Fractional Virasoro algebras

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We show that it is possible to construct a Virasoro algebra as a central extension of the fractional Witt algebra generated by non-local operators of the form, $L_n^a \equiv \left(\frac{\partial f}{\partial z}\right)^a$ where $a \in \mathbb{R}$. The Virasoro algebra is explicitly of the form,

$$(1) \qquad [L_m^a, L_n^a] = A_{m,n}(s) \otimes L_{m+n}^a + \delta_{m,n} h(n) c Z^a$$

where $A_{m,n}(s)$ is a specific meromorphic function c is the central charge (not necessarily a constant), Z^a is in the center of the algebra and $h(n)$ obeys a recursion relation related to the coefficients $A_{m,n}$. In fact, we show that all central extensions which respect the special structure developed here which we term a multimodule Lie-Algebra, are of this form. This result provides a mathematical foundation for non-local conformal field theories, in particular recent proposals in condensed matter in which the current has an anomalous dimension.

1. Introduction

The Virasoro[1, 2] algebra is central to string theory as it underpins the conformal structure of the local current operators. In one of its incarnations, it is constructed as the central extension of the Witt algebra as the space of *local* conformal transformations on the unit disk. Consequently, for any problem controlled by critical scaling, the Virasoro algebra governing the conserved currents is of fundamental importance. In all constructions of the Virasoro algebra thus far, the generators are entirely local. However, there are a number of physical problems in which the currents are inherently non-local and hence require a fundamentally new Virasoro algebra. Consider, for example, three-dimensional bosonization[3] of massless fermions which results in a bosonic non-local Maxwell-Chern-Simons theory in which the kinetic energy operator is the fractional Laplacian, in particular $\Box^{\frac{1}{2}}$. Another example is found in the AdS/CFT correspondence[4–7] in which the boundary operator dual to a bulk massive scalar field is again the fractional Laplacian[8],

where the power is determined by the mass of the scalar field. A possible application of this[8] is the the strange metal in the cuprate high-temperature superconductors. This problem has long been argued to be controlled by quantum critical scaling[9–11]. Recent phenomenology[12] on this problem is based on a vector potential that has an anomalous dimension. Indeed, an anomalous dimension for the vector potential is problematic because the local gauge symmetry of electricity and magnetism, $A_\mu \to A_\mu + \partial_\mu \Lambda$, requires that $[A_\mu] = 1$. While an anomalous dimension for the vector potential can emerge in bulk Lifshitz theories[13], this does not solve the problem of how to maintain gauge invariance in the resultant boundary theory. Regardless of the underlying quantum field theory, squaring an anomalous dimension of the vector potential with gauge invariance necessitates (assuming only that the underlying symmetry is still based on an infinitesimal transformation) a new non-local symmetry[14] of the form

$$(2) \qquad\qquad A_\mu \to A_\mu + d_a \Lambda, \quad [A_\mu] = a$$

where $d_a = (\Delta)^{(a-1)/2}d$, d the complete exterior derivative and $a \in \mathcal{R}$. Such a transformation leads to non-local currents[14, 15], and hence if there is a string formulation of this problem, a Virasoro algebra allowing arbitrary fractional dimensions of the currents must be formulated. It is this task that we perform here. Indeed, fractional generalizations[16, 17] of the Virasoro algebra do exist which do allow central charges that exceed unity. However, such generalizations are not directly applicable to the strange metal as they all have currents that transform without an anomalous dimension.

   To solve this problem and the general phenomena of non-local currents, we construct a family of Lie algebras (which are modules over a certain commutative Lie algebra $\mathcal{H}$ of holomorphic functions over $\mathbb{C}$) $\mathcal{V}_a$, thereby generalizing the Virasoro algebra. These algebras $\mathcal{V}_a$ are defined as central extensions of the algebra $\mathcal{W}_a$ (itself a generalization of the Witt algebra) which consists of operators which are combinations – linear over $\mathcal{H}$– of the form: $\sum_n \phi_n(s) L_n^a$, where $\phi(s) \in \mathcal{H}$ and $L_n^a \equiv \left(\frac{\partial f}{\partial z}\right)^a$ is the fractional z-derivative. We think of these operators as acting on "power series" $\sum_k p_k z^{ak}$ via

$$(3) \qquad \left(\sum_n \phi_n(s) L_n^a\right)\left(\sum_k p_k z^{ak}\right) = \sum_k p_k\, \phi_n(ak)\, L_n^a(z^{ak}).$$

This could be all reformulated in a more analytical and invariant way, but we choose not to because we mean to focus on the algebraic structure of the algebras.

The algebra $\mathcal{W}_a$ has a special structure, which we call Lie multimodule in Definition 3.1; namely, there are operations $\star_{(p,q)}$ on $\mathcal{H}$ and a grading on $\mathcal{W}_a$, such that: $[\phi \otimes L_p, \psi \otimes L_q] = \phi \star_{p,q} \psi [L_p, L_q]$ (cf. eq. (20) for the definition of $\star_{(p,q)}$). We show that all the central extensions which preserve this extra structure

$$(4) \qquad\qquad 0 \to \mathcal{H} \to \mathcal{V}_a \to \mathcal{W}_a \to 0,$$

which are parametrized by a group $H^2_\star(\mathcal{W}_a, \mathcal{H})$ (which we show to be isomorphic to $\mathcal{H}$) are of the form

$$(5) \qquad\qquad [L^a_m, L^a_n] = A_{m,n} L^a_{m+n} + \delta_{m,n} h(n) c Z^a$$

where c is the central charge ($c \in \mathcal{H}$), Z^a is in the center of the algebra and $h(n)$ obeys the recursion relation,

$$h(2) = c$$

$$(6) \qquad \frac{A_{-1,-m}\Gamma_{(-(m+1))} - A_{m,1}\Gamma_{m+1}}{A_{-(m+1),m+1}} h((m+1))$$

$$= \frac{A_{m+1,-1}\Gamma_1 - A_{1,-(m+1)}\Gamma_{-m}}{A_{m,-m}} h(m)$$

Here

$$(7) \qquad A_{p,q}(s) = \frac{\Gamma(a(s+p)+1)}{\Gamma(a(s-1+p)+1)} - \frac{\Gamma(a(s+q)+1)}{\Gamma(a(s-1+q)+1)}$$

and

$$(8) \qquad\qquad \Gamma_p(s) = \frac{\Gamma(a(s+p)+1)}{\Gamma(a(s-1+p)+1)}$$

where Γ is the gamma function. The elements of $\mathcal{W}_a$ are operators acting on $\mathbb{C}[[z^a, z^{-a}]]$ via the prescription

$$(9) \qquad\qquad \left(\phi \otimes L^a_p\right)(z^{ka}) = \phi(k) L^a_p(z^{ka}).$$

The usual $\mathcal{H}$-Lie algebras $\mathcal{V}_a$ are a generalization of the Virasoro algebra in that

$$(10) \qquad\qquad \lim_{a \to 1} \mathcal{V}_a = V.$$

The Lie algebra structure of $\mathcal{V}_a$, on the other hand, does not arise, for $a \neq 1$, as a tensor product of a Lie algebra V with $\mathcal{H}$, reflecting the very non-local nature of the operators in $\mathcal{V}_a$. In this sense, it is a twisted structure, or more properly a Lie multimodule, further indicating the non local nature of non-local conformal field theories.

