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Complete solution of a gauged tensor model

Chethan Krishnan and K. V. Pavan Kumar

Building on a strategy introduced in arXiv:1706.05364, we present exact analytic expressions for all the singlet eigenstates and eigenvalues of the simplest non-linear $(n = 2, d = 3)$ gauged Gurau-Witten tensor model. This solves the theory completely. The ground state eigenvalue is $-2\sqrt{14}$ in suitable conventions. This matches the result obtained for the ground state energy in the ungauged model, via brute force diagonalization on a computer. We find that the leftover degeneracies in the gauged theory, are only partially accounted for by its known discrete symmetries, indicating the existence of previously unidentified "hidden" global symmetries in the system. We discuss the spectral form factor, the beginnings of chaos, and the distinction between theories with $SO(n)$ and $O(n)$ gaugings. Our results provide the complete analytic solution of a non-linear gauge theory in 0+1 dimensions, albeit for a specific value of N. A summary of the main results in this paper were presented in the companion letter `arXiv:1802.02502`.

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1. Introduction

The purpose of this paper is to present the complete analytic solution of a gauged non-linear 0+1 dimensional quantum theory of strongly interacting fermions. More specifically, we will present the eigenvalues and eigenstates of the $n = 2, d = 3$ colored tensor model in the language of [1]. A letter version of this paper with only (some of) the results was presented in [2], here we will present the various technical details of both the method as well as the complete results. Before proceeding, we will present a few words of motivation as well as make some general comments.

Gauged tensor models [3] have been around for a while, but they experienced a resurgence in interest after Witten argued [1] that certain classes of them can mimic the large-N diagrammatics of the SYK model [4, 5] and therefore they might be of interest for holographic purposes (see related discussions in [6, 7]). Unlike the SYK model, these theories are not disorder averaged and therefore are perfectly legitimate quantum mechanical theories. In this paper, we will view these tensor models as a class of strongly coupled gauge theories in 0+1 dimensions, with tunable parameters that capture N. Our goal will be to see how far we can proceed in exactly (non-pertubatively) solving them, and we will stick to relatively small N.

The strategy we present here is in principle general enough to go through for all colored Gurau-Witten tensor models with gauge group $O(n)^4 \times O(2)^2$, as well as possibly other related classes of tensor models. Of course, at larger n the implementatioon of the approach is more complicated. What we have explicitly solved here is the $n = 2$ case. Even though the gauge group is Abelian, because of the quartic self-interactions the theory is still highly non-linear. One of the advantages of 0+1 dimensions is that the gauge field is not dynamical and merely imposes a singlet constraint (see e.g., [8]) for any n. We take systematic advantage of this, as well as of the crucial fact observed in [9] that the Hilbert space is a spinor and that its Clifford structure can be

exploited profitably. We should emphasize that while the ungauged model is more or less straightforwardly diagonalized numerically on a computer, the gauged theory is not. This is because constructing singlets and singlet eigenstates in a useful way in these theories is difficult. In particular, it is unclear (to us) whether having the numerical eigenvalue spectrum of the ungauged theory is helpful in solving the gauged theory. This is what makes this problem interesting. In fact, even the counting of singlets is a non-trivial problem, see [10]. The dimensionality of the gauged Hilbert space we find matches with the count in [10].

It is easy to convince oneself that the eigenvalues of the gauged model should form a subset of the ungauged model. Since the ungauged model can be diagonalized numerically, this offers us a non-trivial test of our solution. Indeed, we find that all the eigenvalues that we find (our eigenvalues are all square roots of integres as it turns out) match with (a subset of) the numerical eigenvalues of [11] upto six decimal places. In particular, the ground state energy is $-2\sqrt{14}$ in units where the coupling $J = 1$[1], and agrees with the ungauged ground state energy. We will offer a qualitative understanding of the origin of the irrational energies in a later section.

Despite the relative smallness of N we find that the theory does lead to some rudimentary large-N features like chaos. In particular, the Spectral Form Factor (SFF) is qualitatively identical to that of the ungauged model found in [11, 12] which in turn was related to the dip-ramp-plateau structure of SYK [13]. This should be contrasted to the simplest uncolored model [14] whose gauged version was solved in [15, 16], but it was found to be a rather trivial two-state system. The ungauged Hilbert space there was only 16 dimensional, and after gauging only two states were left. Here, we start with an ungauged Hilbert space that is 65536 dimensional and after gauging we end up with a Hilbert space that is 140 dimensional. There are 11 distinct eigenvalues in the spectrum. We are able to explain many of the degeneracies in the final Hilbert space in terms of the known [1, 11] discrete global symmetries of the tensor model. However, there remain degeneracies which are unaccounted for by the known symmetries of the

[1]See next section for definition of J. The coupling is dimensionful and setting it to one corresponds merely to a choice of unit, and we will do so in most of the paper. We can always re-instate it by dimensional analysis. Note that the running coupling one expects at long distances in melonic theories should not be directly compared to this. What we are solving is the UV theory. To get IR correlators one should calculate correlators and look at their long time behavior. The effective running coupling will emerge then.

system, suggesting that there are (so far) unidentified hidden symmetries in the Gurau-Witten tensor model. It is clearly of interest to identify them.

Note that it is the *finding* of the solution that is the difficult part in the problem. Once we find the solution it is easy enough to *verify*: by explicitly acting with the Hamiltonian on the eigenstates. This means that we can have quite a bit of confidence that the solution is indeed right. Further tests of the solution include the fact that the eigenvalues match with previous numerical results of a (subset of) eigenvalues in the ungauged tensor model, as well as the match of the dimensionality of the Hilbert space with the indirect count of singlets in [10][2].

We will mostly discuss the $SO(n)$ version of the model in detail in this paper. It is also possible to consider the $O(n)$ model, which will remove many more of the states from the spectrum and for completeness, we present a discussion of that as well in Section 8. But it should be kept in mind that in the rest of the paper we will not emphasize the distinction between the two, even though we always have the $SO(n)$ case in mind.

Comment added. The explicit forms of the singlet states are suppressed in the journal version, but can be found in the appendices of the arXiv version.

2. Gurau-Witten model

Gurau-Witten model is a quantum mechanical model in 0+1 dimensions. The model is constructed using fermionic tensors of the form $\psi_A^{i_1 i_2 \ldots i_d}$. The index A corresponds to color index and take values from 0 to d and the tensor indices take values from 1 to n. That is, the total degrees of freedom is given by $N = (d+1)n^d$.

For each pair of colors (A, B), we assign a symmetry group $O(n)_{AB}$. As there are $(d+1)$ colors in the theory, the overall symmetry group of the theory is given by:

$$G \sim O(n)^{d(d+1)/2} \tag{2.1}$$

Further, we demand that under any group $O(n)_{AB}$, fermions belonging to the colors A and B transform in the vector representation and the fermions of rest of the colors transform trivially. The interaction term of the theory is

[2]Note that [10] appeared after the Letter version of this paper [2] on the arXiv, so this is *not* a retrodiction.

an invariant under the symmetry group G and contains fermions belonging to all the $(d+1)$ colors.

From now on, we work with $d=3$ i.e., we work with a theory that has 4 colors and has a quartic interaction term. The Lagrangian of $d=3$ Gurau-Witten model is given by:

$$\mathcal{L} = \frac{i}{2}\psi_A^{ijk}\partial_t\psi_A^{ijk} + \frac{J}{n^{3/2}}\sum \psi_0^{ijk}\psi_1^{ilm}\psi_2^{njm}\psi_3^{nlk} \tag{2.2}$$

where $(0,1,2,3)$ correspond to the color indices and each of the tensor indices takes values from 1 to n. J is the dimensionful coupling and we set it to unity in the rest of the discussion. Quantizing this theory gives rise to the following anti-commutation relations:

$$\{\psi_A^{ijk}, \psi_B^{pqr}\} = \delta_{AB}\delta^{ip}\delta^{jq}\delta^{kr} \tag{2.3}$$

The kinetic term of $d=3$ Gurau-Witten Lagrangian has $O(4n^3)$ symmetry which is broken down to $O(n)^6$ due to the presence of interaction term. That is, the symmetry group G of the theory is given by:

$$G \sim O(n)_{01} \times O(n)_{02} \times O(n)_{03} \times O(n)_{12} \times O(n)_{13} \times O(n)_{23}. \tag{2.4}$$

More specifically, each of the fermionic tensors transform under G as follows:

$$\begin{aligned}
\psi_0^{ijk} &\to M_{01}^{ii'}\ M_{02}^{jj'}\ M_{03}^{kk'}\ \psi_0^{i'j'k'}\\
\psi_1^{ijk} &\to M_{01}^{ii'}\ M_{13}^{jj'}\ M_{12}^{kk'}\ \psi_1^{i'j'k'}\\
\psi_2^{ijk} &\to M_{23}^{ii'}\ M_{02}^{jj'}\ M_{12}^{kk'}\ \psi_2^{i'j'k'}\\
\psi_3^{ijk} &\to M_{23}^{ii'}\ M_{13}^{jj'}\ M_{03}^{kk'}\ \psi_3^{i'j'k'}
\end{aligned} \tag{2.5}$$

where M_{AB} are the matrices that correspond to the group $O(n)_{AB}$. Using this information, we can compute the Noether charges corresponding to the symmetry group G as follows:

$$Q_{01}^{i_1i_2} = i\left(\psi_0^{i_1jk}\psi_0^{i_2jk} + \psi_1^{i_1jk}\psi_1^{i_2jk}\right) \tag{2.6}$$

$$Q_{23}^{i_1i_2} = i\left(\psi_2^{i_1jk}\psi_2^{i_2jk} + \psi_3^{i_1jk}\psi_3^{i_2jk}\right) \tag{2.7}$$

$$Q_{02}^{j_1j_2} = i\left(\psi_0^{ij_1k}\psi_0^{ij_2k} + \psi_2^{ij_1k}\psi_2^{ij_2k}\right) \tag{2.8}$$

$$Q_{13}^{j_1 j_2} = i\left(\psi_1^{ij_1 k}\psi_1^{ij_2 k} + \psi_3^{ij_1 k}\psi_3^{ij_2 k}\right) \tag{2.9}$$

$$Q_{03}^{k_1 k_2} = i\left(\psi_0^{ijk_1}\psi_0^{ijk_2} + \psi_3^{ijk_1}\psi_3^{ijk_2}\right) \tag{2.10}$$

$$Q_{12}^{k_1 k_2} = i\left(\psi_1^{ijk_1}\psi_1^{ijk_2} + \psi_2^{ijk_1}\psi_2^{ijk_2}\right) \tag{2.11}$$

where the subscripts are the color indices. Note that the upper indices on any of the charges should not be equal.

3. The Clifford basis

Before explaining our strategy to find the singlet spectrum, we define the basis that we work with. The construction of basis is based on the fact that the Hilbert space of our theory forms a spinor representation of $O(n)$.

To start with, following and slightly generalizing [9, 16], we define for even n the "colored" creation and annihilation operators as:

$$\psi_A^{ijk^\pm} = \frac{1}{\sqrt{2}}\left(\psi_A^{ijk} \pm i\psi_A^{ij(k+1)}\right) \tag{3.1}$$

where k takes only odd values and $k^\pm$ is given by the relation:

$$k = 2k^\pm - 1 \tag{3.2}$$

Further, we can show that the $\psi_A^{ijk^\pm}$ obey the following anti-commutation relations:

$$\begin{aligned} &\{\psi_A^{ijk^+}, \psi_B^{lmn^+}\} = 0; \quad \{\psi_A^{ijk^-}, \psi_B^{lmn^-}\} = 0; \\ &\{\psi_A^{ijk^+}, \psi_B^{lmn^-}\} = \delta_{AB}\delta^{il}\delta^{jm}\delta^{k^+,n^-} \end{aligned} \tag{3.3}$$

We define the Clifford vacuum as the state that is annihilated by all ψ^-'s i.e.,

$$\psi_A^{ijk^-}|\,\rangle = 0 \tag{3.4}$$

Acting with the creation operators, we can construct the entire Hilbert space. Since there are $2n^3$ (fermionic) creation operators, we can see that the dimensionality of Hilbert space is 2^{2n^3}.

For forthcoming purposes, we define four level operators L_A, which counts the number of creation operators of each color in a state. These

L_A are defined as:

$$L_A = \psi_A^{ijk+}\psi_A^{ijk-} \tag{3.5}$$

Note that there is no summation over the color index A. The commutation relations with the fermionic tensors are given by:

$$[L_A, \psi_B^{ijk^\pm}] = \pm\psi_B^{ijk^\pm}\delta_{AB} \tag{3.6}$$

There is no summation over B on the RHS.

