

Hence, we should have

$$(1.10) \quad y^2 z = x(x^2 + \tilde{a}_2 s^{1+m} z^2 + \tilde{a}_4 s z^3), \quad m \in \mathbb{Z}_{\geq 0}.$$

The discriminant is $\tilde{a}_4^2 s^3 (4\tilde{a}_4 - \tilde{a}_2^2 s)$. We can therefore take $\tilde{a}_4 = t$ to have a collision of type III+I₂^{ns} on the divisors $S = V(s)$ and $T = V(t)$. Since $a_4 = st$, we have the linear relation $S + T = 4L$. The fiber I₂^{ns} is replaced by a fiber of type III when $v_T(\tilde{a}_2) \geq 1$. That would give

$$(1.11) \quad y^2 z = x(x^2 + \tilde{a}_2 s^{1+m} t^{1+n} z^2 + stz^3), \quad m, n \in \mathbb{Z}_{\geq 0}.$$

If we take the lowest valuations ($m = n = 0$), the coefficient \tilde{a}_2 has to be section of $\mathcal{L}^{-\otimes 2}$ while a_2 is a section of $\mathcal{L}^{\otimes 2}$. A general solution is simply to take $\tilde{a}_2 = 0$, which gives²

$$(1.12) \quad y^2 z = x(x^2 + stz^2).$$

The reduced discriminant $\Delta_{red} = st$ is a normal transverse divisor and the j -invariant is a constant morphism taking the value $j = 1728$ everywhere. After a crepant resolution defined by a sequence of two blow-ups, we get a

²For example, if the base is \mathbb{P}^r , this is the only possibility as a line bundle of \mathbb{P}^r and its inverse cannot have non-trivial sections.

fiber of type III over the generic point of $S = V(s)$ and the generic point of $T = V(t)$. Over the generic point of their intersection $S \cap T$, the fiber degenerates to The fiber III degenerates to a non-Kodaira fiber of type $1 - 2 - 1$, which is a contraction of a fiber of type I_0^* . There are no other singular fibers. This model satisfies all the conditions of a Miranda model. In Miranda regularization, the collision III+III is replaced by a chain of collisions of type III+ I_0^* +III (see [60, Table 13.1]) by blowing up the intersection of the two divisors. The intersection becomes an exceptional divisor of the base over which the generic fiber is of type I_0^{*ss} . Here, we avoid such a blowup of the base since it modifies the canonical class and introduce an additional component in the gauge algebra changing the gauge algebra from type $A_1 \oplus A_1$ to type $A_1 \oplus B_3 \oplus A_1$

2. Summary of results

In this section, we categorize all the possible collisions of the fibers that yield SO(4) and Spin(4)-models, and summarize the results of the paper. We first state the geometrical setup and results including Euler characteristics, Hodge numbers, and the triple intersection polynomials in Section 2.1, and describe their application to the five-dimensional supergravity theories and their six-dimensional uplifted theories in Section 2.2. We then list the collision of singularities in Section 2.3.

2.1. Geometry

Weierstrass equations for SO(4) and Spin(4)-models with minimal valuations of the coefficients are given by Table 1.

Let Y_0 be one of the Weierstrass models considered in Table 1. Then Y_0 has two distinct crepant resolutions $f^\pm : Y^\pm \rightarrow Y_0$. One is given by the sequence of two blowups f^+ . The other crepant resolution is obtained by exchanging the order of the blowups.

$$(2.1) \quad f^+ : X_0 \xleftarrow{(x,y,s|e_1)} X_1^+ \xleftarrow{(x,y,t|w_1)} X_2^+.$$

$$(2.2) \quad f^- : X_0 \xleftarrow{(x,y,t|w_1)} X_1^- \xleftarrow{(x,y,s|e_1)} X_2^-.$$

The two resolutions are connected by an Atiyah flop. These two resolutions are not isomorphic to each other as the triple intersection numbers are not symmetric under the permutation of S and T .

Group	Collision	Mordell–Weil	Weierstrass model
SO(4)	$I_2^{\text{ns}} + I_2^{\text{ns}} \rightarrow I_4^{\text{ns}}$	$\mathbb{Z}/2\mathbb{Z}$	$y^2z = x^3 + a_2x^2z + stxz^2$
	$I_2^{\text{ns}} + I_2^{\text{s}} \rightarrow I_4^{\text{s}}$		$y^2z + a_1xyz = x^3 + \tilde{a}_2tx^2z + stxz^2$
	$I_2^{\text{s}} + I_2^{\text{s}} \rightarrow I_4^{\text{s}}$		$y^2z + a_1xyz = x^3 + \tilde{a}_2stx^2z + stxz^2$
	$\text{III} + I_2^{\text{ns}} \rightarrow 1 - 2 - 1$		$y^2z = x^3 + \tilde{a}_2sx^2z + stxz^2$
	$\text{III} + \text{III} \rightarrow 1 - 2 - 1$		$y^2z = x^3 + \tilde{a}_2stx^2z + stxz^2$
Spin(4)	$I_2^{\text{ns}} + I_2^{\text{ns}} \rightarrow I_4^{\text{ns}}$	trivial	$y^2z = x^3 + a_2x^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2$
	$I_2^{\text{ns}} + I_2^{\text{s}} \rightarrow I_4^{\text{s}}$		$y^2z + a_1xyz = x^3 + \tilde{a}_2tx^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2$
	$I_2^{\text{s}} + I_2^{\text{s}} \rightarrow I_4^{\text{s}}$		$y^2z + a_1xyz = x^3 + \tilde{a}_2stx^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2$
	$\text{III} + I_2^{\text{ns}} \rightarrow 1 - 2 - 1$		$y^2z = x^3 + \tilde{a}_2sx^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2$
	$\text{III} + \text{III} \rightarrow 1 - 2 - 1$		$y^2z = x^3 + \tilde{a}_2stx^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2$
	$\text{IV}^{\text{ns}} + I_2^{\text{ns}} \rightarrow 1 - 2 - (1, 1)$		$y^2z = x^3 + \tilde{a}_2sx^2z + \tilde{a}_4s^2txz^2 + \tilde{a}_6s^2t^2z^3$

Table 1: Weierstrass equations and collision rules.

Theorem 2.1. *The generating polynomial of the Euler characteristic of a Spin(4)-model obtained by a crepant resolution of a Weierstrass model given in Table 1 is*

$$\chi(Y) = 2 \left(\frac{S(-6L^2(4T+3) + L(8(T-1)T-9) + T(5T+4))}{(2L+1)(S+1)(T+1)(-6L+2S+2T-1)} + \frac{S^2(L(8T+6) + 5T+3) + 3(2L+1)(T^2 - L(3T+2))}{(2L+1)(S+1)(T+1)(-6L+2S+2T-1)} \right) c(B).$$

In particular, in the case of a Calabi-Yau threefold, the Euler characteristic of a Spin(4)-model is

$$\chi(Y) = -2(30K^2 + 15K(S+T) + 3S^2 + 3T^2 + 4ST).$$

Theorem 2.2. *The Hodge numbers of a Spin(4)-model given by the crepant resolution of a Weierstrass model given in Table 1 are*

$$h^{1,1}(Y) = 13 - K^2,$$

$$h^{2,1}(Y) = 13 + 29K^2 + 15K(S+T) + 3S^2 + 4ST + 3T^2.$$

For the Euler characteristic and the Hodge numbers of the Spin(4)-models, see their proof and detailed description in Section 5.2.

Theorem 2.3. *The generating polynomial of the Euler characteristic of an $SO(4)$ -model given by the crepant resolution of a Weierstrass model given in Table 1 is*

$$\chi(Y) = \frac{4(3L + 4TL - T^2)}{(1 + T)(1 + 4L - T)}c(B).$$

In particular, if the $SO(4)$ -model is a Calabi-Yau threefold, we have

$$\chi(Y) = -4(9K^2 + 4KT + T^2).$$

Theorem 2.4. *The Hodge numbers of an $SO(4)$ -model given by the crepant resolution of a Weierstrass model given in Table 1 are*

$$h^{1,1}(Y) = 13 - K^2, \quad h^{2,1}(Y) = 13 + 17K^2 + 8KT + 2T^2.$$

For the Euler characteristic and the Hodge numbers of the $SO(4)$ -models, see their proof and detailed description in Section 4.3.

Theorem 2.5. *Let $f^+ : Y^+ \rightarrow Y_0$ be the crepant resolution where Y_0 is any of the $Spin(4)$ -model listed in Table 1. The triple intersection polynomial of Y^+ is*

$$\begin{aligned} \mathcal{F}_{trip}^+ &= \int_Y \pi_* f_* \left[\left(\psi_0 D_0^s + \psi_1 D_1^s + \phi_0 D_0^t + \phi_1 D_1^t \right)^3 \right] \\ &= 2T(-2L + S - T)\phi_1^3 - 6ST\psi_1\phi_1^2 - 2S(2L + S)\psi_1^3 \\ &\quad + 2T(2L - S - 2T)\phi_0^3 + 6T\phi_0^2(\phi_1(-2L + S + T) - S\psi_1) \\ &\quad - 4S(S - L)\psi_0(\psi_0 - \psi_1)^2 - 2S(2L + S)\psi_0\psi_1(\psi_0 - \psi_1) \\ &\quad + 6T\phi_0\left(\phi_1^2(2L - S) - 2S(\psi_0 - \psi_1)^2 + 2S\psi_1\phi_1\right). \end{aligned}$$

The triple intersection polynomial in the fibration Y^- defined by exchanging the order of the blowup is \mathcal{F}_{trip}^- and is obtained from \mathcal{F}_{trip}^+ by the involution $\psi \leftrightarrow \phi$.

The triple intersection polynomial for the $Spin(4)$ -models are derived in Section 5.3. The triple intersection for an $SO(4)$ -model is then derived from the one of a $Spin(4)$ -model by the specialization $S \rightarrow -4K - T$.

Theorem 2.6. *Let $f^+ : Y^+ \rightarrow Y_0$ be the crepant resolution where Y_0 is any of the $SO(4)$ -model listed in Table 1. The triple intersection polynomial of*

Y^+ is

$$\begin{aligned} \mathcal{F}_{trip}^+ &= \pi_* f_* \left(\psi_0 D_0^s + \psi_1 D_1^s + \phi_0 D_0^t + \phi_1 D_1^t \right)^3 \\ &= -2T\phi_0^3(2L + T) - 2(4L - T)(\psi_0 - \psi_1)^2(\psi_0(6L - 2T) + \psi_1(6L - T)) \\ &\quad + 6T\phi_0 \left(-(\psi_0 - \psi_1)^2(4L - T) + 2\psi_1\phi_1(4L - T) + \phi_1^2(T - 2L) \right) \\ &\quad + 6T\phi_0^2(\psi_1(T - 4L) + 2L\phi_1) - 2(24L^2 - 10LT + T^2)\psi_1^3 \\ &\quad + 6T(T - 4L)\psi_1\phi_1^2 + 4T(L - T)\phi_1^3. \end{aligned}$$

This triple intersection polynomial of an $SO(4)$ -model is described in detail in Section 4.4.

2.2. Applications to 5d and 6d supergravity theories

The $\mathbb{Z}/2\mathbb{Z}$ quotients of $Spin(4)$ can be $SO(4)$, or the half-spin groups $SO(3) \times SU(2)$ and $SU(2) \times SO(3)$. But which one is realized by the collisions presented in Table 1? This question is answered by scrutinizing the representation \mathbf{R} associated with the elliptic fibration.

We recall that a representation of $A_1 \oplus A_1$ is given by two spins (j_1, j_2) where j_1 is in $\frac{1}{2}\mathbb{Z}_{\geq 0}$. If we name the representation by two numbers indicating the dimension of each projection, (j_1, j_2) is the same as $(\mathbf{2j}_1 + \mathbf{1}, \mathbf{2j}_2 + \mathbf{1})$. Each representation (j_1, j_2) is a valid representation of $Spin(4)$. But is only a projective representation of the three possible $\mathbb{Z}/2\mathbb{Z}$ quotient of $Spin(4)$. More explicitly, the representations of the semi-spin group $SU(2) \times SO(3)$ are those with spin (j_1, j_2) where j_2 an integer, while the representations of the semi-spin group $SO(3) \times SU(2)$ have spin (j_1, j_2) where j_1 an integer, and the representations of $SO(4)$ have spin (j_1, j_2) such that $j_1 + j_2$ an integer.

Group	Representation (j_1, j_2) or $(\mathbf{2j}_1 + \mathbf{1}, \mathbf{2j}_2 + \mathbf{1})$
$Spin(4)$	$j_1, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$
$Spin^+(4)$	$j_1 \in \mathbb{Z}_{\geq 0}, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$
$Spin^-(4)$	$j_1 \in \frac{1}{2}\mathbb{Z}_{\geq 0}, j_2 \in \mathbb{Z}_{\geq 0}$
$SO(4)$	$j_1, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}, j_1 + j_2 \in \mathbb{Z}_{\geq 0}$

Table 2: Representations for $SO(4)$ and $Spin(4)$ groups.

The determination of the representation \mathbf{R} associated to an elliptic fibration is explained in Section 3.2. In the present case, we find that ³:

$$\begin{aligned} \text{SO}(4)\text{-model:} \quad \mathbf{R} &= (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}), \\ \text{Spin}(4)\text{-model:} \quad \mathbf{R} &= (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}). \end{aligned}$$

The representation $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$ is the adjoint representation of the Lie algebra of type $D_2 \cong A_1 \oplus A_1$ and the representation $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ is the spin representation of D_2 . The representation $(\mathbf{2}, \mathbf{2})$ is the vector representation of D_2 . The collisions listed in Table 1 always produce the bifundamental representation $(\mathbf{2}, \mathbf{2})$ of $\text{spin}(\frac{1}{2}, \frac{1}{2})$ of the Lie algebra of type D_2 . Such a representation is incompatible with the half-spin groups and hence only $\text{SO}(4)$ and $\text{Spin}(4)$ are the possible options. ⁴

From the point of view of the elliptic fibration, the group $\text{Spin}(4)$ requires a trivial Mordell–Weil group while the group $\text{SO}(4)$ requires a Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$. We are not aware of any F-theory construction of a gauge theory with the semi-spin group $\text{SU}(2) \times \text{SO}(3)$.

We denote by g_S and g_T the genus of S and T . The $\text{SO}(4)$ -model only has adjoint representations $\mathbf{Adj}^- = (\mathbf{3}, \mathbf{1})$ and $\mathbf{Adj}^+ = (\mathbf{1}, \mathbf{3})$ and the vector representation $\mathbf{V} = (\mathbf{2}, \mathbf{2})$. For the $\text{Spin}(4)$ -model, there are additional hypermultiplets transforming in the two semi-spin representations $\mathbf{Spin}^- = (\mathbf{2}, \mathbf{1})$ and $\mathbf{Spin}^+ = (\mathbf{1}, \mathbf{2})$ of $D_2 \cong \mathfrak{so}(4)$. If we denote by Δ' the third component of the discriminant locus, then the number of hypermultiplets transforming in the semi-spin representations \mathbf{Spin}^\pm of $D_2 \cong \mathfrak{so}(4)$ are given by the intersection numbers $S \cdot \Delta'$ and $T \cdot \Delta'$. The class of Δ' is $-2(4K + T + S)$. In particular, it is zero when we specialize to the $\text{SO}(4)$ -model.

We show that for an $\text{SO}(4)$ -model and a $\text{Spin}(4)$ -model with the matter content discussed above and summarized on Table 3, all anomalies are canceled in a six-dimensional $\mathcal{N} = (1, 0)$ supergravity theory. Comparing the triple intersection numbers of an $\text{SO}(4)$ -model with the prepotential of a five-dimensional $\mathcal{N} = 1$ theory with matter charged under the same representation \mathbf{R} completely fixes the numbers of charged hypermultiplets. However this is not the case for a $\text{Spin}(4)$ -model, since there are more representations involved; the comparison with the triple intersection numbers only give two linear relations. Using additional information from intersecting branes, we

³We denote a representation by the dimensions $(\mathbf{d}_1, \mathbf{d}_2)$ of its two projections.

⁴The bifundamental representation $(\mathbf{2}, \mathbf{2})$ of $A_1 \oplus A_1$ is the vector representation of $\text{SO}(4)$ of dimension 4, which, in terms of spins of the two $\text{SU}(2)$ forming a $\text{Spin}(4)$, is the representation $(\frac{1}{2}, \frac{1}{2})$. The bifundamental representation rules out the groups $\text{SU}(2) \times \text{SO}(3)$ and $\text{SO}(3) \times \text{SU}(2)$, but is a valid representation of both $\text{SO}(4)$ and $\text{Spin}(4)$.

check that the six-dimensional uplifted theory can be anomaly-free by the Green-Schwarz mechanism.

