ADV. THEOR. MATH. PHYS.
Volume 23, Number 6, 1599–1630, 2019

Covariance of the classical Brink–Schwarz superparticle

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We show that the classical Brink–Schwarz superparticle is a generalized AKSZ field theory. We work in the Batalin–Vilkovisky formalism: the main technical tool is the vanishing of Batalin–Vilkovisky cohomology below degree -1.

1. Introduction

The construction of string theory is based on a nonlinear sigma-model with worldsheet and target a Riemann surface Σ and 26-dimensional Minkowski space respectively. The superstring, a supersymmetric analogue of the string, may be represented in several ways. In the Neveu–Schwarz/Ramond superstring, Σ is a Riemann supersurface, the target is 10-dimensional Minkowski space, and the target-space supersymmetry of the theory is not apparent at the classical level. By contrast, in the Green–Schwarz superstring, Σ is a Riemann surface, while the target is a superspace whose underlying vector space is the same ten-dimensional Minkowski space, but which incorporates an additional fermionic chiral Majorana-Weyl spinor. In this model, the target-space supersymmetry is manifest, at the cost of quantization being considerably more complicated. (There are other approaches to the superstring, such as the pure spinor theory of Berkovits [3].)

There has been a lot of interest in the study of toy models of these two formulations of superstring theory, in which the Riemann surface Σ is replaced by a one-dimensional manifold, the worldline: the resulting toy models of the Neveu–Schwarz/Ramond superstring and Green–Schwarz superstring are respectively known as the spinning particle [6] and the superparticle [7]. In this paper, we focus on the superparticle.

The Batalin–Vilkovisky formalism for classical field theories provides a powerful way of encoding symmetries: it is especially adapted to extending on-shell symmetries, such as extended supersymmetry and supergravity, off-shell. Poisson geometry studies Maurer–Cartan elements in the Schouten algebra of a manifold: the classical Batalin–Vilkovisky master equation generalizes this to graded supermanifolds, and then further generalizes from finite dimensional calculus to variational calculus.

One of the principles in the construction of solutions of the classical master equation of Batalin and Vilkovisky is that the cohomology of the Batalin–Vilkovisky differential on local functionals, known as the (classical) Batalin–Vilkovisky cohomology, is bounded below. In the absence of worldsheet supersymmetry, it appears to be a general phenomenon that these cohomology groups vanish for ghost number less than $-d$, where d is the dimension of the worldsheet. This condition may be violated in the presence of worldsheet supersymmetry: as shown in [8], the classical Batalin–Vilkovisky cohomology of the spinning particle is nontrivial at arbitrarily large negative ghost number.

The superparticle does not have worldsheet supersymmetry, unlike the spinning particle. One of the main results of this paper is that the classical Batalin–Vilkovisky cohomology for the superparticle, suitably interepreted, vanishes below ghost number -1.

Using this result, we show that the classical superparticle is a covariant field theory, in the sense of [10]. Let $\int \mathsf{S}$ be the action of the superparticle, which satisfies the classical Batalin–Vilkovisky master equation

$$\tfrac{1}{2}(\textstyle\int \mathsf{S}, \int \mathsf{S}) = 0.$$

We show that there is a power series

$$\mathsf{S}_u = \mathsf{S} + \sum_{n=0}^{\infty} u^{n+1}\,\mathsf{G}_n,$$

where G_n is a density of ghost number $-2n-2$ and u is a formal variable of ghost number 2, which satisfies the curved Maurer–Cartan equation

$$\delta\mathsf{S}_u + \tfrac{1}{2}(\textstyle\int\mathsf{S}_u, \int\mathsf{S}_u) = -u\int\mathsf{D}. \tag{1.1}$$

Here, D is the density

$$\mathsf{D} = x_\mu^+\partial x^\mu + p^{+\mu}\partial p_\mu - e\partial e^+ + c^+\partial c + \sum_{n=0}^{\infty}\mathsf{T}(\theta_n^+, \partial\theta_n),$$

whose associated Batalin–Vilkovisky Hamiltonian vector field equals ∂, the generator of time translation along the worldline. In particular,

$$(\delta + \mathsf{s})\big(\textstyle\int\mathsf{G}_0\big) = -\int\mathsf{D}. \tag{1.2}$$

Possessing a solution to (1.1) is a property that the superparticle shares with AKSZ theories [1]: in this sense, the superparticle may be viewed as a generalized AKSZ theory.

We solve (1.1) over the open subset of the phase space where the momentum p_μ is nonzero. For the sheaves that we consider, this subset has nontrivial cohomology. The usual way to deal with this would be to work with Čech cochains. Unfortunately, there is no way to extend the Batalin–Vilkovisky antibracket to the space of Čech cochains: this is related to the failure of the cup-product to be graded commutative. We circumvent this difficulty by working with Sullivan's Thom–Whitney complex [12]: this replaces simplicial cochains by differential forms on simplices, and allows the definition of the antibracket at the cochain level. We also find explicit formulas in this complex for the cohomology of the Batalin–Vilkovisky differential. In passing, we note that in Berkovits's description of superstrings using pure spinors [3], a similar issue arises, involving the cone of pure spinors with the origin removed.

The action of the superparticle S in first-order formalism is globally defined, and was found by Lindström et al. [11]. We construct the next term in the expansion of S_u explicitly as a solution to (1.2). For $n > 0$, the coefficient G_n of u^{n+1} in S_u has ghost number $-2n-2$, and is a solution to the equation

$$(\delta+\mathsf{s})\big(\textstyle\int \mathsf{G}_n\big) = -\frac{1}{2}\sum_{j+k=n-1}(\textstyle\int \mathsf{G}_j, \int \mathsf{G}_k).$$

Since the cohomology of the operator $\delta+\mathsf{s}$ vanishes below degree -1, we may solve this equation.

Let $C^*(\mathfrak{so}(9,1))$ be the graded commutative algebra of the Lie algebra $\mathfrak{so}(9,1)$ of the Lorentz group (the exterior algebra generated by $\mathfrak{so}(9,1)^\vee$), with differential d. The action of the Lie algebra $\mathfrak{so}(9,1)$ on the space of fields of the superparticle is generated by the Batalin–Vilkovisky currents

$$M^{\mu\nu} = \eta^{\lambda[\mu}x^{\nu]}x^+_\lambda - \eta^{\lambda[\mu}p^{+\nu]}p_\lambda - \sum_{n=0}^\infty \mathsf{T}^{\mu\nu}(\theta^+_n,\theta_n). \tag{1.3}$$

Let $\mathsf{S}(\epsilon)=\mathsf{S}+M^{\mu\nu}\epsilon_{\mu\nu}$, where $\epsilon_{\mu\nu}$ is the dual basis of $\mathfrak{so}(9,1)^\vee$; this is an element of total degree 0 in the tensor product of $C^*(\mathfrak{so}(9,1))$ and the Batalin–Vilkovisky graded Lie algebra. The Lorentz invariance of the action S may be expressed by the following extension of the classical master equation:

$$d\textstyle\int \mathsf{S}(\epsilon) + \frac{1}{2}(\int \mathsf{S}(\epsilon), \int \mathsf{S}(\epsilon)) = 0.$$

The Lorentz group does not act on Thom–Whitney complex, because the open cover itself is not Lorentz invariant; in particular, G_0 is not invariant under the action of $\mathfrak{so}(9,1)$. Nevertheless, it may be proved that S_u has an enhancement

$$\mathsf{S}_u(\epsilon) = \mathsf{S}(\epsilon) + \sum_{n=0}^\infty u^{n+1}\mathsf{G}_n(\epsilon), \tag{1.4}$$

where $\mathsf{G}_n(\epsilon)$ is an element of total degree $-2n-2$ in the tensor product of $C^*(\mathfrak{so}(9,1))$ and the Thom–Whitney extension of the Batalin–Vilkovisky graded Lie algebra, such that the following extension of (1.1) holds:

$$(d+\delta)\textstyle\int \mathsf{S}_u(\epsilon) + \frac{1}{2}(\int \mathsf{S}_u(\epsilon), \int \mathsf{S}_u(\epsilon)) = -u\int \mathsf{D}. \tag{1.5}$$

In mathematical terms, this equation, which is nothing but the BRST formalism for the global symmetry Lie algebra $\mathfrak{so}(9,1)$, expresses that the covariant field theory is Lorentz invariant up to homotopy.

It may be verified that S and G_0 are invariant under supersymmetry. We also show that the terms $\mathsf{G}_n(\epsilon)$, $n\ge 0$, may be chosen to be invariant under supersymmetry.

2. The classical Batalin–Vilkovisky master equation

Consider a Batalin–Vilkovisky model with fields $\{\xi^a\}_{a\in I}$, of ghost number $\mathrm{gh}(\xi^a)\geq 0$ and parity $p(\xi^a)$ equal to 0 and 1 for bosonic and fermionic fields respectively. Denote the antifield corresponding to the field ξ^a by ξ_a^+: it has ghost number $\mathrm{gh}(\xi_a^+)=-1-\mathrm{gh}(\xi^a)<0$ and opposing parity $p(\xi_a^+)=1-p(\xi^a)$ to ξ^a. Denote by $\mathcal{A}$ the algebra of functions in the variables ξ^a and ξ_a^+ and their derivatives

$$\{\partial^\ell\xi^a\}_{\ell\geq 0}\cup\{\partial^\ell\xi_a^+\}_{\ell\geq 0}$$

with respect to the generators of negative ghost number. The bosonic fields of ghost number 0 play a special role in the theory: they are coordinates on a manifold M, and we will view $\mathcal{A}$ as a sheaf over M.