The Hamiltonian can be written in terms of the creation and annihilation operators as follows:

$$\begin{aligned} H = \sum \psi_0^{ijk^+}\psi_1^{ilm^+}\psi_2^{njm^-}\psi_3^{nlk^-} + \psi_0^{ijk^+}\psi_1^{ilm^-}\psi_2^{njm^+}\psi_3^{nlk^-} \\ + \psi_0^{ijk^-}\psi_1^{ilm^+}\psi_2^{njm^-}\psi_3^{nlk^+} + \psi_0^{ijk^-}\psi_1^{ilm^-}\psi_2^{njm^+}\psi_3^{nlk^+} \end{aligned} \tag{3.7}$$

Note that each of the four terms in the Hamiltonian is manifestly invariant under $O(n)^4 \times U(\frac{n}{2})^2$. From the explicit form of the Hamiltonian, it is clear that it does not commute with the level operators corresponding to individual colors. As we will explain later, this makes finding the eigenstates of the Hamiltonian more difficult in the case of $n = 2$ as compared to that of identifying the singlets.

4. The singlet spectrum

In this section, we describe our strategy to find the singlet spectrum of the Gurau-Witten model. Our strategy here is a "colored" generalization of the one presented in [16] for uncolored models and in principle can be implemented to identify singlet spectrum of Gurau-Witten model with arbitrary d and n. Note that one can gauge first and then solve the theory or solve the theory and then gauge it afterwards. The second approach leads to complications related to Young tableaux proliferation [17], so we will stick to the first.

We start by noting that the singlet states by definition are the states that transform trivially under the symmetry group. This definition can be operationally implemented by demanding that the singlet states are annihilated by the Noether charges (2.6)–(2.11) corresponding to the symmetry group G i.e.,

$$Q_{AB}\ |\text{singlet}\rangle = 0 \tag{4.1}$$

So, our goal is to find the linear combination of our basis states that are annihilated by all the Noether charges. Since the number of basis states are exponentially large, this seems a daunting task. But the following simplification mitigates the situation partially and in particular makes the model of our interest ($n = 2$ model) tractable. We emphasize that the last statement does *not* mean that our method of finding singlet states using equation (4.1) is restricted to $O(2)^6$ GW model alone and indeed can be extended to $O(n)^6$ model. Further, the techniques and simplifications that we discuss in this paper for $O(2)^6$ model will be applicable to the case of $O(n)^4 \times O(2)^2$ model as well and work is in progress along this direction.

The simplification is that all the gauge singlet states are present in the mid-Clifford level. It can be shown as follows. Taking $k_2 = k_1 + 1$ in the last two Noether charges (2.10) and (2.11) and summing over all the odd k_1's, we get:

$$\begin{aligned} \left(L_0 + L_3 - \frac{n^3}{2}\right) \ |\text{singlet}\rangle = 0 \\ \left(L_1 + L_2 - \frac{n^3}{2}\right) \ |\text{singlet}\rangle = 0 \end{aligned} \tag{4.2}$$

These conditions imply that the singlets are at the mid-Clifford ($= n^3$) level in which $\frac{n^3}{2}$ creation operators belong to the colors $A = 0, 3$ and the other half of them belong to the colors $A = 1, 2$.

Even with this simplification, identifying all the singlets is still a non-trivial task and we currently[3] do not have a solution for Gurau-Witten model with arbitrary n. As we will explain later, $n = 2$ case has some extra simplifications which helps us in identifying all the singlets.

Once we have a strategy to identify the singlets, the next step is to find linear combinations of singlets such that they form eigenstates of the Hamiltonian. As the Noether charges commute with the Hamiltonian, acting on any singlet state with the Hamiltonian necessarily gives a combination of singlet states. This is in general a hard task and the judicious application of residual symmetries of the Hamiltonian helps us by reducing the number of computations that we need to do. We will elaborate on the residual symmetries in the next section.

[3]Finding all the singlets in uncolored model with arbitrary n is comparatively simpler problem and work towards this is in progress [18].

5. Discrete (residual) symmetries

The two tasks we have at hand- identifying the singlets and constructing singlet eigenstates of the Hamiltonian are conceptually simple. But the computations involved are often tedious and can not be evaded. But, we can reduce their number by exploiting the residual symmetries[1] of the Gurau-Witten Hamiltonian. In this section, we identify these set of (discrete) residual symmetries. These symmetries are related to the permutation of colors and are not part of the $O(n)^6$ group that we will gauge later in the paper. Thus, these symmetries will be helpful in identifying the singlet eigenstates of the Hamiltonian and also in explaining some of the degeneracies that we find in the singlet spectrum.

The first set of these symmetries are denoted as:

$$S_{01;23};\ S_{02;13};\ S_{03;12} \tag{5.1}$$

The action of these symmetries is as follows. The operator $S_{AB;CD}$ acting on a state exchanges the colors $A \leftrightarrow B$ and $C \leftrightarrow D$ simultaneously. The action of these operators on the Noether charges is as follows:

$$\begin{aligned}
&S_{01;23}\ Q_{01}\ S^{-1}_{01;23} = Q_{01}; \quad S_{01;23}\ Q_{23}\ S^{-1}_{01;23} = Q_{23};\\
&S_{01;23}\ Q_{02}\ S^{-1}_{01;23} = Q_{13}; \quad S_{01;23}\ Q_{13}\ S^{-1}_{01;23} = Q_{02}\\
&S_{02;13}\ Q_{01}\ S^{-1}_{02;13} = Q_{23}; \quad S_{02;13}\ Q_{23}\ S^{-1}_{02;13} = Q_{01};\\
&S_{02;13}\ Q_{02}\ S^{-1}_{02;13} = Q_{13}; \quad S_{02;13}\ Q_{13}\ S^{-1}_{02;13} = Q_{02}\\
&S_{03;12}\ Q_{01}\ S^{-1}_{03;12} = Q_{23}; \quad S_{03;12}\ Q_{23}\ S^{-1}_{03;12} = Q_{01};\\
&S_{03;12}\ Q_{02}\ S^{-1}_{03;12} = Q_{13}; \quad S_{03;12}\ Q_{13}\ S^{-1}_{03;12} = Q_{02}.
\end{aligned} \tag{5.2}$$

From these relations, it is easy to see that if $|a\rangle$ is a singlet state then $S_{AB;CD}|a\rangle$ is also a singlet state. Before moving ahead, we note that the operators $S_{01;23}$, $S_{02;13}$ and $S_{03;12}$ commute with the Hamiltonian.

The next set of operators are:

$$S_{03}; S_{12} \tag{5.3}$$

The operator S_{AB} exchanges the colors $A \leftrightarrow B$ along with the exchange of first two indices on each ψ. For instance,

$$S_{01}\psi_0^{ij+}|\ \rangle = \psi_1^{ji+}|\ \rangle \tag{5.4}$$

The action of these symmetries on the Noether charges is given as follows:

$$
\begin{aligned}
&S_{03}\ Q_{01}\ S_{03}^{-1} = Q_{13}; \quad S_{03}\ Q_{23}\ S_{03}^{-1} = Q_{02};\\
&S_{03}\ Q_{02}\ S_{03}^{-1} = Q_{23}; \quad S_{03}\ Q_{13}\ S_{03}^{-1} = Q_{01};\\
&S_{12}\ Q_{01}\ S_{12}^{-1} = Q_{02}; \quad S_{12}\ Q_{23}\ S_{12}^{-1} = Q_{13};\\
&S_{12}\ Q_{02}\ S_{12}^{-1} = Q_{01}; \quad S_{12}\ Q_{13}\ S_{12}^{-1} = Q_{23}.
\end{aligned} \tag{5.5}
$$

From the action of S_{AB} on Noether charges, we can see that if $|a\rangle$ is a singlet then $S_{AB}|a\rangle$ is also a singlet state.

The next set of operators we define are quite non-trivial as they do not commute with the level operators L_A defined in (3.5). The first such symmetry operator is S_{23}. S_{23} exchanges the colors 2 and 3 along with exchanging the second and third indices on each fermion. For instance,

$$
S_{23}\psi_2^{ijk}S_{23}^{-1} = \psi_3^{ikj} \tag{5.6}
$$

As this operator exchanges second and third indices and since we constructed our basis by breaking the $O(n)^2$ symmetry corresponding to the third indices, S_{23} can be thought of as an operator that relates our basis (the one obtained by breaking $O(n)$'s of the third indices) to another basis that needs breaking of $O(n)$'s corresponding to the second indices. As a result, the action of S_{23} on our Clifford vacuum is quite non-trivial. To understand the action of S_{23} on the Clifford vacuum, we need to know the operator S_{23} explicitly. The construction of S_{23} is straightforward[4] and uses the following identities:

$$
\begin{aligned}
&\frac{1}{2}\left(\psi_A^{ijk} + \psi_B^{i'jk}\right)\psi_A^{ijk}\left(\psi_A^{ijk} + \psi_B^{i'j'k'}\right) = \psi_B^{i'j'k'}\\
&\frac{1}{2}\left(\psi_A^{ijk} + \psi_B^{i'jk}\right)\psi_B^{i'j'k'}\left(\psi_A^{ijk} + \psi_B^{i'j'k'}\right) = \psi_A^{ijk}
\end{aligned} \tag{5.7}
$$

Using these identities, we can write down the explicit form of S_{23} operator as:

$$
\begin{aligned}
S_{23} = {} & \psi_0^{111}\psi_0^{122}\psi_0^{211}\psi_0^{222}\left(\psi_0^{112} + \psi_0^{121}\right)\left(\psi_0^{212} + \psi_0^{221}\right)\\
& \psi_1^{111}\psi_1^{122}\psi_1^{211}\psi_1^{222}\left(\psi_1^{112} + \psi_1^{121}\right)\left(\psi_1^{212} + \psi_1^{221}\right)\\
& \left(\psi_2^{111} + \psi_3^{111}\right)\left(\psi_2^{112} + \psi_3^{121}\right)\left(\psi_2^{121} + \psi_3^{112}\right)\left(\psi_2^{122} + \psi_3^{122}\right)\\
& \left(\psi_2^{211} + \psi_3^{211}\right)\left(\psi_2^{212} + \psi_3^{221}\right)\left(\psi_2^{221} + \psi_3^{212}\right)\left(\psi_2^{222} + \psi_3^{222}\right)
\end{aligned} \tag{5.8}
$$

[4]For more details, see [12]

The action of this operator on the Clifford vacuum is given by:

$$S_{23}|\,\rangle = \big[\,\big(\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+}\big) - i\,\big(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+}\big)\,\big]\,[0 \to 1,2,3]\,|\,\rangle \tag{5.9}$$

The operator S_{23} permutes the Noether charges (2.6)–(2.11) among themselves. More precisely, we have:

$$\begin{aligned} &S_{23}Q_{01}S_{23}^{-1} = Q_{01}; \quad S_{23}Q_{23}S_{23}^{-1} = Q_{23} \\ &S_{23}Q_{12}S_{23}^{-1} = Q_{13}; \quad S_{23}Q_{13}S_{23}^{-1} = Q_{12} \\ &S_{23}Q_{02}S_{23}^{-1} = Q_{03}; \quad S_{23}Q_{03}S_{23}^{-1} = Q_{02} \end{aligned} \tag{5.10}$$

From these relations, it is clear that if $|a\rangle$ is a singlet, then $S_{23}|a\rangle$ is also a singlet.

Likewise, we can define an operator S_{13} that exchanges the colors 1 and 3 along with exchanging the first and third indices. Similar to the case of S_{23}, we can write down the explicit form of S_{13}. The action of S_{13} on the Clifford vacuum is given as:

$$S_{13}|\,\rangle = \big[\,\big(\psi_0^{11+}\psi_0^{12+} - \psi_0^{21+}\psi_0^{22+}\big) - i\,\big(\psi_0^{11+}\psi_0^{22+} - \psi_0^{12+}\psi_0^{21+}\big)\,\big]\,[0 \to 1,2,3]\,|\,\rangle \tag{5.11}$$

Note that S_{13} is not an independent operator and can be obtained from the operators we have already defined. For instance, it can be written as a combination of S_{23} and S_{12} as follows:

$$S_{13} = S_{12}\; S_{23}\; S_{12}^{-1} \tag{5.12}$$

We can also define operators S_{01} and S_{02} that have an action analogous to S_{23} and S_{13} respectively. These operators are also not independent and can be obtained from the operators we have defined already. Note that the operators of the form S_{AB} anti-commute with the Hamiltonian.