2. Fractional (holomorphic) derivatives

2.1. Holomorphic and anti-holomorphic fractional derivatives

In this section we intend to emphasize that the type of operators we consider have various nice analytic incarnations. Nonetheless, we ultimately take a purely algebraic approach to the definition of the operators and the vector spaces (see the next section) they act upon. Recall that the fractional Laplacian in $\mathbb{R}^n$ can be defined as (a regularization of)

$$(-\Delta_x)^a f(x) = C_{n,a} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{\mid x - \xi \mid^{n+2a}} \, d\xi$$

for some constant $C_{n,a}$ or equivalently in terms of its Fourier transform and hence as a *pseudo-differential* operator. Given a complex valued function f, let us denote by $\hat{f}$ its Fourier transform. There are various ways to generalize the concept of the holomorphic derivative. One possibility is to consider the following approach analogous to the one used for the fractional Laplacian.

Definition 2.1. *The fractional holomorphic derivative is*[1]

$$(11) \qquad \left(\frac{\hat{\partial}}{\partial z} \right)^a f = c_a \xi^a \hat{f},$$

where $\xi = \xi_1 + i\,\xi_2$ is the complex momentum and analogously the fractional antiholomorphic derivative is

$$(12) \qquad \left(\widehat{\frac{\partial f}{\partial \bar{z}}} \right)^a = \bar{c}_a \bar{\xi}^a \hat{f}.$$

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[1] This is the same as defining $\left(\frac{\partial}{\partial z} \right)^a f = \frac{\partial}{\partial z}(-\Delta)^{\frac{a-1}{2}} f$ and the analogous expression for $\left(\frac{\partial f}{\partial \bar{z}} \right)^a$

As a result,

$$\left(\frac{\partial}{\partial z}\right)^{\widehat{a}} \left(\left(\frac{\partial f}{\partial \bar{z}}\right)^a\right) = |c_a|^2 \, |\xi|^{2a} \, \hat{f}$$

or equivalently, we have shown the following Lemma.

Lemma 2.1. *The fractional holomorphic and antiholomorphic derivatives are such that*

$$\left(\frac{\partial}{\partial z}\right)^a \left(\frac{\partial}{\partial \bar{z}}\right)^a f = |c_a|^2 \, (-\Delta)^a f,$$

where $(-\Delta)^a$ is the Riesz fractional Laplacian.

One could also take a slightly different tack and consider instead the Louiville approach to fractional calculus as our starting point, which in turn has its origins in the classical Cauchy integral formula for the holomorphic derivative of analytic functions. One could then define,

$$\left(\frac{\partial}{\partial z}\right)^a f(z) = \frac{\Gamma(a+1)}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{(1+a)}} \, d\xi,$$

for any loop γ around z (at any point on the cut, i.e., the negative real axis, the loop must be thought of as lifted on the universal covering). This definition takes into account that the kernel of the integral, namely $\frac{1}{(\xi-z)^{(1+a)}}$, has now a branch rather than a pole at the origin, and therefore one must consider introducing a cut in the complex plane. For example, removing the non-positive semi-axis along the real line will do. These pseudo-differential operators will be the building blocks for the fractional Witt algebra.

2.2. Fractional derivatives: algebraic formulation

We consider the universal cover $\mu : R \to \mathbb{C}^*$ where $\mu(z) = e^z$. It is well known that $R \simeq \mathbb{C}$. For $a \in \mathbb{R}$, w^a defines a map from $\mathbb{C}^*$ to itself through $w^a = e^{ax} e^{iay}$, where $w = e^z$ for $z = x + iy$. The map $w \mapsto w^a$ is defined on $\mathbb{C}^*$ and it is covered by the homothety of R: $z \mapsto az$. From now on we consider z^a as a symbolic expression, with the understanding that it has the incarnation described above. Consider the algebra,

$$V^a := \mathbb{C}[[z^{-a}, z^a]],$$

of formal power series in z^a and z^{-a}. We denote by V_k^a, the $\mathbb{C}$-vector space spanned by z^{ak} and we define the operator $\left(\frac{\partial}{\partial z}\right)^a$ on V_k^a as the linear operator,

$$\left(\frac{\partial}{\partial z}\right)^a : V_k^a \to V_{k-1}^a,$$

defined by (for $ak > -1$)[2]

$$(13) \qquad \left(\frac{\partial}{\partial z}\right)^a (z^{ak}) = A_k c_a \frac{\Gamma(ak+1)}{\Gamma(a(k-1)+1)} z^{a(k-1)},$$

and

$$\left(\frac{\partial}{\partial z}\right)^a (\bar{z}^{ak}) = 0,$$

where Γ is the Gamma function, c_a is an unspecified constant and A_k only depends on k (which in this note, we will take to equal 1 for notational convenience). After the change of coordinates, $\zeta = z^a$, it is apparent that the action of $\left(\frac{\partial}{\partial z}\right)^a$ on V_k^a for $k \neq 0$ is equivalent to the action of

$$c_a \frac{\Gamma(ak+1)}{\Gamma(a(k-1)+1)} \frac{1}{k} \frac{\partial}{\partial \zeta},$$

on the degree k−subspace (which we denote by V_k) of $V := \mathbb{C}[[\zeta^{-1}, \zeta]]$. The operator $\left(\frac{\partial}{\partial z}\right)^a$ thus defined is definitely not a derivation because it fails the Leibnitz rule, as in fact

$$\left(\frac{\partial}{\partial z}\right)^a (1) = A_0 c_a \frac{z^{-a}}{\Gamma(1-a)},$$

which is non-zero unless $a \in \mathbb{N}$ (or $A_0 = 0$). In the rest, for the sake of notation, we will assume $A_k = 1$ for every k, and thus take

$$(14) \qquad \left(\frac{\partial}{\partial z}\right)^a = \bigoplus_k c_a \frac{\Gamma(ak+1)}{\Gamma(a(k-1)+1)} P_k$$

where $P_k : V_k \to V_{k-1}$ is defines as $P_k(z^{ak})) = z^{a(k-1)}$, or equivalently, after the change of coordinates, $\zeta = z^a$,

$$(15) \qquad \left(\frac{\partial}{\partial z}\right)^a = \bigoplus_k c_a \frac{\Gamma(ak+1)}{\Gamma(a(k-1)+1)} \frac{1}{k} \frac{\partial}{\partial \zeta}.$$

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[2]The choice of $A_k = 1$ that we make in this note clashes a bit with the analytic definition we gave in the previous section, depending on the value of a, but it is easy to adjust the rest of the paper with an arbitrary A_k in the rest of the paper.

thus showing that after the change of variables $\zeta = z^a$, the operator $\left(\frac{\partial}{\partial z}\right)^a$ does not equal $\frac{1}{k}\frac{\partial}{\partial \zeta}$, even up to multiples.