The algebra $\mathcal{A}$ is filtered by the ghost number of the antifields. Let $F^k\mathcal{A}$ be the ideal generated by monomials

$$\partial^{\ell_1}\xi_{a_1}^+\cdots\partial^{\ell_n}\xi_{a_n}^+$$

such that $\mathrm{gh}(\xi_{a_1}^+)+\cdots+\mathrm{gh}(\xi_{a_n}^+)+k\leq 0$. The subspaces $F^k\mathcal{A}$ define a decreasing filtration of $\mathcal{A}$, with $F^0\mathcal{A}=\mathcal{A}$ and $F^j\mathcal{A}\cdot F^k\mathcal{A}\subset F^{j+k}\mathcal{A}$. Denote by $\widehat{\mathcal{A}}$ the completion of $\mathcal{A}$ with respect to this filtration:

$$\widehat{\mathcal{A}}=\varprojlim_k\mathcal{A}/F^k\mathcal{A}.$$

Introduce the partial derivatives

$$\partial_{k,a}=\frac{\partial}{\partial(\partial^k\xi^a)}:\widehat{\mathcal{A}}^j\to\widehat{\mathcal{A}}^{j-\mathrm{gh}(\xi^a)},\qquad \partial_k^a=\frac{\partial}{\partial(\partial^k\xi_a^+)}:\widehat{\mathcal{A}}^j\to\widehat{\mathcal{A}}^{j-\mathrm{gh}(\xi_a^+)}.$$

Let ∂ be the total derivative with respect to t:

$$\partial=\sum_{k=0}^{\infty}\Big((\partial^{k+1}\xi^a)\partial_{k,a}+(\partial^{k+1}\xi_a^+)\partial_k^a\Big).$$

An **evolutionary** vector field X is a graded derivation of $\mathcal{A}$ that commutes with ∂; such a vector field has the form

$$\begin{aligned}X&=\mathrm{pr}\,(X^a\partial_a+X_a\partial^a)\\&=\sum_{k=0}^{\infty}\left(\partial^kX^a\partial_{k,a}+\partial^kX_a\partial_k^a\right).\end{aligned}$$

The Soloviev antibracket on $\mathcal{A}$ is defined by the formula

$$
\begin{aligned}
(\!(f,g)\!) = \sum_a (-1)^{(p(f)+1)\,p(\xi^a)} \\
\times \sum_{k,\ell=0}^{\infty} \Big(\partial^\ell(\partial_{a,k} f)\, \partial^k(\partial_\ell^a g) + (-1)^{p(f)} \partial^\ell(\partial_k^a f)\, \partial^k(\partial_{a,\ell} g) \Big).
\end{aligned}
$$

This bracket, and its extension to $\widehat{\mathcal{A}}$, satisfies the axioms for a graded Lie superalgebra: it is graded antisymmetric

$$(\!(y,x)\!) = -(-1)^{(p(x)+1)(p(y)+1)} (\!(x,y)\!),$$

and satisfies the Jacobi relation

$$(\!(x,(\!(y,z)\!)\!)\!) = (\!((\!(x,y)\!),z)\!) + (-1)^{(p(x)+1)(p(y)+1)} (\!(y,(\!(x,z)\!)\!)\!).$$

Furthermore, it is linear over ∂:

$$(\!(\partial f,g)\!) = (\!(f,\partial g)\!) = \partial(\!(f,g)\!).$$

In this paper, all graded Lie superalgebras are 1-shifted: the antibracket has ghost number 1.

The superspace $\mathcal{F} = \widehat{\mathcal{A}}/\partial\widehat{\mathcal{A}}$ of functionals is the graded quotient of $\widehat{\mathcal{A}}$ by the subspace $\partial\widehat{\mathcal{A}}$ of total derivatives. Denote the image of $f \in \widehat{\mathcal{A}}$ in $\mathcal{F}$ by $\int f$. The Soloviev antibracket $(\!(f,g)\!)$ descends to an antibracket

$$(\textstyle\int f, \int g)$$

on $\mathcal{F}$, called the Batalin–Vilkovisky antibracket. Thus, $\mathcal{F}$ is a sheaf of graded Lie superalgebras over M.

Introduce the variational derivatives

$$\delta_a = \sum_{k=0}^{\infty} (-\partial)^k \partial_{k,a} : \mathcal{F}^j \to \widehat{\mathcal{A}}^{j-\mathrm{gh}(\xi^a)}, \quad \delta^a = \sum_{k=0}^{\infty} (-\partial)^k \partial_k^a : \mathcal{F}^j \to \widehat{\mathcal{A}}^{j-\mathrm{gh}(\xi_a^+)}.$$

The Batalin–Vilkovisky antibracket is given by the formula

$$(\textstyle\int f, \int g) = (-1)^{(p(f)+1)\,p(\xi^a)} \int \big((\delta_a f)\,(\delta^a g) + (-1)^{p(f)} (\delta^a f)\,(\delta_a g) \big).$$

The classical Batalin–Vilkovisky master equation is the equation for an element $S \in \mathcal{F}$ of ghost number 0 and even parity

$$\tfrac{1}{2}(\textstyle\int S, \int S) = 0. \tag{2.1}$$

The Batalin–Vilkovisky differential is the Hamiltonian vector field

$$s = \sum_a (-1)^{p(\xi^a)} \operatorname{pr}\big((\delta_a S)\partial^a + (\delta^a S)\partial_a\big).$$

This is a graded derivation of ghost number 1, and satisfies the equation $s^2 = 0$ precisely when S satisfies the classical master equation [10, Section 3].

3. The particle

Before recalling the Batalin–Vilkovisky approach to the superparticle, we review the simpler case of the particle. Consider the d-dimensional Minkowski space $V = \mathbb{R}^{d-1,1}$ with basis $\{v_\mu\}_{0\le\mu<d}$ and inner product

$$\langle v_\mu, v_\nu\rangle = \eta_{\mu\nu}.$$

The particle has physical fields x^μ, and Lagrangian density $S = \frac{1}{2}\eta_{\mu\nu}\partial x^\mu \partial x^\nu$. For technical reasons, we prefer to work in a first-order formulation of this theory, which has additional physical fields p_μ, and the action

$$S = p_\mu \partial x^\mu - \tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu.$$

In order to have a theory with local reparametrization invariance, we may couple the particle to "gravity" on the world-line, represented by a nowhere-vanishing 1-form field e, the graviton. Of course, the gravitational field in dimension 1 has no dynamical content. The modified action for the particle is

$$S_{[0]} = p_\mu \partial x^\mu - \tfrac{1}{2} e \eta^{\mu\nu} p_\mu p_\nu.$$

The associated differential is

$$s_{[0]} = \operatorname{pr}\left((\partial x^\mu - \eta^{\mu\nu} e p_\nu)\frac{\partial}{\partial p^{+\mu}} - \partial p_\mu \frac{\partial}{\partial x^+_\mu} - \tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+}\right).$$

The variation $\mathsf{s}_{[0]} e^+ = -\frac{1}{2}\eta^{\mu\nu} p_\mu p_\nu$ may be recognized as the $d = 1$ stress-energy tensor.

The local gauge symmetries of this model correspond to cohomology classes of $s_{[0]}$ at ghost number -1:

$$s_{[0]}\big(\partial e^+ - \eta^{\mu\nu} x^+_\mu p_\nu\big) = 0.$$

This cohomology class is killed by the introduction of a ghost field c, with ghost-number 1, transforming as a scalar on the worldline, and the addition to the action of the term

$$S_{[1]} = \big(\partial e^+ - \eta^{\mu\nu} x^+_\mu p_\nu\big) c.$$

This adds the following terms to the differential:

$$s_{[1]} = \operatorname{pr}\left(\eta^{\mu\nu} c x^+_\nu \frac{\partial}{\partial p^{+\mu}} + (\partial e^+ - \eta^{\mu\nu} x^+_\mu p_\nu) \frac{\partial}{\partial c^+} - \eta^{\mu\nu} c p_\nu \frac{\partial}{\partial x^\mu} - \partial c \frac{\partial}{\partial e} \right).$$

We see that the bosonic fields of ghost number 0 of the theory are the position x^μ and the momentum p_μ, and the manifold M is the cotangent bundle $T^\vee V$ of V. For definiteness, we take the structure sheaf $\mathcal{O}$ of M to be functions with analytic dependence on x^μ and algebraic dependence on p_μ, but our results are actually insensitive to the regularity as functions of x^μ. The sheaf $\mathcal{A}$ is the graded commutative algebra generated over $\mathcal{O}$ by the variables

$$\{\partial^\ell x^\mu, \partial^\ell p_\mu\}_{\ell>0} \cup \{\partial^\ell e, e^{-1}, \partial^\ell c, \partial^\ell x^+_\mu, \partial^\ell p^{+\mu}, \partial^\ell e^+, \partial^\ell c^+\}_{\ell\ge 0}.$$

As in the last section, we denote its completion by $\widehat{\mathcal{A}}$.

The sum $S = S_{[0]} + S_{[1]}$ satisfies the classical master equation, and the cohomology of the differential $s = s_{[0]} + s_{[1]}$ on the space of functionals $\mathcal{F}$ vanishes below degree -1.

In preparation for the proof, we recall a criterion of Boardman for the convergence of a spectral sequence. Let V be a complex, with differential $d : V^i \to V^{i+1}$. A decreasing filtration on V is a sequence of subcomplexes

$$\cdots \supset F^{-1}V \supset F^0 V \supset F^1 V \supset \cdots$$

The associated graded complex is

$$\operatorname{gr}^k_F V = F^k V / F^{k+1} V.$$

The filtration is **exhaustive** if for each $i \in \mathbb{Z}$,

$$\bigcup_k F^k V^i = V^i.$$

The filtration is **Hausdorff** if for each $i \in \mathbb{Z}$,

$$\bigcap_k F^k V^i = 0.$$

The filtration is **complete** if

$$V = \varprojlim_k V/F^kV.$$

The filtration $F^\bullet$ induces a filtration on the cohomology $H^*(V)$, which we denote by the same letter. The spectral sequence associated to the filtration converges if for all $(p,q) \in \mathbb{Z}^2$ the induced morphism

$$\mathrm{gr}^p_F H^{p+q}(V) \longrightarrow E^{pq}_\infty$$

is an isomorphism, and the induced filtration on $H^*(V)$ is complete, exhaustive and Hausdorff. The spectral sequence degenerates if $E_\infty = E_r$ for $r \gg 0$.