Further, we can define operators of the form:

$$S_A = \prod_{i,j,k=1}^{n} \psi_A^{ijk} \tag{5.13}$$

From the anti-commutation relations, we see that the action of this operator is as follows:

$$S_A \psi_B S_A^{-1} = (-1)^{n-1} \psi_B \quad \text{if} \quad A = B \tag{5.14}$$

$$= (-1)^n \psi_B \quad \text{if} \quad A \neq B \tag{5.15}$$

The operators S_A commute with the Noether charges but anti-commute with the Hamiltonian.

6. $n = 2$

From now on, we specialize to the case of $n = 2$. The Clifford levels are $2n^3 = 16$ in number and thus the Hilbert space is 2^{16} dimensional. From the mid-level condition (4.2), we find that all the singlets should be at 8th Clifford level. Out of the 8 creation operators, four of them should have $A = \{0, 3\}$ color indices and the remaining four should have $A = \{1, 2\}$ color indices. A generic candidate singlet state satisfying these constraints is of the form:

$$\sum \alpha^{0/3\ldots,0/3,1/2\ldots1/2}_{i_1 j_1; i_2 j_2; \ldots i_8 j_8} \psi^{i_1 j_1 1^+}_{0/3} \psi^{i_2 j_2 1^+}_{0/3} \psi^{i_3 j_3 1^+}_{0/3} \psi^{i_4 j_4 1^+}_{0/3} \psi^{i_5 j_5 1^+}_{1/2} \psi^{i_6 j_6 1^+}_{1/2} \psi^{i_7 j_7 1^+}_{1/2} \psi^{i_8 j_8 1^+}_{1/2} | \rangle \tag{6.1}$$

The total number of α's are given by $\binom{8}{4}\binom{8}{4} = 4900$. These α's can be divided into 25 different groups based on the bi-partitions of 4 which are given by:

$$4 = 4 + 0 = 3 + 1 = 2 + 2 = 1 + 3 = 0 + 4 \tag{6.2}$$

We call these partitions as $p_1, \ldots p_5$. Each of the candidate singlet states[5] belong to a unique group that can be denoted by an ordered pair (p_a, p_b) where p_a denotes the partition of the colors 0 and 3 whereas p_b denotes the partition corresponding to other two colors. For instance, the group of states denoted by (p_1, p_2) has four ψ_0^+'s; three ψ_1^+'s and a ψ_2^+. The number of states in each of the groups is given in Table 1.

[5]By candidate singlet state, we mean a state in the Clifford basis that satisfies the mid-level condition.

(p_a, p_b)	Number of states	Number of Singlets
(p_1, p_1)	1	1
(p_1, p_2)	16	0
(p_1, p_3)	36	4
(p_1, p_4)	16	0
(p_1, p_5)	1	1
(p_2, p_1)	16	0
(p_2, p_2)	256	16
(p_2, p_3)	576	0
(p_2, p_4)	256	16
(p_2, p_5)	16	0
(p_3, p_1)	36	4
(p_3, p_2)	576	0
(p_3, p_3)	1296	16+8+8+8+8+4+4=56
(p_3, p_4)	576	0
(p_3, p_5)	36	4
(p_4, p_1)	16	0
(p_4, p_2)	256	16
(p_4, p_3)	576	0
(p_4, p_4)	256	16
(p_4, p_5)	16	0
(p_5, p_1)	1	1
(p_5, p_2)	16	0
(p_5, p_3)	36	4
(p_5, p_4)	16	0
(p_5, p_5)	1	1
Total	4900	$1 \times 4 + 4 \times 4 + 16 \times 4 + 56 = 140$

Table 1: Number of states and singlets in various groups (p_a, p_b).

For the case of $n = 2$, the mid-level condition exhausts the information with respect to the equations:

$$\begin{aligned} Q_{03}^{k_1 k_2} \ |\text{singlet}\rangle &= 0 \\ Q_{12}^{k_1 k_2} \ |\text{singlet}\rangle &= 0 \end{aligned} \tag{6.3}$$

The remaining four independent charges in the language of creation and annihilation operators can be written as[6]:

$$
\begin{aligned}
Q^{12}_{01} &= i\left(\psi_0^{1j+}\psi_0^{2j-} - \psi_0^{2j+}\psi_0^{1j-} + \psi_1^{1j+}\psi_1^{2j-} - \psi_1^{2j+}\psi_1^{1j-}\right) \\
&\equiv R^{12}_0 + R^{12}_1 \\
Q^{12}_{23} &= i\left(\psi_2^{1j+}\psi_2^{2j-} - \psi_2^{2j+}\psi_2^{1j-} + \psi_3^{1j+}\psi_3^{2j-} - \psi_3^{2j+}\psi_3^{1j-}\right) \\
&\equiv R^{12}_2 + R^{12}_3 \\
Q^{12}_{02} &= i\left(\psi_0^{i1+}\psi_0^{i2-} - \psi_0^{i2+}\psi_0^{i1-} + \psi_2^{i1+}\psi_2^{i2-} - \psi_1^{i2+}\psi_1^{i1-}\right) \\
&\equiv S^{12}_0 + S^{12}_2 \\
Q^{12}_{13} &= i\left(\psi_1^{i1+}\psi_1^{i2-} - \psi_1^{i2+}\psi_1^{i1-} + \psi_3^{i1+}\psi_3^{i2-} - \psi_3^{i2+}\psi_3^{i1-}\right) \\
&\equiv S^{12}_1 + S^{12}_3
\end{aligned}
\tag{6.4}
$$

where the charges R and S are the colored analogues of the charges[7] Q_1 and Q_2 in $n = 2$ uncolored model. As the level operators L_A commute with the above charges, we can find the singlet states in each of the groups separately. Note that this simplification is unique to $n = 2$ case. More generally, this simplification happens whenever we construct the Clifford basis by breaking a $O(2)$ group. For instance, this simplification also occurs in $O(n)^4 \times O(2)^2$ Gurau-Witten model and hence the singlets in that model can be written down straightforwardly following our method-I to construct singlets. We leave further details to a future work.

7. Singlets of $n = 2$

In this section, we identify the singlets of $n = 2$ Gurau-Witten model in all the groups using two different methods. In the first method, we will make use of the group-theoretic facts about the orthogonal group to list down the singlets. In the second method, we solve the equations $Q_{AB}|\text{singlets}\rangle = 0$ explicitly to find the singlet states. We find that there are 140 singlets in $n = 2$ Gurau-Witten model spreading over only 13 out of the total 25 groups. See Table 1 for details. Note that this number of singlets matches exactly with that of [10] where a systematic way to count the number of singlet states in the Gurau-Witten and uncolored tensor models is presented.

[6]In the rest of the paper we denote the last index as $\pm$ instead of $1^{\pm}$.

[7]See appendix for more details.

7.1. Method-I

In this method, we take advantage of various facts about orthogonal groups. Firstly, we note that there are only two invariant tensors of $SO(n)$ that are given by:

- Kronecker Delta- Due to the following property of the orthogonal group:

$$M^T M = I \tag{7.1}$$

  where $M \in SO(n)$ or $O(n)$. Note that Kronecker delta is the only invariant tensor of $O(n)$.

- Levi-Civita tensor - Because the determinant of $SO(n)$ matrices is equal to $+1$.

Secondly, note that the Clifford vacuum is invariant under $O(2)^4 \times U(1)^2$ by definition and thus is annihilated by the charges (6.4). Further, the Noether charges (6.4) correspond to those four orthogonal groups. This implies that the quantity that appears before the Clifford vacuum of any singlet state should be an invariant of $O(2)^4$. Combining all these observations with the mid-level condition, we can list down all the singlet states. More operationally, if we start with a generic singlet state of the form (6.1), then the α's are made up of Kronecker deltas and Levi-Civita tensors. Concerning our future work [18], we mention that the singlets under the generalizations of the charges (6.4) to arbitrary n case can be found in a similar way.

We have listed down all the 140 independent singlet states that can be constructed using this method in an appendix of the arXiv version of this paper. Note that not all possible contractions lead to different singlets.

7.2. Method-II

Our strategy here is to start with a generic linear combination of states in a particular[8] group of the form (6.1) and then demand that the Noether charges (6.4) annihilate this state to find the numerical coefficients α in (6.1). This gives us the singlets. Instead of identifying all the singlets using this strategy, we can also use the discrete symmetry operators defined in

[8]This can be done because the Noether charges in $n = 2$ case commute with the level operators L_A as explained in the last section.

the last section to identify some of the singlets. As an example, consider the following singlet state:

$$\begin{aligned}(7.2)\quad &\left[(\psi_2^{11+}\psi_2^{12+}-\psi_2^{21+}\psi_2^{22+})(2\to 3)+(\psi_2^{11+}\psi_2^{22+}-\psi_2^{12+}\psi_2^{21+})(2\to 3)\right]\\ &(\psi_0^{11+}\psi_0^{12+}+\psi_0^{21+}\psi_0^{22+})(\psi_1^{11+}\psi_1^{12+}+\psi_1^{21+}\psi_1^{22+})\end{aligned}$$

From the last section, we know that $S_{02;13}$ acting on the above state is also a singlet i.e., the following state is also a singlet[9]:

$$\begin{aligned}(7.3)\quad &\left[(\psi_0^{11+}\psi_0^{12+}-\psi_0^{21+}\psi_0^{22+})(0\to 1)+(\psi_0^{11+}\psi_0^{22+}-\psi_0^{12+}\psi_0^{21+})(0\to 1)\right]\\ &(\psi_2^{11+}\psi_2^{12+}+\psi_2^{21+}\psi_2^{22+})(\psi_3^{11+}\psi_3^{12+}+\psi_3^{21+}\psi_3^{22+})\end{aligned}$$

Likewise, we can identify three more singlets in this particular case using other symmetry operators. We can also use these symmetries as a (rough) check that we have identified a complete set of singlets. We take any singlet state and act with these discrete symmetry operators and then verify whether the resultant state is also present in our singlet spectrum. The singlets we have listed in the next section are indeed closed under the action of these operators. Now, we move on to finding the singlets.

We start by noting that the charges R_A and S_A act non-trivially on the fermions belonging to color A and acts trivially on the objects of other colors. Also, for a specific color A, there are four singlets[10] with respect to R_A and S_A and are given by:

$$\begin{aligned}(7.4)\quad &1.f(\psi_B)|\,\rangle\\ &2.\left(\psi_A^{11+}\psi_A^{12+}+\psi_A^{21+}\psi_A^{22+}\right)f(\psi_B)|\,\rangle\\ &3.\left(\psi_A^{11+}\psi_A^{21+}+\psi_A^{12+}\psi_A^{22+}\right)f(\psi_B)|\,\rangle\\ &4.\ \psi_A^{11+}\psi_A^{12+}\psi_A^{21+}\psi_A^{22+}f(\psi_B)|\,\rangle\end{aligned}$$

where $f(\psi_B)$ denote functions of fermions that do not belong to the color A. Also, we note that the charges R_B and S_B act trivially on ψ_A's when $A \neq B$. This information along with the mid-level conditions (6.3) is sufficient to show that the single state present in each of the four groups $(p_{1,5}, p_{1,5})$ is a singlet state. Also, it helps us to list down the first 16 singlet states in the group (p_3, p_3) as we discuss later in the section.

[9]This can be verified by doing an explicit computation.

[10]Just to avoid any confusions, we emphasize that singlets under R_A and S_A (for a particular A) are not necessarily singlets under the Noether charges (6.4).

Let us now consider the group (p_1, p_2) which has 16 states. The states have four ψ_0's, three ψ_1's and one ψ_2. The generic form of a singlet state is given by:

$$\begin{aligned}\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\big[&\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\left(\alpha_1\psi_2^{11+}+\alpha_2\psi_2^{12+}+\alpha_3\psi_2^{21+}+\alpha_4\psi_2^{22+}\right)\\&+\psi_1^{11+}\psi_1^{12+}\psi_1^{22+}\left(\alpha_5\psi_2^{11+}+\alpha_6\psi_2^{12+}+\alpha_7\psi_2^{21+}+\alpha_8\psi_2^{22+}\right)\\&+\psi_1^{11+}\psi_1^{21+}\psi_1^{22+}\left(\alpha_9\psi_2^{11+}+\alpha_{10}\psi_2^{12+}+\alpha_{11}\psi_2^{21+}+\alpha_{12}\psi_2^{22+}\right)\\&+\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\left(\alpha_{13}\psi_2^{11+}+\alpha_{14}\psi_2^{12+}+\alpha_{15}\psi_2^{21+}+\alpha_{16}\psi_2^{22+}\right)\big]|\,\rangle\end{aligned}\tag{7.5}$$

Noting that $\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}$ is a singlet under S_0, and imposing the condition that $Q_{02} = S_0 + S_2$ should annihilate this state, we see that all α's need to be zero. That is, there are no singlets in this group. In a similar way, we can show that there are no singlets in (p_1, p_4), $(p_5, p_{2,4})$ and $(p_{2,4}, p_{1,5})$ groups.