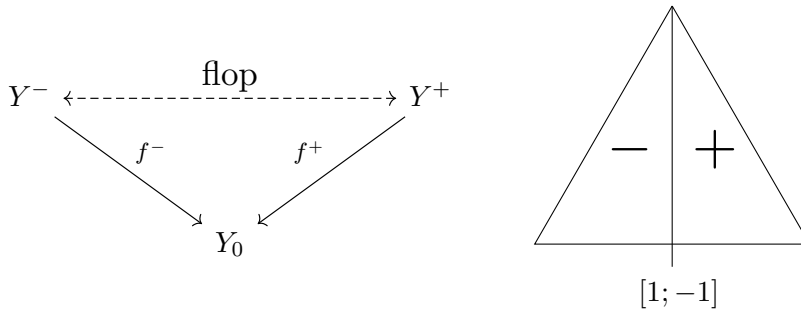


Figure 1: Coulomb phases of an $SO(4)$ -model or a $Spin(4)$ -model with matter in the representation $(\mathbf{2}, \mathbf{2})$. The addition of the representation $(\mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3})$, $(\mathbf{2}, \mathbf{1})$, or $(\mathbf{1}, \mathbf{2})$ do not change the chamber structure. The only weight defining an interior wall is the weight $[1; -1]$ of the representation $(\mathbf{2}, \mathbf{2})$. Geometrically, the chambers are identified by the presence of a curve with weights $\pm[1, -1]$ with respect to the fibral divisors D_1^s and D_1^t that project respectively to the divisors S and T and do not touch the zero section of the elliptic fibration.

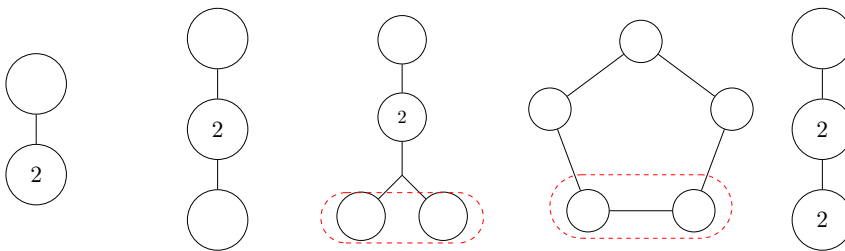


Figure 2: Non-Kodaira fibers. Nodes surrounded by a red ellipsis are obtained only after a field extension. For example, all these fibers appear in the collision $I_2^s + I_2^s$.

SO(4)	$\mathbf{R} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2})$ $\chi(CY_3) = -4(9K^2 + 4KT + T^2)$ $h^{1,1}(CY_3) = 13 - K^2, \quad h^{2,1}(CY_3) = 13 + 17K^2 + 8KT + 2T^2$ $\mathcal{F}_{triple}^+ = -2(24L^2 - 10LT + T^2)\psi_1^3 + 6T(T - 4L)\psi_1\phi_1^2 + 4T(L - T)\phi_1^3$ $n_{\mathbf{1},\mathbf{3}} = g_T, \quad n_{\mathbf{3},\mathbf{1}} = g_S, \quad n_{\mathbf{2},\mathbf{2}} = S \cdot T$
Spin(4)	$\mathbf{R} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ $\chi(CY_3) = -2(30K^2 + 15KS + 15KT + 3S^2 + 4ST + 3T^2)$ $\mathcal{F}_{triple}^+ = 2T(-2L + S - T)\phi_1^3 - 6ST\psi_1\phi_1^2 - 2S(2L + S)\psi_1^3$ $h^{1,1}(CY_3) = 13 - K^2, \quad h^{2,1}(CY_3) = 13 + 29K^2 + 15KS + 15KT + 3S^2 + 4ST + 3T^2$ $n_{\mathbf{3},\mathbf{1}} = g_S, \quad n_{\mathbf{1},\mathbf{3}} = g_T, \quad n_{\mathbf{2},\mathbf{2}} = S \cdot T, \quad n_{\mathbf{2},\mathbf{1}} = S \cdot V(\tilde{b}_8), \quad n_{\mathbf{1},\mathbf{2}} = T \cdot V(\tilde{b}_8)$

Table 3: Data of the low energy effective theory of a SO(4) and a Spin(4)-model compactified on a Calabi-Yau threefolds. \mathbf{R} is the representation, χ is the Euler characteristic of the Calabi-Yau threefold, S and T are the two divisors supporting the reducible fibers, g_S and g_T are the genus of these divisors. The number of representations in the irreducible representation \mathbf{R}_i is denoted $n_{\mathbf{R}_i}$. \mathcal{F}_{triple}^+ is the triple intersection polynomial for the variety Y^+ defined by the crepant resolution of equation (2.2). In the resolution Y^- , we have a distinct \mathcal{F}_{triple}^- obtained by exchanging ψ_1 and ϕ_1 .

2.3. Collision of singularities

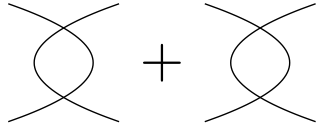
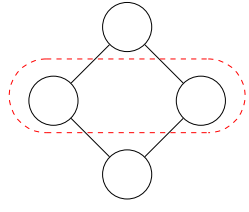
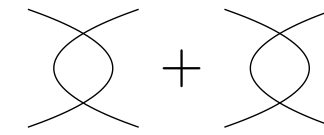
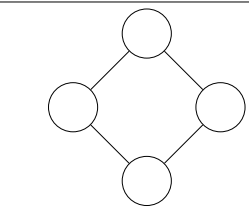
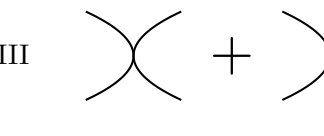
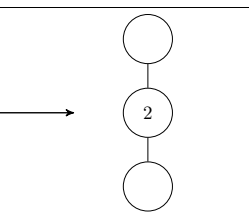
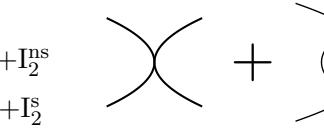
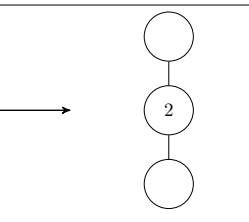
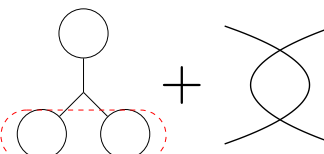
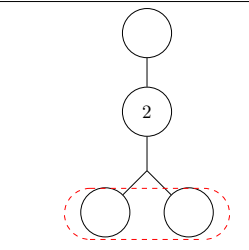
$I_2^{ns} + I_2^{ns}$		\longrightarrow	
$I_2^{ns} + I_2^s$ or $I_2^s + I_2^s$		\longrightarrow	
$III + III$		\longrightarrow	
$III + I_2^{ns}$ or $III + I_2^s$		\longrightarrow	
$IV^{ns} + I_2^{ns}$		\longrightarrow	

Table 4: Collisions at the intersections of two Kodaira fibers with dual graph \tilde{A}_1 . The representation produced is always the bifundamental representation $(\mathbf{2}, \mathbf{2})$ of $A_1 \oplus A_1$. Nodes surrounded by a red ellipse form a non-split node and are obtained individually only after a field extension (they are “related by monodromies”).

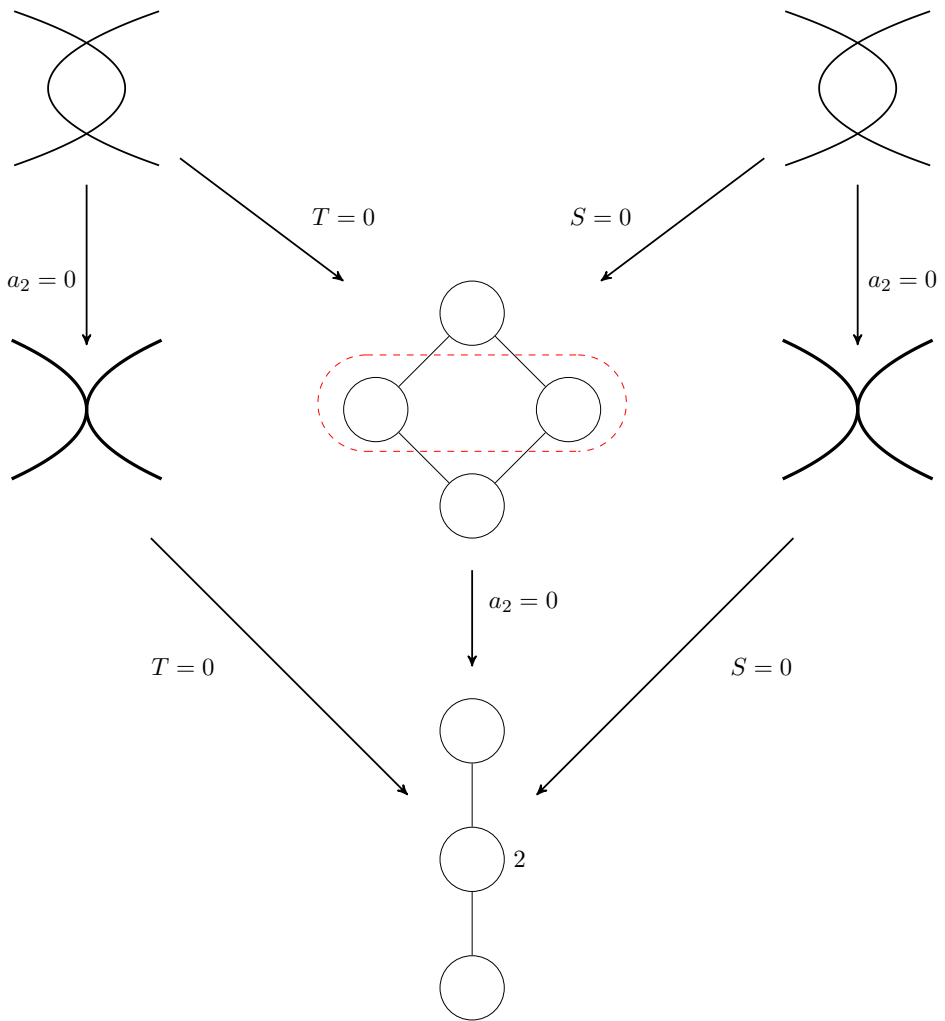


Figure 3: Fiber structure of an $SO(4)$ -model with collision $I_2^{ns} + I_2^{ns}$ and Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$.

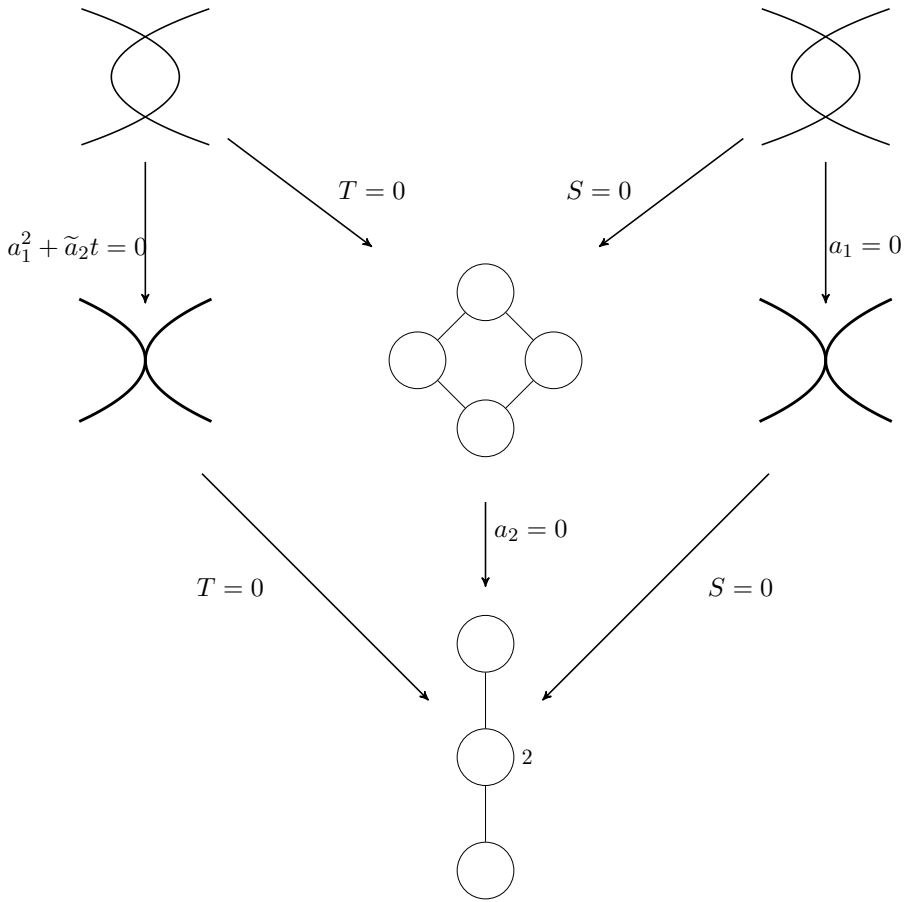


Figure 4: Fiber structure of an $SO(4)$ -model with collision $I_2^{\text{ns}} + I_2^{\text{s}}$ and Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$.

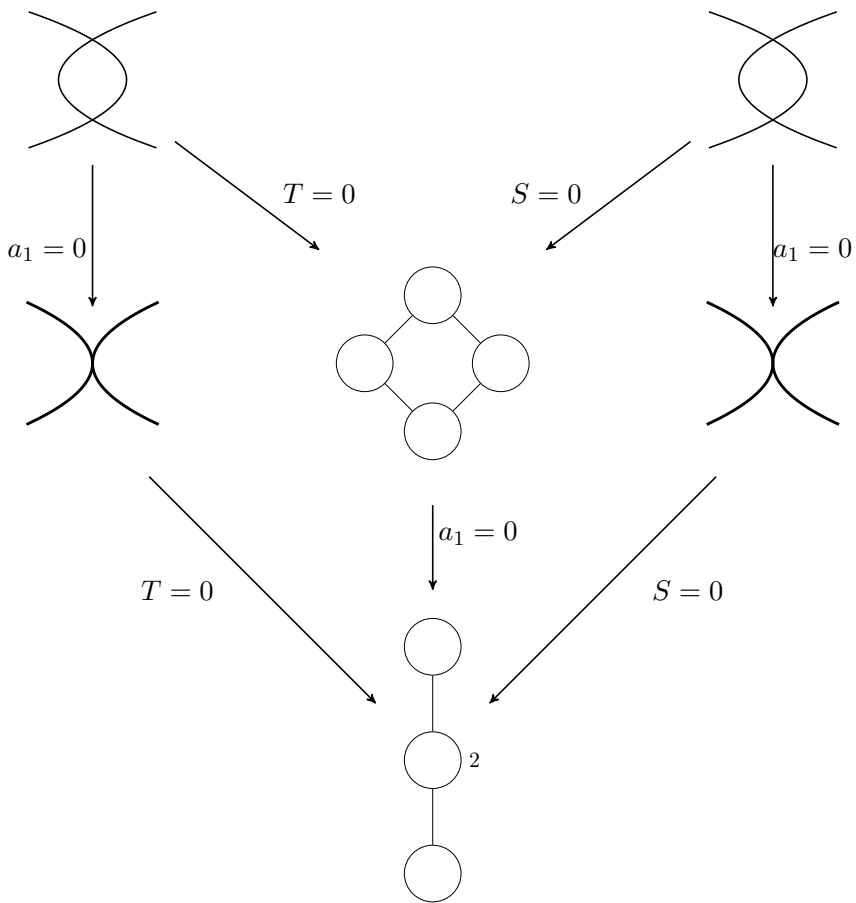


Figure 5: Fiber structure of an $SO(4)$ -model with collision $I_2^s + I_2^s$ and Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$.

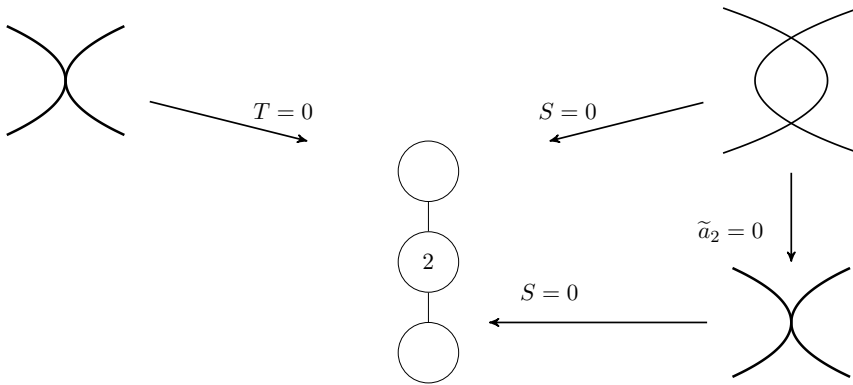


Figure 6: Fiber structure of an $SO(4)$ -model with $III+I_2^{ns}$. The Weierstrass equation is $y^2z = x^3 + a_2st^2z + stxz^2$.

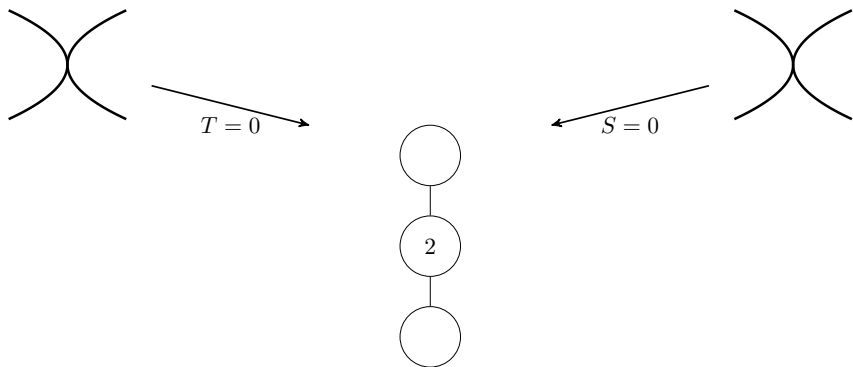


Figure 7: Fiber structure of an $SO(4)$ -model with $III+III$. The Weierstrass equation is $y^2z = x^3 + stxz^2$ and there are moduli coming from the Weierstrass coefficients since we necessarily have $S + T = 4L$ and $a_2 = 0$. The discriminant $\Delta \propto s^3t^3$ is a divisor with simple normal crossing and the j -invariant is constant and equal to 1728.