Theorem 3.1 (Boardman [4]). *If the spectral sequence associated to a complete, exhaustive Hausdorff filtration (V, d, F^kV) degenerates, then it is convergent.*

Proof. Combine the following results from [4]: Theorems 8.2 and 9.2, the remark after Theorem 7.1, and Lemma 8.1. □

A filtration on a differential graded algebra A is a filtration on the underlying complex such that $F^jA \cdot F^kA \subset F^{j+k}A$. In this case, the pages (E_r, d_r) of the spectral sequence are themselves differential graded algebras, and the product on $E_{r+1} \cong H^*(E_r, d_r)$ is induced by the product on E_r.

Introduce the light-cone

$$C = \{(x^\mu, p_\mu) \in M \mid \eta^{\mu\nu} p_\mu p_\nu = 0\}.$$

Theorem 3.2. *The sheaf $H^i(\widehat{\mathcal{A}}, s)$ vanishes for $i < 0$, and is concentrated on the light-cone C.*

Let $\widetilde{\mathcal{A}}$ be the quotient of $\widehat{\mathcal{A}}$ by constant multiples of the identity. The sheaf $H^i(\widetilde{\mathcal{A}}, s)$ also vanishes for $i < 0$, and is concentrated on the light-cone.

Proof. Introduce an auxilliary grading on the sheaf of algebras $\mathcal{A}$. The structure sheaf $\mathcal{O}$ of the manifold M is placed in degree 0, and the generators of

$\mathcal{A}$ over $\mathcal{O}$ are assigned the degrees in the following table:

Φ	$\partial^\ell x^\mu$	$\partial^\ell p_\mu$	$\partial^\ell e$	e^{-1}	$\partial^\ell c$	$\partial^\ell x^+_\mu$	$\partial^\ell p^{+\mu}$	$\partial^\ell e^+$	$\partial^\ell c^+$
$\deg(\Phi)$	0	0	0	0	3	0	0	-1	-1

Write $\mathrm{gh}(f) = \mathrm{gh}_+(f) - \mathrm{gh}_-(f)$, where $\mathrm{gh}_+(f)$ and $\mathrm{gh}_-(f)$ are the contributions of the fields, respectively antifields, to the ghost number. Rearranging, we see that

$$\mathrm{gh}_-(f) = \mathrm{gh}_+(f) - \mathrm{gh}(f).$$

Since

$$\mathrm{gh}_+(f) + 2\,\mathrm{gh}(f) \le \deg(f) \le 3\,\mathrm{gh}_+(f),$$

we see that

$$\tfrac{1}{3}\big(\deg(f) - 3\,\mathrm{gh}(f)\big) \le \mathrm{gh}_-(f) \le \deg(f) - 3\,\mathrm{gh}(f). \tag{3.1}$$

From this grading, we construct an exhaustive and Hausdorff descending filtration on $\mathcal{A}$: $G^k\mathcal{A}^i$ is the span of elements $f \in \mathcal{A}^i$ such that $\deg(f) \ge k$. By (3.1), the completion of this filtration is isomorphic to $\widehat{\mathcal{A}}$.

We now consider the spectral sequence for the filtration induced by G on $\widehat{\mathcal{A}}$. We will show that $E^{pq}_\infty = E^{pq}_2$, that the sheaf E^{pq}_2 vanishes if $p + q < 0$, and that it is supported on the light-cone C. This establishes the theorem.

The differential s_0 of the zeroth page E^{pq}_0 of the spectral sequence equals

$$s_0 = \mathrm{pr}\left(\big(\partial x^\mu - \eta^{\mu\nu} e p_\nu\big) \frac{\partial}{\partial p^{+\mu}} - \partial p_\mu \frac{\partial}{\partial x^+_\mu} + \partial e^+ \frac{\partial}{\partial c^+} \right).$$

This is a Koszul differential and its cohomology E_1 is the graded commutative algebra generated over $\mathcal{O}$ by the variables

$$\{\partial^\ell e, e^{-1}, \partial^\ell c, e^+\}_{\ell \ge 0}.$$

The differential s_1 of the first page E_1 of the spectral sequence equals

$$s_1 = \mathrm{pr}\left(-\tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+} \right).$$

The element $\eta^{\mu\nu} p_\mu p_\nu \in E^{00}_1$ is not a zero divisor in E_1. We conclude that E_2 vanishes in negative degrees and is concentrated on the zero-locus of $\eta^{\mu\nu} p_\mu p_\nu$ in M, namely the light-cone C.

We see that the second page E_2 of the spectral sequence is a graded commutative algebra, generated over $\mathcal{O}_C$ by the variables

$$\{\partial^\ell e, e^{-1}\}_{\ell\geq 0} \in E_2^{00}, \qquad \{\partial^\ell c\}_{\ell\geq 0} \in E_2^{3,-2}.$$

We see that s_2 vanishes, so that $E_3^{pq} = E_2^{pq}$, and that the differential s_3 of the third page E_3^{pq} equals

$$s_3 = \operatorname{pr}\left(-\eta^{\mu\nu} c p_\mu \frac{\partial}{\partial x^\nu} + \partial c \frac{\partial}{\partial e}\right).$$

Using an auxilliary filtration, we see that the differentials s_r vanish for $r > 3$, and E_3^{pq} is quasi-isomorphic to the quotient complex obtained by taking the variables $\{\partial^\ell e, \partial^\ell c\}_{\ell>0}$ to 0 and the variables e and e^{-1} to 1:

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{-\eta^{\mu\nu} c p_\mu \partial/\partial x^\nu} c\,\mathcal{O}_C \longrightarrow 0.$$

Turning to the case of the sheaf $\widetilde{\mathcal{A}}$, we have a long exact sequence for cohomology sheaves

$$0 \longrightarrow H^{-1}(\widetilde{\mathcal{A}}, s) \longrightarrow \mathbb{R} \longrightarrow H^0(\widehat{\mathcal{A}}, s) \longrightarrow \cdots$$

But the above proof shows that the morphism $\mathbb{R} \to E_\infty^{00}$ is an injection, and hence that $H^{-1}(\widetilde{\mathcal{A}}, s) = 0$. $\square$

Corollary 3.3. *Let $\mathcal{F} = \widehat{\mathcal{A}}/\partial\widehat{\mathcal{A}}$. The cohomology sheaf $H^i(\mathcal{F}, \mathsf{s})$ vanishes for $i < -1$, and is concentrated on the light-cone C.*

Proof. The sheaf $\mathcal{F}$ has a resolution

$$0 \longrightarrow \widetilde{\mathcal{A}} \xrightarrow{\partial} \widehat{\mathcal{A}} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The associated long exact sequence implies that $H^i(M, \mathcal{F}) = 0$ for $i < -1$. $\square$

In [10], we show that the covariance of a field theory with respect to reparametrization of the worldline may be expressed by introducing the graded Lie superalgebra $\mathcal{F}[[u]]$ of power series in a formal variable u of ghost number 2 and even parity. Consider the global section of $\mathcal{A}$

$$D = x^+_\mu \partial x^\mu + p^{+\mu}\partial p_\mu - e\partial e^+ + c^+\partial c,$$

of ghost number -1 and odd parity. Its associated Hamiltonian vector field is ∂, and its image $\int D$ in $\mathcal{F}$ is central. Let $G = x^+_\mu p^{+\mu} + ec^+$. Covariance of the theory is expressed by the curved Maurer–Cartan equation

$$\tfrac{1}{2}(\textstyle\int S_u, \int S_u) = -u \int D,$$

where

$$S_u = S + uG.$$

If $\int f \in \mathcal{F}^k$ is a cocycle, then

$$(\textstyle\int G, \int f) \in \mathcal{F}^{k-1}$$

is again a cocycle, called the **transgression** of f. In particular, the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H^{-1}(\widehat{\mathcal{A}}, s) \arrow[r] & H^{-1}(\mathcal{F}, s) \arrow[r] & H^0(\widetilde{\mathcal{A}}, s) \arrow[dll, "\partial"] \\
& H^0(\widehat{\mathcal{A}}, s) \arrow[r] & H^0(\mathcal{F}, s) \arrow[r] & H^1(\widetilde{\mathcal{A}}, s) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

splits, in the sense that the morphisms ∂ vanish.

The particle is actually an AKSZ model (Alexandrov et al. [1]), associated to the symplectic supermanifold $T^\vee(V \times \mathbb{R}[1])$, and G may be interpreted as the Poisson tensor on this supermanifold. The main result of this paper is to find the analogue of S_u for the superparticle, establishing that the superparticle is a "generalized AKSZ model."

4. The superparticle

The Batalin–Vilkovisky action S for the superparticle is based on the above action for the particle, in the special case where $V = \mathbb{R}^{9,1}$. Recall some properties of Majorana–Weyl spinors in this signature of space-time: for further details, see the Appendix. The spin group $\mathrm{Spin}(9,1)$ is the universal cover of the identity component of $\mathrm{SO}(9,1)$. It has two real irreducible sixteen-dimensional representations, the left and right-handed Majorana–Weyl spinors $\mathbb{S}_+$ and $\mathbb{S}_-$. The γ-matrices $\gamma^\mu : \mathbb{S}_\pm \to \mathbb{S}_\mp$ satisfy the relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}.$$

The Lie algebra of the group $\mathrm{Spin}(9,1)$ is spanned by the quadratic expressions

$$\gamma^{\mu\nu} = \tfrac{1}{2}\big(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\big).$$

There is a non-degenerate symmetric bilinear form $\mathsf{T}(\alpha,\beta)$ on $\mathbb{S}$, which vanishes on $\mathbb{S}_\pm \otimes \mathbb{S}_\pm$ and places $\mathbb{S}_\pm$ in duality with $\mathbb{S}_\mp$. We have

$$\mathsf{T}^\mu(\alpha,\beta) = \mathsf{T}(\gamma^\mu\alpha,\beta) = \mathsf{T}(\alpha,\gamma^\mu\beta).$$

In particular, we see that

$$\mathsf{T}(\gamma^{\mu\nu}\alpha,\beta) = -\mathsf{T}(\alpha,\gamma^{\mu\nu}\beta).$$

Hence, the pairing $\mathsf{T}(\alpha,\beta)$ is invariant under the action of the Lie group $\mathrm{Spin}(9,1)$, and, in particular, $\mathbb{S}_- \cong (\mathbb{S}_+)^\vee$ as a representation of $\mathrm{Spin}(9,1)$.