Let us now consider the states in the group (p_2, p_2). A generic singlet state is of the form:

$$\begin{aligned}&\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{21+}\psi_1^{22+}\\&\times\left[\psi_2^{11+}\left(\alpha_1\psi_3^{11+}+\alpha_2\psi_3^{12+}+\alpha_3\psi_3^{21+}+\alpha_4\psi_3^{22+}\right)+\ldots\right]\\&+\psi_0^{11+}\psi_0^{12+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\\&\times\left[\psi_2^{11+}\left(\alpha_{17}\psi_3^{11+}+\alpha_{18}\psi_3^{12+}+\alpha_{19}\psi_3^{21+}+\alpha_{20}\psi_3^{22+}\right)+\ldots\right]+\cdots|\,\rangle\end{aligned}\tag{7.6}$$

Let us start by imposing the condition that $Q_{01} \equiv R_0 + R_1$ annihilates the state. We note that a linear combination of $\psi_0\psi_0\psi_0$ (or $\psi_1\psi_1\psi_1$) can not form a singlet under R_0 (or R_1). Further, since the charge Q_{01} does not affect the colors 2 & 3 , the state can be annihilated by Q_{01} only by some suitable combinations of ψ_0's and ψ_1's. The important point is that under the action of Q_{01}, the cancellations can only happen between the terms of the form $\psi_0\psi_0\psi_0\psi_1\psi_1\psi_1$ and $R_0\left(\psi_0\psi_0\psi_0\right)R_1\left(\psi_1\psi_1\psi_1\right)$. This can be shown as follows. The action of Q_{01} on $\psi_0\psi_0\psi_0\psi_1\psi_1\psi_1$ is given by:

$$Q_{01}\left(\psi_0\psi_0\psi_0\psi_1\psi_1\psi_1\right) = R_0\left(\psi_0\psi_0\psi_0\right)\psi_1\psi_1\psi_1 + \psi_0\psi_0\psi_0 R_1\left(\psi_1\psi_1\psi_1\right)\tag{7.7}$$

From the appendix of the arXiv version of this paper, we see that acting with R_0 twice on $\psi_0\psi_0\psi_0$ gives us the negative of the same state. Hence the action of Q_{01} on $R_0\left(\psi_0\psi_0\psi_0\right)R_1\left(\psi_1\psi_1\psi_1\right)$ is the only other possibility that can reproduce[11] the terms in the RHS of (7.7). The final message from this

[11]The action of Q_{01} on $R_0\left(\psi_0\psi_0\psi_0\right)R_1\left(\psi_1\psi_1\psi_1\right)$ is given as:

$$Q_{01}\left[R_0\left(\psi_0\psi_0\psi_0\right)R_1\left(\psi_1\psi_1\psi_1\right)\right] = -\psi_0\psi_0\psi_0 R_1\left(\psi_1\psi_1\psi_1\right) - R_0\left(\psi_0\psi_0\psi_0\right)\psi_1\psi_1\psi_1$$

discussion is that a singlet under Q_{01} is of the form:

$$[\psi_0\psi_0\psi_0\psi_1\psi_1\psi_1 + R_0\,(\psi_0\psi_0\psi_0)\,R_1\,(\psi_1\psi_1\psi_1)]\,f(\psi_2,\psi_3) \tag{7.8}$$

Similar arguments go through for the other three charges as well. Putting this all together, we can see that a singlet in the group (p_2, p_2) is of the following form:

$$\begin{aligned}
&[\psi_0\psi_0\psi_0\psi_1\psi_1\psi_1 + R_0\,(\psi_0\psi_0\psi_0)\,R_1\,(\psi_1\psi_1\psi_1)]\,[\psi_2\psi_3 + R_2(\psi_2)R_3(\psi_3)]\\
&+ [\psi_0\psi_0\psi_0 S_1(\psi_1\psi_1\psi_1) + R_0\,(\psi_0\psi_0\psi_0)\,S_1R_1\,(\psi_1\psi_1\psi_1)]\\
&\times [\psi_2 S_3(\psi_3) + R_2(\psi_2)S_3R_3(\psi_3)]\\
&+ [S_0(\psi_0\psi_0\psi_0)\psi_1\psi_1\psi_1 + S_0R_0\,(\psi_0\psi_0\psi_0)\,R_1\,(\psi_1\psi_1\psi_1)]\\
&\times [S_2(\psi_2)\psi_3 + S_2R_2(\psi_2)R_3(\psi_3)]\\
&+ [S_0(\psi_0\psi_0\psi_0)S_1(\psi_1\psi_1\psi_1) + S_0R_0\,(\psi_0\psi_0\psi_0)\,S_1R_1\,(\psi_1\psi_1\psi_1)]\\
&\times [S_2(\psi_2)S_3(\psi_3) + S_2R_2(\psi_2)S_3R_3(\psi_3)]
\end{aligned} \tag{7.9}$$

From the structure of the singlet, it is clear that starting from any one of the terms, one can construct the entire singlet uniquely. This observation suggests that there are 16 singlet states in the group (p_2, p_2). In a similar way, we can show that there are 16 singlets in each of (p_2, p_4), (p_4, p_2) and (p_4, p_4).

Now, we move on to the group (p_1, p_3). The states in this group comprises of four ψ_0's, two ψ_1's and two ψ_2's. Hence the singlets are of the form:

$$\psi_0\psi_0\psi_0\psi_0 f(\psi_1\psi_1, \psi_2\psi_2) \tag{7.10}$$

Since $\psi_0\psi_0\psi_0\psi_0$ is a singlet under the charges (6.4), $f(\psi_1\psi_1, \psi_2\psi_2)$ should also be made up of singlets under those charges. From the list of singlets (7.4), we see that there are 2 singlets that can be constructed using two ψ's of same color. So, there are totally four singlets in this group. Similar arguments show that there are four singlets in each of (p_3, p_1), (p_3, p_5) and (p_5, p_3).

We now move on to the group (p_2, p_3). The states have three ψ_0's, one ψ_3 and two of each of ψ_1 and ψ_2. Hence the singlets are of the form:

$$\psi_0\psi_0\psi_0 f_1(\psi_1\psi_1, \psi_2\psi_2, \psi_3) + \psi_0\psi_0\psi_0 f_2(\psi_1\psi_1, \psi_2\psi_2, \psi_3) + \cdots \tag{7.11}$$

Since a linear combination of $\psi_0\psi_0\psi_0$ can not form a singlet under R_0 or S_0, the function $f_i(\psi_1\psi_1, \psi_2\psi_2, \psi_3)$ can not include singlets of $\{1, 2\}$ colors

under $R_{1,2}$ or $S_{1,2}$. This suggests that the singlets are of the form:

$$\begin{aligned}(7.12)\quad &\psi_0\psi_0\psi_0\big[\alpha_1(\psi_1^{11+}\psi_1^{21+}-\psi_1^{12+}\psi_1^{22+})+\alpha_2(\psi_1^{11+}\psi_1^{12+}-\psi_1^{21+}\psi_1^{22+})\\ &+\alpha_3(\psi_1^{11+}\psi_1^{22+}-\psi_1^{12+}\psi_1^{21+})+\alpha_4(\psi_1^{11+}\psi_1^{22+}+\psi_1^{12+}\psi_1^{21+})\big]\\ &+R_0(\psi_0\psi_0\psi_0)\left[\alpha_5(\psi_1^{11+}\psi_1^{21+}-\psi_1^{12+}\psi_1^{22+})+\cdots\right]|\,\rangle+\cdots\end{aligned}$$

Demanding that Q_{01} annihilates the above state sets all the α's to be zero and hence there are no singlets in this group. Similarly, we can show that there are no singlets in (p_4, p_3), (p_3, p_2) and (p_3, p_4).

We now consider the last group: (p_3, p_3). The states in this group include two fermions of each color. From the appendix, we have the following observations:

- Under the charge R_A, the following four states are singlets:

$$\begin{aligned}(7.13)\qquad &\psi_A^{11+}\psi_A^{21+}f(\psi_B);\quad(\psi_A^{11+}\psi_A^{12+}+\psi_A^{21+}\psi_A^{22+})f(\psi_B)\\ &\psi_A^{12+}\psi_A^{22+}f(\psi_B);\quad(\psi_A^{11+}\psi_A^{22+}+\psi_A^{12+}\psi_A^{21+})f(\psi_B)\end{aligned}$$

  where $f(\psi_B)$ denotes some function of fermions of all the colors except the ones that belong to color A. Note that there is no summation over A in the above states.

- Under the charge S_A, the following four states are singlets:

$$\begin{aligned}(7.14)\qquad &\psi_A^{11+}\psi_A^{12+}f(\psi_B);\quad(\psi_A^{11+}\psi_A^{21+}+\psi_A^{12+}\psi_A^{22+})f(\psi_B)\\ &\psi_A^{21+}\psi_A^{22+}f(\psi_B);\quad(\psi_A^{11+}\psi_A^{22+}-\psi_A^{12+}\psi_A^{21+})f(\psi_B)\end{aligned}$$

- Under both the charges R_A and S_A, the following two states are singlets:

$$(7.15)\qquad(\psi_A^{11+}\psi_A^{21+}+\psi_A^{12+}\psi_A^{22+})f(\psi_B);\quad(\psi_A^{11+}\psi_A^{12+}+\psi_A^{21+}\psi_A^{22+})f(\psi_B)$$

There are different types of singlets (with respect to the charges (6.4)) in this group. The first type of singlets is obtained by taking a product of singlets under both R_A and S_A for each of the colors $A = \{0, 1, 2, 3\}$. For example, the following singlet belongs to this type:

$$\begin{aligned}(7.16)\quad &(\psi_0^{11+}\psi_0^{21+}+\psi_0^{12+}\psi_0^{22+})(\psi_1^{11+}\psi_1^{21+}+\psi_1^{12+}\psi_1^{22+})\\ &\times(\psi_2^{11+}\psi_2^{21+}+\psi_2^{12+}\psi_2^{22+})(\psi_3^{11+}\psi_3^{21+}+\psi_3^{12+}\psi_3^{22+})\end{aligned}$$

There are 16 singlets of this type. Singlets of this type range from 21 to 36 in the list given in the next section.

Let us now take three of the colors to have singlets of the form (7.15) i.e., we consider states of the form:

$$
\begin{aligned}
(7.17)\quad & (u_A/v_A)(u_B/v_B)(u_C/v_C)\big[\alpha_1(\psi_D^{11+}\psi_D^{12+}-\psi_D^{21+}\psi_D^{22+}) \\
& +\alpha_2(\psi_D^{11+}\psi_D^{21+}-\psi_D^{12+}\psi_D^{22+})+\alpha_3(\psi_D^{11+}\psi_D^{22+}-\psi_D^{12+}\psi_D^{21+}) \\
& +\alpha_4(\psi_D^{11+}\psi_D^{22+}+\psi_D^{12+}\psi_D^{21+})\big]
\end{aligned}
$$

where $u_A \equiv (\psi_A^{11+}\psi_A^{21+}+\psi_A^{12+}\psi_A^{22+})$ and $v_A \equiv (\psi_A^{11+}\psi_A^{12+}+\psi_A^{21+}\psi_A^{22+})$. Demanding that the Noether charges (6.4) annihilates this state sets all the α's to be zero and hence there are no singlets of this form.