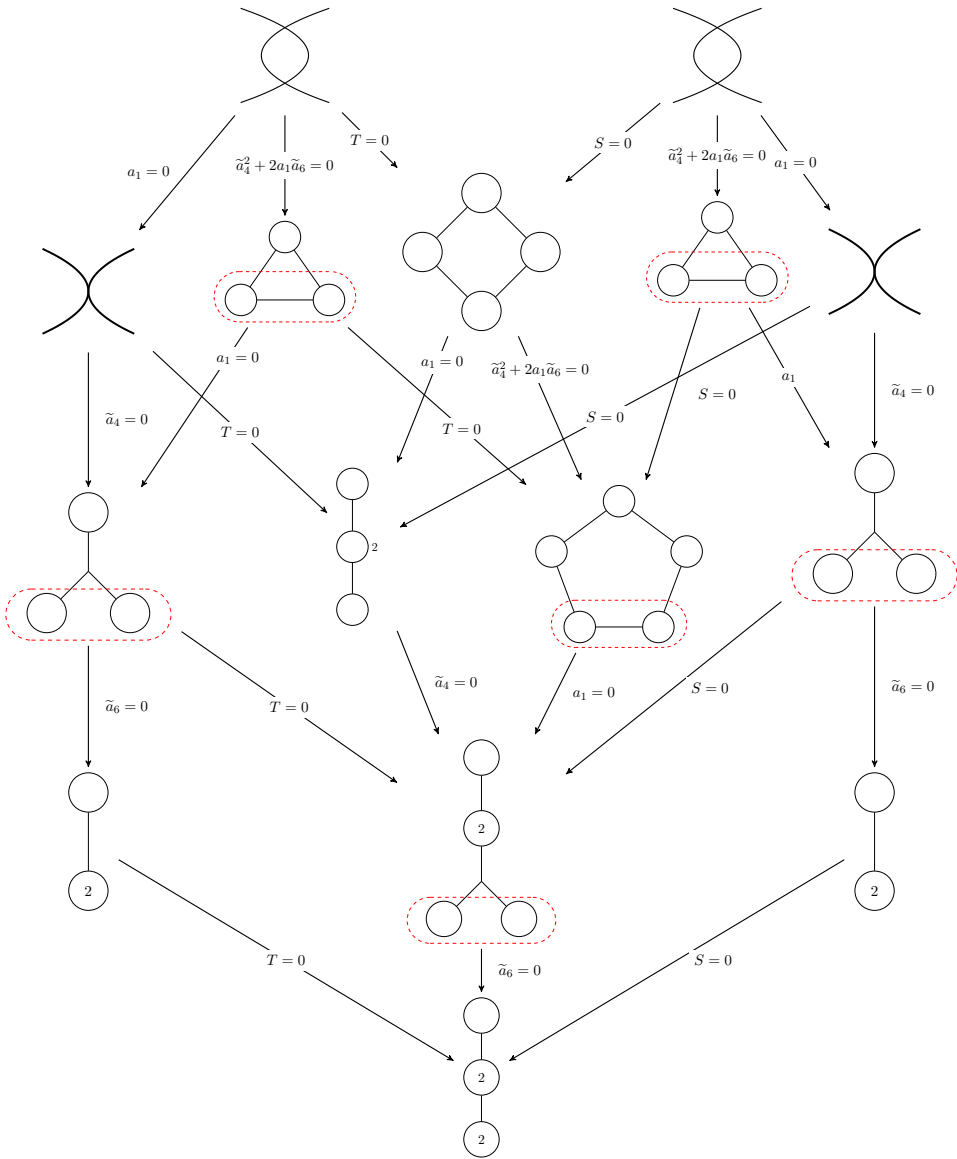


Figure 8: Fiber structure of a $Spin(4)$ -model with $I_2^5 + I_2^5$.

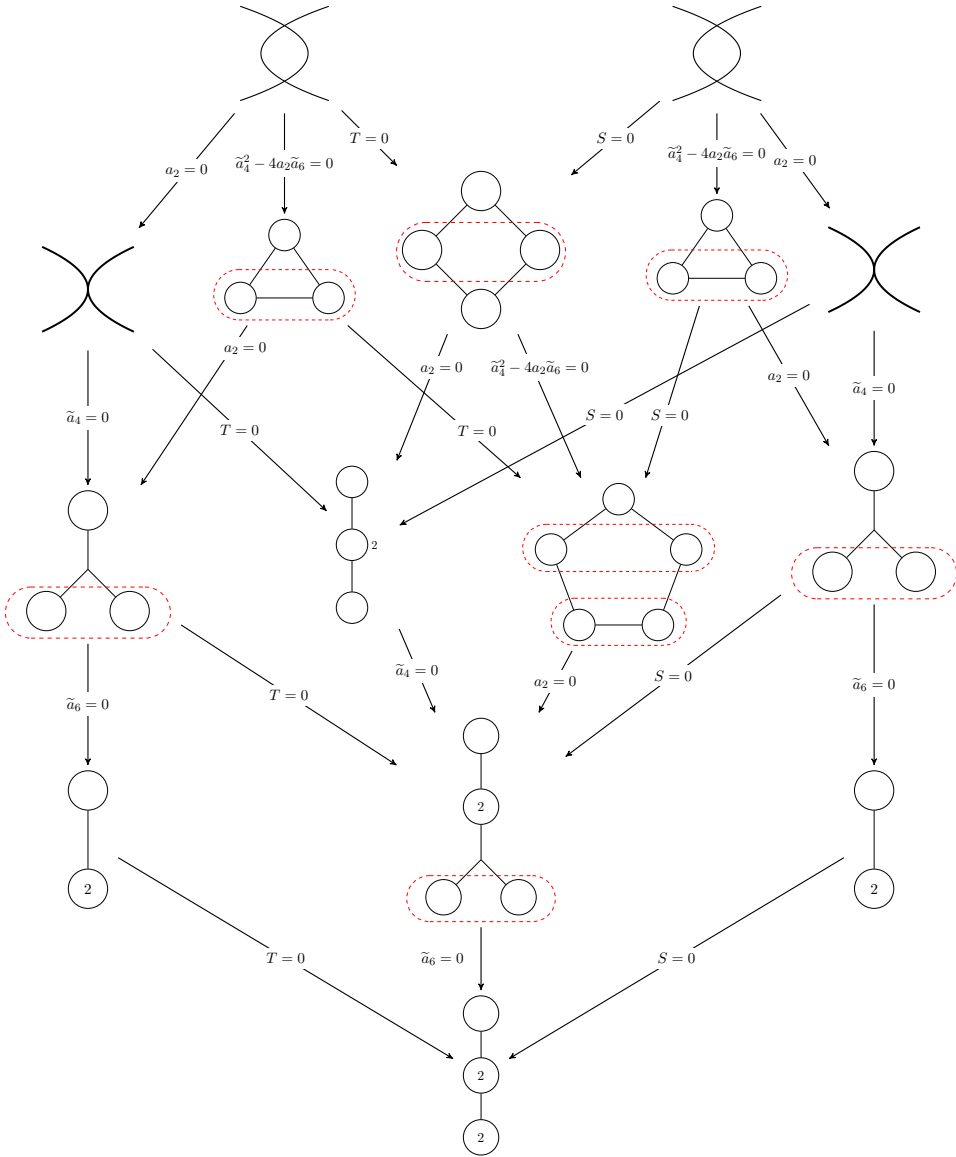


Figure 9: Fiber structure of a Spin(4)-model with $I_2^{ns} + I_2^{ns}$.

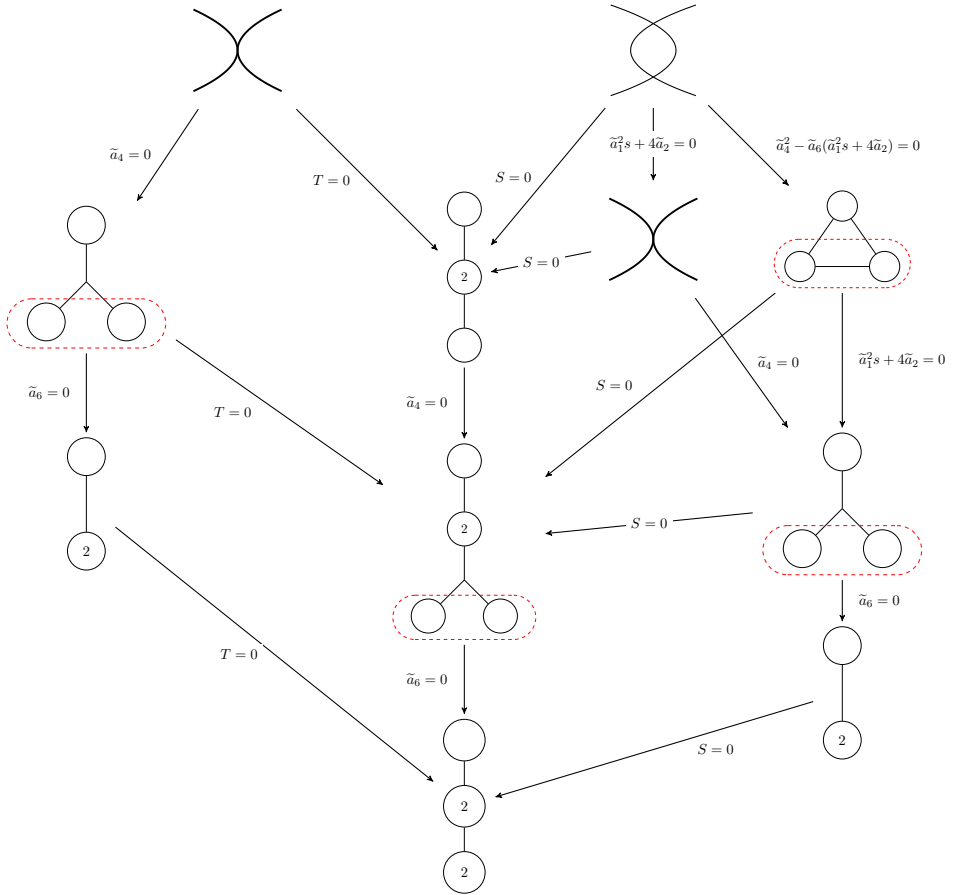


Figure 10: Fiber structure of a $Spin(4)$ -model with $III+I_2^{ns}$.

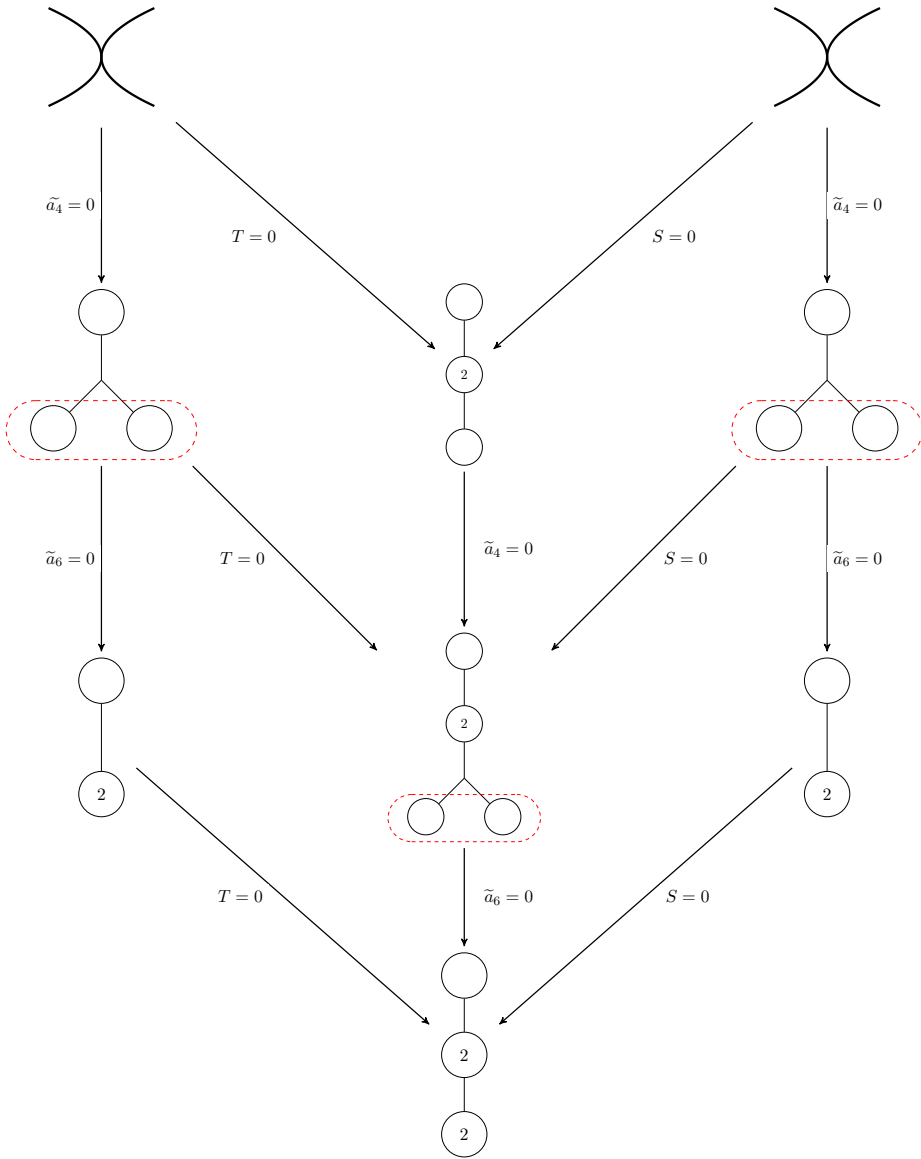


Figure 11: Fiber structure of a Spin(4)-model with III+III.

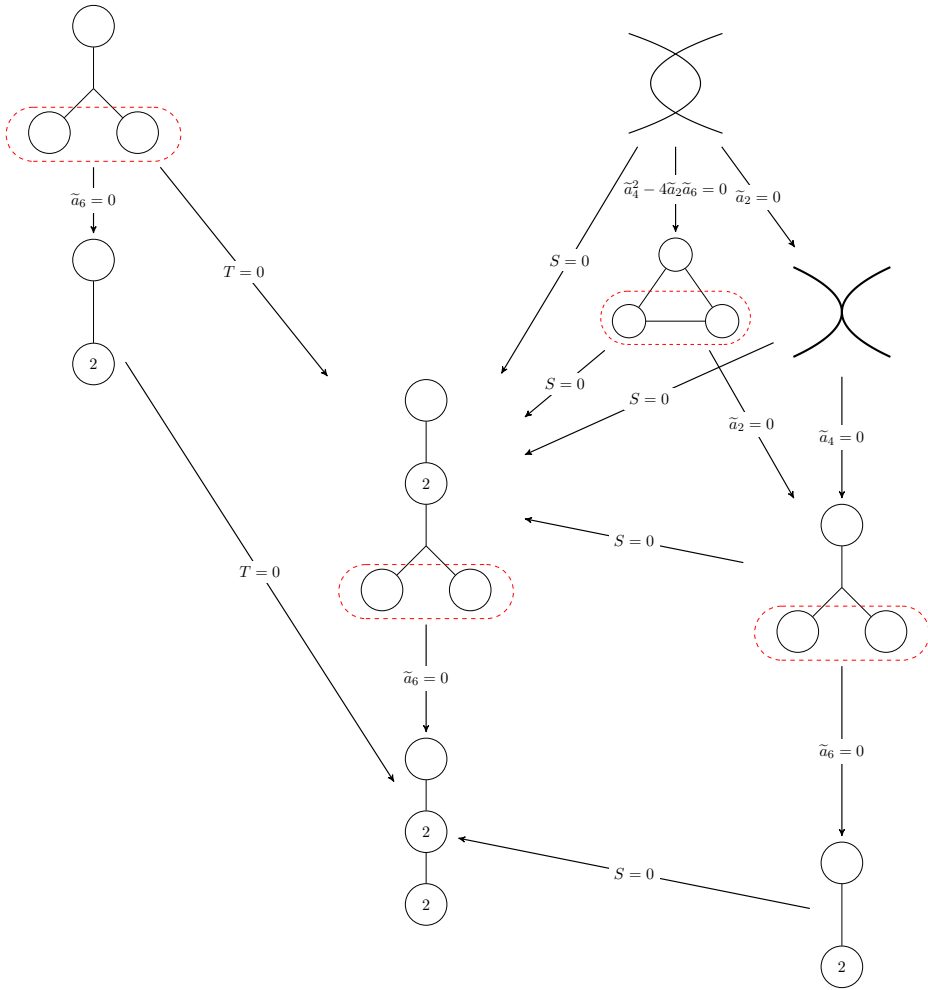


Figure 12: Fiber structure of a $Spin(4)$ -model with $IV^{ns} + I_2^{ns}$.

3. Preliminaries

Definition 3.1. An elliptic fibration is a proper surjective morphism $\varphi : Y \rightarrow B$ between algebraic varieties such that the generic fiber of φ is a smooth algebraic curve of genus one and φ is endowed with a rational section.