To define the superparticle, we adjoin to the particle a series of fields θ_n, $n \geq 0$, of ghost number n, which are left-handed Majorana–Weyl spinors if n is even, and right-handed Majorana–Weyl spinors if n is odd: the parity of θ_n is the opposite of the parity of n. As functions on the worldline, these fields all transform as scalars.

For the correct definition of the superparticle, it is necessary to exclude the states of vanishing momentum. To this end, we let M_0 be the complement in $M = T^\vee V$ of the zero-section. Denote by $j : M_0 \to M$ the open embedding, and by $\mathcal{O}_0 = j^*\mathcal{O}$ the structure sheaf of M_0. Denote by $\mathbb{A}$ the algebra generated over $\mathcal{A}_0 = j^*\mathcal{A}$ by the variables

$$\{\partial^\ell\theta_n, \partial^\ell\theta_n^+ \mid n \geq 0\}_{\ell\geq 0}.$$

We denote the completion of $\mathbb{A}$ with respect to antifields by $\widehat{\mathbb{A}}$.

We see that $\mathbb{A}$ is a sheaf of graded commutative algebras over M_0. Let

$$C_0 = C \cap M_0$$

be the intersection of the light-cone with open submanifold M_0 of the cotangent bundle.

Consider the composite fields

$$\Psi_n = \begin{cases} (-1)^{\binom{n+1}{2}}\,\theta^+_{-n-1}, & n < -1, \\ \theta^+_0 + \tfrac{1}{2}x^+_\mu\gamma^\mu\theta_0 + 2c^+\theta_1, & n = -1, \\ \partial\theta_n + (-1)^{n+1}x^+_\mu\gamma^\mu\theta_{n+1} + 2c^+\theta_{n+2}, & n \geq 0. \end{cases} \tag{4.1}$$

Denote by S the full Batalin–Vilkovisky action of the superparticle, and by s the associated Batalin–Vilkovisky differential. The formula for S may be found in Lindström et al. [11]: we content ourselves here with the following characterization, in terms of the differential s. Let S be the Batalin–Vilkovisky action of the particle

$$S = p_\mu\partial x^\mu - \tfrac{1}{2}e\eta^{\mu\nu}p_\mu p_\nu + \big(\partial e^+ - \eta^{\mu\nu}x^+_\mu p_\nu\big)c,$$

Proposition 4.1. *The Batalin–Vilkovisky action S of the superparticle is characterized by the following conditions:*

i) S satisfies the classical master equation;

ii) $\mathsf{S} - S$ depends only on the fields and antifields $\{p_\mu, \theta_n, x^+_\mu, e^+, c^+, \theta^+_n\}$ and their derivatives;

iii) for all $n \in \mathbb{Z}$, the differential s acts on the composite fields Ψ_n as follows:

$$\mathsf{s}\Psi_n = (-1)^{n+1}p_\mu\gamma^\mu\Psi_{n+1} - 2e^+\Psi_{n+2}.$$

Using the vanishing result Theorem 3.2, we now establish the analogous result for the superparticle.

Theorem 4.2. *The sheaf $H^i(\widehat{\mathbb{A}}, \mathsf{s})$ vanishes for $i < 0$, and is concentrated on the light-cone C_0.*

Let $\widetilde{\mathbb{A}}$ be the quotient of $\widehat{\mathbb{A}}$ by constant multiples of the identity. The sheaf $H^i(\widetilde{\mathbb{A}}, \mathsf{s})$ also vanishes for $i < 0$, and is concentrated on the light-cone C_0

In the proof of Theorem 4.2, we need the formula for the differential s on fields and antifields of the theory. We see that s equals s on the fields and antifields $\{p_\mu, x^+_\mu, e^+, c^+\}$, and

$$\begin{aligned}
\mathsf{s}\theta_n &= (-1)^{n+1} p_\mu \gamma^\mu \theta_{n+1} - 2e^+\theta_{n+2} \\
\mathsf{s}x^\mu &= -\eta^{\mu\nu} c p_\nu + \tfrac{1}{2} p_\nu \mathsf{T}(\gamma^\nu\gamma^\mu\theta_0, \theta_1) + e^+\mathsf{T}^\mu(\theta_1,\theta_1) - e^+\mathsf{T}^\mu(\theta_0,\theta_2) \\
\mathsf{s}c &= -p_\mu \mathsf{T}^\mu(\theta_1,\theta_1) - 4e^+\mathsf{T}(\theta_1,\theta_2) \\
\mathsf{s}e &= -\partial c + x^+_\mu \mathsf{T}^\mu(\theta_1,\theta_1) - 4c^+\mathsf{T}(\theta_1,\theta_2) + 2\sum_{n=0}^\infty (-1)^{\binom{n}{2}} \mathsf{T}(\Psi_{-n}, \theta_{n+1}) \\
\mathsf{s}p^{+\mu} &= \partial x^\mu - \eta^{\mu\nu} e p_\nu + \tfrac{1}{2} x^+_\nu \mathsf{T}(\gamma^\nu\gamma^\mu\theta_0,\theta_1) - c^+\mathsf{T}^\mu(\theta_1,\theta_1) + c^+\mathsf{T}^\mu(\theta_0,\theta_2) \\
&\quad + \tfrac{1}{2}\mathsf{T}^\mu(\Psi_0,\theta_0) + \sum_{n=1}^\infty (-1)^{\binom{n}{2}} \mathsf{T}^\mu(\Psi_{-n},\theta_n).
\end{aligned}$$

The infinite sums in the formulas for $\mathsf{s}e$ and $\mathsf{s}p^{+\mu}$ make sense by the completeness property of $\widehat{\mathbb{A}}$.

Proof of Theorem 4.2. We define an auxilliary grading on $\mathbb{A}$ extending the grading on $\mathcal{A}_0$ used in the proof of Theorem 3.2: the generators of $\mathbb{A}$ over $\mathcal{A}_0$ are assigned the degrees in the following table:

Φ	θ_n	$\partial^\ell\Psi_n$	$\partial^\ell\Psi_{-n}$
$\deg(\Phi)$	$3n+1$	$3n$	$-2n$

Observe that

$$\deg(f) \leq 4\,\mathrm{gh}_+(f) + 16;$$

the factor 4 accounts for the field θ_1, which has ghost number 1 and degree 4, while the constant 16 accounts for the 16 modes of the fermionic field θ_0, which have ghost number 0 and degree 1. In the other direction, we have

$$\mathrm{gh}_+(f) + 2\,\mathrm{gh}(f) \leq \deg(f).$$

Combining these two inequalities, we see that

$$\tfrac{1}{4}\deg(f) - \mathrm{gh}(f) - 4 \leq \mathrm{gh}_-(f) \leq \deg(f) - 3\,\mathrm{gh}(f). \tag{4.2}$$

From this grading, we construct an exhaustive and Hausdorff descending filtration on $\mathbb{A}$: $G^k\mathbb{A}^i$ is the span of elements $f \in \mathbb{A}^i$ such that $\deg(f) \geq k$. By (4.2), the completion of this filtration is isomorphic to $\widehat{\mathbb{A}}$.

The differential s_0 on the zeroth page of the spectral sequence E_0^{pq} equals

$$\mathsf{s}_0 = \operatorname{pr}\left(\partial x^\mu \frac{\partial}{\partial p^{+\mu}} - \partial p_\mu \frac{\partial}{\partial x^+_\mu} - \partial e^+ \frac{\partial}{\partial c^+}\right).$$

This is a Koszul differential and its cohomology E_1 is the graded commutative algebra generated over $\mathcal{O}_0$ by the variables

$$\{\partial^\ell e, e^{-1}, \partial^\ell c, e^+\}_{\ell\ge 0} \cup \{\theta_n \mid n \ge 0\} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}\}_{\ell \ge 0}.$$

The differential s_1 on the first page E_1 of the spectral sequence is given by the formula

$$\mathsf{s}_1 = \operatorname{pr}\left(-\tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+}\right).$$

The element $\eta^{\mu\nu}p_\mu p_\nu$ is not a zero divisor in E_1: its zero-locus in M_0 is the light-cone C_0, with structure sheaf $\mathcal{O}_{C_0}$.

We see that the second page E_2 of the spectral sequence is a graded commutative algebra generated over $\mathcal{O}_{C_0}$ by the variables

$$\{\partial^\ell e, e^{-1}, \partial^\ell c\}_{\ell\ge 0} \cup \{\theta_n \mid n \ge 0\} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}\}_{\ell \ge 0}.$$

The differential s_2 on the second page E_2 of the spectral sequence is given by the formula

$$\mathsf{s}_2 = \sum_{n=1}^\infty (-1)^{n+1} \operatorname{pr}\left(p_\mu \mathsf{T}^\mu\left(\Psi_{1-n}, \frac{\partial}{\partial \Psi_{-n}}\right)\right).$$

On the light-cone C_0, the operator

$$p_\mu \gamma^\mu : \mathbb{S}_\pm \to \mathbb{S}_\mp$$

has square zero, since $(p_\mu\gamma^\mu)^2 = \eta^{\mu\nu}p_\mu p_\nu = 0$. The cohomology of this operator vanishes, in the sense that

$$\ker(p_\mu\gamma^\mu) = \operatorname{im}(p_\mu\gamma^\mu).$$

To see this, choose a vector q_μ such that $\eta^{\mu\nu}p_\mu q_\nu > 0$: then $q_\mu\gamma^\mu$ yields a contracting homotopy for the differential $p_\mu\gamma^\mu$. (This is where in the proof we need to have localized away from the zero section of M.)