Now, we consider states of the form:

$$
\begin{aligned}
(7.18)\quad & (u_A/v_A)(u_B/v_B)\Big[(\psi_C^{11+}\psi_C^{12+}-\psi_C^{21+}\psi_C^{22+})\big\{\alpha_1(\psi_D^{11+}\psi_D^{12+}-\psi_D^{21+}\psi_D^{22+}) \\
& +\alpha_2(\psi_D^{11+}\psi_D^{21+}-\psi_D^{12+}\psi_D^{22+})+\alpha_3(\psi_D^{11+}\psi_D^{22+}-\psi_D^{12+}\psi_D^{21+}) \\
& +\alpha_4(\psi_D^{11+}\psi_D^{22+}+\psi_D^{12+}\psi_D^{21+})\big\} \\
& +(\psi_C^{11+}\psi_C^{21+}-\psi_C^{12+}\psi_C^{22+})\big\{\alpha_5(\psi_D^{11+}\psi_D^{12+}-\psi_D^{21+}\psi_D^{22+}) \\
& +\alpha_6(\psi_D^{11+}\psi_D^{21+}-\psi_D^{12+}\psi_D^{22+})+\alpha_7(\psi_D^{11+}\psi_D^{22+}-\psi_D^{12+}\psi_D^{21+}) \\
& +\alpha_8(\psi_D^{11+}\psi_D^{22+}+\psi_D^{12+}\psi_D^{21+})\big\} \\
& +(\psi_C^{11+}\psi_C^{22+}-\psi_C^{12+}\psi_C^{21+})\big\{\alpha_9(\psi_D^{11+}\psi_D^{12+}-\psi_D^{21+}\psi_D^{22+}) \\
& +\alpha_{10}(\psi_D^{11+}\psi_D^{21+}-\psi_D^{12+}\psi_D^{22+})+\alpha_{11}(\psi_D^{11+}\psi_D^{22+}-\psi_D^{12+}\psi_D^{21+}) \\
& +\alpha_{12}(\psi_D^{11+}\psi_D^{22+}+\psi_D^{12+}\psi_D^{21+})\big\} \\
& +(\psi_C^{11+}\psi_C^{22+}+\psi_C^{12+}\psi_C^{21+})\big\{\alpha_{13}(\psi_D^{11+}\psi_D^{12+}-\psi_D^{21+}\psi_D^{22+}) \\
& +\alpha_{14}(\psi_D^{11+}\psi_D^{21+}-\psi_D^{12+}\psi_D^{22+})+\alpha_{15}(\psi_D^{11+}\psi_D^{22+}-\psi_D^{12+}\psi_D^{21+}) \\
& +\alpha_{16}(\psi_D^{11+}\psi_D^{22+}+\psi_D^{12+}\psi_D^{21+})\big\}\Big]
\end{aligned}
$$

Depending on the different choices of the colors $\{A, B, C, D\}$, we obtain different values for α's such that the above state is a singlet. We will outline the calculation for determining α's for one such choice and we just mention the singlets directly for the rest of the choices. Let us take

$$A = 0; B = 1; C = 2; D = 3$$

. The above state is trivially a singlet under Q_{01}. For it to be a singlet under Q_{23}, we need:

$$
\begin{aligned}
(7.19)\quad & \alpha_2 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_{10} = \alpha_{12} = \alpha_{13} = \alpha_{15} = 0 \\
& \alpha_1 = \alpha_{11}; \quad \alpha_3 = -\alpha_9
\end{aligned}
$$

Demanding that this state is annihilated by Q_{02} will give us:

$$\alpha_6 = \alpha_8 = \alpha_{14} = \alpha_{16} = 0 \tag{7.20}$$

The charge Q_{13} does not give any new conditions. Hence we have two different[12] types of independent singlet states in this case:

$$\begin{aligned}(u_0/v_0)(u_1/v_1)\big[&(\psi_2^{11+}\psi_2^{12+} - \psi_2^{21+}\psi_2^{22+})(2 \to 3) \\ &+ (\psi_2^{11+}\psi_2^{22+} - \psi_2^{12+}\psi_2^{21+})(2 \to 3)\big]\end{aligned} \tag{7.21}$$

$$\begin{aligned}(u_0/v_0)(u_1/v_1)\big[&(\psi_2^{11+}\psi_2^{12+} - \psi_2^{21+}\psi_2^{22+}) \\ &\times (\psi_3^{11+}\psi_3^{22+} - \psi_3^{12+}\psi_3^{21+}) - (2 \leftrightarrow 3)\big]\end{aligned} \tag{7.22}$$

We can follow a similar strategy and obtain the singlets for other choices of colors. See (37)-(68) in the list of singlets in the next section for explicit expressions of these type of singlets.

Now consider the states of the form:

$$(u_A/v_A) f(\psi_B, \psi_C, \psi_D) \tag{7.23}$$

where $f(\psi_B, \psi_C, \psi_D)$ does not contain either $u_{B,C,D}$ or $v_{B,C,D}$. We can show that there are no singlets that can be constructed from these states.

Lastly, we consider the states that does not include either $u_{A,B,C,D}$ or $v_{A,B,C,D}$. Following a similar analysis as presented for the states of the form (7.18), we find that there are eight singlets in this final set. These singlets are listed from (69) to (76) in the next section.

We should have ideally done this entire analysis for the (p_3, p_3) group starting from the most general state that can be written down for this group. Instead we have identified singlets by considering different type of states in this group. This is consistent because the Noether charges indeed do not mix these different type of states and hence our results are not affected.

8. $SO(n)$ vs $O(n)$

In the last section, we found the singlets by demanding that the Noether charges (6.4) annihilate the singlet states. The Noether charges we have computed takes into account only the continuous part of the group i.e., the part of the group that is continuously connected to the identity. This

[12]One of them is obtained if we take $\alpha_1 = \alpha_{11} = 0$ and the other one for the choice $\alpha_3 = -\alpha_9 = 0$

means that we have obtained the singlets of $SO(n)$ in the last section. In this section, we give a strategy on how to obtain the $O(n)$ singlets starting from the $SO(n)$ singlets.

To begin with, note that the $O(n)$ group contains an extra parity transformation i.e.,

$$O(n) \sim SO(n) \times Z_2 \tag{8.1}$$

We denote the parity transformation by P^i_{AB} where A/B are color indices and its action is given by:

$$P^i_{AB}\psi^{i'j'k'}_{A/B}\,{P^i_{AB}}^{-1} = -\psi^{i'j'k'}_{A/B} \qquad \text{if } i = i' \tag{8.2}$$

$$= +\psi^{i'j'k'}_{A/B} \qquad \text{if } i \neq i' \tag{8.3}$$

That is, the action of parity operator P^i_{AB} changes the sign of fermions $\psi^i_{A/B}$ and leaves the rest of them unchanged. Also, the product of two such parity transformations within the same orthogonal group $O(n)$ corresponds to a $SO(n)$ rotation. Hence, we need to consider only one such parity transformation to obtain the singlets of $O(n)$. There are five more such parity operators, corresponding to the ${}^4C_2$ different pairs of colors.

On $SO(n)$ singlets we obtained in the last section, we need to impose extra constraint that the $O(n)$ singlets are invariant under the parity transformations. This constraint is easier to implement in the method-I of construction of singlets. Note that under the parity transformations, the $SO(n)$ singlets constructed using Kronecker deltas are invariant whereas the ones constructed using Levi-Civita tensor change sign. So, the $O(n)$ singlets are the ones that are constructed using only[13] the Kronecker deltas. This strategy works for the groups $O(n)_{01}$, $O(n)_{23}$, $O(n)_{02}$ and $O(n)_{13}$.

For the groups $O(n)_{03}$ and $O(n)_{12}$, things are bit involved as the Clifford vacuum that we are working with transforms non-trivially under these groups. To obtain the action of parity operators corresponding to these two groups on a generic state, we need to know their action on the Clifford vacuum. To that end, we construct explicit forms of the parity operators using their definitions. For $n = 2$, the parity operators are given by:

$$P^{k=1}_{03} = 2^4\ \psi^{111}_0\psi^{121}_0\psi^{211}_0\psi^{221}_0\psi^{111}_3\psi^{121}_3\psi^{211}_3\psi^{221}_3 \tag{8.4}$$

$$P^{k=1}_{12} = 2^4\ \psi^{111}_1\psi^{121}_1\psi^{211}_1\psi^{221}_1\psi^{111}_2\psi^{121}_2\psi^{211}_2\psi^{221}_2 \tag{8.5}$$

[13]This is consistent with the fact that Kronecker delta is the only invariant tensor of $O(n)$.

From the explicit forms, it is easy to verify that they satisfy the following as expected:

$$P_{03}\psi^{ij1}_{0/3}P^{-1}_{03} = -\psi^{ij1}_{0/3};\quad P_{03}\psi^{ij2}_{0/3}P^{-1}_{03} = +\psi^{ij2}_{0/3} \tag{8.6}$$

$$P_{12}\psi^{ij1}_{1/2}P^{-1}_{12} = -\psi^{ij1}_{1/2};\quad P_{12}\psi^{ij2}_{1/2}P^{-1}_{12} = +\psi^{ij2}_{1/2} \tag{8.7}$$

From these relations, we can find the action of these parity operators on the creation and annihilation operators as:

$$P_{03}\psi^{ij\pm}_{0/3}P^{-1}_{03} = -\psi^{ij\mp}_{0/3};\quad P_{12}\psi^{ij\pm}_{1/2}P^{-1}_{12} = -\psi^{ij\mp}_{1/2} \tag{8.8}$$

Further, we can find the action of these parity operators on the Clifford vacuum as:

$$P_{03}|\,\rangle = \psi^{11+}_0\psi^{12+}_0\psi^{21+}_0\psi^{22+}_0\psi^{11+}_3\psi^{12+}_3\psi^{21+}_3\psi^{22+}_3|\,\rangle \tag{8.9}$$

$$P_{12}|\,\rangle = \psi^{11+}_1\psi^{12+}_1\psi^{21+}_1\psi^{22+}_1\psi^{11+}_2\psi^{12+}_2\psi^{21+}_2\psi^{22+}_2|\,\rangle \tag{8.10}$$

Now, we have all the information needed to identify the singlets of $O(2)_{03}$ and $O(n)_{12}$. Among the $SO(2)_{03}$ (and $SO(2)_{12}$) singlets that we already have, the $O(2)_{03}$ (and $O(2)_{12}$) singlets can be identified as the ones that are left invariant under the respective parity operators.

Out of the 140 $SO(2)^6$ singlets, only six of them are invariant under the parity transformations corresponding to all the $O(2)^6$ groups. More explicitly, following are the $O(2)^6$ singlets:

$$|1\rangle + |2\rangle + |3\rangle + |4\rangle;\quad |21\rangle;\quad |36\rangle;\quad |69\rangle;\quad |73\rangle;\quad |77\rangle + |93\rangle + |109\rangle + |125\rangle \tag{8.11}$$

where $|i\rangle$ correspond to the i^{th} singlet in the $SO(2)^6$ singlet list in the appendix. It is interesting to note that all the $O(2)^6$ singlets are from[14] the first independent set. Further, the $O(2)^6$ invariant eigenstates of the Hamiltonian are given as follows:

$$\begin{aligned} &|36\rangle - |21\rangle;\quad |73\rangle - 4|21\rangle;\quad |69\rangle - 4|36\rangle;\quad |36\rangle - (|1\rangle + |2\rangle + |3\rangle + |4\rangle) \\ &4\,(|1\rangle + |2\rangle + |3\rangle + |4\rangle) + |36\rangle + |21\rangle + |69\rangle + |73\rangle \\ &\pm\sqrt{\frac{7}{2}}\,(|77\rangle + |93\rangle + |109\rangle + |125\rangle) \end{aligned} \tag{8.12}$$

[14]See the next section for more details on the independent sets.

The eigenstates in the first line have zero energy and the ones in the next line have eigenvalues of $\pm 2\sqrt{14}$. Note that the latter eigenvalues correspond to the highest energy state and the lowest energy state (ground state).

9. Singlet eigenstates of the Hamiltonian

In the last section, we have identified all the singlets in the theory. Now, we want to identify the energy eigenstates among those singlets. Before doing that, we note that the Hamiltonian is a singlet of $O(2)^6$ and hence commutes with the Noether charges (6.4). As a result, we have:

$$H|\text{singlet}\rangle = \sum_a \beta_a |\text{singlet}\rangle_a \tag{9.1}$$

where a denotes the singlets in the theory and runs from 1 to 140 and β_a are some numerical coefficients. The point we emphasize here is that the singlets are a closed set under the action of the Hamiltonian.