We work over the complex numbers and assume that the base B is a smooth projective variety. The locus of points of B over which the fiber is singular is called the discriminant locus of φ . Under mild assumptions, the discriminant is a Cartier divisor [22].

3.1. Singular fibers over generic points

Kodaira has classified the possible types of singular fibers of a minimal elliptic surface. Néron arrived at the same classification in an arithmetic setting based on Weierstrass models. For higher-dimensional elliptic fibrations, one can define a notion of minimality and Kodaira's classification continues to hold for geometric fibers over generic points of the discriminant locus.

We recall that a fiber over a point p is scheme defined with respect to the residue field of the point p . If the point p is a closed point, its residue field is just the ground field and the fiber is always geometric when the ground field is algebraically closed. But if the point p is the generic point of an irreducible variety, its residue field might not be algebraically closed. In that case, some components of the singular fiber that are irreducible with respect to the residue field of p might become reducible after a field extension. Such a component is said to not be geometrically irreducible.

The fibers over generic points are classified by decorated Kodaira fibers where the decoration (“split”, “non-split”, “semi-split”) keeps track of the minimal field extension needed to make all component of the singular fiber (and the divisors defined by intersection of components) geometrically irreducible. It is enough to consider only quadratic and cubic field extension. The split case corresponds to the case in which the fiber has components that are geometrically irreducible. Kodaira fibers of type I_1 , II , III , III^* , and II^* are always geometric. Kodaira fibers of type $I_{n \geq 2}$, IV , IV^* , $I_{n \geq 0}^*$ are said to be split when they have a geometrically irreducible component or non-split otherwise. For these fibers, a quadratic field extension is required to make them geometrically irreducible. For the fiber I_0^* , the non-split and split cases require respectively a cubic and quadratic field extension.

The fiber of type I_2 is very special in the sense that its non-split and split type have the same number of irreducible components, but distinguished themselves by the splitting properties of their intersection points. The fiber of type I_2 is composed of two rational curves intersecting transversally at two distinct geometric points. Let κ be the residue field of the point p over which the fiber I_2 is considered. When the two points of intersection form a κ -irreducible divisor on each component, the curve (with respect to the residue field of p), the fiber is said to be non-split, otherwise, the divisor is

the sum of two points that are rational with respect to the residue field of p and the fiber is said to be split. At the collision of two I_2 fibers, the generic fiber can be I_4^{ns} or I_4^s . The difference between I_4^{ns} or I_4^s over a codimension-two loci evaporates when the base is a surface since then these fibers are located over closed points with residue fields that are algebraically closed. That means that the fibers over codimension-two points of an elliptic threefolds are always geometric.

While geometric fibers have dual graphs that are always ADE affine Dynkin diagrams, the fiber types over generic points are twisted affine Dynkin diagram. We denoted an affine Dynkin diagram by \tilde{g} and its Langlands dual by \tilde{g}^t , this is the unique twisted Dynkin diagram whose Cartan matrix is the transpose of the Cartan matrix of \tilde{g} .

If K_i are decorated Kodaira fibers and S_i are irreducible divisors of the base, a model of type $K_1 + K_2 + \dots + K_n$ is an elliptic fibration for which there exists two irreducible components S_1 and S_2 in the discriminant locus such that the type of the fiber over the generic point of S_i is K_i and the fiber over the generic point of any other component of the discriminant locus is irreducible (Kodaira type II or I_1).

3.2. Lie algebra, representations, and Lie groups from an elliptic fibrations

If the dual fibers of K_1 and K_2 are affine Dynkin diagrams of type \tilde{g}_1^t and \tilde{g}_2^t , then the Lie algebra associated with the $K_1 + K_2$ -model is

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Let C be a vertical curve, i.e. a curve contained in a fiber of the elliptic fibration. Let S be an irreducible component of the reduced discriminant of the elliptic fibration $\varphi : Y \rightarrow B$. The pullback of φ^*S has irreducible components D_0, D_1, \dots, D_n , where D_0 is the component touching the section of the elliptic fibration. The divisors D_a are called fibral divisors. The *weight vector* of C over S is by definition the vector $\varpi_S(C) = (-D_1 \cdot C, \dots, -D_n \cdot C)$ of intersection numbers $D_i \cdot C$ for $i = 1, \dots, n$.

The irreducible curves of the degenerations over codimension-two loci give weights of a representation \mathbf{R} . However, they only give a subset of weights. Hence, we need an algorithm that retrieves the full representation \mathbf{R} given only a few of its weights. This problem can be addressed systematically using the notion of a saturated set of weights introduced by Bourbaki [16, Chap.VIII.§7. Sect. 2].

Definition 3.2 (Saturated set of weights). A set Π of integral weights is *saturated* if for any weight $\varpi \in \Pi$ and any simple root α , the weight $\varpi - i\alpha$ is also in Π for any i such that $0 \leq i \leq \langle \varpi, \alpha \rangle$. A saturated set has *highest weight* λ if $\lambda \in \Lambda^+$ and $\mu \prec \lambda$ for any $\mu \in \Pi$.

Definition 3.3 (Saturation of a subset). Any subsets Π of weights is contained in a unique smallest saturated subset. We call it the saturation of Π .

Proposition 3.4.

- (a) *A saturated set of weights is invariant under the action of the Weyl group.*
- (b) *The saturation of a set of weights Π is finite if and only if the set Π is finite.*
- (c) *A saturated set with highest weight λ consists of all dominant weights lower than or equal to λ and their conjugates under the Weyl group.*

Proof. See [51, Chap. III §13.4]. □

Theorem 3.5 ([16, Chap. VIII. §7. Sect. 2, Corollary to Prop. 5]). *Let Π be a finite saturated set of weights. Then there exists a finite dimensional \mathfrak{g} -module whose set of weights is Π .*

Definition 3.6. To a G -model, we associate a representation \mathbf{R} of the Lie algebra \mathfrak{g} as follows. The weight vectors of the irreducible vertical rational curves of the fibers over codimension-two points form a set Π whose saturation defines uniquely a representation \mathbf{R} by Theorem 3.5. We call this representation \mathbf{R} the representation of the G -model.⁵

The unique compact, connected, and simply connected Lie group with Lie algebra \mathfrak{g} is

$$\tilde{G} = \exp(\mathfrak{g}).$$

Assuming that the Mordell–Weil group has rank r and torsion subgroup H , the gauge group attached to the elliptic fibration requires the specification

⁵This definition is a formalization of the method of Aspinwall and Gross [6, §4]. See also [56]. Note that we always get the adjoint representation as a summand of \mathbf{R} .

of an embedding of H in the center of \tilde{G}

$$H \cong \tilde{H} \subset Z(\tilde{G}).$$

Then

$$G = U(1)^r \times \tilde{G}/\tilde{H}, \quad H \cong \tilde{H}, \quad \tilde{H} \trianglelefteq Z(\tilde{G}).$$

Two different isomorphic subgroups of $Z(\tilde{G})$ can give two different quotients \tilde{G}/\tilde{H} . The choice of the correct embedded is restricted by the representation \mathbf{R} attached to the elliptic fibration since not all representation of Lie algebra of a group G is a representation of G .

3.3. Hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$

Let \mathfrak{g} be a semi-simple Lie algebra and \mathbf{R} a representation of \mathfrak{g} . The kernel of each weight ϖ of \mathbf{R} defines a hyperplane ϖ^\perp through the origin of the Cartan sub-algebra of \mathfrak{g} .

Definition 3.7 (hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$). The hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$ is defined inside the dual fundamental Weyl chamber of \mathfrak{g} , i.e. the dual cone of the fundamental Weyl chamber of \mathfrak{g} , and its hyperplanes are the set of kernels of the weights of \mathbf{R} .

For each G -model, we associate the hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$ using the representation \mathbf{R} induced by the weights of vertical rational curves produced by degenerations of the generic fiber over codimension-two points of the base. We then study the incidence structure of the hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$ [25–27, 33, 50].

3.4. Non-simply-connected simple groups in F-theory

The classification of connected Lie groups over the complex or real numbers is based on the following theorem whose content can be traced back to Cartan and Lie. In what follows, all groups and Lie algebras are defined over the complex numbers or the real numbers. We refer to [15, Chap III, §6] for more information.

Theorem 3.8.

- 1) Any connected Lie group G defined over the complex numbers or the real numbers is isomorphic to a quotient \tilde{G}/K of its universal covering group \tilde{G} by a discrete central subgroup K of \tilde{G} isomorphic to the fundamental group of G .
- 2) Two connected Lie groups having the same Lie algebra are locally isomorphic. Two simply connected Lie groups having the same Lie algebra are isomorphic.
- 3) (Cartan–Lie theorem) If \mathfrak{g} is a finite dimensional Lie algebra, there exists a simply connected Lie group whose associated Lie algebra is isomorphic to \mathfrak{g} .

Proof. See [15, Chap III, §6.3, Theorem 3]. □

The center of G is isomorphic to the quotient $Z(\tilde{G})/K$. In particular, this implies that the fundamental group of any connected group is Abelian. The third assertion is the Cartan–Lie theorem (usually called Lie’s third fundamental theorem).⁶ The second assertion (without the simply connected specialization) is Lie’s second fundamental theorem. The following definition is inspired by the Cartan–Lie theorem (second assertion of Theorem 3.8).

Definition 3.9 (simply connected Lie group associated with a Lie algebra). Given a finite-dimensional Lie algebra \mathfrak{g} , the simply connected Lie group whose Lie algebra is isomorphic to \mathfrak{g} is called the *simply connected Lie group associated with \mathfrak{g}* and is denoted by $\exp(\mathfrak{g})$.

A direct consequence of Theorem 3.8 is that the classification of connected simple Lie groups reduces to the classification of simply connected simple Lie groups and of the subgroups of their centers. The centers of simply connected simple complex Lie groups are given in Table 5.

If \mathfrak{g} is \mathfrak{G}_2 , \mathfrak{F}_4 , or \mathfrak{E}_8 , there is a unique connected, simple, complex compact Lie group with Lie algebra \mathfrak{g} . If \mathfrak{g} is \mathfrak{A}_{p-1} (with p a prime number), \mathfrak{B}_{3+n} , \mathfrak{C}_{2+n} , \mathfrak{D}_{5+2n} , \mathfrak{E}_6 , \mathfrak{E}_7 , there are two compact connected Lie groups with Lie algebra \mathfrak{g} , namely the simply connected group $G = \exp(\mathfrak{g})$ and the centerless group $G_{ad} := G/Z(G)$. For symplectic groups, we denote by $\mathrm{Sp}(2n)$ for the compact and connected simple complex Lie group with Lie algebra \mathfrak{C}_n , that is $\mathrm{Sp}(2n) \cong \mathrm{USp}(2n)$.

⁶Lie proved the local existence of a Lie group. Cartan proved the global existence and the simply connected property.

\mathfrak{g}	$G = \exp(\mathfrak{g})$	Center $Z(G)$	$G_{Ad} := G/Z(G)$	Quotient with non-trivial center
A_{n-1}	$SU(n)$	$\mathbb{Z}/n\mathbb{Z}$	$PSU(n)$	$SU(n)/(\mathbb{Z}/r\mathbb{Z})$ with $r n$ and $r \neq 1, n$
B_{3+2n}	$Spin(7 + 2n)$	$\mathbb{Z}/2\mathbb{Z}$	$SO(7 + 2n)$	-
C_{2+2n}	$Sp(4 + 2n)$	$\mathbb{Z}/2\mathbb{Z}$	$PSp(4 + 2n)$	-
D_{4+2n}	$Spin(8 + 4n)$	$(\mathbb{Z}/2\mathbb{Z})^2$	$PSO(8 + 4n)$	$SO(8 + 4n), HSpin^\pm(8 + 4n)$
D_{5+2n}	$Spin(10 + 4n)$	$\mathbb{Z}/4\mathbb{Z}$	$PSO(10 + 4n)$	$SO(10 + 4n)$
E_6	E_6	$\mathbb{Z}/3\mathbb{Z}$	$E_6/(\mathbb{Z}/3\mathbb{Z})$	-
E_7	E_7	$\mathbb{Z}/2\mathbb{Z}$	$E_7/(\mathbb{Z}/2\mathbb{Z})$	-
E_8	E_8	trivial	E_8	-
F_4	F_4	trivial	F_4	-
G_2	G_2	trivial	G_2	-

Table 5: Classification of connected compact simple complex Lie groups. The group $G = \exp(\mathfrak{g})$ is the unique simply connected compact group with Lie algebra \mathfrak{g} . We denote its center by Z . The group $G_{Ad} = G/Z$ is the unique centerless connected simple group with Lie algebra \mathfrak{g} . The group G and G_{Ad} are isomorphic when $Z(G)$ is the trivial group, that is for $\mathfrak{g} = G_2, F_4,$ or E_8 . G and G_{Ad} are the only connected compact simple groups with Lie algebra \mathfrak{g} except in the case of $\mathfrak{g} = A_{n-1}$ with n a non-prime number and $\mathfrak{g} = D_{4+2n}$ where the orthogonal group $SO(8 + 4n)$ and the half spin group $HSpin(8 + 4n)$ are non-simply connected with center $\mathbb{Z}/2\mathbb{Z}$. The orthogonal group $SO(2m)$ is the quotient of $Spin(2n)$ by ± 1 . The groups $HSpin^\pm(8 + 4m)$ are the $\mathbb{Z}/2\mathbb{Z}$ quotient of $Spin(8 + 4m)$ for which one of the half-spin representations is faithful. The group $HSpin^\pm(8 + 4m)$ are isomorphic to each other. The half-spin groups $HSpin^\pm(8 + 4m)$ are isomorphic to $SO(8 + 4m)$ if and only if $m = 0$.

The case of A_{n-1} involves the classification of subgroups of $\mathbb{Z}/n\mathbb{Z}$. The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are the cyclic groups $\mathbb{Z}/r\mathbb{Z}$ such that r is a divisor of n . There are three compact connected Lie groups with Lie algebra D_{5+2n} and four with Lie algebra D_{4+2n} .⁷ Namely, the simply connected group $Spin(8 + 2n)$, its $\mathbb{Z}/2\mathbb{Z}$ quotient $SO(8 + 2n)$, and its centerless quotient $PSO(8 + 2n)$.

⁷The Lie algebra D_{5+2n} has three distinct compact groups: $Spin(10 + 4n)$, its $\mathbb{Z}/2\mathbb{Z}$ quotient $SO(10 + 4n)$, and the centerless group $PSO(10 + 4n)$. The center of $Spin(8 + 4n)$ is $(\mathbb{Z}/2\mathbb{Z})^2 = \{\pm 1, \pm \Gamma_*\}$ with $\Gamma_*^2 = 1$. We have 4 distinct subgroups: the trivial group, the full group $(\mathbb{Z}/2\mathbb{Z})^2$, and three proper subgroups—each generated by a non-neutral element of $(\mathbb{Z}/2\mathbb{Z})^2$ —each isomorphic to $\mathbb{Z}/2\mathbb{Z}$. As a result there are give compact groups with Lie algebra D_{4+2n} : the simply connected group $Spin(8 + 4n)$, the orthogonal group $SO(8 + 4n)$, the center less group $PSO(8 + 4n)$, and two groups $HSpin(8 + 4n)$ that are isomorphic to each other. One is a quotient of $Spin(8 + 4n)$ by $\{1, \Gamma_*\}$ and the other is a quotient by $\{1, -\Gamma_*\}$.

3.5. Weierstrass models and Deligne's formulaire

Definition 3.10 (Weierstrass model [24]). Consider a variety B endowed with a line bundle $\mathcal{L} \rightarrow B$. A Weierstrass model $\mathcal{E} \rightarrow B$ over B is a hypersurface cut out by the zero locus of a section of the line bundle of $\mathcal{O}(3) \otimes \pi^* \mathcal{L}^{\otimes 6}$ in the projective bundle $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \rightarrow B$. We denote by $\mathcal{O}(1)$ the dual of the tautological line bundle of the projective bundle, and denote by $\mathcal{O}(n)$ ($n > 0$) its n th-tensor product. The relative projective coordinates of the \mathbb{P}^2 bundle are denoted by $[x : y : z]$. In particular, x is a section of $\mathcal{O}(1) \otimes \pi^* \mathcal{L}^{\otimes 2}$, y is a section of $\mathcal{O}(1) \otimes \pi^* \mathcal{L}^{\otimes 3}$, and z is a section of $\mathcal{O}(1)$. Following Tate and Deligne's notation, the defining equation of a Weierstrass model is

$$\mathcal{E} : \quad zy(y + a_1x + a_3z) - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3) = 0,$$

where the coefficient a_i ($i = 1, 2, 3, 4, 6$) is a section of $\mathcal{L}^{\otimes i}$ on B . Such a hypersurface is an elliptic fibration since over the generic point of the base, the fiber is a nonsingular cubic planar curve with a rational point ($x = z = 0$). We use the convention of Deligne's formulaire [19]:

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, & b_4 &= a_1a_3 + 2a_4, & b_6 &= a_3^2 + 4a_6, \\ b_8 &= b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \\ c_4 &= b_2^2 - 24b_4, & c_6 &= -b_2^3 + 36b_2b_4 - 216b_6, \\ \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, & j &= \frac{c_4^3}{\Delta}. \end{aligned}$$

These quantities are related by the relations

$$4b_8 = b_2b_6 - b_4^2 \quad \text{and} \quad 1728\Delta = c_4^3 - c_6^2.$$

The discriminant locus is the subvariety of B cut out by the equation $\Delta = 0$, and is the locus of points p of the base B such that the fiber over p (i.e. Y_p) is singular. Over a generic point of Δ , the fiber is a nodal cubic that degenerates to a cuspidal cubic over the codimension-two locus $c_4 = c_6 = 0$. Up to isomorphisms, the j -invariant $j = c_4^3/\Delta$ uniquely characterizes nonsingular elliptic curves.