The third page E_3^{pq} is generated over $\mathcal{O}_{C_0}$ by the variables

$$\begin{aligned} &\{\partial^\ell e, e^{-1}\}_{\ell\geq 0} \in E_3^{00}, \quad \{\theta_n \mid n \geq 0\} \in E_3^{3n+1,-2n-1}, \\ &\{\partial^\ell c\}_{\ell\geq 0} \in E_3^{3,-2}, \quad \{\partial^\ell \Psi_n \mid n \geq 0\}_{\ell\geq 0} \in E_3^{3n,-2n}, \end{aligned}$$

modulo relations

$$\{\partial^\ell(p_\mu \gamma^\mu \Psi_0)\}_{\ell\geq 0} \in E_3^{00}.$$

Thus, E_r^{pq} vanishes unless $p \geq 0$, $p+q \geq 0$, and $3p+4q \geq -16$; this last inequality is saturated by the product of the 16 components of the field θ_0, located in $E_0^{16,-16}$. The differential s_r of the rth page of the spectral sequence vanishes for $r > 20$, and hence the spectral sequence degenerates, proving the first part of the theorem.

The proof of the vanishing of the cohomology sheaves $H^i(\widetilde{\mathbb{A}}, \mathsf{s})$ follows the same lines as the proof of the analogous result for the particle. □

Corollary 4.3. *Let* $\mathbb{F} = \widehat{\mathbb{A}}/\partial\widehat{\mathbb{A}}$. *The sheaf* $H^i(\mathbb{F}, \mathsf{s})$ *vanishes for* $i < -1$, *and is concentrated on the light-cone* C_0.

5. The Thom–Whitney normalization

Let X be a manifold with cover

$$\mathcal{U} = \{U_\alpha\}_{\alpha\in I}.$$

The nerve $N_k\mathcal{U}$ of the cover is the sequence of manifolds indexed by $k \geq 0$

$$N_k\mathcal{U} = \bigsqcup_{\alpha_0\cdots\alpha_k\in I^{k+1}} U_{\alpha_0\cdots\alpha_k},$$

where

$$U_{\alpha_0\cdots\alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}.$$

Denote by $\epsilon : N_0\mathcal{U} \to X$ the map which on each summand U_α equals the inclusion $U \hookrightarrow X$.

In order to globalize the classical master equation, we have to replace the manifold X by a sequence of manifolds of the form $\{N_k\mathcal{U}\}$. To do this, we will use the formalism of simplicial and cosimplicial objects, and we now review their definition.

Let Δ be the category whose objects are the totally ordered sets

$$[k] = (0 < \cdots < k), \quad k \in \mathbb{N},$$

and whose morphisms are the order-preserving functions. A simplicial manifold $M_\bullet$ is a contravariant functor from Δ to the category of manifolds. (We leave open here whether we are working in the smooth, analytic or algebraic setting.) Here, M_k is the value of $M_\bullet$ at the object $[k]$, and $f^* : M_\ell \to M_k$ is the action of the arrow $f : [k] \to [\ell]$ of Δ. The arrow $d_i : [k] \to [k+1]$ which takes $j < i$ to j and $j \geq i$ to $j+1$ is known as a face map, while the arrow $s_i : [k] \to [k-1]$ which takes $j \leq i$ to j and $j > i$ to $j-1$ is known as a degeneracy map.

The simplicial manifolds used in this paper are the Čech nerves $N_\bullet\mathcal{U}$ of covers $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$. The face map $\delta_i = d_i^* : N_{k+1}\mathcal{U} \to N_k\mathcal{U}$ corresponds to the inclusion of the open subspace

$$U_{\alpha_0\cdots\alpha_{k+1}} \subset N_{k+1}\mathcal{U}$$

into the open subspace

$$U_{\alpha_0\cdots\widehat{\alpha}_i\cdots\alpha_{k+1}} \subset N_k\mathcal{U},$$

and the degeneracy map $\sigma_i = s_i^* : N_{k-1}U \to N_kU$ corresponds to the identification of the open subspace

$$U_{\alpha_0\cdots\alpha_k} \subset N_k\mathcal{U}$$

with the open subspace

$$U_{\alpha_0\cdots\alpha_i\alpha_i\cdots\alpha_{k+1}} \subset N_{k+1}\mathcal{U}.$$

Any simplicial map $f^* : M_\ell \to M_k$ is the composition of a sequence of face maps followed by a sequence of degeneracy maps. In particular, we see that in the case $M_\bullet = N_\bullet\mathcal{U}$ of the nerve of a cover, all of these maps are local embeddings.

A covariant functor $X^\bullet$ from Δ to a category $\mathcal{C}$ is called a cosimplicial object of $\mathcal{C}$. These arise as the result of applying a contravariant functor to a simplicial space: for example, applying the functor $\mathcal{F}(-)$ to the simplicial graded supermanifold $N_\bullet\mathcal{U}$, we obtain the cosimplicial graded Lie superalgebra

$$\mathcal{F}(N_\bullet\mathcal{U})$$

with the Batalin–Vilkovisky antibracket.

We now generalize the classical master equation of Batalin–Vilkovisky theory to a Maurer–Cartan equation for the cosimplicial graded Lie superalgebra $\mathcal{F}(N_\bullet\mathcal{U})$. We use a construction introduced in rational homotopy theory by Sullivan [12] (see also Bousfield and Guggenheim [5]), the Thom–Whitney normalization.

Let Ω_k be the free graded commutative algebra with generators $\{t_i\}_{i=0}^k$ of degree 0 and $\{dt_i\}_{i=0}^k$ of degree 1, and relations

$$t_0 + \cdots + t_k = 1$$

and $dt_0 + \cdots + dt_k = 0$. There is a unique differential δ on Ω_k such that $\delta(t_i) = dt_i$, and $\delta(dt_i) = 0$.

The differential graded commutative algebras Ω_k are the components of a simplicial differential graded commutative algebra $\Omega_\bullet$ (that is, contravariant functor from Δ to the category of differential graded commutative algebras): the arrow $f : [k] \to [\ell]$ in Δ acts by the formula

$$f^* t_i = \sum_{f(j)=i} t_j, \quad 0 \le i \le n.$$

The Thom–Whitney normalization $\Omega_\bullet \otimes_\Delta V^\bullet$ of a cosimplicial superspace is the equalizer of the maps

$$\prod_{k=0}^{\infty} \Omega_k \otimes V^k \overset{f^*\otimes 1}{\underset{1\otimes f_*}{\rightrightarrows}} \prod_{k,\ell=0}^{\infty} \prod_{f:[k]\to[\ell]} \Omega_k \otimes V^\ell$$

If the superspaces V^k making up the cosimplicial superspace are themselves graded V^{k*}, the Thom–Whitney totalization of $V^{\bullet *}$ is the product superspace

$$\|V\|^n = \prod_{k=0}^{\infty} \Omega_\bullet^k \otimes_\Delta V^{\bullet, n-k}.$$

The Thom–Whitney normalization takes cosimplicial 1-shifted graded Lie superalgebras to 1-shifted graded Lie superalgebras. The reason is simple: if L^k is a 1-shifted graded Lie superalgebra, then so is $\Omega_k \otimes L^k$, with differential δ and antibracket

$$[\alpha_1 \otimes x_1, \alpha_2 \otimes x_2] = (-1)^{j_2\, p(x_1)+1}\, \alpha_1\alpha_2\, [x_1, x_2],$$

where $\alpha_\ell \in \Omega_k^{i_\ell}$ and $x_\ell \in L^{k,j_\ell}$. The Thom–Whitney totalization $\|L\|$ is a subspace of the product of 1-shifted graded superalgebras $\Omega_k \otimes L^k$, and this subspace is preserved by the differential and by the antibracket.

The construction of $\|\mathcal{F}(N_\bullet\mathcal{U})\|$ behaves well under refinement of covers. A refinement $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ of a cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is determined by a function of indexing sets $\varphi : J \to I$, such that for all $\beta \in J$, V_β is a subset of $U_{\varphi(\beta)}$. There is a morphism of cosimplicial 1-shifted graded Lie superalgebras $\Phi^* : \mathcal{F}(N_\bullet\mathcal{U}) \to \mathcal{F}(N_\bullet\mathcal{V})$, obtained by sections of $\mathcal{F}$ on $U_{\varphi(\alpha_0)\cdots\varphi(\alpha_k)}$ to sections on $V_{\alpha_0\cdots\alpha_k}$. Applying the totalization functor, we obtain a morphism of complexes

$$\Phi^* : \|\mathcal{F}(N_\bullet\mathcal{U})\| \to \|\mathcal{F}(N_\bullet\mathcal{V})\|.$$

If we have a further refinement $\mathcal{W} = \{W_\gamma\}_{\gamma\in K}$ of $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ with $\psi : K \to J$, we may define a composition of these refinements $\varphi\psi : K \to I$, and we obtain a commuting triangle of morphisms of complexes

$$\begin{tikzcd} & \|\mathcal{F}(N_\bullet\mathcal{U})\| \arrow[dl, "\Phi^*"'] \arrow[dr, "\Psi^*\Phi^*"] & \\ \|\mathcal{F}(N_\bullet\mathcal{V})\| \arrow[rr, "\Psi^*"'] & & \|\mathcal{F}(N_\bullet\mathcal{W})\| \end{tikzcd}$$

The arrows in this diagram are morphisms of differential graded 1-shifted Lie superalgebras.

The analogue of the classical master equation (2.1) in the global setting is the Maurer–Cartan equation for the differential graded 1-shifted Lie superalgebra $\|\mathcal{F}(N_\bullet\mathcal{U})\|$:

$$\delta{\textstyle\int}\mathsf{S} + \tfrac{1}{2}({\textstyle\int}\mathsf{S}, {\textstyle\int}\mathsf{S}) = 0.$$

Here, S is a consistent collection of elements $\mathsf{S}^j_{\alpha_0\cdots\alpha_k} \in \Omega_k^j \otimes \mathcal{F}^{-j}(U_{\alpha_0\cdots\alpha_k})$ of total degree 0 which satisfies the sequence of Maurer–Cartan equations

$$\delta{\textstyle\int}\mathsf{S}^{j-1} + \frac{1}{2}\sum_{i=0}^{j}({\textstyle\int}\mathsf{S}^i, {\textstyle\int}\mathsf{S}^{j-i}) = 0.$$

6. The superparticle as a covariant field theory

We now return to the superparticle. In this section, using the Thom–Whitney formalism of the previous section, we will show that the superparticle is a global covariant field theory, in the terminology of [10]. We also find explicit formulas in the Thom–Whitney complex for the cohomology of the Batalin–Vilkovisky differential.