Remark 2.1. *We would like to emphasize that the actual form of the coefficients in the defining equation* (13) *is not important, we could choose different coefficients. That is we could define:* $\left(\frac{\partial}{\partial z}\right)^a (z^{ak}) = C_{a,k}\, z^{a(k-1)}$ *and have a similar algebra. The constructions to follow will all go through, mutatis mutandis.*

3. Fractional Virasoro algebra

3.1. Lie multimodules

In this section we define the notion of a Lie multimodule over an algebra. Let $\mathcal{A}$ be an algebra endowed with a family of operations: $\star_{p,q}$ parametrized by $p, q \in \mathbb{Z}$.

Definition 3.1. *A Lie* **multi-module** *over* $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}, p, q})$ *(or $\mathcal{A}$-Lie algebra for short) is a graded $\mathcal{A}$-module $\mathcal{W} = \oplus_k \mathcal{W}_k$ endowed with a Lie bracket*

$$[\cdot, \cdot] : \mathcal{W} \times \mathcal{W} \to \mathcal{W},$$

such that

$$[a_1 v, a_2 w] = a_1 \star_{p,q} a_2 [v, w],$$

for any $v \in \mathcal{W}_p$, $w \in \mathcal{W}_q$.

A *trivial* example of such a structure is the standard Lie-module over a Lie algebra. For instance, the trivial one obtained by tensoring a Lie algebra $(V, [\cdot, \ ])$ with a commutative Lie algebra $\mathcal{A}$ where the Lie bracket of $V \otimes \mathcal{A}$ is given by

(16) $$[v \otimes \phi, w \otimes \psi] = \phi\psi\,[v, w]$$

for every $v, w \in V$ and $\phi, \psi \in \mathcal{A}$ is such an example. In the next section we will construct a more complex example of such a structure.

3.2. The fractional Witt algebra

We now define the infinite dimensional Lie algebra of pseudo-differential operators which is a generalization of the standard Witt algebra as follows.

We consider

$$(17) \qquad L_n^a = -z^{a(n+1)} \left(\frac{\partial}{\partial z} \right)^a, \qquad \bar{L}_n^a := -\bar{z}^{a(n+1)} \left(\frac{\partial}{\partial \bar{z}} \right)^a$$

acting on $V^a := \mathbb{C}[[z^{-a}, z^a]]$. In this algebraic description we think of z^a merely as a formal expression as in Puiseaux series, if $a \in \mathbb{Q}$ [18].

For any integer p, we define the following functions

$$(18) \qquad \Gamma_p(s) := \frac{\Gamma(a(s+p)+1)}{\Gamma(a(s+p-1)+1)}$$

and

$$(19) \qquad A_{p,q}(s) = \Gamma_p(s) - \Gamma_q(s)$$
$$= \left(\frac{\Gamma(a(s+p)+1)}{\Gamma(a(s-1+p)+1)} - \frac{\Gamma(a(s+q)+1)}{\Gamma(a(s-1+q)+1)} \right).$$

Clearly

$$\Gamma_p(s) = \Gamma_0(s+p).$$

Let $\mathcal{M}(\mathbb{C})$ be the algebra of meromorphic functions on $\mathbb{C}$ (when needed, we will denote by $\mathcal{M}_k(\mathbb{C})$ the set of meromorphic functions which only have poles of order k) and set[3]

$$\mathcal{F} := \{ f \in \mathcal{M}(\mathbb{C}) : f(z) \text{ is holomorphic in a neighborhood of } z \text{ if } Re(z) \in \mathbb{Z} \},$$

which we think of as a sub-algebra of meromorphic functions on $\mathbb{C}$. We endow $\mathcal{F}$ with the family of operations

$$(20) \qquad \phi \star_{p,q} \psi := \frac{\psi(p+s)\phi(s)\,\Gamma_p - \phi(q+s)\psi(s)\Gamma_q}{A_{p,q}}$$

and the related family of "brackets"

$$(21) \qquad [\phi(s), \psi(s)]_{\mathcal{H},p,q} = \begin{cases} \psi(p+s)\phi(s)\,\Gamma_p - \phi(q+s)\psi(s)\Gamma_q \text{ for } p \neq q \\ 0 \text{ for } p = q. \end{cases}$$

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[3]We could actually take functions in $\mathcal{F}$ to have more regularity, by taking $\mathcal{F} := \mathcal{M}(\mathbb{C}) \cap \mathcal{H}(\mathbb{C} \setminus S)$ where $S := \{ z \in \mathbb{C} : Re(z) \in -\frac{1}{a}\mathbb{N} - \mathbb{N}, Im(z) = 0 \}$ and $H(\mathbb{C} \setminus S)$ is the space of holomorphic functions on $\mathbb{C} \setminus S$

Definition 3.2. *We define the algebra $\mathcal{H}$ to be the subalgebra of $\mathcal{F}$ generated by $\mathbb{C}$ closed under $[\phi(s), \psi(s)]_{\mathcal{H},p,q} = \psi(p+s)\phi(s)\,\Gamma_p - \phi(q+s)\psi(s)\Gamma_q$, for every $p, q \in \mathbb{Z}$.*

Clearly $\mathcal{H}$ is contained in the algebra $\mathbb{C}[A_{p,q}, \Gamma_\ell]_{p,q,\ell \in \mathbb{Z}}$ of polynomials in $A_{p,q}, \Gamma_\ell$, but it is generally smaller. In fact, we will see that for $a = 1$, $\mathcal{H} = \mathbb{C}$, thus highlighting, at the same time, the a-dependence of $\mathcal{H}$.

We then define the algebra

$$\mathcal{W}_a := \bigoplus_{n \in \mathbb{Z}} \mathcal{H} \otimes L_n,$$

evidently an $\mathcal{H}$-module, which we think of as acting on $V^a := \mathbb{C}[[z^{-a}, z^a]]$ by considering the action of a typical generator on a basis for V^a via

$$(\phi(w) \otimes L_n)\,(z^{ak}) = \phi(k)L_n(z^{ak}),$$

for $k \neq 0$. This is clearly a representation by construction; that is $\phi(w) \otimes L_n$ acts as a linear operator on V^a and the bracket of elements of $\mathcal{W}_a$ is defined as the commutator of endomorphisms of V^a.

Remark 3.1. *Let us emphasize that, because of equation (14) or equivalently eq. (15), the fractional Witt algebra thus constructed (and therefore the fractional Virasoro algebra) are not isomorphic to the Witt algebra.*

A consequence of Eq. (13) is the following Lemma.