3.6. Pushforward formulas

Many of the intersection theory computations performed on elliptic fibrations come down to the following three theorems. The first one is a theorem

of Aluffi which gives the Chern class after a blowup along a local complete intersection. The second theorem is a pushforward theorem that provides a user-friendly method to compute invariant of the blowup space in terms of the original space. The last theorem is a direct consequence of functorial properties of the Segre class and gives a simple method to pushforward analytic expressions in the Chow ring of a projective bundle to the Chow ring of its base.

Definition 3.11 (Resolution of singularities). A resolution of singularities of a variety Y is a proper birational morphism $\varphi : \tilde{Y} \rightarrow Y$ such that \tilde{Y} is nonsingular and φ is an isomorphism away from the singular locus of Y . In other words, \tilde{Y} is nonsingular and if U is the singular locus of Y , φ maps $\varphi^{-1}(Y \setminus U)$ isomorphically onto $Y \setminus U$.

Definition 3.12 (Crepant birational map). A birational map $\varphi : \tilde{Y} \rightarrow Y$ between two algebraic varieties with \mathbb{Q} -Cartier canonical classes is said to be *crepant* if it preserves the canonical class, i.e. $K_{\tilde{Y}} = \varphi^*K_Y$.

Theorem 3.13 (Aluffi, [1, Lemma 1.3]). Let $Z \subset X$ be the complete intersection of d nonsingular hypersurfaces Z_1, \dots, Z_d meeting transversally in X . Let $f : \tilde{X} \rightarrow X$ be the blowup of X centered at Z . We denote the exceptional divisor of f by E . The total Chern class of \tilde{X} is then:

$$c(T\tilde{X}) = (1 + E) \left(\prod_{i=1}^d \frac{1 + f^*Z_i - E}{1 + f^*Z_i} \right) f^*c(TX).$$

Theorem 3.14 (Esole–Jefferson–Kang, see [29]). Let the nonsingular variety $Z \subset X$ be a complete intersection of d nonsingular hypersurfaces Z_1, \dots, Z_d meeting transversally in X . Let E be the class of the exceptional divisor of the blowup $f : \tilde{X} \rightarrow X$ centered at Z . Let $\tilde{Q}(t) = \sum_a f^*Q_a t^a$ be a formal power series with $Q_a \in A_*(X)$. We define the associated formal power series $Q(t) = \sum_a Q_a t^a$, whose coefficients pullback to the coefficients of $\tilde{Q}(t)$. Then the pushforward $f_*\tilde{Q}(E)$ is

$$f_*\tilde{Q}(E) = \sum_{\ell=1}^d Q(Z_\ell)M_\ell, \quad \text{where} \quad M_\ell = \prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{Z_m}{Z_m - Z_\ell}.$$

Theorem 3.15 (See [29] and [2, 3, 32, 44]). Let \mathcal{L} be a line bundle over a variety B and $\pi : X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \rightarrow B$ a projective bundle

over B . Let $\tilde{Q}(t) = \sum_a \pi^* Q_a t^a$ be a formal power series in t such that $Q_a \in A_*(B)$. Define the auxiliary power series $Q(t) = \sum_a Q_a t^a$. Then

$$\pi_* \tilde{Q}(H) = -2 \frac{Q(H)}{H^2} \Big|_{H=-2L} + 3 \frac{Q(H)}{H^2} \Big|_{H=-3L} + \frac{Q(0)}{6L^2},$$

where $L = c_1(\mathcal{L})$ and $H = c_1(\mathcal{O}_{X_0}(1))$ is the first Chern class of the dual of the tautological line bundle of $\pi : X_0 = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \rightarrow B$.

3.7. Euler characteristic of crepant resolutions

Using p -adic integration and the Weil conjecture, Batyrev proved the following theorem.

Theorem 3.16 (Batyrev, [10]). *Let X and Y be irreducible birational smooth n -dimensional projective algebraic varieties over \mathbb{C} . Assume that there exists a birational rational map $\varphi : X \dashrightarrow Y$ that does not change the canonical class. Then X and Y have the same Betti numbers.*

Batyrev’s result was strongly inspired by string dualities, in particular by the work of Dixon, Harvey, Vafa, and Witten [20]. As a direct consequence of Batyrev’s theorem, the Euler characteristic of a crepant resolution of a variety with Gorenstein canonical singularities is independent on the choice of resolution. We identify the Euler characteristic as the degree of the total (homological) Chern class of a crepant resolution $f : \tilde{Y} \rightarrow Y$ of a Weierstrass model $Y \rightarrow B$:

$$\chi(\tilde{Y}) = \int c(\tilde{Y}).$$

We then use the birational invariance of the degree under the pushforward to express the Euler characteristic as a class in the Chow ring of the projective bundle X_0 . We subsequently push this class forward to the base to obtain a rational function depending upon only the total Chern class of the base $c(B)$, the first Chern class $c_1(\mathcal{L})$, and the class S of the divisor in B :

$$\chi(\tilde{Y}) = \int_B \pi_* f_* c(\tilde{Y}).$$

In view of Theorem 3.16, this Euler characteristic is independent of the choice of a crepant resolution.

3.8. Hodge numbers for Calabi-Yau elliptic threefolds

Using motivic integration, Kontsevich shows in his famous “String Cohomology” Lecture at Orsay that birational equivalent Calabi-Yau varieties have the same class in the completed Grothendieck ring [55]. Hence, birational equivalent Calabi-Yau varieties have the same Hodge-Deligne polynomial, Hodge numbers, and Euler characteristic. In this section, we compute the Hodge numbers of crepant resolutions of Weierstrass models in the case of Calabi-Yau threefolds.

Theorem 3.17 (Kontsevich, (see [55])). *Let X and Y be birational equivalent Calabi-Yau varieties over the complex numbers. Then X and Y have the same Hodge numbers.*

Remark 3.18. In Kontsevich’s theorem, a Calabi-Yau variety is a non-singular complete projective variety of dimension d with a trivial canonical divisor. To compute Hodge numbers in this section, we use the following stronger definition of a Calabi-Yau variety.

Definition 3.19. A *Calabi-Yau variety* is a smooth compact projective variety Y of dimension n with a trivial canonical class and such that $H^i(Y, \mathcal{O}_X) = 0$ for $1 \leq i \leq n - 1$.

We first recall some basic definitions and relevant classical theorems.

Theorem 3.20 (Noether’s formula). *If B is a smooth compact, connected, complex surface with canonical class K_B and Euler number c_2 , then*

$$\chi(\mathcal{O}_B) = 1 - h^{0,1}(B) + h^{0,2}(B), \quad \chi(\mathcal{O}_B) = \frac{1}{12}(K^2 + c_2).$$

When B is a smooth compact rational surface, we have a simple expression of $h^{1,1}(B)$ as a function of K^2 using the following lemma.

Lemma 3.21. *Let B be a smooth compact rational surface with canonical class K . Then*

$$(3.1) \quad h^{1,1}(B) = 10 - K^2.$$

Proof. Since B is a rational surface, $h^{0,1}(B) = h^{0,2}(B) = 0$. Hence $c_2 = 2 + h^{1,1}(B)$ and the lemma follows from Noether’s formula. \square

We now compute $h^{1,1}(Y)$ using the Shioda–Tate–Wazir theorem [73, Corollary 4.1].

Theorem 3.22. *Let Y be a smooth Calabi–Yau threefold elliptically fibered over a smooth variety B with Mordell–Weil group of rank zero. Then,*

$$h^{1,1}(Y) = h^{1,1}(B) + f + 1, \quad h^{2,1}(Y) = h^{1,1}(Y) - \frac{1}{2}\chi(Y),$$

where f is the number of geometrically irreducible fibral divisors not touching the zero section. In particular, if Y is a G -model with G being a semi-simple group, then f is the rank of G .

3.9. Prepotential of a $5d \mathcal{N} = 1$ supergravity theory

The compactification of M-theory on a Calabi–Yau threefold Y yields a five dimensional supergravity theory with eight supercharges coupled to $h^{1,1}(Y)$ vector multiplets and $h^{2,1}(Y) + 1$ neutral hypermultiplets [17]. The gravity multiplet also contains a gauge field called the graviphoton. The kinetic terms of the vector multiplets and the graviphoton, together with the coefficients of the Chern Simons terms, are derived from the prepotential $\mathcal{F}(\phi)$, which is a real function of the scalar fields of the vector multiplets. After integrating out massive charged vector and matter fields, the prepotential receives a one-loop quantum correction protected from additional quantum corrections by supersymmetry. The vector multiplets transform in the adjoint representation of the gauge group while the hypermultiplets transform in representation $\mathbf{R} = \bigoplus_i \mathbf{R}_i$ of the gauge group, where \mathbf{R}_i are irreducible components of \mathbf{R} .

The Coulomb branches of the theory correspond to the chambers of the hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$. By matching the crepant resolutions with the chambers of $I(\mathfrak{g}, \mathbf{R})$, we determine which resolutions correspond to which phases of the Coulomb branch. The triple intersection numbers of the fibral divisors correspond to the coefficient of the Chern–Simons couplings of the five dimensional gauge theory and can be compared with the Intrilligator–Morrison–Seiberg (IMS) prepotential, which is the one-loop quantum contribution to the prepotential of the five-dimensional gauge theory. Since in field theory the Chern–Simons couplings are linear in the numbers $n_{\mathbf{R}_i}$ of hypermultiplets transforming in the irreducible representation \mathbf{R}_i such that $\mathbf{R} = \bigoplus_i \mathbf{R}_i$, computing the triple intersection numbers provides a way to determine the numbers $n_{\mathbf{R}_i}$ from the topology of the elliptic fibration. We observe by direct computation in each chamber that the numbers we find do

not depend on the choice of the crepant resolution. This idea of using the triple intersection numbers to determine the number of multiplets transforming in a given representation was used previously in [39] for SU(n)-models and most recently for F_4 -models in [33]. This technique has been advocated by Grimm and Hayashi in [48].

Multiplet	Fields
Graviton	$(g_{\mu\nu}, A_\mu, \psi_\mu)$
Vector	$(A_\mu, \varphi, \lambda)$
Hyper	(q, ζ)

Table 6: Supermultiplets for $\mathcal{N} = 1$ five-dimensional supergravity theories. The indices μ and ν refer to the five-dimensional spacetime coordinates. The tensor $g_{\mu\nu}$ is the metric of the five dimensional spacetime. The fields ψ_μ, λ, ζ are symplectic Majorana spinors. The field ψ_μ is the gravitino and A_μ is the graviphoton. The hyperscalar q is a quaternion composed of four real scalar fields.

The Intrilligator–Morrison–Seiberg (IMS) prepotential is the one-loop quantum contribution to the prepotential of a five-dimensional gauge theory with the matter fields in the representations \mathbf{R}_i of the gauge group. Let ϕ denote an element of the Cartan subalgebra of the Lie algebra \mathfrak{g} , α_ℓ the fundamental roots, ϖ a weight, and $\langle \varpi, \phi \rangle$ the evaluation of a weight ϖ on an element ϕ of the Cartan subalgebra. The Intrilligator–Morrison–Seiberg (IMS) prepotential is [53]

$$6\mathcal{F}_{\text{IMS}}(\phi) = \frac{1}{2} \left(\sum_{\ell} |\langle \alpha_\ell, \phi \rangle|^3 - \sum_i \sum_{\varpi \in \mathbf{R}_i} n_{\mathbf{R}_i} |\langle \varpi, \phi \rangle|^3 \right).$$

The full prepotential also contains a contribution proportional to the third Casimir invariant of the Lie algebra \mathfrak{g} ; for simple groups, it is only nonzero for SU(N) groups with $N \geq 3$.

For a given choice of a Lie algebra \mathfrak{g} , choosing a dual fundamental Weyl chamber resolves the absolute values in the sum over the roots. We then consider the arrangement of hyperplanes $\langle \varpi, \phi \rangle = 0$, where ϖ runs through all the weights of all the representations \mathbf{R}_i . They define the hyperplane arrangement $\mathbf{I}(\mathfrak{g}, \mathbf{R} = \bigoplus_i \mathbf{R}_i)$ restricted to the dual fundamental Weyl chamber. If none of these hyperplanes intersect the interior of the dual fundamental Weyl chamber, we can safely remove the absolute values in the sum over the weights. Otherwise, we have hyperplanes partitioning the fundamental

Weyl chamber into subchambers. Each of these subchambers is defined by the signs of the linear forms $\langle \varpi, \phi \rangle$. Two such subchambers are adjacent when they differ by the sign of a unique linear form.

Each of the subchambers is called a *Coulomb phase* of the gauge theory. The transition from one chamber to an adjacent chamber is a *phase transition* that geometrically corresponds to a flop between different crepant resolutions of the same singular Weierstrass model. The number of chambers of such a hyperplane arrangement is physically the number of phases of the Coulomb branch of the gauge theory.

3.10. Anomaly Cancellations in $6d \mathcal{N} = (1, 0)$ supergravity theories

The matter content of the six-dimensional $\mathcal{N} = (1, 0)$ supergravity theory are given by [47] and reviewed in Table 7. The scalar manifold of the tensor multiplets is the symmetric space $SO(1, n_T)/SO(n_T)$ where n_T is the number of tensor multiplets. The scalar manifold of the hypermultiplet is a quaternionic-Kähler manifold of quaternionic dimension n_H , where n_H is the number of hypermultiplets.

supergravity multiplets:	$(g_{\mu\nu}, B_{\mu\nu}^-, \psi_\mu^{A-})$
tensor multiplets:	$(B_{\mu\nu}^+, \chi^{A+}, \sigma)$
vector multiplets:	(A_μ, λ^{A-})
hypermultiplets:	$(4\phi, \zeta^+)$

Table 7: The indices $\mu, \nu = 0, \dots, 5$ label spacetime indices, $A = 1, 2$ labels the fundamental representation of the R -symmetry $SU(2)$, and \pm denotes the chirality of Weyl spinors or the self-duality (+) or anti-self-duality (-) of the field strength of antisymmetric two-forms. The gravitini ψ_μ^{A-} , the tensorini χ^{A+} , and gaugini λ^{A-} are symplectic Majorana Weyl spinors. The hyperino ζ^+ is a Weyl spinor invariant under the R -symmetry group $SU(2)_R$.

Consider a gauged six-dimensional $\mathcal{N} = (1, 0)$ supergravity theory with a semi-simple gauge group $G = \prod_a G_a$, $n_V^{(6)}$ vector multiplets, n_T tensor multiplets, and n_H hypermultiplets consisting of n_H^0 neutral hypermultiplets and n_H^{ch} charged under a representation $\bigoplus_i \mathbf{R}_i$ of the gauge group with $\mathbf{R}_i = \bigotimes_a \mathbf{R}_{i,a}$, where $\mathbf{R}_{i,a}$ is an irreducible representation of the simple component G_a of the semi-simple group G . The vector multiplets belongs to the

adjoint of the gauge group (hence $n_V = \dim G$). CPT invariance requires the representation $\bigoplus_i \mathbf{R}_i$ to be quaternionic.

By denoting the zero weights of a representation \mathbf{R}_i as $\mathbf{R}_i^{(0)}$, the charged dimension of the hypermultiplets in representation \mathbf{R}_i is given by $\dim \mathbf{R}_i - \dim \mathbf{R}_i^{(0)}$, as the hypermultiplets of zero weights are considered neutral. For a representation \mathbf{R}_i , $n_{\mathbf{R}_i}$ denotes the multiplicity of the representation \mathbf{R}_i . Then the number of charged hypermultiplets is simply [45, 46]

$$(3.2) \quad n_H^{ch} = \sum_i n_{\mathbf{R}_i} \left(\dim \mathbf{R}_i - \dim \mathbf{R}_i^{(0)} \right).$$

The total number of hypermultiplets is the sum of the neutral hypermultiplets and the charged hypermultiplets. For a compactification on a Calabi-Yau threefold Y , the number of neutral hypermultiplets is $n_H^0 = h^{2,1}(Y) + 1$ [17]. The number of each multiplet is

$$(3.3) \quad n_V^{(6)} = \dim G, \quad n_T = h^{1,1}(B) - 1 = 9 - K^2,$$

$$(3.4) \quad n_H = n_H^0 + n_H^{ch} = h^{2,1}(Y) + 1 + \sum_i n_{\mathbf{R}_i} \left(\dim \mathbf{R}_i - \dim \mathbf{R}_i^{(0)} \right),$$

where the (elliptically fibered) base B is a rational surface.