We now proceed to identify the eigenstates. First of all, let us write down the Hamiltonian for the $n = 2$ case explicitly:

(9.2)

$$\begin{aligned}
H &= \psi_0^{ij+}\psi_1^{il+}\psi_2^{nj-}\psi_3^{nl-} + \psi_0^{ij+}\psi_1^{il-}\psi_2^{nj+}\psi_3^{nl-} + \psi_0^{ij-}\psi_1^{il+}\psi_2^{nj-}\psi_3^{nl+} \\
&\quad + \psi_0^{ij-}\psi_1^{il-}\psi_2^{nj+}\psi_3^{nl+} \\
&= \psi_0^{11+}\psi_1^{11+}\psi_2^{11-}\psi_3^{11-} + \psi_0^{11+}\psi_1^{11+}\psi_2^{21-}\psi_3^{21-} + \psi_0^{11+}\psi_1^{12+}\psi_2^{11-}\psi_3^{12-} \\
&\quad + \psi_0^{11+}\psi_1^{12+}\psi_2^{21-}\psi_3^{22-} + \psi_0^{12+}\psi_1^{11+}\psi_2^{12-}\psi_3^{11-} + \psi_0^{12+}\psi_1^{11+}\psi_2^{22-}\psi_3^{21-} \\
&\quad + \psi_0^{12+}\psi_1^{12+}\psi_2^{12-}\psi_3^{12-} + \psi_0^{12+}\psi_1^{12+}\psi_2^{22-}\psi_3^{22-} + \psi_0^{21+}\psi_1^{21+}\psi_2^{11-}\psi_3^{11-} \\
&\quad + \psi_0^{21+}\psi_1^{21+}\psi_2^{21-}\psi_3^{21-} + \psi_0^{21+}\psi_1^{22+}\psi_2^{11-}\psi_3^{12-} + \psi_0^{21+}\psi_1^{22+}\psi_2^{21-}\psi_3^{22-} \\
&\quad + \psi_0^{22+}\psi_1^{21+}\psi_2^{12-}\psi_3^{11-} + \psi_0^{22+}\psi_1^{21+}\psi_2^{22-}\psi_3^{21-} + \psi_0^{22+}\psi_1^{22+}\psi_2^{12-}\psi_3^{12-} \\
&\quad + \psi_0^{22+}\psi_1^{22+}\psi_2^{22-}\psi_3^{22-} \\
&= \psi_0^{11+}\psi_1^{11-}\psi_2^{11+}\psi_3^{11-} + \psi_0^{11+}\psi_1^{11-}\psi_2^{21+}\psi_3^{21-} + \psi_0^{11+}\psi_1^{12-}\psi_2^{11+}\psi_3^{12-} \\
&\quad + \psi_0^{11+}\psi_1^{12-}\psi_2^{21+}\psi_3^{22-} + \psi_0^{12+}\psi_1^{11-}\psi_2^{12+}\psi_3^{11-} + \psi_0^{12+}\psi_1^{11-}\psi_2^{22+}\psi_3^{21-} \\
&\quad + \psi_0^{12+}\psi_1^{12-}\psi_2^{12+}\psi_3^{12-} + \psi_0^{12+}\psi_1^{12-}\psi_2^{22+}\psi_3^{22-} + \psi_0^{21+}\psi_1^{21-}\psi_2^{11+}\psi_3^{11-} \\
&\quad + \psi_0^{21+}\psi_1^{21-}\psi_2^{21+}\psi_3^{21-} + \psi_0^{21+}\psi_1^{22-}\psi_2^{11+}\psi_3^{12-} + \psi_0^{21+}\psi_1^{22-}\psi_2^{21+}\psi_3^{22-} \\
&\quad + \psi_0^{22+}\psi_1^{21-}\psi_2^{12+}\psi_3^{11-} + \psi_0^{22+}\psi_1^{21-}\psi_2^{22+}\psi_3^{21-} + \psi_0^{22+}\psi_1^{22-}\psi_2^{12+}\psi_3^{12-} \\
&\quad + \psi_0^{22+}\psi_1^{22-}\psi_2^{22+}\psi_3^{22-}
\end{aligned}$$

$$
\begin{aligned}
&= \psi_0^{11-}\psi_1^{11+}\psi_2^{11-}\psi_3^{11+} + \psi_0^{11-}\psi_1^{11+}\psi_2^{21-}\psi_3^{21+} + \psi_0^{11-}\psi_1^{12+}\psi_2^{11-}\psi_3^{12+} \\
&\quad + \psi_0^{11-}\psi_1^{12+}\psi_2^{21-}\psi_3^{22+} + \psi_0^{12-}\psi_1^{11+}\psi_2^{12-}\psi_3^{11+} + \psi_0^{12-}\psi_1^{11+}\psi_2^{22-}\psi_3^{21+} \\
&\quad + \psi_0^{12-}\psi_1^{12+}\psi_2^{12-}\psi_3^{12+} + \psi_0^{12-}\psi_1^{12+}\psi_2^{22-}\psi_3^{22+} + \psi_0^{21-}\psi_1^{21+}\psi_2^{11-}\psi_3^{11+} \\
&\quad + \psi_0^{21-}\psi_1^{21+}\psi_2^{21-}\psi_3^{21+} + \psi_0^{21-}\psi_1^{22+}\psi_2^{11-}\psi_3^{12+} + \psi_0^{21-}\psi_1^{22+}\psi_2^{21-}\psi_3^{22+} \\
&\quad + \psi_0^{22-}\psi_1^{21+}\psi_2^{12-}\psi_3^{11+} + \psi_0^{22-}\psi_1^{21+}\psi_2^{22-}\psi_3^{21+} + \psi_0^{22-}\psi_1^{22+}\psi_2^{12-}\psi_3^{12+} \\
&\quad + \psi_0^{22-}\psi_1^{22+}\psi_2^{22-}\psi_3^{22+} \\
&= \psi_0^{11-}\psi_1^{11-}\psi_2^{11+}\psi_3^{11+} + \psi_0^{11-}\psi_1^{11-}\psi_2^{21+}\psi_3^{21+} + \psi_0^{11-}\psi_1^{12-}\psi_2^{11+}\psi_3^{12+} \\
&\quad + \psi_0^{11-}\psi_1^{12-}\psi_2^{21+}\psi_3^{22+} + \psi_0^{12-}\psi_1^{11-}\psi_2^{12+}\psi_3^{11+} + \psi_0^{12-}\psi_1^{11-}\psi_2^{22+}\psi_3^{21+} \\
&\quad + \psi_0^{12-}\psi_1^{12-}\psi_2^{12+}\psi_3^{12+} + \psi_0^{12-}\psi_1^{12-}\psi_2^{22+}\psi_3^{22+} + \psi_0^{21-}\psi_1^{21-}\psi_2^{11+}\psi_3^{11+} \\
&\quad + \psi_0^{21-}\psi_1^{21-}\psi_2^{21+}\psi_3^{21+} + \psi_0^{21-}\psi_1^{22-}\psi_2^{11+}\psi_3^{12+} + \psi_0^{21-}\psi_1^{22-}\psi_2^{21+}\psi_3^{22+} \\
&\quad + \psi_0^{22-}\psi_1^{21-}\psi_2^{12+}\psi_3^{11+} + \psi_0^{22-}\psi_1^{21-}\psi_2^{22+}\psi_3^{21+} + \psi_0^{22-}\psi_1^{22-}\psi_2^{12+}\psi_3^{12+} \\
&\quad + \psi_0^{22-}\psi_1^{22-}\psi_2^{22+}\psi_3^{22+}
\end{aligned}
$$

We need to act with this Hamiltonian on each of the singlet states and then identify appropriate linear combinations such that:

$$
H \sum_a \alpha_a |\text{singlet}\rangle_a = \lambda \sum_a \alpha_a |\text{singlet}\rangle_a \tag{9.3}
$$

Even though this is conceptually straightforward, the calculations are tedious. Additionally, the Hamiltonian does not[15] commute with the level operators (3.5) of specific colors. As a result, the Hamiltonian mixes the states from different groups and hence we lost the simplification that happened in the case of identifying singlets.

The discrete symmetry operators we have defined earlier make things a bit easier. To begin with, we note that the Hamiltonian commutes with the operators $S_{01;23}$, $S_{02;13}$ and $S_{03;12}$ whereas it anti-commutes with the operators S_{AB} and S_A. We will describe the usefulness of these symmetry operators via an example. Consider the action of the Hamiltonian on the

[15]Note that the Hamiltonian commutes with the overall level operator defined by the sum of level operators of individual colors.

following singlet state:

(9.4)
$$
\begin{aligned}
&H\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} \equiv H|a\rangle \\
&= \left(\psi_2^{11+}\psi_3^{11+} + \psi_2^{21+}\psi_3^{21+}\right) \\
&\quad \times \left(\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{22+}\right) \\
&\quad - \left(\psi_2^{11+}\psi_3^{12+} + \psi_2^{21+}\psi_3^{22+}\right) \\
&\quad \times \left(\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\right) \\
&\quad - \left(\psi_2^{12+}\psi_3^{11+} + \psi_2^{22+}\psi_3^{21+}\right) \\
&\quad \times \left(\psi_0^{11+}\psi_0^{21+}\psi_0^{22+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_1^{11+}\psi_1^{12+}\psi_1^{22+}\right) \\
&\quad + \left(\psi_2^{12+}\psi_3^{12+} + \psi_2^{22+}\psi_3^{22+}\right) \\
&\quad \times \left(\psi_0^{11+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\right) \\
&\equiv |(12,21,22),(12,21,22),(11),(11)\rangle.
\end{aligned}
$$

Suppose we now want to find the action of Hamiltonian on

$$|b\rangle \equiv \psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+}.$$

Since $|b\rangle = S_{02;13}|a\rangle$, we see that $H|b\rangle = S_{02;13}\,(H|a\rangle)$. More explicitly, we find that:

(9.5)
$$
\begin{aligned}
&H\psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+} \equiv H|b\rangle \\
&= \left(\psi_0^{11+}\psi_1^{11+} + \psi_0^{21+}\psi_1^{21+}\right) \\
&\quad \times \left(\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+} + \psi_2^{11+}\psi_2^{12+}\psi_2^{22+}\psi_3^{11+}\psi_3^{12+}\psi_3^{22+}\right) \\
&\quad - \left(\psi_0^{11+}\psi_1^{12+} + \psi_0^{21+}\psi_1^{22+}\right) \\
&\quad \times \left(\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\psi_3^{11+}\psi_3^{21+}\psi_3^{22+} + \psi_2^{11+}\psi_2^{12+}\psi_2^{22+}\psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\right) \\
&\quad - \left(\psi_0^{12+}\psi_1^{11+} + \psi_0^{22+}\psi_1^{21+}\right) \\
&\quad \times \left(\psi_2^{11+}\psi_2^{21+}\psi_2^{22+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+} + \psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_3^{11+}\psi_3^{12+}\psi_3^{22+}\right) \\
&\quad + \left(\psi_0^{12+}\psi_1^{12+} + \psi_0^{22+}\psi_1^{22+}\right) \\
&\quad \times \left(\psi_2^{11+}\psi_2^{21+}\psi_2^{22+}\psi_3^{11+}\psi_3^{21+}\psi_3^{22+} + \psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\right) \\
&\equiv |(11),(11),(12,21,22),(12,21,22)\rangle.
\end{aligned}
$$

By using the other symmetry operators, we can find the action of Hamiltonian on three more states in a similar way. Since many of the singlet states are related by these operators, the number of calculations we need to do are considerably reduced.

The symmetry operators are further helpful in identifying the eigenstates. Let $|E\rangle$ be an eigenstate of the Hamiltonian with eigenvalue E. Then the states $S_{01;23}|E\rangle$, $S_{02;13}|E\rangle$ and $S_{03;12}|E\rangle$ are also eigenstates with the same eigenvalue whereas the states $S_{AB}|E\rangle$ and $S_A|E\rangle$ are eigenstates of the Hamiltonian with eigenvalue $-E$.

We found that the 140 singlet eigenstates we have fall into 16 independent sets. By independent sets, we mean that the action of Hamiltonian on any of the states in an independent set produces the states in that set. From the appendix, it is clear that each of these independent sets has only one singlet from the (p_2, p_2) group. This fact is useful to organize our calculations. We go about identifying the independent sets of eigenstates following the steps listed below:

- Start with any one of the singlet states of the (p_2, p_2) group. Let us denote it by $|a\rangle$.
- Act with the Hamiltonian on $|a\rangle$ and organize the result in terms of singlets i.e.,

$$H|a\rangle = \sum_i \beta_i |b_i\rangle \tag{9.6}$$

  where $|b_i\rangle$ are some singlets that depend on our choice of singlet state $|a\rangle$ and β_i are numerical coefficients.
- Now act with the Hamiltonian on $|b_i\rangle$'s and organize the result in terms of singlets.
- Repeat this until we have a set of singlet states that closes under the action of the Hamiltonian. We call this set of states as an independent set.
- Take appropriate linear combination of states in the independent sets to form eigenstates.

We demonstrate these steps using an example in the following subsection. We will choose the example such that the ground state is a part of it. By ground state we mean the lowest energy state of the entire theory not just the gauged sector of it. The ground state being a part of our singlet spectrum is expected since we know that it is unique as found numerically in [11]. This is also consistent with the discussions in [16].

Before we proceed further, we emphasize that the reader should not be surprised by the fact that we find some of the eigenvalues to be irrational. We present here a simple case where the eigenvalues can be irrational. This

example mirrors the situation that arises while finding the eigenvalues. Consider an operator K whose action on two states $|p\rangle$ and $|q\rangle$ is given as following:

$$K|p\rangle = \eta|q\rangle; \quad K|q\rangle = \zeta|p\rangle \tag{9.7}$$

for some positive integers η and ζ. Two of the eigenstates of K can then be constructed as:

$$K\left(|p\rangle \pm \sqrt{\frac{\eta}{\zeta}}|q\rangle\right) = \pm\sqrt{\eta\zeta}\left(|p\rangle \pm \sqrt{\frac{\eta}{\zeta}}|q\rangle\right) \tag{9.8}$$

As we can see from this simple example, even though η and ζ are integers, the eigenvalues $\pm\sqrt{\eta\zeta}$ can be irrational. This is similar to how we get some of the eigenvalues of the Gurau-Witten Hamiltonian to be irrational and it will become clearer after the following example.