Lemma 3.1. *The $\mathcal{H}$-module $\mathcal{W}_a$ spanned by the pseudodifferential operators L_n is an $\mathcal{H}$-Lie algebra, when endowed with brackets $[\cdot, \cdot]$ consisting of commutators. In fact, one has*

$$(22) \quad [L_n, L_m]f(z) = \sum_k a_k A^a_{n,m}(k)\, L_{n+m}(z^k) = (A^a_{n,m}(s) \otimes L_{n+m})(f(z))$$

for any $f(z) = \sum_k a_k z^k$, where $A^a_{n,m}(k)$ is the evaluation at k of the meromorphic function $A^a_{n,m}(s)$ defined in (19) clearly $A^a_{n,m}(s) \in \mathcal{H}$, if $a \notin \mathbb{Z}$. More generally,

$$(23) \quad [\phi \otimes L_n, \psi \otimes L_m] = (\psi(n+s)\phi(s)\Gamma^a_n - \psi(s)\phi(m+s)\Gamma^a_m) \otimes L_{m+n}$$

.

Proof. First observe that

$$(24) \qquad L_p(z^{a\ell}) = -\Gamma_0(\ell)\, z^{a(\ell+p)},$$

as one can verify via

$$L_p(z^{a\ell}) = -z^{a(p+1)} \left(\frac{\partial}{\partial z}\right)^a (z^{a\ell})$$

$$= -z^{a(p+1)} \frac{\Gamma(a\ell+1)}{\Gamma(a(\ell-1)+1)}\, z^{a(\ell-1)} = -\Gamma_0(\ell)\, z^{a(\ell+p)}.$$

Using this, we find that

$$L_n(L_m\, z^{ak}) = z^{a(n+1)} \left(\frac{\partial}{\partial z}\right)^a \left(\Gamma_0(k)\, z^{a(k+m)}\right)$$

$$= \frac{\Gamma(a(k+m)+1)}{\Gamma(a(k-1+m)+1)}\, \Gamma_0(k) z^{a(k+m+n)}$$

$$= \Gamma_m(k)\, L_{n+m}(z^{ak}) = \left(\Gamma_m(s) \otimes L_{n+m}(z^{ak})\right),$$

whence

$$(25) \qquad L_n \circ L_m = \Gamma_m(s) \otimes L_{n+m},$$

where $\circ$ denotes composition. From this (using (24) again), we obtain

$$(\phi \otimes L_n)(\psi \otimes L_m)(z^{ak}) = (\phi \otimes L_n)\left(-\psi(k)\Gamma_0(k)\, z^{a(k+m)}\right)$$

$$= \phi(m+k)\psi(k)\Gamma_m(k)_m\Gamma_0(k)\, z^{a(k+m+n)}$$

$$= -\phi(m+k)\psi(k)\Gamma_m(k)L_{n+m}(z^{ak}).$$

Thus,

$$(26) \qquad (\phi \otimes L_n) \circ (\psi \otimes L_m) = -\left(\phi(m+s)\psi(s)\Gamma_m(s)\right) \otimes L_{n+m}.$$

As a result, we have

$$(27) \qquad [\phi \otimes L_n, \psi \otimes L_m] = \left(\psi(n+s)\phi(s)\Gamma_n^a - \psi(s)\phi(m+s)\Gamma_m^a\right) \otimes L_{m+n},$$

which is Eq. (23). From this, Eq. (22) follows directly, taking $\phi = \psi = 1$.

Clearly the properties of the Γ-function imply that $A_{n,m}^a(s) \in \mathcal{H}$. $\qquad \square$

We next intend to show that indeed the Lie algebra structure is simple for $a = 1$.

Lemma 3.2. *One has*

$$\lim_{a \to 1} \mathcal{W}_a = W.$$

Proof. The $\mathcal{H}$-Lie algebra structure of $\mathcal{W}_a$ is determined by the formula in Eq. (22)

$$[L_n, L_m]\phi(z) = \sum_k a_k A^a_{n,m}(k) L_{n+m}(z^k),$$

where the holomorphic functions $A^a_{n,m}(s)$ are given by Eq. (19). We next observe that, since

$$\Gamma(z + 1) = z \, \Gamma(z),$$

for $a = 1$ and any s

$$\Gamma_p(s) = \frac{\Gamma((s + n) + 1)}{\Gamma((s - 1 + n) + 1)} = \frac{(s + n)\Gamma(s + n)}{\Gamma(s + n)} = s + n.$$

Using this computation, we can show that $\mathcal{H} = \mathbb{C}$, when $a = 1$. In fact, by definition, $\mathcal{H}$ is the algebra generated by $\mathbb{C}$, closed with respect to

$$\phi \star_{p,q} \psi := \psi(p + s)\phi(s) \, \Gamma_p - \phi(q + s)\psi(s)\Gamma_q.$$

A straightforward computation, using the formula we obtained for Γ_p then yields

(28) $$\phi \star_{p,q} \psi = \psi(p + s)\phi(s) \, (s + p) - \psi(q + s)\phi(s)(s + q).$$

Since $A^1_{n,m}(s)$ is s-independent, it follows that for $a = 1$,

$$[L_n, L_m]\phi(z) = (n - m) \, L_{n+m}(\phi(z)),$$

which is the standard structure of the Witt algebra. Also for elements in $\mathcal{H}$, by definition of the family of Lie brackets and the computations above, for

$p \neq q$,

$$[\phi(s), \psi(s)]_{\mathcal{H},p,q} = \frac{\psi(p+s)\phi(s)\,\Gamma_p - \psi(q+s)\phi(s)\Gamma_q}{A_{p,q}}$$

$$= \psi(s)\phi(s)\frac{\Gamma_p - \Gamma_q}{A_{p,q}} = \psi(s)\phi(s),$$

and for $p = q$

$$[\phi(s), \psi(s)]_{\mathcal{H},p,p} = 0,$$

since $\mathcal{H}$ is the smallest algebra which is a Lie algebra for every $[\cdot, \cdot]_{\mathcal{H},p,q}$ and which contains $\mathbb{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.2. *We make the observation that the $\mathcal{H}$-Lie algebra structure is necessary, and that in general it does not arise from a tensor product of a Lie algebra over $\mathcal{C}$ (a standard Lie algebra) and the commutative Lie algebra $\mathcal{H}$, unless $a = 1$. This can be seen from the fact that $A_{n,m}^a(k)$ is in general k-dependent.*

Here we record some useful identities about the functions $A_{m,n}(s)$.