The anomaly polynomial I_8 has a pure gravitational contribution of the form $\text{tr } R^4$ where R is the Riemann tensor thought of as a 6×6 real matrix of two-form values. To apply the Green-Schwarz mechanism, the coefficient of $\text{tr } R^4$ in the anomaly polynomial is required to vanish [66]

$$(3.5) \quad n_H - n_V^{(6)} + 29n_T - 273 = 0.$$

In order to define the remainder terms of the anomaly polynomial I_8 , we define

$$(3.6) \quad \begin{aligned} X_a^{(n)} &= \text{tr}_{\text{adj}} F_a^n - \sum_i n_{\mathbf{R}_{i,a}} \text{tr}_{\mathbf{R}_{i,a}} F_a^n, \\ Y_{ab} &= \sum_{i,j} n_{\mathbf{R}_{i,a}, \mathbf{R}_{j,b}} \text{tr}_{\mathbf{R}_{i,a}} F_a^2 \text{tr}_{\mathbf{R}_{j,b}} F_b^2, \end{aligned}$$

where $n_{\mathbf{R}_{i,a}, \mathbf{R}_{j,b}}$ is the number of hypermultiplets transforming in the representation $(\mathbf{R}_{i,a}, \mathbf{R}_{j,b})$ of $G_a \times G_b$. The trace identities for a representation

$\mathbf{R}_{i,a}$ of a simple group G_a are

$$(3.7) \quad \begin{aligned} \text{tr}_{\mathbf{R}_{i,a}} F_a^2 &= A_{\mathbf{R}_{i,a}} \text{tr}_{\mathbf{F}_a} F_a^2, \\ \text{tr}_{\mathbf{R}_{i,a}} F_a^4 &= B_{\mathbf{R}_{i,a}} \text{tr}_{\mathbf{F}_a} F_a^4 + C_{\mathbf{R}_{i,a}} (\text{tr}_{\mathbf{F}_a} F_a^2)^2 \end{aligned}$$

with respect to a reference representation \mathbf{F}_a for each simple component G_a . To define the anomaly polynomial I_8 , we introduce the following expressions [69]:

$$(3.8) \quad X_a^{(2)} = \overbrace{\left(A_{a,\text{adj}} - \sum_i n_{\mathbf{R}_{i,a}} A_{\mathbf{R}_{i,a}} \right)}^{A_a} \text{tr}_{\mathbf{F}_a} F_a^2,$$

$$(3.9) \quad X_a^{(4)} = \overbrace{\left(B_{a,\text{adj}} - \sum_i n_{\mathbf{R}_{i,a}} B_{\mathbf{R}_{i,a}} \right)}^{B_a} \text{tr}_{\mathbf{F}_a} F_a^4 + \overbrace{\left(C_{a,\text{adj}} - \sum_i n_{\mathbf{R}_{i,a}} C_{\mathbf{R}_{i,a}} \right)}^{C_a} (\text{tr}_{\mathbf{F}_a} F_a^2)^2,$$

$$(3.10) \quad Y_{ab} = \sum_{i,j} n_{\mathbf{R}_{i,a},\mathbf{R}_{j,b}} A_{\mathbf{R}_{i,a}} A_{\mathbf{R}_{j,b}} \text{tr}_{\mathbf{F}_a} F_a^2 \text{tr}_{\mathbf{F}_b} F_b^2.$$

For each simple component G_a , the anomaly polynomial I_8 has a pure gauge contribution proportional to the quartic term $\text{tr} F_a^4$ that is required to vanish in order to factorize I_8 :

$$(3.11) \quad B_{a,\text{adj}} - \sum_i n_{\mathbf{R}_{i,a}} B_{\mathbf{R}_{i,a}} = 0.$$

When the coefficients of all quartic terms ($\text{tr} R^4$ and $\text{tr} F_a^4$) vanish, the remaining part of the anomaly polynomial I_8 is [67]

$$(3.12) \quad I_8 = \frac{K^2}{8} (\text{tr} R^2)^2 + \frac{1}{6} \sum_a X_a^{(2)} \text{tr} R^2 - \frac{2}{3} \sum_a X_a^{(4)} + 4 \sum_{a < b} Y_{ab}.$$

The anomalies are canceled by the Green-Schwarz mechanism when I_8 factorizes [47, 68, 69].

There is a subtlety on the representations that are charged on more than a simple component of the group, as it affects not only Y_{ab} but also $X_a^{(2)}$ and $X_a^{(4)}$. Consider a representation $(\mathbf{R}_1, \mathbf{R}_2)$ for of a semi-simple group with

two simple components $G = G_1 \times G_2$, where \mathbf{R}_a is a representation of G_a . Then this representation contributes to $n_{\mathbf{R}_1} \dim \mathbf{R}_2$ times, and contributes to $n_{\mathbf{R}_2} \dim \mathbf{R}_1$ times:

$$(3.13) \quad n_{\mathbf{R}_1} = \dim \mathbf{R}_2 \, n_{\mathbf{R}_1, \mathbf{R}_2}, \quad n_{\mathbf{R}_2} = \dim \mathbf{R}_1 \, n_{\mathbf{R}_1, \mathbf{R}_2}.$$

If the coefficients of $\text{tr } R^4$ and $\text{tr } F_a^4$ vanish and $G = G_1 \times G_2$, the remainder of the anomaly polynomial is given by

$$(3.14) \quad I_8 = \frac{K^2}{8} (\text{tr } R^2)^2 + \frac{1}{6} (X_1^{(2)} + X_2^{(2)}) \text{tr } R^2 - \frac{2}{3} (X_1^{(4)} + X_2^{(4)}) + 4Y_{12}.$$

If I_8 factors as $\frac{1}{2} \Omega_{ij} X_i^{(4)} X_j^{(4)}$, then the anomaly is cancelled by adding the counter term $\Omega_{ij} B_i \wedge X_j^{(4)}$ to the Lagrangian. The modification of the field strength $H^{(i)}$ of the antisymmetric tensor $B^{(i)}$ are $H^{(i)} = dB^{(i)} + \omega^{(i)}$, where $\omega^{(i)}$ is a proper combination of Yang-Mills and gravitational Chern-Simons terms.

4. SO(4)-model

The generic SO(4)-model is defined as an $I_2^{\text{ns}} + I_2^{\text{ns}}$ -model with a $\mathbb{Z}/2\mathbb{Z}$ Mordell–Weil group. We assume that the first and second I_2^{ns} are the generic fibers of two smooth divisors $S = V(s)$ and $T = V(t)$ intersecting transversally. Using the work of Miranda or Nakayama, we expect a geometric fiber of type I_4 at the collision of S and T . All the other SO(4)-model are then obtained as degenerations of this one.

4.1. Weierstrass equation, crepant resolutions and flops

We recall that the generic Weierstrass model with a $\mathbb{Z}/2\mathbb{Z}$ Mordell–Weil group is

$$(4.1) \quad y^2 z = x^3 + a_2 x^2 z + a_4 x z^2,$$

where a_2 and a_4 are respectively sections of $\mathcal{L}^{\otimes 2}$ and of $\mathcal{L}^{\otimes 4}$. The discriminant locus of this elliptic fibration is $\Delta = 16a_4^2(a_2^2 - 4a_4)$. The generic fiber over $V(a_4)$ is of type I_2^{ns} while the generic fiber over $V(a_2^2 - 4a_4)$ is of type I_1 . An SO(4)-model is obtained by requiring a_4 to factor into two irreducible

components

$$(4.2) \quad a_4 = st.$$

We assume that $S = V(s)$ and $T = V(t)$ are two smooth divisors intersecting transversally. Since a_4 is a section of $\mathcal{L}^{\otimes 4}$, the equation $a_4 = st$ imposes the following linear constraint on the class of S and T :

$$(4.3) \quad S + T = 4L,$$

where we denote the first Chern class of \mathcal{L} as $L = c_1(\mathcal{L})$. The Weierstrass model of this $SO(4)$ -model is then

$$(4.4) \quad y^2z = x^3 + a_2x^2z + stxz^2.$$

Its discriminant locus is

$$(4.5) \quad \Delta = 16s^2t^2(a_2^2 - 4st),$$

which consists of three components

$$(4.6) \quad S = V(s), \quad T = V(t), \quad \text{and} \quad \Delta' = a_2^2 - 4st.$$

When the base is of dimension three or higher, the divisor Δ' has double point singularities at $V(a_2, s, t)$. The divisors S and T do not intersect Δ' transversally as the intersection is non-reduced.

We can easily deform the elliptic fibration to produce other fibers still giving an $SO(4)$ -model.

$$(4.7) \quad I_2^{\text{ns}} + I_2^{\text{s}} : y^2z + a_1yxz = x^3 + \tilde{a}_2tx^2z + stxz^2$$

$$(4.8) \quad I_2^{\text{s}} + I_2^{\text{s}} : y^2z + a_1yxz = x^3 + \tilde{a}_2stx^2z + stxz^2$$

$$(4.9) \quad \text{III} + I_2^{\text{s}} : y^2z = x^3 + \tilde{a}_2sx^2z + stxz^2$$

$$(4.10) \quad \text{III} + I_2^{\text{s}} : y^2z + \tilde{a}_1sxyz = x^3 + \tilde{a}_2sx^2z + stxz^2$$

$$(4.11) \quad \text{III} + \text{III} : y^2z = x^3 + \tilde{a}_2stx^2z + stxz^2.$$

We give a crepant resolution via following sequence of two blowups

$$(4.12) \quad X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(x,y,t|w_1)} X_2.$$

The other crepant resolution connected by a flop is obtained by exchanging S and T . For that reason, we work with the current resolution as the other

one is described by a simple exchange of S and T . The relative “projective coordinates” of X_2 are then given by

$$(4.13) \quad [e_1 w_1 x : e_1 w_1 y : z = 1][w_1 x : w_1 y : s][x : y : t].$$

The proper transform of the original Weierstrass model is

$$(4.14) \quad y^2 z = e_1 w_1 x^3 + a_2 x^2 z + stx z^2.$$

By the Jacobian criterion, assuming that S and T are transverse is enough to prove that this proper transform is smooth since (x, y, st) is an empty ideal after the two blowups.

In the SO(4)-model, the Atiyah flop is easily seen by considering a partial resolution that is a common blowdown for both resolutions.

$$(4.15) \quad \begin{array}{ccc} & X_0 & \\ & \downarrow (x,y|e_1) & \\ & X_1 & \\ \swarrow (e_1,s|e_2) & & \searrow (e_1,t|w_2) \\ X_2^- & \dashleftarrow & X_2^+ \end{array}$$

After the first blow-up, we get the partial resolution

$$(4.16) \quad \mathcal{E}_1 : e_1(y^2 - e_1 x^3 - a_2 x^2) = stx.$$

Since (x, y) cannot vanish at the same time, the previous equation is of the type $u_1 u_2 - u_3 u_4 = 0$, which is exactly the equation of the singularity whose resolutions determine the Atiyah’s flop. The two crepant resolutions are then obtained by blowing up (e_1, s) or (e_1, t) .

4.2. Fiber structure

We can now study the fiber structure of this elliptic fibration. We assume that the base is of arbitrary dimension. We have four fibral divisors:

$$(4.17) \quad \begin{cases} D_0^s & : & s = y^2 - e_1 w_1 x^3 - a_2 x^2 = 0, \\ D_1^s & : & e_1 = y^2 - a_2 x^2 - stx = 0, \\ D_0^t & : & t = y^2 - e_1 w_1 x^3 - a_2 x^2 = 0, \\ D_1^t & : & w_1 = y^2 - a_2 x^2 - stx = 0. \end{cases}$$

The divisors D_0^s and D_1^s (resp. D_0^t and D_1^t) project to S (resp. T). All these divisors are conic bundles. Over the generic point of S , we have two fibers C_0^s and C_1^s intersecting at

$$(4.18) \quad C_0^s \cap C_1^s : s = e_1 = y^2 - a_2x^2 = 0.$$

The intersection points are the two roots of $y^2 - a_2x^2 = 0$. Thus, this does represent an I_2^{ns} since computing the roots requires taking the square root of a_2 , the two points coincide when $a_2 = 0$ yielding a fiber of type III. The same is true over T , where the generic fiber I_2^{ns} degenerates to a fiber III over $T \cap V(a_2)$. At the intersection of S and T , we get

$$(4.19) \quad \begin{cases} D_0^s \cap D_0^t & : s = t = y^2 - e_1w_1x^3 - a_2x^2 = 0 \rightarrow \eta^{00}, \\ D_1^s \cap D_0^t & : e_1 = t = y^2 - a_2x^2 = 0 \rightarrow \eta^{10\pm}, \\ D_1^s \cap D_1^t & : e_1 = w_1 = y^2 - a_2x^2 - stx = 0 \rightarrow \eta^{11}, \end{cases}$$

where the intersections of the new fibers are given by

$$(4.20) \quad \eta^{00} \cap \eta^{10\pm} : s = e_1 = t = y^2 - a_2x^2 = 0,$$

$$(4.21) \quad \eta^{10\pm} \cap \eta^{11} : e_1 = t = w_1 = y^2 - a_2x^2 = 0.$$

This gives a fiber of type I_4^{ns} (see Figure 7). Over $V(S, T, a_2)$, the fiber I_4^{ns} degenerates further to a non-Kodaira fiber of type $1 - 2 - 1$ as the two rational curves $\eta^{10\pm}$ coincide. There are no other degenerations.

At the collision $S \cap T$, the splitting of the curves $C_0^s, C_1^s, C_0^t,$ and C_1^t are given by

$$(4.22) \quad \begin{cases} C_0^s \rightarrow \eta^{00}, \\ C_1^s \rightarrow \eta^{10+} + \eta^{10-} + \eta^{11}, \\ C_0^t \rightarrow \eta^{00} + \eta^{10+} + \eta^{10-}, \\ C_1^t \rightarrow \eta^{11}, \end{cases}$$

This induces linear relations that we exploit to compute the intersection numbers between the curves of the collision and the fibral divisors

$$(4.23) \quad \begin{array}{c|cccc} & D_0^s & D_1^s & D_0^t & D_1^t \\ \hline \eta^{00} & -2 & 2 & 0 & 0 \\ \eta^{10+} + \eta^{10-} & 2 & -2 & -2 & 2 \\ \eta^{10\pm} & 1 & -1 & -1 & 1 \\ \eta^{11} & 0 & 0 & 2 & -2. \end{array}$$

Hence, we see that we get the weight $[1; -1]$ from the curves $\eta^{10\pm}$. Since both $[1]$ and $[-1]$ are from the representation $\mathbf{2}$ of $\mathfrak{su}(2)$, we can deduce that the matter content we get is a bifundamental, $(\mathbf{2}, \mathbf{2})$, corresponding to the vector representation of $SO(4)$. Hence the matter contents of the $I_2^{ns} + I_2^{ns}$ -model with a $\mathbb{Z}/2\mathbb{Z}$ Mordell–Weil group is given by

$$(4.24) \quad \mathbf{R} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}).$$

4.3. Euler characteristic

Now we compute the Euler characteristic using the blowup maps for getting this crepant resolution. The following theorems are direct specializations of Theorem 2.1 and 2.2 for $Spin(4)$ -models after imposing the condition $S + T = 4L$.

Theorem 2.3. *The generating polynomial of the Euler characteristic of an $SO(4)$ -model given by the crepant resolution of a Weierstrass model given in Table 1 is*

$$\chi(Y) = \frac{4(3L + 4TL - T^2)}{(1 + T)(1 + 4L - T)} c(B).$$

In particular, if the $SO(4)$ -model is a Calabi–Yau threefold, we have

$$\chi(Y) = -4(9K^2 + 4KT + T^2).$$

Theorem 2.4. *The Hodge numbers of an $SO(4)$ -model given by the crepant resolution of a Weierstrass model given in Table 1 are*

$$h^{1,1}(Y) = 13 - K^2, \quad h^{2,1}(Y) = 13 + 17K^2 + 8KT + 2T^2.$$

4.4. Triple intersection and the prepotential

In this subsection, we compute the triple intersection polynomial of the $SO(4)$ -model.

$$(4.25) \quad \begin{aligned} \mathcal{F}_{trip}^+ &= \pi_* f_* \left(\psi_0 D_0^s + \psi_1 D_1^s + \phi_0 D_0^t + \phi_1 D_1^t \right)^3 \\ &= -2T\phi_0^3(2L+T) - 2(4L-T)(\psi_0 - \psi_1)^2(\psi_0(6L-2T) + \psi_1(6L-T)) \\ &\quad + 6T\phi_0 \left(-(\psi_0 - \psi_1)^2(4L-T) + 2\psi_1\phi_1(4L-T) + \phi_1^2(T-2L) \right) \\ &\quad + 6T\phi_0^2(\psi_1(T-4L) + 2L\phi_1) - 2(24L^2 - 10LT + T^2)\psi_1^3 \\ &\quad + 6T(T-4L)\psi_1\phi_1^2 + 4T(L-T)\phi_1^3. \end{aligned}$$

In the flop, \mathcal{F}_{trip}^- is obtained by the involution $\psi \leftrightarrow \phi$. Using the Intrilligator-Morrison-Seiberg approach, we compute the prepotential of the five-dimensional theory to be

$$(4.26) \quad 6\mathcal{F}_{\text{IMS}} = -8(n_{\mathbf{1},\mathbf{3}} - 1)|\phi_1|^3 - n_{\mathbf{2},\mathbf{2}}\left(|\phi_1 - \psi_1|^3 - |\phi_1 + \psi_1|^3\right) - 8(n_{\mathbf{3},\mathbf{1}} - 1)|\psi_1|^3.$$

In the chamber $\phi_1 - \psi_1 > 0$, the prepotential is

$$(4.27) \quad 6\mathcal{F}_{\text{IMS}}^+ = -8(n_{\mathbf{1},\mathbf{3}} - 1)\phi_1^3 - 2(n_{\mathbf{2},\mathbf{2}} + 4n_{\mathbf{3},\mathbf{1}} - 4)\psi_1^3 - 6n_{\mathbf{2},\mathbf{2}}\phi_1^2\psi_1.$$

In the chamber $\phi_1 - \psi_1 < 0$, the prepotential is

$$(4.28) \quad 6\mathcal{F}_{\text{IMS}}^- = -8(n_{\mathbf{1},\mathbf{3}} - 1)\psi_1^3 - 2(n_{\mathbf{2},\mathbf{2}} + 4n_{\mathbf{3},\mathbf{1}} - 4)\phi_1^3 - 6n_{\mathbf{2},\mathbf{2}}\psi_1^2\phi_1.$$

4.5. Counting charged hypermultiplets in 5d

Matching the 5d prepotential with the triple intersection numbers, we can determine the number of representations. We find a perfect match between the triple intersection number and the prepotential when

$$(4.29) \quad n_{\mathbf{2},\mathbf{2}} = -T(-4L + T), \quad n_{\mathbf{1},\mathbf{3}} = \frac{1}{2}(-LT + T^2 + 2), \\ n_{\mathbf{3},\mathbf{1}} = \frac{1}{2}(12L^2 - 7LT + T^2 + 2)$$

Using the relation $T + S = 4L$, we can rewrite these numbers as

$$(4.30) \quad n_{\mathbf{2},\mathbf{2}} = TS, \quad n_{\mathbf{1},\mathbf{3}} = g_T, \quad n_{\mathbf{3},\mathbf{1}} = g_S,$$

where g_T and g_S are respectively the genus of T and S .