Let $\mathsf{D} \in \Gamma(M_0, \widehat{\mathbb{A}}^{-1})$ be the element

$$\mathsf{D} = x^+_\mu \partial x^\mu + p^{+\mu} \partial p_\mu - e\partial e^+ + c^+ \partial c + \sum_{n=0}^{\infty} \mathsf{T}(\theta_n^+, \partial\theta_n).$$

Definition 6.1. A global covariant field theory is a solution of the curved Maurer–Cartan equation in $\|\mathbb{F}(N_\bullet\mathcal{U})[u]\|$, where $\mathcal{U}$ is a cover of M_0:

$$\delta{\textstyle\int}\mathsf{S}_u + \tfrac{1}{2}({\textstyle\int}\mathsf{S}_u, {\textstyle\int}\mathsf{S}_u) = -u{\textstyle\int}\mathsf{D}.$$

If S_u is a covariant field theory with respect to a cover $\mathcal{U}$ of M_0 and $(\mathcal{V}, \varphi)$ is a refinement of $\mathcal{U}$, $\Phi^*\mathsf{S}_u$ is again a global covariant field theory with respect to the refined cover.

Theorem 6.2. *There is a global covariant field theory*

$$\mathsf{S}_u = \mathsf{S} + \sum_{n=0}^{\infty} u^{n+1}\mathsf{G}_n$$

such that S *is the solution of the classical master equation for the superparticle.*

Proof. Consider the open affine cover $\mathcal{U} = \{U_\mu\}_{0\le\mu\le 9}$ of M_0, where

$$U_\mu = \{p_\mu \neq 0\}.$$

We must construct a series of cochains

$${\textstyle\int}\mathsf{G}_n \in \|\mathbb{F}(N_\bullet\mathcal{U})\|^{-2n-2},$$

in the Thom–Whitney totalization $\|\mathbb{F}(N_\bullet\mathcal{U})\|$ of the cosimplicial graded Lie superalgebra $\mathbb{F}(N_\bullet\mathcal{U})$, satisfying the curved Maurer–Cartan equation

$$\delta{\textstyle\int}\mathsf{S}_u + \tfrac{1}{2}({\textstyle\int}\mathsf{S}_u, {\textstyle\int}\mathsf{S}_u) = -u{\textstyle\int}\mathsf{D}.$$

Equivalently, we must find a solution G_0 of the equation

$$(\delta + \mathsf{s}){\textstyle\int}\mathsf{G}_0 = -{\textstyle\int}\mathsf{D}, \tag{6.1}$$

and for $n > 0$, solutions of the equations

$$(\delta + \mathsf{s}) \textstyle\int \mathsf{G}_n = -\frac{1}{2} \sum_{j+k=n-1} (\int \mathsf{G}_j, \int \mathsf{G}_k). \tag{6.2}$$

Assuming that we have solved these equations for $(\mathsf{G}_0, \cdots, \mathsf{G}_{n-1})$, we see that

$$\begin{aligned} &\frac{1}{2} \sum_{j+k=n-1} (\delta + \mathsf{s})(\textstyle\int \mathsf{G}_j, \int \mathsf{G}_k) \\ &= -(\textstyle\int \mathsf{D}, \int \mathsf{G}_{n-1}) - \sum_{i+j+k=n-2} \big((\int \mathsf{G}_i, \int \mathsf{G}_j), \int \mathsf{G}_k\big). \end{aligned}$$

The first term vanishes since $\int \mathsf{D}$ lies in the centre of $\mathbb{F}$, while the second term vanishes by the Jacobi relation for graded Lie superalgebras. Thus, the right-hand side of (6.2) is a cocycle. Since the cohomology of the complex $\|\mathbb{F}(N_\bullet \mathcal{U})\|$ vanishes below degree -1 by Theorem 4.2, we may solve the equation for G_n.

Rewrite the formula for D, using the definition (4.1) of Ψ_n and the formula for the action of s:

$$\mathsf{D} = -\mathsf{s}(x^+_\mu p^{+\mu} + ec^+) + \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}(\Psi_{-n}, \Psi_{n-1}). \tag{6.3}$$

Introduce the variable

$$q_\alpha = \frac{t_\alpha}{2\eta^{\alpha\mu} p_\mu},$$

and its de Rham differential δq_α. We will show that the expression

$$\begin{aligned} \mathsf{G}_0 = x^+_\mu p^{+\mu} + ec^+ + \frac{1}{2} \sum_{k\geq 0} \sum_{\alpha_0 \cdots \alpha_k} (-1)^k \, q_{\alpha_0} \delta q_{\alpha_1} \cdots \delta q_{\alpha_k} \\ \times \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \alpha_k} (\Psi_{-n}, \Psi_{n-k-2}) \end{aligned} \tag{6.4}$$

in $\|\mathbb{A}(N_\bullet \mathcal{U})\|^{-2}$ gives a solution of the equation

$$(\delta + \mathsf{s})\mathsf{G}_0 = -\mathsf{D}, \tag{6.5}$$

yielding (6.1). (In the definition of G_0, we understand the Einstein summation convention for the index μ, but not α_i.) By (6.3), it suffices to show that

$$(6.6)\quad \mathsf{s} \sum_{\alpha_0 \cdots \alpha_k} q_{\alpha_0} \delta q_{\alpha_1} \cdots \delta q_{\alpha_k} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \alpha_k}(\Psi_{-n}, \Psi_{n-k-2})$$
$$= \begin{cases} \displaystyle\sum_{\alpha_0 \cdots \alpha_{k-1}} \delta q_{\alpha_0} \cdots \delta q_{\alpha_{k-1}} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \alpha_{k-1}}(\Psi_{-n}, \Psi_{n-k-1}), & k > 0, \\ \displaystyle -\sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}(\Psi_{-n}, \Psi_{n-1}), & k = 0. \end{cases}$$

We have

$$\begin{aligned} &\mathsf{s} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \alpha_k}(\Psi_{-n}, \Psi_{n-k-2}) \\ &= \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \alpha_k}((-1)^{n+1} p_\mu \gamma^\mu \Psi_{-n+1} - 2e^+ \Psi_{-n+2}, \Psi_{n-k-2}) \\ &\quad + \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}+n+1} \mathsf{T}^{\alpha_0 \cdots \alpha_k}(\Psi_{-n}, (-1)^{n+k+1} p_\mu \gamma^\mu \Psi_{n-k-1} - 2e^+ \Psi_{n-k}) \\ &= p_\mu \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \Big(\mathsf{T}^{\alpha_0 \cdots \alpha_k}(\gamma^\mu \Psi_{-n}, \Psi_{n-k-1}) \\ &\qquad\qquad + (-1)^k \mathsf{T}^{\alpha_0 \cdots \alpha_k}(\Psi_{-n}, \gamma^\mu \Psi_{n-k-1}) \Big) \\ &\quad - 2e^+ \sum_{n=-\infty}^{\infty} \Big((-1)^{\binom{n+2}{2}} + (-1)^{\binom{n}{2}} \Big) \mathsf{T}^{\alpha_0 \cdots \alpha_k}(\Psi_{-n}, \Psi_{n-k}). \end{aligned}$$

The sum on the last line vanishes, since $(-1)^{\binom{n+2}{2}} = -(-1)^{\binom{n}{2}}$. We conclude that

$$\begin{aligned} &\mathsf{s} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \alpha_k}(\Psi_{-n}, \Psi_{n-k-2}) \\ &= 2 \sum_{j=0}^{k} (-1)^{k-j} \eta^{\mu \alpha_j} p_\mu \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}^{\alpha_0 \cdots \widehat{\nu}_j \cdots \alpha_k}(\Psi_{-n}, \Psi_{n-k-1}), \end{aligned}$$

from which (6.6) follows. □

Corollary 6.3. *The long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H^{-1}(\widehat{\mathbb{A}},\mathsf{s}) \arrow[r] & H^{-1}(\mathbb{F},\mathsf{s}) \arrow[r] & H^{0}(\widetilde{\mathbb{A}},\mathsf{s}) \arrow[dll, "\partial"] & \\
& H^{0}(\widehat{\mathbb{A}},\mathsf{s}) \arrow[r, "\partial"] & H^{0}(\mathbb{F},\mathsf{s}) \arrow[r] & H^{1}(\widetilde{\mathbb{A}},\mathsf{s}) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

splits, in the sense that the morphisms ∂ vanish.

By an extension of this method, we may show that the space of solutions of (1.1) is a contractible simplicial set. This amounts to showing that for each $n > 0$, any solution of (1.1) in $\Omega(\partial\Delta^n) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})[[u]]\|$ may be extended to a solution of (1.1) in

$$\Omega(\Delta^n) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})[[u]]\| = \Omega_n \otimes \|\mathbb{F}(N_\bullet\mathcal{U})[[u]]\|.$$

In particular, the case $n = 1$ shows that there is a solution of (1.1) in $\Omega_1 \otimes \|\mathbb{F}(N_\bullet\mathcal{U})[[u]]\|$ interpolating between any pair of solutions of (1.1) in $\|\mathbb{F}(N_\bullet\mathcal{U})[[u]]\|$.