Lemma 3.3.   • *For every m, n, $A_{m,n}(s) = A_{m,0}(s) + A_{0,n}(s)$.*

　　　• *For every m, $A_{m,0}(s) = A_{0,-m}(s + m)$.*

　　　• *One has*

$$(29) \quad A_{m+\ell,n+\ell}(s)A_{\ell,0}(s) - A_{m,n}(s)A_{m+n,0}(s) + A_{n+m,\ell+m}(s)A_{m,0}(s)$$
$$- A_{n,\ell}(s)A_{n+\ell,0}(s) + A_{\ell+n,m+n}(s)A_{n,0}(s) - A_{\ell,m}(s)A_{\ell+m,0}(s)$$
$$+ \Gamma_0(s)\big(A_{m,n}(s) - A_{m+\ell,n+\ell}(s) + A_{n,\ell}(s)$$
$$- A_{n+m,\ell+m} + A_{\ell,m}(s) - A_{\ell+n,m+n}(s)\big) = 0.$$

Proof. The first two follow readily from the definition of $A_{m,n}(s)$. The last one is simply a reformulation of the Jacobi identity,

$$(30) \qquad ([L_\ell, [L_m, L_n]] + [L_m, [L_n, L_\ell]] + [L_n, [L_\ell, L_m]])\, z^{as} = 0.$$

On the other hand, applying the operator $[L_r, [L_p, L_q]]$ to z^{as} leads to the expression

$$[L_r, [L_p, L_q]] \, z^{as} = L_r \left(A_{p,q}(s) L_{p+q}(z^{as}) \right) - A_{p,q} \otimes L_{p+q} \left(L_r(z^{as}) \right)$$

$$= A_{p,q}(s) L_r \left(-\Gamma_0(s) \, z^{a(p+q+s)} \right)$$

$$- A_{p,q} \otimes L_{p+q} \left(-\Gamma_0(s) \, z^{a(r+s)} \right)$$

$$= A_{p,q}(s) \left(\Gamma_0(s) \frac{\Gamma(a(p+q+s)+1)}{\Gamma(a(p+q+s-1)+1)} \, z^{a(r+p+q+s)} \right)$$

$$- A_{p,q}(s+r) \left(\Gamma_0(s) \frac{\Gamma(a(r+s)+1)}{\Gamma(a(r+s-1)+1)} \, z^{a(r+s+p+q)} \right)$$

$$= -A_{p,q}(s) \frac{\Gamma(a(p+q+s)+1)}{\Gamma(a(p+q+s-1)+1)} L_{p+q+r}(z^{as})$$

$$+ A_{p,q}(s+r) \Gamma_0(s) L_{p+q+r}(z^{as}),$$

whence

$$(31) \quad [L_r, [L_p, L_q]] \, z^{as} = \big(A_{p+r,q+r}(s) A_{r,0}(s) - A_{p,q}(s) A_{p+q,0}(s)$$
$$+ \Gamma_0(s) \left(A_{p+r,q+r}(s) - A_{p,q}(s) \right) \big) L_{p+q+r}(z^{as}).$$

Hence, recalling Eq. (30), in light of the identity in Eq. (31), one has

$$0 = A_{m+\ell,n+\ell}(s) A_{\ell,0}(s) - A_{m,n}(s) A_{m+n,0}(s)$$
$$+ \Gamma_0(s) \left(A_{m,n}(s) - A_{m+\ell,n+\ell}(s) \right)$$
$$+ A_{n+m,\ell+m}(s) A_{m,0}(s) - A_{n,\ell}(s) A_{n+\ell,0}(s)$$
$$+ \Gamma_0(s) \left(A_{n,\ell}(s) - A_{n+m,\ell+m}(s) \right)$$
$$+ A_{\ell+n,m+n}(s) A_{n,0}(s) - A_{\ell,m}(s) A_{\ell+m,0}(s)$$
$$+ \Gamma_0(s) \left(A_{\ell,m}(s) - A_{\ell+n,m+n}(s) \right).$$

Thus the identity of Eq. (29) must hold. $\square$

3.3. Central extensions and cohomology

We now consider the question of central extensions

$$0 \to \mathcal{H} \to \mathcal{V}_a \to \mathcal{W}_a \to 0,$$

which preserve the multimodule structure as in Definition 3.1. We will show in this section that these are parametrized by the cohomology group,

$$H^2_\star(\mathcal{W}_a, \mathcal{H}) = Z^2_\star(\mathcal{W}_a, \mathcal{H}) / B^2_\star(\mathcal{W}_a, \mathcal{H}).$$

We note that the Lie algebra structure of $\mathcal{H}$ is given by $[\phi, \psi] = 0$ consistently with the fact that we identify ϕ and ψ, respectively with $\phi \otimes 1$ and $\psi \otimes 1$ so that $[\phi, \psi] = [\phi \otimes 1, \psi \otimes 1] = [\phi, \psi]_{\mathcal{H},0,0} = 0$ since 1 is degree 0. Given a bilinear map,

$$\omega : \mathcal{W}_a \times \mathcal{W}_a \to \mathcal{H}$$

we say that is is $\star_{p,q}$-*bilinear* if

(32) $$\omega(\phi \otimes L_p, \psi \otimes L_q) = \phi \star_{p,q} \psi \omega(L_p, L_q).$$

Here the set of *2-cycles* is by definition

$$Z_\star^2(\mathcal{W}_a, \mathcal{H}) = \left\{ \omega : \mathcal{W}_a \times \mathcal{W}_a \to \mathcal{H} : \begin{array}{l} (1)\ \omega \text{ is } \star_{p,q}\text{-bilinear} \\ (2)\ \omega \text{ is alternate} \\ (3)\ \omega(g_1, [g_2, g_3]) + \omega(g_2, [g_3, g_1]) + \omega(g_3, [g_1, g_2]) = 0 \end{array} \right\}.$$

Property (3) above is called the Jacobi identity.

The subgroup $B^2(\mathcal{W}_a, \mathcal{H})$ of $Z^2(\mathcal{W}_a, \mathcal{H})$ is defined as the image via the *co-boundary* map δ of $Z^1(\mathcal{W}_a, \mathcal{H})$,

$$B_\star^1(\mathcal{W}_a, \mathcal{H}) := \{\mathcal{H}\text{-linear maps } \lambda : \mathcal{W}_a \to \mathcal{H}\},$$

and $\delta(\lambda)$ is defined as the 2-chain,

$$\delta(\lambda)(g_1, g_2) = \lambda([g_1, g_2]).$$

Clearly, because of the Jacobi identity, which reads (cf. Eq. (30)),

$$[L_\ell, [L_m, L_n]] + [L_m, [L_n, L_\ell]] + [L_n, [L_\ell, L_m]],$$

one has that $\delta(\lambda)$ satisfies identity (3) in the definition of $Z_\star^2(\mathcal{W}_a, \mathcal{H})$. We will prove the following generalization in our context.