4.6. Anomaly cancellations

The number of vector multiplets $n_V^{(6)}$, tensor multiplets n_T , and hypermultiplets n_H are (see equations (3.3)):

$$(4.31) \quad n_V^{(6)} = 6, \quad n_T = 9 - K^2, \\ n_H = h^{2,1}(Y) + 1 + n_{\mathbf{3},\mathbf{1}}(3 - 1) + n_{\mathbf{2},\mathbf{2}}(4 - 0) + n_{\mathbf{1},\mathbf{3}}(3 - 1) \\ = 18 + 29K^2.$$

Thus, we can conclude that the purely gravitational anomaly is canceled:

$$(4.32) \quad n_H - n_V^{(6)} + 29n_T - 273 = 0.$$

Now consider the remaining parts of the anomaly polynomial

$$(4.33) \quad I_8 = \frac{9 - n_T}{8} (\text{tr } R^2)^2 + \frac{1}{6} (X_1^{(2)} + X_2^{(2)}) \text{tr } R^2 - \frac{2}{3} (X_1^{(4)} + X_2^{(4)}) + 4Y_{12},$$

where

$$(4.34) \quad X_1^{(2)} = (A_3(1 - n_3) - n_2 A_2) \text{tr}_2 F_1^2 = -6K(4K + T) \text{tr}_2 F_1^2$$

$$(4.35) \quad \begin{aligned} X_1^{(4)} &= (B_3(1 - n_3) - n_2 B_2) \text{tr}_2 F_1^4 + (C_3(1 - n_3) - n_2 C_2) (\text{tr}_2 F_1^2)^2 \\ &= -3(4K + T)^2 (\text{tr}_2 F_1^2)^2 \end{aligned}$$

$$(4.36) \quad X_2^{(2)} = (A_3(1 - n_3) - n_2 A_2) \text{tr}_2 F_1^2 = 6KT \text{tr}_2 F_1^2$$

$$(4.37) \quad \begin{aligned} X_2^{(4)} &= (B_3(1 - n_3) - n_2 B_2) \text{tr}_2 F_1^4 + (C_3(1 - n_3) - n_2 C_2) (\text{tr}_2 F_1^2)^2 \\ &= -3T^2 (\text{tr}_2 F_1^2)^2 \end{aligned}$$

$$(4.38) \quad Y_{12} = (n_{2,4} + n_{2,\bar{4}}) \text{tr}_2 F_1^2 \text{tr}_4 F_2^2 = -T(4K + T) \text{tr}_2 F_1^2 \text{tr}_4 F_2^2.$$

Hence the anomaly polynomial becomes simply

$$(4.39) \quad I_8 = \frac{1}{2} \left(\frac{1}{2} K \text{tr } R^2 + 2S \text{tr}_2 F_1^2 + 2T \text{tr}_2 F_2^2 \right)^2,$$

where $S + T = -4K$. Since the anomaly polynomial is a perfect square, we can conclude that the anomalies are all canceled.

5. Spin(4)-model

A Weierstrass equation for Spin(4) is given by the collision of type $I_2^{ns} + I_2^{ns}$ in an elliptic fibration with a trivial Mordell–Weil group.

$$(5.1) \quad Y_0 : \quad y^2 z = x^3 + a_2 x^2 z + \tilde{a}_4 s t x z^2 + \tilde{a}_6 s^2 t^2 z^3.$$

This is a good starting point as the other possibilities (see Table 1) can be described as degenerations of it. The discriminant locus of this model is

$$(5.2) \quad \Delta = -s^2 t^2 (4a_2^3 \tilde{a}_6 - a_2^2 \tilde{a}_4^2 - 18a_2 \tilde{a}_4 \tilde{a}_6 s t + 4\tilde{a}_4^3 s t + 27\tilde{a}_6^2 s^2 t^2).$$

Theorem 5.1. *Assuming that $S = V(s)$ and $T = V(t)$ intersect transversally, the Weierstrass model*

$$Y_0 : y^2z = x^3 + a_2x^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2z^3.$$

has two distinct crepant resolutions Y^\pm . One is given by the sequence of blowups

$$(5.3) \quad X_0 \xleftarrow{(x,y,s|e_1)} X_1^+ \xleftarrow{(x,y,t|w_1)} X_2^+.$$

The other crepant resolution is obtained by exchanging the order of the blowup.

$$(5.4) \quad X_0 \xleftarrow{(x,y,t|w_1)} X_1^- \xleftarrow{(x,y,s|e_1)} X_2^-.$$

The proper transforms of the Weierstrass model Y_0 is then a smooth elliptic fibration Y^+ or Y^- are crepant resolutions of Y_0 connected by an Atiyah flop.

The proper transforms are

$$(5.5) \quad Y^\pm : y^2z = e_1w_1x^3 + a_2x^2z + \tilde{a}_4stxz^2 + \tilde{a}_6s^2t^2z^3.$$

The relative projective coordinates for X_2^\pm are then given by

$$(5.6) \quad X_2^+ \quad [e_1w_1x : e_1w_1y : z = 1][w_1x : w_1y : s][x : y : t].$$

and for the second sequence of blowups:

$$(5.7) \quad X_2^- \quad [e_1w_1x : e_1w_1y : z = 1][e_1x : e_1y : t][x : y : s].$$

The same blowups are also giving crepant resolutions connected by an Atiyah flop.

5.1. Fiber structure

We have four fibral divisors where two (D_0^s and D_1^s) are over S and the other two (D_0^t and D_1^t) are over T :

$$(5.8) \quad D_0^s : s = y^2 - e_1 w_1 x^3 - a_2 x^2 = 0,$$

$$(5.9) \quad D_1^s : e_1 = y^2 - a_2 x^2 - \tilde{a}_4 s t x - \tilde{a}_6 s^2 t^2 = 0$$

$$(5.10) \quad D_0^t : t = y^2 - e_1 w_1 x^3 - a_2 x^2 = 0,$$

$$(5.11) \quad D_1^t : w_1 = y^2 - a_2 x^2 - \tilde{a}_4 s t x - \tilde{a}_6 s^2 t^2 = 0.$$

First we can understand the fiber structure of this chamber. Over S , we have two fibers C_0^s and C_1^s , where they intersect at two points given by

$$(5.12) \quad C_0^s \cap C_1^s : s = e_1 = y^2 - a_2 x^2 = 0.$$

This is the same intersection from the $I_2^{ns} + I_2^{ns}$ -model with $\mathbb{Z}/2\mathbb{Z}$. This enhances to a type III fiber when $a_2 = 0$ as well. This has another enhancement over S when $\tilde{a}_4^2 - 4a_2\tilde{a}_6 = 0$, because the conic curve D_1^s splits into two curves

$$(5.13) \quad C_1^s \rightarrow C_{1+}^s + C_{1-}^s,$$

which yields the fiber I_3^{ns} . The intersection numbers between the new curves and the fibral divisors are given by

	D_0^s	D_1^s	D_0^t	D_1^t
C_0^s	-2	2	0	0
C_{1+}^s	1	-1	0	0
C_{1-}^s	1	-1	0	0

From the intersection numbers, we get the weight $[1; 0]$ from both curves $C_{1\pm}^s$. Since $[1]$ corresponds to the representation $\mathbf{2}$ of $\mathfrak{su}(2)$ and it is only charged over S , we get the matter content in the representation $(\mathbf{2}, \mathbf{1})$.

Over T , the same enhancements $I_2 \rightarrow I_3$ and $I_2 \rightarrow III$ exist as the two divisors are the same when $s \leftrightarrow t$ and $e_1 \leftrightarrow w_1$. For the former enhancement, the intersection numbers between the new curves via the splitting

$$(5.15) \quad C_1^t \rightarrow C_{1+}^t + C_{1-}^t,$$

and the fibral divisors are given by

$$(5.16) \quad \begin{array}{c|cccc} & D_0^s & D_1^s & D_0^t & D_1^t \\ \hline C_0^t & 0 & 0 & -2 & 2 \\ C_{1+}^t & 0 & 0 & 1 & -1 \\ C_{1-}^t & 0 & 0 & 1 & -1 \end{array}$$

From these intersection numbers of the curves $C_{1\pm}^t$, we get the weight $[0; 1]$. Since $[1]$ corresponds to the representation $\mathbf{2}$ of $\mathfrak{su}(2)$ and it is only charged over T , we get the matter content in the representation $(\mathbf{1}, \mathbf{2})$.

Over both S and T , we get four following curves:

$$(5.17) \quad \begin{cases} D_0^s \cap D_0^t & : s = t = y^2 - e_1 w_1 x^3 - a_2 x^2 = 0 \rightarrow \eta^{00}, \\ D_1^s \cap D_0^t & : e_1 = t = y^2 - a_2 x^2 = 0 \rightarrow \eta^{10}, \\ D_1^s \cap D_1^t & : e_1 = w_1 = y^2 - a_2 x^2 - \tilde{a}_4 s t x - \tilde{a}_6 s^2 t^2 = 0 \rightarrow \eta^{11}. \end{cases}$$

The curve η^{10} is not geometrically irreducible and consists of two geometrically irreducible curves $\eta^{10\pm}$ that requires taking the square-root of a_2 . Clearly, if in the Weierstrass equation, a_2 was a perfect square modulo t or s , the fiber η^{10} will factorize into $\eta^{10\pm}$ without the need of a field extension. The intersections of the new curves are

$$(5.18) \quad \eta^{00} \cap \eta^{10\pm} : s = e_1 = t = y^2 - a_2 x^2 = 0,$$

$$(5.19) \quad \eta^{10\pm} \cap \eta^{11} : e_1 = t = w_1 = y^2 - a_2 x^2 = 0.$$

This gives a fiber of type I_4^{ns} . When $a_2 = 0$, this specializes into a new fiber, as represented in Figure 9, where the nonsplit curves $\eta^{10\pm}$ become a single curve η^{10y} with degeneracy two, where

$$(5.20) \quad \eta^{10y} : e_1 = t = y = 0.$$

From these splittings of the curve,

$$(5.21) \quad \begin{cases} C_0^s \rightarrow \eta^{00}, \\ C_1^s \rightarrow \eta^{10+} + \eta^{10-} + \eta^{11}, \\ C_0^t \rightarrow \eta^{00} + \eta^{10+} + \eta^{10-}, \\ C_1^t \rightarrow \eta^{11}, \end{cases}$$

we compute the intersection numbers between the curves on the collision with the divisors of two I_2^{ns} fibers to be

$$(5.22) \quad \begin{array}{c|cccc} & D_0^s & D_1^s & D_0^t & D_1^t \\ \hline \eta^{00} & -2 & 2 & 0 & 0 \\ \eta^{10+} + \eta^{10-} & 2 & -2 & -2 & 2 \\ \eta^{10\pm} & 1 & -1 & -1 & 1 \\ \eta^{11} & 0 & 0 & 2 & -2 \end{array}$$

Hence we see that we get the weight $[1; -1]$ from the curves $\eta^{10\pm}$. Since both $[1]$ and $[-1]$ is from the representation $\mathbf{2}$ of $\mathfrak{su}(2)$, we can deduce that the matter content we get is a bifundamental, $(\mathbf{2}, \mathbf{2})$.

Hence, the matter content of the $I_2^{ns} + I_2^{ns}$ -model with a trivial Mordell–Weil group is

$$(5.23) \quad \mathbf{R} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}).$$

5.2. Euler characteristic and Hodge numbers

We compute the Euler characteristic using the blowup maps and the push-forward theorems of [29]. Let $S = V(s), T = V(t)$ be the two smooth divisors that support the fibers with dual graph \tilde{A}_1 . We assume that S and T intersect transversally. Let us recall that $L = c_1(\mathcal{L})$ is the first Chern class of the fundamental line bundle of the Weierstrass model.

Theorem 2.1. *The generating polynomial of the Euler characteristic of a $Spin(4)$ -model obtained by a crepant resolution of a Weierstrass model given in Table 1 is*

$$\begin{aligned} \chi(Y) = 2 & \left(\frac{S(-6L^2(4T+3) + L(8(T-1)T - 9) + T(5T+4))}{(2L+1)(S+1)(T+1)(-6L+2S+2T-1)} \right. \\ & \left. + \frac{S^2(L(8T+6) + 5T+3) + 3(2L+1)(T^2 - L(3T+2))}{(2L+1)(S+1)(T+1)(-6L+2S+2T-1)} \right) c(B). \end{aligned}$$

In particular, in the case of a Calabi-Yau threefold, the Euler characteristic of a $Spin(4)$ -model is

$$\chi(Y) = -2(30K^2 + 15K(S+T) + 3S^2 + 3T^2 + 4ST).$$

Proof. The Euler characteristic of a variety Y is the degree of its homological total Chern class. The homological Chern class of a Weierstrass model is

$$\frac{(1 + H)(1 + H + 2\pi^*L)(1 + H + 3\pi^*L)}{1 + 3H + 6\pi^*L} \pi^*c(TB).$$

The ambient space for the Weierstrass model is the projective bundle is $\pi : X_0 \rightarrow B$. We denote by $f_1 : X_1^+ \rightarrow X_0$ the first blowup with center (x, y, s) and by $f_2 : X_2^+ \rightarrow X_1^+$ the second blowup with center (x, y, t) . The center of the first and second blowups have centered of class:

$$\begin{aligned} Z_1^{(1)} &= H + 2\pi^*L, & Z_2^{(1)} &= H + 3\pi^*L, & Z_3^{(1)} &= S \\ Z_1^{(2)} &= f_1^*H + 2f_1^*\pi^*L - E_1, & Z_2^{(2)} &= f_1^*H + 3f_1^*\pi^*L - E_1, & Z_3^{(2)} &= T \end{aligned}$$

The total Chern class of X_0, X_1 , and X_2 are

$$\begin{aligned} c(TX_0) &= (1 + H)(1 + H + 2\pi^*L)(1 + H + 3\pi^*L)\pi^*c(TB), \\ c(TX_1^+) &= (1 + E_1) \frac{(1 + Z_1^{(1)})(1 + Z_2^{(1)})(1 + Z_3^{(1)})}{(1 + Z_1^{(1)})(1 + Z_2^{(1)})(1 + Z_3^{(1)})} f_1^*c(TX_0), \\ c(TX_2^+) &= (1 + W_1) \frac{(1 + Z_1^{(2)})(1 + Z_2^{(2)})(1 + Z_3^{(2)})}{(1 + Z_1^{(2)})(1 + Z_2^{(2)})(1 + Z_3^{(2)})} f_2^*c(TX_1^+). \end{aligned}$$

After each blowup, we can pull out two powers of the exceptional divisor. It follows that the defining equation is a section of a line bundle whose first Chern class is

$$3f_1^*H + 6f_1^*\pi^*L - 2E_1.$$

After the second blowup, the proper transform of the defining equation is a section of a line bundle whose first Chern class is

$$3f_2^*f_1^*H + 6f_2^*f_1^*\pi^*L - 2f_2^*E_1 - 2W_1$$

It follows that the total Chern class of the proper transform of the elliptic fibration is

$$c(TY_2^+) = \frac{c(TX_2^+)}{1 + 3f_2^*f_1^*H + 6f_2^*f_1^*\pi^*L - 2f_2^*E_1 - 2W_1}.$$

The Euler characteristic is then

$$\chi(Y_2^+) = \int \frac{(3f_2^*f_1^*H + 6f_2^*f_1^*\pi^*L - 2f_2^*E_1 - 2W_1)}{1 + 3f_2^*f_1^*H + 6f_2^*f_1^*\pi^*L - 2f_2^*E_1 - 2W_1} c(TX_1^+),$$

where $\int A$ is the degree of A . Since the degree is invariant under a pushforward, we get

$$\chi(Y_2^+) = \int \pi_* f_{1*} f_{2*} \frac{(3f_2^* f_1^* H + 6f_2^* f_1^* \pi^* L - 2f_2^* E_1 - 2W_1)}{1 + 3f_2^* f_1^* H + 6f_2^* f_1^* \pi^* L - 2f_2^* E_1 - 2W_1} c(TX_2^+).$$

The pushforwards f_{1*} and f_{2*} are computed using Theorem 3.14. The final pushforward π_* is computed using Theorem 3.15. These three pushforwards are purely algebraic computations.