The cover $\mathcal{U}$ of M_0 may be used to give explicit formulas for cohomology classes in the hypercohomology of the complexes of sheaves $\mathbb{F}$ and $\widehat{\mathbb{A}}$. The 1-cochain in the Thom-Whitney complex of $\widehat{\mathbb{A}}$

$$\mathsf{c} = c - \sum_{\alpha=0}^{9} q_\alpha \big(p_\mu \mathsf{T}(\gamma^\mu\gamma^\alpha\theta_0,\theta_1) + 2e^+\mathsf{T}^\alpha(\theta_1,\theta_1) - 2e^+\mathsf{T}^\alpha(\theta_0,\theta_2)\big)$$

is a cocycle, and the 0-cochain

$$\mathsf{x}^\mu = x^\mu - \tfrac{1}{2}\sum_{\alpha=0}^{9} q_\alpha \big(p_\nu \mathsf{T}(\gamma^\alpha\gamma^\nu\gamma^\mu\theta_0,\theta_0) - 4e^+\mathsf{T}^{\alpha\mu}(\theta_0,\theta_1)\big)$$

satisfies the formula $(\delta+\mathsf{s})\mathsf{x}^\mu = -\eta^{\mu\nu}\mathsf{c}p_\nu$, analogous to the formula $sx^\mu = -\eta^{\mu\nu}cp_\nu$ for the particle. (In the definitions of c and x^μ, we understand the Einstein summation convention for the indices μ and ν, but not α.) We see that if $f(x,p,\theta)$ is a function of x^μ, p_μ and $\theta \in \mathbb{S}_-$, then

$$(\delta+\mathsf{s})f(\mathsf{x},p,\uptheta) = -\mathsf{c}p_\mu \frac{\partial f}{\partial x^\mu}(\mathsf{x},p,\uptheta),$$

and $(\delta+\mathsf{s})\mathsf{c}f(\mathsf{x},p,\uptheta) = 0$, where $\uptheta = p_\mu\gamma^\mu\theta - 2e^+\theta_1$. This may be compared to those of Section 2 of Bergshoeff et al. [2]; the formulas presented there only apply outside the hypersurface $p_0 - p_9 = 0$.

7. Supersymmetry and Lorentz invariance of the solution

The reason for the interest of the superparticle, and of the Green–Schwarz superstring for which it is a toy model, is that it is manifestly supersymmetric. The supersymmetry is generated by the functional $\int Q$, where

$$Q = \theta_0^+ - \tfrac{1}{2} x_\mu^+ \gamma^\mu \theta_0 \in \mathbb{S}_- \otimes \mathbb{A}^{-1}.$$

The formula $\mathsf{s}Q = \partial\big(p_\mu \gamma^\mu \theta_0 + 2e^+\theta_1\big)$ implies the vanishing of the Batalin–Vilkovisky antibracket

$$(\textstyle\int Q, \int \mathsf{S}) = 0. \tag{7.1}$$

Let $\mathbb{A}_\star$ be the subalgebra of the sheaf $\mathbb{A}$ generated by the fields

$$\{\partial^\ell p_\mu, \partial^\ell x_\mu^+, \partial^\ell e^+, \partial^\ell c^+\}_{\ell \geq 0} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}\}_{\ell \geq 0}.$$

Let $\widehat{\mathbb{A}}_\star \subset \widehat{\mathbb{A}}$ be its associated completion, with respect to the fields of negative degree

$$\{\partial^\ell x_\mu^+, \partial^\ell e^+, \partial^\ell c^+\}_{\ell \geq 0} \cup \{\partial^\ell \Psi_n \mid n < 0\}_{\ell \geq 0}.$$

Both $\mathbb{A}_\star$ and $\widehat{\mathbb{A}}_\star$ may be viewed as sheaves over the fibre $M_\star$ of M_0 over the point $x^\mu = 0$.

Lemma 7.1. *The subsheaf $\widehat{\mathbb{A}}_\star \subset \widehat{\mathbb{A}}$ satisfies $\partial[\widehat{\mathbb{A}}_\star] \subset \widehat{\mathbb{A}}_\star$ and is closed under the Soloviev bracket.*

Proof. It follows directly from its definition that $\widehat{\mathbb{A}}_\star$ is preserved by the action of ∂. In order for $\widehat{\mathbb{A}}_\star$ to be closed under the Soloviev bracket, it suffices to observe that for all fields Φ that generate $\mathbb{A}_\star$, we have

$$\frac{\partial \Phi}{\partial(\partial^k x^\mu)} = \frac{\partial \Phi}{\partial(\partial^k p^{+\mu})} = \frac{\partial \Phi}{\partial(\partial^k e^+)} = \frac{\partial \Phi}{\partial(\partial^k c)} = 0.$$

This implies that

$$\begin{aligned}(\!(f,g)\!) = \sum_{n=0}^\infty (-1)^{(n+1)(p(f)+1)} \sum_{k,\ell=0}^\infty \Bigg(\partial^\ell \left(\frac{\partial f}{\partial(\partial^k \theta_n)} \right) \partial^k \left(\frac{\partial g}{\partial(\partial^\ell \theta_n^+)} \right) \\ + (-1)^{p(f)} \partial^\ell \left(\frac{\partial f}{\partial(\partial^k \theta_n^+)} \right) \partial^k \left(\frac{\partial g}{\partial(\partial^\ell \theta_n)} \right) \Bigg).\end{aligned}$$

It only remains to observe that for all $m \in \mathbb{Z}$ and $n \geq 0$, and all $k, \ell \geq 0$, the partial derivatives $\partial(\partial^\ell \Psi_m)/\partial(\partial^k \theta_n)$ and $\partial(\partial^\ell \Psi_m)/\partial(\partial^k \theta_n^+)$ are in $\widehat{\mathbb{A}}_\star$. $\square$

We now have the following analogue of Theorem 4.3. The proof follows the same lines, but is actually somewhat simpler.

Lemma 7.2. *Let $\mathbb{F}_\star = \widehat{\mathbb{A}}_\star/\partial\widehat{\mathbb{A}}_\star$. The cohomology sheaf $H^i(\mathbb{F}_\star, \mathsf{s})$ vanishes unless $i \in \{-1, 0\}$.*

Proof. The sheaf $\widehat{\mathbb{A}}_\star$ is an algebra over the momentum space $M_\star$, whose structure sheaf is the algebra of rational functions in the variables $\{p_\mu\}$. The filtration of $\widehat{\mathbb{A}}$ induces a filtration of $\widehat{\mathbb{A}}_\star$, and the differential s_0 on the zeroth page of the associated spectral sequence E_0^{pq} equals

$$\mathsf{s}_0 = -\operatorname{pr}\left(\partial p_\mu \frac{\partial}{\partial x_\mu^+} + \partial e^+ \frac{\partial}{\partial c^+}\right).$$

This is a Koszul differential and its cohomology E_1 is the graded commutative algebra freely generated over the structure sheaf $\mathcal{O}_{M_\star}$ by the variables

$$\{e^+\} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}\}_{\ell \geq 0}.$$

The differential s_1 on the first page E_1 of the spectral sequence is given by the formula

$$\mathsf{s}_1 = \operatorname{pr}\left(-\tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+}\right).$$

The element $\eta^{\mu\nu} p_\mu p_\nu$ is not a zero divisor in E_1: its zero-locus is the light-cone

$$C_\star = \{p_\mu \neq 0 \mid \eta^{\mu\nu} p_\mu p_\nu = 0\}.$$

We see that the second page E_2 of the spectral sequence is a sheaf of graded commutative algebras generated over $\mathcal{O}_{C_\star}$ by the variables

$$\{\partial^\ell \Psi_n \mid n \in \mathbb{Z}\}_{\ell \geq 0}.$$

The differential s_2 on the second page E_2 of the spectral sequence is given by the formula

$$\mathsf{s}_2 = \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{pr}\left(p_\mu \mathsf{T}^\mu \left(\Psi_{1-n}, \frac{\partial}{\partial \Psi_{-n}}\right)\right).$$

On the light-cone, the operator

$$p_\mu \gamma^\mu : \mathbb{S}_\pm \to \mathbb{S}_\mp$$

has vanishing cohomology. We see that the third page E_3^{p+q} is generated over $\mathcal{O}_{C_\star}$ by the variables

$$\{\partial^\ell \Psi_n \mid n \geq 0\}_{\ell \geq 0} \in E_3^{3n,-2n},$$

modulo relations $\{p_\mu \gamma^\mu \partial^\ell \Psi_0\}_{\ell \geq 0} \in E_3^{00}$ and hence the differential s_r of the rth page of the spectral sequence vanishes for $r > 3$.

The remainder of the proof follows the proof of Theorem 4.3. □

Theorem 7.3. *There is a choice of the solution* S_u *to the equation* (1.1) *such that*

$$(\textstyle\int Q, \int \mathsf{S}_u) = 0.$$

Proof. Let q be the Hamiltonian vector field associated to $\int Q$. It is easily seen that $\mathsf{q}\Psi_n = 0$, and hence that q annihilates $\widehat{\mathbb{A}}_\star$. It is easily seen that $\mathsf{q}\mathsf{G}_0 = 0$. We prove the theorem by showing that for all $n > 0$, G_n may be chosen in $\|\mathbb{F}_\star(N_\bullet \mathcal{U})\|^{-2n-2}$. In view of Lemma 7.2, it suffices to show that the cocycle

$$-\frac{1}{2} \sum_{j+k=n-1} (\textstyle\int \mathsf{G}_j, \int \mathsf{G}_k)$$

lies in $\|\mathbb{F}_\star(N_\bullet \mathcal{U})\|^{-2n-1}$ for $n > 0$. By induction, we may assume that this holds for all of the terms of this sum with $j, k > 0$. It remains to check that

$$(\textstyle\int \mathsf{G}_0, \int \mathsf{G}_{n-1}) \in \|\mathbb{F}_\star(N_\bullet \mathcal{U})\|^{-2n-1}.$$

But modulo $\widehat{\mathbb{A}}_\star$, $\mathsf{G}_0 = x^+_\mu p^{+\mu} + ec^+$, and it is easily seen that

$$(\!(x^+_\mu p^{+\mu} + ec^+, \widehat{\mathbb{A}}_\star)\!) \subset \widehat{\mathbb{A}}_\star.$$

Indeed, on restriction to $\widehat{\mathbb{A}}_\star$, the Soloviev bracket $\mathrm{ad}(x^+_\mu p^{+\mu} + ec^+)$ is given by the evolutionary vector field

$$\mathrm{pr}\left(-x^+_\mu \frac{\partial}{\partial p_\mu} + c^+ \frac{\partial}{\partial e^+}\right),$$

which preserves $\widehat{\mathbb{A}}_\star$. □

We now turn to the question of Lorentz invariance of the solution S_u to (1.1) that we have obtained. Let $\mathfrak{so}(9,1)$ be the Lie algebra of the Lorentz group $\mathrm{SO}(9,1)$, with basis $\rho^{\mu\nu} = -\rho^{\nu\mu}$, $0 \le \mu < \nu \le 9$, satisfying the commutation relations

$$[\rho^{\kappa\lambda}, \rho^{\mu\nu}] = \eta^{\lambda\mu}\rho^{\kappa\nu} + \eta^{\kappa\nu}\rho^{\lambda\mu} - \eta^{\lambda\nu}\rho^{\kappa\mu} - \eta^{\kappa\mu}\rho^{\lambda\nu}.$$

This action is realized on $\mathbb{F}$ by the currents of (1.3).