9.1. An example

Here, we describe our strategy to find independent sets by choosing the singlet state from the group (p_2, p_2) to be:

$$\begin{aligned}
&|(12,21,22),(12,21,22),(11),(11)\rangle \\
&= \left(\psi_2^{11+}\psi_3^{11+} + \psi_2^{21+}\psi_3^{21+}\right) \\
&\quad \times \left(\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{22+}\right) \\
&\quad - \left(\psi_2^{11+}\psi_3^{12+} + \psi_2^{21+}\psi_3^{22+}\right) \\
&\quad \times \left(\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\right) \\
&\quad - \left(\psi_2^{12+}\psi_3^{11+} + \psi_2^{22+}\psi_3^{21+}\right) \\
&\quad \times \left(\psi_0^{11+}\psi_0^{21+}\psi_0^{22+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_1^{11+}\psi_1^{12+}\psi_1^{22+}\right) \\
&\quad + \left(\psi_2^{12+}\psi_3^{12+} + \psi_2^{22+}\psi_3^{22+}\right) \\
&\quad \times \left(\psi_0^{11+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{21+}\psi_1^{22+} + \psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\right)
\end{aligned} \tag{9.9}$$

As can be seen from the appendix, this choice corresponds to the independent set-I. Before finding the action of the Hamiltonian on this state, let us

define the following for convenience:

$$|a_1\rangle = \left(\psi_0^{11+}\psi_0^{21+} + \psi_0^{12+}\psi_0^{22+}\right)(0 \to 1,2,3)\,|\,\rangle \tag{9.10}$$

$$|a_2\rangle = \left(\psi_0^{11+}\psi_0^{12+} + \psi_0^{21+}\psi_0^{22+}\right)(0 \to 1,2,3)\,|\,\rangle \tag{9.11}$$

$$\begin{aligned}|a_3\rangle = \big[&\left(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+}\right)(0 \leftrightarrow 2)\\ &+ \left(\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+}\right)(0 \leftrightarrow 2)\big]\\ &\times \left(\psi_1^{11+}\psi_1^{21+} + \psi_1^{12+}\psi_1^{22+}\right)\left(\psi_3^{11+}\psi_3^{21+} + \psi_3^{12+}\psi_3^{22+}\right)|\,\rangle\end{aligned} \tag{9.12}$$

$$\begin{aligned}|a_4\rangle = \big[&\left(\psi_1^{11+}\psi_1^{22+} + \psi_1^{12+}\psi_1^{21+}\right)(1 \leftrightarrow 3)\\ &+ \left(\psi_1^{11+}\psi_1^{21+} - \psi_1^{12+}\psi_1^{22+}\right)(1 \leftrightarrow 3)\big]\\ &\times \left(\psi_0^{11+}\psi_0^{21+} + \psi_0^{12+}\psi_0^{22+}\right)\left(\psi_2^{11+}\psi_2^{21+} + \psi_2^{12+}\psi_2^{22+}\right)|\,\rangle\end{aligned} \tag{9.13}$$

$$\begin{aligned}|a_5\rangle = \big[&\left(\psi_0^{11+}\psi_0^{22+} - \psi_0^{12+}\psi_0^{21+}\right)(0 \leftrightarrow 1)\\ &+ \left(\psi_0^{11+}\psi_0^{12+} - \psi_0^{21+}\psi_0^{22+}\right)(0 \leftrightarrow 1)\big]\\ &\times \left(\psi_2^{11+}\psi_2^{12+} + \psi_2^{21+}\psi_2^{22+}\right)\left(\psi_3^{11+}\psi_3^{12+} + \psi_3^{21+}\psi_3^{22+}\right)|\,\rangle\end{aligned} \tag{9.14}$$

$$\begin{aligned}|a_6\rangle = \big[&\left(\psi_2^{11+}\psi_2^{22+} - \psi_2^{12+}\psi_2^{21+}\right)(2 \leftrightarrow 3)\\ &+ \left(\psi_2^{11+}\psi_2^{12+} - \psi_2^{21+}\psi_2^{22+}\right)(2 \leftrightarrow 3)\big]\\ &\times \left(\psi_0^{11+}\psi_0^{12+} + \psi_0^{21+}\psi_0^{22+}\right)\left(\psi_1^{11+}\psi_1^{12+} + \psi_1^{21+}\psi_1^{22+}\right)|\,\rangle\end{aligned} \tag{9.15}$$

$$\begin{aligned}|a_7\rangle = \big[&\left(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+}\right)(0 \leftrightarrow 2)\\ &+ \left(\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+}\right)(0 \leftrightarrow 2)\big]\\ &\times \big[\left(\psi_1^{11+}\psi_1^{22+} + \psi_1^{12+}\psi_1^{21+}\right)(1 \leftrightarrow 3)\\ &+ \left(\psi_1^{11+}\psi_1^{21+} - \psi_1^{12+}\psi_1^{22+}\right)(1 \leftrightarrow 3)\big]|\,\rangle\end{aligned} \tag{9.16}$$

$$\begin{aligned}|a_8\rangle = \big[&\left(\psi_0^{11+}\psi_0^{22+} - \psi_0^{12+}\psi_0^{21+}\right)(0 \leftrightarrow 1)\\ &+ \left(\psi_0^{11+}\psi_0^{12+} - \psi_0^{21+}\psi_0^{22+}\right)(0 \leftrightarrow 1)\big]\\ &\times \big[\left(\psi_2^{11+}\psi_2^{22+} - \psi_2^{12+}\psi_2^{21+}\right)(2 \leftrightarrow 3)\\ &+ \left(\psi_2^{11+}\psi_2^{12+} - \psi_2^{21+}\psi_2^{22+}\right)(2 \leftrightarrow 3)\big]|\,\rangle\end{aligned} \tag{9.17}$$

$$|a_9\rangle = |(12,21,22),(12,21,22),(11),(11)\rangle \tag{9.18}$$

$$|a_{10}\rangle = |(12,21,22),(11),(12,21,22),(11)\rangle \tag{9.19}$$

$$|a_{11}\rangle = |(11),(12,21,22),(11),(12,21,22)\rangle \tag{9.20}$$

$$|a_{12}\rangle = |(11),(11),(12,21,22),(12,21,22)\rangle \tag{9.21}$$

$$|a_{13}\rangle = \psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} \tag{9.22}$$

$$|a_{14}\rangle = \psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}\psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_2^{22+} \tag{9.23}$$

$$|a_{15}\rangle = \psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+}\psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+} \tag{9.24}$$

$$|a_{16}\rangle = \psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+} \tag{9.25}$$

The action of the Hamiltonian on the state (9.9) gives:

$$H|a_9\rangle = 16|a_{13}\rangle + |a_1\rangle + |a_2\rangle - |a_3\rangle - |a_4\rangle + |a_5\rangle + |a_6\rangle + |a_7\rangle + |a_8\rangle \tag{9.26}$$

Now, we need to act with the Hamiltonian on the singlets on RHS. This would give us:

$$4H|a_1\rangle = 4H|a_2\rangle = H|a_7\rangle = H|a_8\rangle = +4|a_9\rangle + 4|a_{10}\rangle + 4|a_{11}\rangle + 4|a_{12}\rangle \tag{9.27}$$

$$H|a_3\rangle = H|a_4\rangle = -2|a_9\rangle + 2|a_{10}\rangle + 2|a_{11}\rangle - 2|a_{12}\rangle \tag{9.28}$$

$$H|a_5\rangle = H|a_6\rangle = +2|a_9\rangle - 2|a_{10}\rangle - 2|a_{11}\rangle + 2|a_{12}\rangle \tag{9.29}$$

$$H|a_{13}\rangle = |a_9\rangle \tag{9.30}$$

Acting with the Hamiltonian on the extra singlets that appeared here leads to:

$$H|a_{10}\rangle = 16|a_{14}\rangle + |a_1\rangle + |a_2\rangle + |a_3\rangle + |a_4\rangle - |a_5\rangle - |a_6\rangle + |a_7\rangle + |a_8\rangle \tag{9.31}$$

$$H|a_{11}\rangle = 16|a_{15}\rangle + |a_1\rangle + |a_2\rangle + |a_3\rangle + |a_4\rangle - |a_5\rangle - |a_6\rangle + |a_7\rangle + |a_8\rangle \tag{9.32}$$

$$H|a_{12}\rangle = 16|a_{16}\rangle + |a_1\rangle + |a_2\rangle - |a_3\rangle - |a_4\rangle + |a_5\rangle + |a_6\rangle + |a_7\rangle + |a_8\rangle \tag{9.33}$$

Lastly, the action of the Hamiltonian on $|a_{14}\rangle$, $|a_{15}\rangle$ and $|a_{16}\rangle$ is given as follows:

$$H|a_{14}\rangle = |a_{10}\rangle;\ H|a_{15}\rangle = |a_{11}\rangle;\ H|a_{16}\rangle = |a_{12}\rangle \tag{9.34}$$

As can be seen from the explicit expressions, these singlet states $|a_1\rangle$ to $|a_{16}\rangle$ are closed under the action of Hamiltonian.

From all the information at hand here, it is easy to construct the eigenstates and are given by:

- Zero energy eigenstates: (8)

$$\begin{aligned} &|a_1\rangle - |a_2\rangle;\ |a_5\rangle - |a_6\rangle;\ |a_7\rangle - |a_8\rangle; \\ &|a_3\rangle - |a_4\rangle; |a_7\rangle - 4|a_1\rangle; |a_3\rangle + |a_5\rangle \\ &|a_1\rangle - |a_{13}\rangle - |a_{14}\rangle - |a_{15}\rangle - |a_{16}\rangle; \\ &|a_3\rangle + 2|a_{13}\rangle - 2|a_{14}\rangle - 2|a_{15}\rangle + 2|a_{16}\rangle \end{aligned} \tag{9.35}$$

- Eigenvalue of ± 4: (2×2)

$$4(|a_{13}\rangle - |a_{16}\rangle) \pm (|a_9\rangle - |a_{12}\rangle) \tag{9.36}$$

$$4(|a_{14}\rangle - |a_{15}\rangle) \pm (|a_{10}\rangle - |a_{11}\rangle) \tag{9.37}$$

Eigenvalue	$-2\sqrt{14}$	$-4\sqrt{3}$	$-2\sqrt{6}$	-4	$-2\sqrt{2}$	0	$2\sqrt{2}$	4	$2\sqrt{6}$	$4\sqrt{3}$	$2\sqrt{14}$
Degeneracy	1	3	4	6	31	50	31	6	4	3	1

Table 2: Eigenvalues and corresponding degeneracy of the singlet eigenstates.

- Eigenvalue of $\pm 4\sqrt{3}$: (1×2)

$$4\left(|a_{13}\rangle - |a_{14}\rangle - |a_{15}\rangle + |a_{16}\rangle\right) - |a_3\rangle - |a_4\rangle + |a_5\rangle + |a_6\rangle \pm \sqrt{3}\left(|a_9\rangle - |a_{10}\rangle - |a_{11}\rangle + |a_{12}\rangle\right) \tag{9.38}$$

- Eigenvalue of $\pm 2\sqrt{14}$: (1×2)

$$4\left(|a_{13}\rangle + |a_{14}\rangle + |a_{15}\rangle + |a_{16}\rangle\right) + |a_1\rangle + |a_2\rangle + |a_7\rangle + |a_8\rangle \pm \sqrt{\frac{7}{2}}\left(|a_9\rangle + |a_{10}\rangle + |a_{11}\rangle + |a_{12}\rangle\right) \tag{9.39}$$

The ground state corresponds to the eigenvalue of $-2\sqrt{14}$ and this value exactly matches with the one that is obtained via numerical diagonalization. In the numerical diagonallization, it was found that the ground state is unique. So, we verified that the ground state we obtained here is unique with respect to all the discrete symmetries that we have defined. This is a non-trivial test as some of the discrete symmetry operators act quite non-trivially on the singlet states.

In the appendix, we give a list of all the eigenstates along with their eigenvalues. The eigenvalues and their degeneracies are summarized in the Table 2.