In the Calabi–Yau case, we have $L = -K$. For a Calabi-Yau threefold, we just keep the terms of degree two in the base. □

Theorem 2.2. *The Hodge numbers of a Spin(4)-model given by the crepant resolution of a Weierstrass model given in Table 1 are*

$$\begin{aligned} h^{1,1}(Y) &= 13 - K^2, \\ h^{2,1}(Y) &= 13 + 29K^2 + 15K(S + T) + 3S^2 + 4ST + 3T^2. \end{aligned}$$

Proof. For a Calabi-Yau threefold, we have $h^{1,0} = 0$ and $\chi(Y) = 2(h^{1,1}(Y) - h^{2,1}(Y))$. Since the variety is elliptically fibered, we can compute $h^{1,1}(Y)$ by the Shioda–Tate–Wazir theorem (see Theorem 3.22) and Noether’s formula. We then have $h^{1,1}(Y) = h^{1,1}(B) + f + 1$ with $f = 2$ and $h^{1,1}(B) = 10 - K^2$ (see Lemma 3.21). Finally, $h^{2,1}(Y) = h^{1,1}(Y) - \chi(Y)/2$, where $\chi(Y)$ is given in Theorem 2.1. □

5.3. Triple intersection numbers

In this subsection, we compute the triple intersection polynomial of the Spin(4)-model. In contrast to the Euler characteristic, the triple intersection polynomial does depend on the choice of a crepant resolution. In particular, the two varieties Y^\pm have distinct topologies as they have distinct triple intersection numbers.

Theorem 2.5. *Let $f^+ : Y^+ \rightarrow Y_0$ be the crepant resolution where Y_0 is any of the Spin(4)-model listed in Table 1. The triple intersection polynomial of*

Y^+ is

$$\begin{aligned} \mathcal{F}_{trip}^+ &= \int_Y \pi_* f_* \left[\left(\psi_0 D_0^s + \psi_1 D_1^s + \phi_0 D_0^t + \phi_1 D_1^t \right)^3 \right] \\ &= 2T(-2L + S - T)\phi_1^3 - 6ST\psi_1\phi_1^2 - 2S(2L + S)\psi_1^3 \\ &\quad + 2T(2L - S - 2T)\phi_0^3 + 6T\phi_0^2(\phi_1(-2L + S + T) - S\psi_1) \\ &\quad - 4S(S - L)\psi_0(\psi_0 - \psi_1)^2 - 2S(2L + S)\psi_0\psi_1(\psi_0 - \psi_1) \\ &\quad + 6T\phi_0 \left(\phi_1^2(2L - S) - 2S(\psi_0 - \psi_1)^2 + 2S\psi_1\phi_1 \right). \end{aligned}$$

The triple intersection polynomial in the fibration Y^- defined by exchanging the order of the blowup is \mathcal{F}_{trip}^- and is obtained from \mathcal{F}_{trip}^- by the involution $\psi \leftrightarrow \phi$.

Proof. The ambient space for the Weierstrass model is the projective bundle is $\pi : X_0 \rightarrow B$. We denote by $f_1 : X_1^+ \rightarrow X_0$ the first blowup with center (x, y, s) and by $f_2 : X_2^+ \rightarrow X_1^+$ the second blowup with center (x, y, t) .

The class of the fibral divisors $D_0^s, D_1^s, D_0^t, D_1^t$ are

$$\begin{aligned} [D_0^s] &= f_2^* f_1^* \pi^* S - f_2^* E_1, & [D_1^s] &= f_2^* E_1, \\ [D_0^t] &= f_2^* f_1^* \pi^* T - W_1, & [D_1^t] &= W_1. \end{aligned}$$

$$\begin{aligned} &\int_Y (f_2^* E_1 a + W_1 b + f_2^* f_1^* H b)^3 \\ &= \int_{X_2} (f_2^* E_1 a + W_1 b + f_2^* f_1^* H c)^3 (3f_2^* f_1^* H + 6f_2^* f_1^* \pi^* L - 2f_2^* E_1 - 2W_1) \\ &= -2S(S + 2L)a^3 - 6STab^2 + 2(S - 2L - T)Tb^3 + 27L^2c^3 \end{aligned}$$

which gives the following non-vanishing triple intersection numbers

$$\begin{aligned} \int_Y f_2^* E_1^3 &= -2S(S + 2L), & \int_Y f_2^* E_1 W_1^2 &= -2ST, \\ \int_Y W_1^3 &= 2T(S - 2L - T), & \int_Y f_2^* f_1^* H^3 &= 27L^2. \end{aligned}$$

The triple intersection numbers of the fibral divisors are then computed from these equations. □

5.4. Counting charged hypermultiplets in 5d

M-theory compactified on a Calabi-Yau threefold yields a five-dimensional theory with eight supercharges. This is reviewed in Section 3.9. The dynamics of the vector fields and the scalar fields of the vector multiplets depends on the prepotential \mathcal{F} , which gets a one-loop correction protected from additional corrections by supersymmetry. In the present case, the Lie algebra is $D_2 = A_1 \oplus A_1$ and charged hypermultiplets transform in the representation $\mathbf{R} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ of D_2 . In the Coulomb phase, all massive particles have been integrated out; in particular, the prepotential depends only on fields in the Cartan subalgebra of the Lie algebra.

Let (ψ_1, ϕ_1) be a parametrization of the coroots written in the basis of simple coroots of the Lie algebra $D_2 = A_1 \oplus A_1$. The prepotential of the five-dimensional supergravity theory with Lie algebra D_2 and matter in the representation \mathbf{R} defined above is [53]

$$(5.24) \quad 6\mathcal{F}_{\text{IMS}} = -(8n_{\mathbf{1},\mathbf{3}} + n_{\mathbf{1},\mathbf{2}} - 8) |\phi_1|^3 - n_{\mathbf{2},\mathbf{2}} \left(|\phi_1 - \psi_1|^3 - |\phi_1 + \psi_1|^3 \right) - (8n_{\mathbf{3},\mathbf{1}} + n_{\mathbf{2},\mathbf{1}} - 8) |\psi_1|^3,$$

where $n_{\mathbf{R}_1, \mathbf{R}_2}$ is the number of hypermultiplets transforming in the irreducible representation $(\mathbf{R}_1, \mathbf{R}_2)$ of D_2 . The dual fundamental Weyl chamber is $\psi_1 > 0$ and $\phi_1 > 0$. The only weights of the representation \mathbf{R} are the weights $\pm[1; -1]$. It follows that there are two Coulomb chambers, each characterized by the sign of $\phi_1 - \psi_1$.

In the chamber $\phi_1 - \psi_1 > 0$, the prepotential is

$$(5.25) \quad 6\mathcal{F}_{\text{IMS}}^+ = \phi_1^3 (-n_{\mathbf{1},\mathbf{2}} - 8n_{\mathbf{1},\mathbf{3}} - 2n_{\mathbf{2},\mathbf{2}} + 8) + \psi_1^3 (-n_{\mathbf{2},\mathbf{1}} - 8n_{\mathbf{3},\mathbf{1}} + 8) - 6n_{\mathbf{2},\mathbf{2}} \psi_1^2 \phi_1.$$

In the chamber $\phi_1 - \psi_1 < 0$, the prepotential is

$$(5.26) \quad 6\mathcal{F}_{\text{IMS}}^- = \phi_1^3 (-n_{\mathbf{1},\mathbf{2}} - 8n_{\mathbf{1},\mathbf{3}} + 8) + \psi_1^3 (-n_{\mathbf{2},\mathbf{1}} - 2n_{\mathbf{2},\mathbf{2}} - 8n_{\mathbf{3},\mathbf{1}} + 8) - 6n_{\mathbf{2},\mathbf{2}} \psi_1 \phi_1^2.$$

Matching the 5d prepotential with the triple intersection numbers gives constraints on the number of representations that are sometimes enough to completely fix them. However, in the present case, they do determine only the number of bifundamental $(\mathbf{2}, \mathbf{2})$ while giving two linear relations for the

four other numbers:

$$(5.27) \quad \begin{aligned} n_{\mathbf{1},\mathbf{2}} + 8n_{\mathbf{1},\mathbf{3}} &= T(4L - 2S + 2T) + 8, \\ n_{\mathbf{2},\mathbf{1}} + 8n_{\mathbf{3},\mathbf{1}} &= S(4L - 2T + 2S) + 8. \end{aligned}$$

This can be fixed using two different methods.

Firstly, by using the intersecting brane picture, we compute the number of fundamentals as the number of intersection points between appropriate components of the reduced discriminant. Since the split curve that gives the representation $(\mathbf{2}, \mathbf{1})$ is from $s = a_4^2 - 4a_2a_6 = 0$, whose class is $[s](2[a_4]) = 2S(4L - S - T)$, we deduce that $n_{\mathbf{2},\mathbf{1}} = 2S(4L - S - T)$. Likely, for the representation $(\mathbf{1}, \mathbf{2})$ is from $t = a_4^2 - 4a_2a_6 = 0$, whose class is $[t](2[a_4]) = 2T(4L - S - T)$, we can also deduce that $n_{\mathbf{1},\mathbf{2}} = T(4L - S - T)$. This fixes all the number of representations to be

$$(5.28) \quad \begin{aligned} n_{\mathbf{1},\mathbf{2}} &= 2T(4L - S - T), & n_{\mathbf{2},\mathbf{1}} &= 2S(4L - S - T), & n_{\mathbf{2},\mathbf{2}} &= ST, \\ n_{\mathbf{1},\mathbf{3}} &= \frac{1}{2}(-LT + T^2 + 2) = g_T, & n_{\mathbf{3},\mathbf{1}} &= \frac{1}{2}(-LS + S^2 + 2) = g_S, \end{aligned}$$

where $g_C = (K \cdot C + C^2 + 2)/2$ is the arithmetic genus of a curve C .

Secondly, using that Witten’s genus formula holds in this context, the number of adjoint representations $n_{\mathbf{3},\mathbf{1}}$ and $n_{\mathbf{1},\mathbf{3}}$ are given respectively by the genus of the supporting curve S and T as in the last two equations of (5.28). We then determine $n_{\mathbf{2},\mathbf{1}}$ and $n_{\mathbf{1},\mathbf{2}}$ using the linear relations (5.27) and thereby reproducing (5.28) after imposing the Calabi-Yau condition $-K = L$.

The vanishing of the coefficients of $\text{tr } R^4$ and $\text{tr } F_i^4$ are necessary conditions for ensuring that the six-dimensional supergravity theory is anomaly free. Since here we are dealing with two $SU(2)$, we never have a fourth Casimir.

5.5. Counting hypermultiplets in 6d and anomaly cancellations

Sadov’s F-theory geometric interpretation of the Green-Schwarz anomaly conditions gives the following system of five equations

$$(5.29) \quad \begin{cases} A_3(1 - n_{\mathbf{3},\mathbf{1}}) - A_2(n_{\mathbf{2},\mathbf{1}} + 2n_{\mathbf{2},\mathbf{2}}) = 6K \cdot S, \\ C_3(1 - n_{\mathbf{3},\mathbf{1}}) - C_2(n_{\mathbf{2},\mathbf{1}} + 2n_{\mathbf{2},\mathbf{2}}) = -3S^2, \\ A_3(1 - n_{\mathbf{1},\mathbf{3}}) - A_2(n_{\mathbf{1},\mathbf{2}} + 2n_{\mathbf{2},\mathbf{2}}) = 6K \cdot T, \\ C_3(1 - n_{\mathbf{1},\mathbf{3}}) - C_2(n_{\mathbf{1},\mathbf{2}} + 2n_{\mathbf{2},\mathbf{2}}) = -3T^2, \\ A_1A_1n_{\mathbf{2},\mathbf{2}} = S \cdot T, \end{cases}$$

which give⁸

$$(5.30) \quad \begin{aligned} 4(1 - n_{\mathbf{3},\mathbf{1}}) - (n_{\mathbf{2},\mathbf{1}} + 2n_{\mathbf{2},\mathbf{2}}) &= 6K \cdot S, \\ 16(1 - n_{\mathbf{3},\mathbf{1}}) - (n_{\mathbf{2},\mathbf{1}} + 2n_{\mathbf{2},\mathbf{2}}) &= -6S^2, \end{aligned}$$

$$(5.31) \quad \begin{aligned} 4(1 - n_{\mathbf{1},\mathbf{3}}) - (n_{\mathbf{1},\mathbf{2}} + 2n_{\mathbf{2},\mathbf{2}}) &= 6K \cdot T, \\ 16(1 - n_{\mathbf{1},\mathbf{3}}) - (n_{\mathbf{1},\mathbf{2}} + 2n_{\mathbf{2},\mathbf{2}}) &= -6T^2, \end{aligned}$$

$$(5.32) \quad n_{\mathbf{2},\mathbf{2}} = S \cdot T,$$

The solutions of these linear equations provide a direct computation of the numbers of representations (see equation (5.28)) from a purely six-dimensional perspective.

Witten’s genus formula for S and T states that the number of hypermultiplets transforming in the adjoint representation is the arithmetic genus of the curve supporting the corresponding fiber. Here it becomes a direct consequence of equations (5.30) and (5.31). Equation (5.32) is equivalent to the condition obtained by comparing the triple intersection numbers and the one-loop prepotential (the coefficient of $\psi_1^2 \phi_1$ of $\phi_1^2 \psi_1$) depending on the chamber) or by counting the number of intersection points of S and T .

Since we have geometrically computed the Hodge numbers and the number of representations, we can check if the anomalies are canceled in the uplifted six-dimensional supergravity theory. The number of vector multiplets $n_V^{(6)}$, tensor multiplets n_T , and hypermultiplets n_H are

$$(5.33) \quad \begin{aligned} n_V^{(6)} &= 6, \quad n_T = 9 - K^2, \\ n_H &= h^{2,1}(Y) + 1 + n_{\mathbf{3},\mathbf{1}}(3 - 1) + n_{\mathbf{2},\mathbf{1}}(2 - 0) + n_{\mathbf{2},\mathbf{2}}(4 - 0) \\ &\quad + n_{\mathbf{1},\mathbf{2}}(2 - 0) + n_{\mathbf{1},\mathbf{3}}(3 - 1) \\ &= 29K^2 + 15KS + 15KT + 3S^2 + 4ST + 3T^2 + 13. \end{aligned}$$

Thus, a direct computation shows that the purely gravitational anomalies are also canceled:

$$(5.34) \quad n_H - n_V^{(6)} + 29n_T - 273 = 0.$$

The remaining parts of the anomaly polynomial is

$$(5.35) \quad I_8 = \frac{K^2}{8}(\text{tr } R^2)^2 + \frac{1}{6}(X_1^{(2)} + X_2^{(2)}) \text{tr } R^2 - \frac{2}{3}(X_1^{(4)} + X_2^{(4)}) + 4Y_{12}.$$

⁸For the Lie algebra of SU(2), we recall that $A_3 = 4$, $B_3 = B_2 = 0$, $A_2 = 1$, $C_3 = 8$, $C_2 = \frac{1}{2}$. See [8, 23, 69].

Since A_1 does not have quartic Casimir invariants, the coefficients of $\text{tr } F_1^4$ and $\text{tr } F_2^4$ vanish. We also have

$$(5.36) \quad X_1^{(2)} = (A_3(1 - n_{3,1}) - (n_{2,1} + 2n_{2,2})A_2) \text{tr}_2 F_1^2 = 6KS \text{tr}_2 F_1^2,$$

$$(5.37) \quad X_2^{(2)} = (A_3(1 - n_{1,3}) - (n_{1,2} + 2n_{2,2})A_2) \text{tr}_2 F_2^2 = 6KT \text{tr}_2 F_2^2,$$

$$(5.38) \quad X_1^{(4)} = (C_3(1 - n_3) - (n_{2,1} + 2n_{2,2})C_2) (\text{tr}_2 F_1^2)^2 = -3S^2(\text{tr}_2 F_1^2)^2,$$

$$(5.39) \quad X_2^{(4)} = (C_3(1 - n_3) - (n_{1,2} + 2n_{2,2})C_2) (\text{tr}_2 F_1^2)^2 = -3T^2(\text{tr}_2 F_1^2)^2,$$

$$(5.40) \quad Y_{12} = n_{2,2} \text{tr}_2 F_1^2 \text{tr}_4 F_2^2 = ST \text{tr}_2 F_1^2 \text{tr}_2 F_2^2.$$

It follows that the anomaly polynomial becomes

$$(5.41) \quad I_8 = \frac{K^2}{8} (\text{tr } R^2)^2 + K(S \text{tr}_2 F_1^2 + T \text{tr}_2 F_2^2) \text{tr } R^2 \\ + 2S^2(\text{tr}_1 F_1^2)^2 (\text{tr}_2 F_1^2)^2 + 4ST \text{tr}_2 F_1^2 \text{tr}_2 F_2^2,$$

which is a perfect square

$$(5.42) \quad I_8 = \frac{1}{2} \left(\frac{1}{2} K \text{tr } R^2 + 2S \text{tr}_2 F_1^2 + 2T \text{tr}_2 F_2^2 \right)^2.$$

Since the anomaly polynomial is a perfect square, we can safely deduce that all the anomalies are canceled by the Green–Schwarz mechanism.

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