Theorem 3.1. *Let $\mathcal{W}_a$ be a Lie multimodule over $\mathcal{H}$ (in the sense of Definition 3.1) endowed with $[\cdot, \cdot]_{\mathcal{H},p,q}$. Isomorphism classes of central extensions of the Lie multimodule $\mathcal{W}_a$ by $\mathcal{H}$ which are still multimodules are in one-to-one correspondence with $H_\star^2(\mathcal{W}_a, \mathcal{H}) = Z_\star^2(\mathcal{W}_a, \mathcal{H})/B_\star^1(\mathcal{W}_a, \mathcal{H})$.*

Proof. Standard theory about central extensions yields that these are parametrized by $H^2(\mathcal{W}_a, \mathcal{H})$. In fact, it is a standard fact that the central

extension

$$0 \to \mathcal{H} \to \mathcal{V}_a \to \mathcal{W}_a \to 0$$

is equivalent to providing a bilinear, alternate map $\omega : \mathcal{W}_a \times \mathcal{W}_a \to \mathcal{H}$ which satisfies the Jacobi identity; furthermore, such a bilinear map produces a Lie bracket on $\mathcal{W}_a \oplus \mathcal{H}$ by the ansatz,

$$[X \oplus Z, Y \oplus Z]_{\mathcal{V}_a} = [X, Y]_{\mathcal{W}_a} + \omega(X, Y).$$

All such ω's are in one-to-one correspondence splittings of $\pi : V_a \to \mathcal{W}_a$, i.e., linear maps $\alpha : \mathcal{W}_a \to \mathcal{V}_a$ such that $\alpha \circ \pi = id_{\mathcal{W}_a}$. This correspondence follows the prescription

$$\omega(X, Y) = [\alpha(X), \alpha(Y)] - \alpha([X, Y]).$$

Clearly the multimodule properties of $[\phi \otimes L_p, \psi \otimes L_q]$ imply that ω must be $\star_{p,q}$-bilinear. $\square$

3.4. The Virasoro algebra

Theorem 3.2. *One has that*

$$H^2(\mathcal{W}_a, \mathcal{H}) \simeq \mathcal{H}.$$

In other words central $\mathcal{H}$-extensions of $\mathcal{W}_a$ are parametrized by $\mathcal{H}$. Furthermore $\mathcal{V}_a$ is spanned by $\{L_m^a, Z^a\}$ with commutator relations

(33) $$[L_m, L_n] = A_{m,n} L_{m+n} + \delta_{m+n} h(n) \, c \, Z^a$$

where $h(n), c \in \mathcal{H}$ and $h(n)$ is defined by the recursive relation

(34)
$$\begin{cases} h(2) = c \\ \frac{A_{-1,-m}\Gamma_{(-(m+1))} - A_{m,1}\Gamma_{m+1}}{A_{-(m+1),m+1}} h((m+1)) \\ = \frac{A_{m+1,-1}\Gamma_1 - A_{1,-(m+1)}\Gamma_{-m}}{A_{m,-m}} h(m). \end{cases}$$

Proof. Let us analyze the meaning of condition (3) (the Jacobi like identity) in the definition of $Z^2(\mathcal{W}_a, \mathcal{H})$. By bilinearity of ω, it is enough to study the relation on the generators of $\mathcal{W}_a$ as an $\mathcal{H}$-module. The Jacobi identity then

reads

$$(35) \qquad \omega\left(L_\ell, [L_m, L_n]\right) + \omega\left(L_m[L_n, L_\ell]\right) + \omega\left(L_n, [L_\ell, L_m]\right) = 0.$$

Using Eq. (22), that is,

$$[L_n, L_m]\phi(z) = \sum_k a_k A^a_{n,m}(k)\, L_{n+m}(z^k) = (A^a_{n,m}(s) \otimes L_{n+m})(\phi(z))$$

and observing that $\star_{\mathcal{H},p,q}$-bilinearity implies that

$$\omega(L_p, \psi \otimes L_q) = \frac{\psi(s+p)\Gamma_p - \psi(s)\Gamma_q}{A_{p,q}}\, \omega(L_p, L_q),$$

lead us to rewrite the Jacobi identity as

$$(36) \qquad 0 = \frac{A_{m+\ell,n+\ell}\Gamma_\ell - A_{m,n}\Gamma_{m+n}}{A_{\ell,m+n}}\omega_{\ell,m+n}$$
$$+ \frac{A_{n+m,\ell+m}\Gamma_m - A_{n,\ell}\Gamma_{n+\ell}}{A_{m,n+\ell}}\omega_{m,n+\ell}$$
$$+ \frac{A_{\ell+n,m+n}\Gamma_n - A_{\ell,m}\Gamma_{\ell+m}}{A_{n,\ell+m}}\omega_{n,\ell+m},$$

where we have denoted by $\omega_{p,q} := \omega(L_p, L_q) \in \mathcal{H}$.

Setting $\ell = 0$ in the equation above and using that $\omega_{m,n} = -\omega_{n,m}$ gives rise to the relation,

$$(37) \qquad 0 = \frac{A_{m,n}\Gamma_0 - A_{m,n}\Gamma_{m+n}}{A_{0,m+n}}\, \omega_{0,m+n}$$
$$+ \left(\frac{A_{n,m+n}\Gamma_n - A_{0,m}\Gamma_m}{A_{n,m}} - \frac{A_{n+m,m}\Gamma_m - A_{n,0}\Gamma_n}{A_{m,n}}\right)\omega_{n,m}.$$

From $A_{p,q}(s) = \Gamma_p(s) - \Gamma_q(s)$ and Lemma 3.4 (below), we have that

$$\frac{A_{n,m+n}\Gamma_n - A_{0,m}\Gamma_m}{A_{n,m}} - \frac{A_{n+m,m}\Gamma_m - A_{n,0}\Gamma_n}{A_{m,n}} = A_{n+m,0},$$

and that

$$\frac{A_{m,n}\Gamma_0 - A_{m,n}\Gamma_{m+n}}{A_{0,m+n}} = A_{m,n}\frac{\Gamma_0 - \Gamma_{m+n}}{A_{0,m+n}} = A_{m,n}.$$

Thus, Eq. (37) reduces to

$$(38) \qquad \omega_{n,m} = \frac{A_{m,n}}{A_{n+m,0}} \, \omega_{0,m+n},$$

when $m \neq -n$. Since an element of $H^2(\mathcal{W}, \mathcal{H})$ is defined as an element of $Z^2(\mathcal{W}, \mathcal{H})$ modulo co-boundaries (i.e. elements in $B^2(\mathcal{W}, \mathcal{H})$), we can modify the representative ω of the class $[\omega] \in H^2(\mathcal{W}, \mathcal{H})$ by adding a co-boundary and not change the class $[\omega]$. It is this redundancy of description that leads to central extension of the Witt algebra to form the Virasoro algebra.