Consider the complex

$$C^*(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})\|.$$

The antibracket on $\|\mathbb{F}(N_\bullet\mathcal{U})\|$ induces a graded Lie bracket on $C^*(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})\|$, making it into a differential graded Lie algebra. Denote the dual basis of $\mathfrak{so}(9,1)^\vee$ by $\epsilon_{\mu\nu}$, and consider the element

$$\mathsf{S}(\epsilon) = \mathsf{S} + M^{\mu\nu}\epsilon_{\mu\nu} \in C^*(\mathfrak{so}(9,1)) \otimes \mathbb{F}(M_0) \subset C^*(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})\|.$$

The invariance of S under the action of the Lorentz group implies that this is a Maurer–Cartan element of $C^*(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})\|$; see (1.4).

We will construct a sequence of elements

$$\mathsf{G}_n(\epsilon) \in C^*(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})\|,$$

of total degree $-2n-2$, supersymmetric $\mathsf{q}\mathsf{G}_n(\epsilon) = 0$, such that the series

$$\mathsf{S}_u(\epsilon) = \mathsf{S}(\epsilon) + \sum_{n=0}^{\infty} u^{n+1}\mathsf{G}_n(\epsilon) \in C^*(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}(N_\bullet\mathcal{U})\|[[u]]$$

satisfies the equation (1.5).

We solve (1.5) inductively, by an extension of the method used to prove Theorem 7.3. Write

$$\mathsf{G}_n(\epsilon) = \sum_{k=0}^{10} \mathsf{G}_{n,k},$$

where $\mathsf{G}_{0,0}$ equals the explicit solution $\mathsf{G}_0 \in \|\mathbb{F}(N_\bullet\mathcal{U})\|^{-2}$ of (6.4), and $\mathsf{G}_{n,k} \in C^k(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}_\star(N_\bullet\mathcal{U})\|^{-2n-k-2}$ for $n > 0$ or $k > 0$. Assuming we have found $\mathsf{G}_{m,\ell}$ for $m < n$ or $m = n$ and $\ell < k$, we must solve the equation

$$(7.2)\qquad \begin{aligned}(\delta + \mathsf{s})\mathsf{G}_{n,k} &= -d\mathsf{G}_{n,k-1} - (M^{\mu\nu}\epsilon_{\mu\nu}, \mathsf{G}_{n,k-1}) \\ &\quad - \frac{1}{2}\sum_{m=0}^{n-1}\sum_{\ell=0}^{k}(\mathsf{G}_{m,\ell}, \mathsf{G}_{n-m-1,k-\ell}) \\ &\in C^k(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}_\star(N_\bullet\mathcal{U})\|^{-2n-k-1}.\end{aligned}$$

By the Lorentz invariance of S, $\mathsf{s}M^{\mu\nu} = 0$. For $n > 0$ or $k > 0$, this is sufficient to imply that the right-hand side of (7.2) is a cocycle. In the case $n = 0$ and $k = 1$, we need in addition the formula

$$(M^{\mu\nu}, \mathsf{D}) = 0.$$

By Lemma 7.2, there is a solution

$$\mathsf{G}_{n,k} \in C^k(\mathfrak{so}(9,1)) \otimes \|\mathbb{F}_\star(N_\bullet\mathcal{U})\|^{-2n-k-2}.$$

Thus there exists a supersymmetric solution to the equation (1.5).

Appendix A. Spinors in signature (9, 1)

Let $\mathrm{Spin}(9,1)$ be the double cover of the proper Lorentz group $\mathrm{SO}_+(9,1)$. Let $\mathbb{S}_+$ and $\mathbb{S}_-$ be the left and right-handed Majorana–Weyl spinor representations: these are 16-dimensional real representations of $\mathrm{Spin}(9,1)$. Let us review their construction.

Let W be a finite-dimensional oriented real vector space with a non-degenerate inner product (v, w) of dimension n. Let $\mathbb{T} : \Lambda^n W \to \mathbb{R}$ be the linear map which takes the wedge product of an oriented unimodular frame of W to 1. This induces an inner product on the exterior algebra $\Lambda^* W$, defined on $\alpha \in \Lambda^k W$ and $\beta \in \Lambda^\ell W$ by the formula

$$\langle \alpha, \beta \rangle = (-1)^{\binom{k}{2}} \mathbb{T}\big(\alpha \wedge \beta\big).$$

Thus $\langle \alpha, \beta \rangle$ vanishes unless $k + \ell = n$, and the form satisfies the symmetry

$$\langle \beta, \alpha \rangle = (-1)^{\binom{n}{2}} \langle \alpha, \beta \rangle.$$

The endomorphism algebra of the exterior algebra $\Lambda^* W$ is the Clifford algebra generated by the operators $c^\pm(v) = \iota(v) \pm \epsilon(v)$, where $\epsilon(v) : \Lambda^* W \to \Lambda^{*+1} W$ is exterior multiplication by $v \in W$ and $\iota(v) : \Lambda^{*-1} W \to \Lambda^* W$ is contraction with v. It is easily seen that $\epsilon(v)$ and $\iota(v)$, and hence $c^{\pm}(v)$, are self-adjoint for the bilinear form $\langle \alpha, \beta \rangle$.

The endomorphism algebra of the real vector space underlying the quaternions $\mathbb{H}$ is the tensor product of the two commuting subalgebras of left and right multiplication in $\mathbb{H}$. (This is one way of seeing the isomorphism $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.) Denote left, respectively right, multiplication by an element $a \in \mathbb{H}$ by a_L, respectively a_R.

We now give an explicit representation of the Clifford algebra in signature $(9,1)$ acting on the space of spinors $\mathbb{S} = \mathbb{H} \otimes \Lambda^* \mathbb{R}^3$:

$$\begin{array}{llll} \gamma^1 = c_1^+ & \gamma^2 = i_L c_1^- & \gamma^3 = j_L c_1^- & \gamma^4 = k_L c_1^- \\ \gamma^5 = c_2^+ & \gamma^6 = i_R c_2^- & \gamma^7 = j_R c_2^- & \gamma^8 = k_R c_2^- \\ \gamma^9 = c_3^+ & \gamma^0 = c_3^- & & \end{array}$$

The γ-matrices γ^μ exchange the subspaces $\mathbb{S}_\pm \cong \mathbb{R}^{16} = \mathbb{H} \otimes \Lambda^\pm \mathbb{R}^3$ of $\mathbb{H} \otimes \Lambda^* \mathbb{R}^3$ of even, respectively odd, exterior degree in the exterior algebra.

The Lie algebra of the group $\mathrm{Spin}(9,1)$ is spanned by the quadratic expressions in the γ-matrices

$$\gamma^{\mu\nu} = \tfrac{1}{2}\big(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\big).$$

In particular, $\mathrm{Spin}(9,1)$ preserves the subspaces $\mathbb{S}_\pm$ of $\mathbb{S}$. These are the left and right-handed 16-dimensional Majorana–Weyl spinors representations of $\mathrm{Spin}(9,1)$.

There is a non-degenerate symmetric bilinear form on $\mathbb{S}$, given by the formula

$$\mathsf{T}(\alpha, \beta) = \mathbb{T} \operatorname{Re}\big(c_1^- c_2^- \alpha \wedge \overline{\beta}\big) : \mathbb{S} \otimes \mathbb{S} \to \mathbb{R}.$$

From the explicit formulas for γ^μ, we see that

$$\mathsf{T}(\gamma^\mu \alpha, \beta) = \mathsf{T}(\alpha, \gamma^\mu \beta).$$

The Clifford algebra of $\mathbb{R}^{9,1}$ has basis

$$\gamma^{\mu_1 \cdots \mu_k} = \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi \gamma^{\mu_{\pi(1)}} \cdots \gamma^{\mu_{\pi(k)}},$$

where $\mu_1 \cdots \mu_k$ ranges over the set $\{1 \le \mu_1 < \cdots < \mu_k \le 10\}$. Let $\mathcal{V}$ be the vector representation $\mathbb{R}^{9,1}$ of $\mathrm{Spin}(9,1)$, and define pairings $\mathsf{T}^{\mu_1 \cdots \mu_k} : \mathbb{S} \otimes \mathbb{S} \to \Lambda^k \mathcal{V}$ by

$$\mathsf{T}^{\mu_1 \cdots \mu_k}(\alpha, \beta) = \mathsf{T}(\gamma^{\mu_1 \cdots \mu_k} \alpha, \beta) = (-1)^{\binom{k}{2}} \mathsf{T}(\alpha, \gamma^{\mu_1 \cdots \mu_k} \beta).$$

Lemma A.1.

$$\gamma^{\mu_1\cdots\mu_k}\gamma^{\mu} - (-1)^k\,\gamma^{\mu}\gamma^{\mu_1\cdots\mu_k} = 2\sum_{j=1}^{k}(-1)^{k-j}\eta^{\mu\mu_j}\gamma^{\mu_1\cdots\widehat{\mu}_j\cdots\mu_k}$$

Acknowledgments

The first author is grateful to Chris Hull for introducing him to the superparticle. His research is partially supported by a Fellowship of the Simons Foundation, Collaboration Grants 243025 and 524522 of the Simons Foundation, and EPSRC Programme Grant EP/K034456/1 "New Geometric Structures from String Theory." Parts of this paper were written while he was visiting Imperial College, the Yau Mathematical Sciences Center at Tsinghua University and the Department of Mathematics of Columbia University, as a guest of Chris Hull, Si Li and Mohammed Abouzaid respectively.

The research of the second author is supported in part by the National Science Foundation grant "RTG: Analysis on manifolds" at Northwestern University.

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