We then consider the new co-cycle (still in the same class of ω)

$$\omega' = \omega + \delta\mu,$$

where λ is the linear map that is defined (on a basis) by

$$\begin{cases} \mu(L_p) = \dfrac{1}{A_{p,0}} \, \omega(L_0, L_p) & \text{for } p \neq 0 \\ \mu(L_0) = -\dfrac{1}{A_{1,-1}} \omega(L_1, L_{-1}). \end{cases}$$

It is now straightforward to verify that, for $n + m \neq 0$,

$$\omega'(L_n, L_m) = 0.$$

In fact by definition, and using Eq. (38)

$$\begin{aligned} \omega'(L_n, L_m) &= \omega(L_n, L_m) + \mu([L_n, L_m]) \\ &= \frac{A_{m,n}}{A_{n+m,0}} \, \omega_{0,n+m} + \mu(A_{n,m} \otimes L_{n+m}) \\ &= \frac{A_{m,n}}{A_{n+m,0}} \, \omega_{0,n+m} - \frac{A_{m,n}}{A_{n+m,0}} \, \omega_{0,n+m} = 0. \end{aligned}$$

Therefore, $\omega'(L_n, L_m)$ is non-zero only if $n + m = 0$, and thus,

$$\omega'_{m,n} = \delta_{m+n} h_n(s).$$

Now, using that $\mu(L_0) = -\frac{1}{A_{1,-1}} \omega(L_1, L_{-1})$, we have that

$$(39) \qquad\qquad\qquad h(1) = 0.$$

In fact,

$$h(1) = \omega'(L_1, L_{-1}) = \omega(L_1, L_{-1}) + \mu([L_1, L-1])$$
$$= \omega(L_1, L_{-1}) - A_{1,-1}\mu(L_0)$$
$$= \omega(L_1, L_{-1}) - A_{1,-1}\frac{1}{A_{1,-1}}\omega(L_1, L_{-1}) = 0.$$

Also, since ω' is alternate

$$h(0) = 0.$$

We next compute $h(n)$. In order to do this, we plug in $\omega'_{m,n} = \delta_{m+n}h_n(s)$ back into Eq. (36) –which we now use for ω' instead of ω– and obtain,

$$(40) \qquad 0 = \frac{A_{m+\ell,n+\ell}\Gamma_\ell - A_{m,n}\Gamma_{m+n}}{A_{\ell,m+n}}\omega'_{\ell,m+n}$$
$$+ \frac{A_{n+m,\ell+m}\Gamma_m - A_{n,\ell}\Gamma_{n+\ell}}{A_{m,n+\ell}}\omega'_{m,n+\ell}$$
$$+ \frac{A_{\ell+n,m+n}\Gamma_n - A_{\ell,m}\Gamma_{\ell+m}}{A_{n,\ell+m}}\omega'_{n,\ell+m}$$
$$= \frac{A_{m+\ell,n+\ell}\Gamma_\ell - A_{m,n}\Gamma_{m+n}}{A_{\ell,m+n}}\delta_{\ell+m+n}\,h(l)$$
$$+ \frac{A_{n+m,\ell+m}\Gamma_m - A_{n,\ell}\Gamma_{n+\ell}}{A_{m,n+\ell}}\delta_{\ell+m+n}\,h(m)$$
$$+ \frac{A_{\ell+n,m+n}\Gamma_n - A_{\ell,m}\Gamma_{\ell+m}}{A_{n,\ell+m}}\delta_{\ell+m+n}\,h(n).$$

Since $\delta_{\ell+m+n} = 0$ unless $\ell + m + n = 0$, we can set $\ell = -(n+m)$. As a consequence

$$0 = \frac{A_{-n,-m}\Gamma_{-(n+m)} - A_{m,n}\Gamma_{m+n}}{A_{-(n+m),m+n}}\,h(-(n+m))$$
$$+ \frac{A_{n+m,-n}\Gamma_m - A_{n,-(n+m)}\Gamma_{-m}}{A_{m,-m}}\,h(m)$$
$$+ \frac{A_{-m,m+n}\Gamma_n - A_{-(n+m),m}\Gamma_{-n}}{A_{n,-n}}\,h(n)$$
$$= -\frac{A_{-n,-m}\Gamma_{(-n+m)} - A_{m,n}\Gamma_{m+n}}{A_{-(n+m),m+n}}\,h((n+m))$$
$$+ \frac{A_{n+m,-n}\Gamma_m - A_{n,-(n+m)}\Gamma_{-m}}{A_{m,-m}}\,h(m)$$
$$+ \frac{A_{-m,m+n}\Gamma_n - A_{-(n+m),m}\Gamma_{-n}}{A_{n,-n}}\,h(n),$$

where we have used that $h(n)$ is odd: $h(-(n+m)) = -h(n+m)$ (since ω' is alternate). Next, we set also $n = 1$ so as to obtain after using that $h(1) = 0$ as per Eq. (39), the following recursive relation,

$$(41) \qquad \frac{A_{-1,-m}\Gamma_{(-(m+1))} - A_{m,1}\Gamma_{m+1}}{A_{-(m+1),m+1}} \, h((m+1))$$
$$= \frac{A_{m+1,-1}\Gamma_1 - A_{1,-(m+1)}\Gamma_{-m}}{A_{m,-m}} \, h(m),$$

and thus $h(m)$ is completely determined by $c := h(2)$. $\qquad\qquad\qquad$ $\square$

Next, we show a mere algebraic fact that was used in the proof above, namely

Lemma 3.4.

$$\frac{A_{n,m+n}\Gamma_n - A_{0,m}\Gamma_m}{A_{n,m}} - \frac{A_{n+m,m}\Gamma_m - A_{n,0}\Gamma_n}{A_{m,n}} = A_{n+m,0}$$

and

$$\frac{A_{m,n}\Gamma_0 - A_{m,n}\Gamma_{m+n}}{A_{0,m+n}} = A_{m,n}.$$

Proof. Since

$$A_{p,q}(s) = \Gamma_p(s) - \Gamma_q(s),$$

we have that

$$\frac{A_{n,m+n}\Gamma_n - A_{0,m}\Gamma_m}{A_{n,m}} - \frac{A_{n+m,m}\Gamma_m - A_{n,0}\Gamma_n}{A_{m,n}}$$
$$= \frac{\Gamma_n^2 - \Gamma_{m+n}\Gamma_n - \Gamma_0\Gamma_m + \Gamma_m^2 + \Gamma_{n+m}\Gamma_m - \Gamma_m^2 - \Gamma_n^2 + \Gamma_0\Gamma_n}{A_{m,n}},$$
$$= \frac{\Gamma_{n+m}(\Gamma_m - \Gamma_n) - \Gamma_0(\Gamma_m - \Gamma_n)}{A_{m,n}} = \Gamma_{n+m} - \Gamma_0 = A_{n+m,0}$$

which proves the first identity and

$$\frac{A_{m,n}\Gamma_0 - A_{m,n}\Gamma_{m+n}}{A_{0,m+n}} = A_{m,n}\frac{\Gamma_0 - \Gamma_{m+n}}{A_{0,m+n}} = A_{m,n}$$

proves the second. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. Asymptotics of functions in $\mathcal{H}$

Here we show that for $a < 1$ the functions $\Gamma_p(s)$ are sub-linear as $|s| \to +\infty$ and that $A_{p,q}(s) \to 0$ as $|s| \to +\infty$. More precisely, we propose as follows.

Proposition 4.1. *If* $0 < a < 1$,

$$\Gamma_p(s) \sim \left(\frac{a}{e}\right)^a (s+p)^a,$$

and

$$\lim_{|s| \to +\infty} A_{p,q}(s) = 0.$$

Therefore, for any $\phi \in \mathcal{H}$, *either*

$$\phi(s) \sim C \ (s+p)^a$$

for some constant C, or

$$\lim_{|s| \to +\infty} \phi(s) = 0.$$

Proof. Stirling's formula dictates that

$$\Gamma(z+1) \sim \sqrt{2\pi z} \left(\frac{z}{e}\right)^z;$$

whence

$$\Gamma_p(s) = \frac{\Gamma(a(s+p)+1)}{\Gamma(a(s+p-1)+1)}$$

$$\sim \sqrt{\frac{2\pi(a(s+p))}{2\pi(a(s+p-1))}} \left(\frac{(a(s+p))}{e}\right)^{(a(s+p))} \left(\frac{e}{(a(s+p-1))}\right)^{(a(s+p-1))}$$

$$= \sqrt{\frac{s+p}{s+p-1}} \left(\frac{(a(s+p))}{e}\right)^a \sim \left(\frac{(a(s+p))}{e}\right)^a,$$

thereby corroborating our first claim. Next, using this relationship, we find that

$$A_{p,q}(s) \sim \left(\frac{a}{e}\right)^a ((s+p)^a - (s+q)^a),$$

and all we need to show is that

$$\lim_{|s| \to +\infty} ((s+p)^a - (s+q)^a) = 0.$$

In order to do that we appeal to the generalized binomial theorem which establishes that, for $|s| > |p|$,

$$(s + p)^a = \sum_{k=0}^{\infty} \binom{a}{k} s^{a-k} p^k,$$

and likewise, for $|s| > |q|$,

$$(s + q)^a = \sum_{k=0}^{\infty} \binom{a}{k} s^{a-k} q^k,$$

where the binomial coefficients are defines by,

$$\binom{a}{k} = \frac{\Gamma(a + 1)}{k! \, \Gamma(a - k + 1)}.$$

Therefore, for $|s| > \max\{|p|, |q|\}$,

$$(s + p)^a - (s + q)^a = \sum_{k=1}^{\infty} \binom{a}{k} s^{a-k} (p^k - q^k).$$

Since $a < 1$, clearly for $k \geq 1$, $s^{a-k} \to 0$ as $|s| \to +\infty$ and hence the claim is proven. The last statement about functions in $\mathcal{H}$ follows from the previous two statements and the very definition of $\mathcal{H}$, which is generated by the constants to be closed under the brackets, $[\phi(s), \psi(s)]_{\mathcal{H},p,q} = \phi(s + p)\psi(s)\Gamma_p - \psi(q + s)\phi(s)\Gamma_q.$ □

The main application of the study of asymptotics of elements in $\mathcal{H}$ is in showing the following simple structure of the algebra $\mathcal{H}_a$.

Theorem 4.1. $\mathcal{H}_a$ *has a filtration* $\mathcal{H}_0 = \mathbb{C} \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots$ *consisting of finite dimensional vector spaces, for* $a \in \mathbb{Q}$.

Proof. First, we define $\mathcal{H}_1$ as the vector space $\bigoplus_{p,q \in \mathbb{Z}} A_{p,q} \mathbb{C}$ and $\mathcal{H}_2$ the vector space generated by elements of the form:

(42) $$\psi(s + p)\phi(s)\Gamma_p - \phi(s + q)\psi(s)\Gamma_q,$$

where $\psi, \phi \in \mathcal{H}_1$. More generally, we define $\mathcal{H}_\ell$ inductively as the vector space generated by elements of the in equation (42) with $\psi, \phi \in \mathcal{H}_{\ell-1}$.

The main idea is to show that there are finitely many elements in $\mathcal{H}_\ell$ that do not have finitely many assigned poles and which generate via linear

combinations all the other elements with infinitely many poles. Then the asymptotics previously discussed will do the trick.

Observe that because the Gamma function has simple poles at negative integers $\{-k: \quad k \in \mathbb{N}\}$ with residue at $-k$ equal to,

$$a_{-1} = \frac{(-1)^k}{k!},$$

analogously the functions $\Gamma(a(s+p)+1)$ and $\Gamma(a(s+p-1)+1)$, which appear in the ratio $\Gamma_p(s) = \frac{\Gamma(a(s+p)+1)}{\Gamma(a(s+p-1)+1)}$, also have poles of the same type at the points $s_{k,p} = \frac{-k}{a} - p$ and $t_{h,p} = \frac{-h}{a} - p + 1$ respectively, with $k, h \in \mathbb{N} \setminus \{0\}$. One readily verifies that the functions

$$\Gamma\left(a(s+p)+1\right) \quad \text{and} \quad \Gamma\left(a(s+p-1)+1\right)$$

never share the poles $s_{k,p}$ and $t_{h,p}$.

We then concentrate on the difference $A_{p,q} = \Gamma_p - \Gamma_q$. We assume that $a \in \mathbb{Q}$, so we can write $a = \frac{M}{N}$ with $(M, N) = 1$ (i.e., they are coprime). Since Γ_p and Γ_q have the same residues at poles, if we show that for some pole $s_{k,p}$ of Γ_p there is a corresponding pole $s_{q,h}$ for Γ_q such that $s_{k,p} = s_{q,h}$, then at such point $A_{p,q}$ will be regular. One readily verifies that

$$s_{k,p} = s_{h,q}$$

if and only if

$$N(k-h) = M(q-p)$$

which is solvable if and only if $q - p$ is divisible by N. Thus, the $A_{p,q}$ which do not have finitely many poles are the ones for which $p \not\equiv q(\text{mod } N)$. Next, observe that if $p \not\equiv q(\text{mod } N)$, we can write:

$$A_{p,q} = A_{p,n} + A_{n,q}$$

and we can choose n so that $n \equiv q(mod. N)$, thus reducing to the case where we only have finitely many $A_{p,q}$ for any given p. But by the symmetry of $A_{p,q}$, we can apply the same argument to p and reduce ourselves to essential $A_{p',q'}$ with $0 \le p', q' < N$. This shows that $\mathcal{H}_1$ is generated by the finitely many $A_{p',q'}$ with $0 \le p', q' < N$ and a set of $A_{p,q}$'s with $p \equiv q(mod. N)$. The latter $A_{p,q}$'s span a finitely dimensional vector space due to their sublinear asymptotics and the fact that they all have fintely many poles. The arguments for the other $\mathcal{H}_k$'s are similar. $\square$

4.1. Final remarks

We have constructed a Virasoro algebra with non-local operators as the generators of the currents. This algebra should be relevant to any problem in which the effective currents are non-local. The resultant algebra reflects the non-locality as it cannot be defined independent of the basis. Consequently, the central charge is explicitly a function rather than a number. Evaluated on the basis results in a charge for every degree, where the degree is the exponent of the element, z^p. While the next step is to construct explicitly the representations that generate the fractional currents, this algebra opens the possibility of new models in string theory based on such non-local currents as the basis for the conformal sector.

Acknowledgements

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