10. Uniqueness of the ground state and the degeneracies

In this section, we verify that the ground state is unique i.e., we show that it remains unchanged under the action of all the discrete symmetry operators we have defined. Also, we will explain the degeneracy of $+4\sqrt{3}$ energy eigenvalue using those operators. We conclude this section by commenting on the other symmetry operators that might exist in the theory.

To start with, let us write down the ground state explicitly:

(10.1)

$$
\begin{aligned}
|g\rangle \equiv{}& \left(\psi_0^{11+}\psi_0^{21+} + \psi_0^{12+}\psi_0^{22+}\right)(0 \to 1,2,3) \\
&+ \left(\psi_0^{11+}\psi_0^{12+} + \psi_0^{21+}\psi_0^{22+}\right)(0 \to 1,2,3) \\
&+ 4\left(\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+} + \psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+}\right) \\
&\times \left(\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} + \psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\right) \\
&+ \left[\left(\psi_1^{11+}\psi_1^{22+} + \psi_1^{12+}\psi_1^{21+}\right)(1 \leftrightarrow 3) + \left(\psi_1^{11+}\psi_1^{21+} - \psi_1^{12+}\psi_1^{22+}\right)(1 \leftrightarrow 3)\right] \\
&\times \left[\left(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+}\right)(0 \leftrightarrow 2) + \left(\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+}\right)(0 \leftrightarrow 2)\right] \\
&+ \left[\left(\psi_0^{11+}\psi_0^{22+} - \psi_0^{12+}\psi_0^{21+}\right)(0 \leftrightarrow 1) + \left(\psi_0^{11+}\psi_0^{12+} - \psi_0^{21+}\psi_0^{22+}\right)(0 \leftrightarrow 1)\right] \\
&\times \left[\left(\psi_2^{11+}\psi_2^{22+} - \psi_2^{12+}\psi_2^{21+}\right)(2 \leftrightarrow 3) + \left(\psi_2^{11+}\psi_2^{12+} - \psi_2^{21+}\psi_2^{22+}\right)(2 \leftrightarrow 3)\right] \\
&\pm \sqrt{\frac{7}{2}}\big(|(12,21,22),(12,21,22),(11),(11)\rangle \\
&\qquad + |(12,21,22),(11),(12,21,22),(11)\rangle\big) \\
&\pm \sqrt{\frac{7}{2}}\big(|(11),(12,21,22),(11),(12,21,22)\rangle \\
&\qquad + |(11),(11),(12,21,22),(12,21,22)\rangle\big)
\end{aligned}
$$

We want to show that $S|g\rangle = |g\rangle$ where S is one of the discrete symmetry operators. It is easy to verify that under the action of the operators $S_{AB;CD}$, the ground state transforms into itself. Since the operators S_{AB} and S_A anti-commute with the Hamiltonian, we take a product of two such operators to construct operators (S') that commute with the Hamiltonian. As long as we consider the operators S' that include S_{AB} with $(A,B) = (0,3)$ or $(A,B) = (1,2)$, it is straightforward to show that the ground state remains unchanged under its action.

The action of S' becomes non-trivial if it includes the operators S_{AB} with $(A,B) = (2,3)$ or $(A,B) = (0,1)$ or $(A,B) = (1,3)$ or $(A,B) = (0,2)$. This is because these operators have a non-trivial action on the Clifford vacuum. They do not commute with the level operators (3.5) and hence their action on any singlet (in general) mixes singlets of various groups.

For concreteness, let us consider the operator S_{23}. Now, we will consider the action of S_{23} on each of the singlets present in the ground state. Let us start with the singlets of $(p_{1,5}, p_{1,5})$. Before considering the entire singlet, let

us consider the following:

$$
\begin{aligned}
& S_{23}(\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+})|\,\rangle \\
&= -\left[(\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+}) + i(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+})\right] \\
&\quad \times \left[(\psi_1^{11+}\psi_1^{21+} - \psi_1^{12+}\psi_1^{22+})\right. \\
&\qquad \left. - i(\psi_1^{11+}\psi_1^{22+} + \psi_1^{12+}\psi_1^{21+})\right][1 \to 2,3]\,|\,\rangle
\end{aligned} \tag{10.2}
$$

Using this, we can show that:

$$
\begin{aligned}
& \frac{1}{4} S_{23}\left(\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+} + \psi_3^{11+}\psi_3^{12+}\psi_3^{21+}\psi_3^{22+}\right) \\
&\quad \times \left(\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+} + \psi_2^{11+}\psi_2^{12+}\psi_2^{21+}\psi_2^{22+}\right)|\,\rangle \\
&= \left[(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+})(0 \to 2)\right. \\
&\qquad \left. + (\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+})(0 \to 2)\right] \\
&\quad \times \left[(\psi_1^{11+}\psi_1^{22+} + \psi_2^{12+}\psi_2^{21+})(1 \to 3)\right. \\
&\qquad \left. + (\psi_1^{11+}\psi_1^{21+} - \psi_1^{12+}\psi_1^{22+})(2 \to 3)\right]|\,\rangle
\end{aligned} \tag{10.3}
$$

Next, we move on to the singlets of the group (p_3, p_3) in the ground state. To compute the action of S_{23} on these singlets, the following relations are useful:

$$
\begin{aligned}
& S_{23}(\psi_0^{11+}\psi_0^{22+} + \psi_0^{12+}\psi_0^{21+})|\,\rangle \\
&= -2i(\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+} + 1)\ f'(\psi_{1,2,3})|\,\rangle \\
& S_{23}(\psi_0^{11+}\psi_0^{21+} - \psi_0^{12+}\psi_0^{22+})|\,\rangle \\
&= +2(\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+} - 1)\ f'(\psi_{1,2,3})|\,\rangle \\
& S_{23}(\psi_0^{11+}\psi_0^{22+} - \psi_0^{12+}\psi_0^{21+})|\,\rangle \\
&= -2(\psi_0^{11+}\psi_0^{12+} - \psi_0^{21+}\psi_0^{22+})\ f'(\psi_{1,2,3})|\,\rangle \\
& S_{23}(\psi_0^{11+}\psi_0^{12+} - \psi_0^{21+}\psi_0^{22+})|\,\rangle \\
&= +2(\psi_0^{11+}\psi_0^{22+} - \psi_0^{12+}\psi_0^{21+})\ f'(\psi_{1,2,3})|\,\rangle \\
& S_{23}(\psi_0^{11+}\psi_0^{12+} + \psi_0^{21+}\psi_0^{22+})|\,\rangle \\
&= +2i(\psi_0^{11+}\psi_0^{21+} + \psi_0^{12+}\psi_0^{22+})\ f'(\psi_{1,2,3})|\,\rangle \\
& S_{23}(\psi_0^{11+}\psi_0^{21+} + \psi_0^{12+}\psi_0^{22+})|\,\rangle \\
&= +2i(\psi_0^{11+}\psi_0^{12+} + \psi_0^{21+}\psi_0^{22+})\ f'(\psi_{1,2,3})|\,\rangle
\end{aligned} \tag{10.4}
$$

where we have defined the function

$$
f'(\psi_{1,2,3}) = \left[(\psi_1^{11+}\psi_1^{21+} - \psi_1^{12+}\psi_1^{22+}) - i(\psi_1^{11+}\psi_1^{22+} + \psi_1^{12+}\psi_1^{21+})\right][1 \to 2,3]
$$

Using the above relations, we can show that:

$$\begin{aligned}(10.5)\qquad &\frac{1}{16}S_{23}\big[(\psi_0^{11+}\psi_0^{22+}-\psi_0^{12+}\psi_0^{21+})(0\to 1)\\ &\quad+(\psi_0^{11+}\psi_0^{12+}-\psi_0^{21+}\psi_0^{22+})(0\to 1)\big]\\ &\times\big[(\psi_2^{11+}\psi_2^{22+}-\psi_2^{12+}\psi_2^{21+})(2\to 3)\\ &\quad+(\psi_2^{11+}\psi_2^{12+}-\psi_2^{21+}\psi_2^{22+})(2\to 3)\big]|\,\rangle\\ =&\big[(\psi_0^{11+}\psi_0^{22+}+\psi_0^{12+}\psi_0^{21+})(0\to 1)\\ &\quad+(\psi_0^{11+}\psi_0^{12+}-\psi_0^{21+}\psi_0^{22+})(0\to 1)\big]\\ &\times\big[(\psi_2^{11+}\psi_2^{22+}+\psi_2^{12+}\psi_2^{21+})(2\to 3)\\ &\quad+(\psi_2^{11+}\psi_2^{12+}-\psi_2^{21+}\psi_2^{22+})(2\to 3)\big]|\,\rangle\end{aligned}$$

$$\begin{aligned}(10.6)\qquad &\frac{1}{16}S_{23}\big[(\psi_0^{11+}\psi_0^{22+}+\psi_0^{12+}\psi_0^{21+})(0\to 2)\\ &\quad+(\psi_0^{11+}\psi_0^{21+}-\psi_0^{12+}\psi_0^{22+})(0\to 2)\big]\\ &\times\big[(\psi_1^{11+}\psi_1^{22+}+\psi_2^{12+}\psi_2^{21+})(1\to 3)\\ &\quad+(\psi_1^{11+}\psi_1^{21+}-\psi_1^{12+}\psi_1^{22+})(2\to 3)\big]|\,\rangle\\ =&\,4\left(\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}+0\leftrightarrow 3\right)\\ &\times\left(\psi_1^{11+}\psi_1^{12+}\psi_1^{21+}\psi_1^{22+}+1\leftrightarrow 2\right)|\,\rangle\end{aligned}$$

$$\begin{aligned}(10.7)\qquad &\frac{1}{16}S_{23}\left[(\psi_0^{11+}\psi_0^{21+}+\psi_0^{12+}\psi_0^{22+})(0\to 1,2,3)\right]\\ =&\left[(\psi_0^{11+}\psi_0^{12+}+\psi_0^{21+}\psi_0^{22+})(0\to 1,2,3)\right]\end{aligned}$$

$$\begin{aligned}(10.8)\qquad &\frac{1}{16}S_{23}\left[(\psi_0^{11+}\psi_0^{12+}+\psi_0^{21+}\psi_0^{22+})(0\to 1,2,3)\right]\\ =&\left[(\psi_0^{11+}\psi_0^{21+}+\psi_0^{12+}\psi_0^{22+})(0\to 1,2,3)\right]\end{aligned}$$

Lastly, we move to the singlets belonging to the groups $(p_{2,4}, p_{2,4})$. Before finding the action of S_{23} on these singlets, we need the following:

(10.9)

$$\begin{aligned}S_{23}\psi_0^{11+}|\,\rangle &= -\left(\psi_0^{11+}\psi_0^{12+}+i\right)\left(i\psi_0^{21+}+\psi_0^{22+}\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{12+}|\,\rangle &= -\left(\psi_0^{11+}\psi_0^{12+}-i\right)\left(\psi_0^{21+}-i\psi_0^{22+}\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{21+}|\,\rangle &= +\left(i\psi_0^{11+}+\psi_0^{12+}\right)\left(\psi_0^{21+}\psi_0^{22+}+i\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{22+}|\,\rangle &= +\left(\psi_0^{11+}-i\psi_0^{12+}\right)\left(\psi_0^{21+}\psi_0^{22+}-i\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{12+}\psi_0^{21+}\psi_0^{22+}|\,\rangle &= -\left(\psi_0^{11+}\psi_0^{12+}-i\right)\left(i\psi_0^{21+}-\psi_0^{22+}\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{11+}\psi_0^{21+}\psi_0^{22+}|\,\rangle &= +\left(\psi_0^{11+}\psi_0^{12+}+i\right)\left(\psi_0^{21+}+i\psi_0^{22+}\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{11+}\psi_0^{12+}\psi_0^{22+}|\,\rangle &= +\left(i\psi_0^{11+}-\psi_0^{12+}\right)\left(\psi_0^{21+}\psi_0^{22+}-i\right)\,f'(\psi_{1,2,3})|\,\rangle\\ S_{23}\psi_0^{11+}\psi_0^{12+}\psi_0^{21+}|\,\rangle &= -\left(\psi_0^{11+}+i\psi_0^{12+}\right)\left(\psi_0^{21+}\psi_0^{22+}+i\right)\,f'(\psi_{1,2,3})|\,\rangle\end{aligned}$$

where we have defined the function $f'(\psi_{1,2,3})$ above. Computing the action of S_{23} on the singlets of $(p_{2,4}, p_{2,4})$ is now straightforward. The explicit expressions are as follows:

(10.10)

$$
\begin{aligned}
&-\frac{1}{4}S_{23}|(12,21,22),(12,21,22),(11),(11)\rangle \\
&=|(12,21,22),(12,21,22),(11),(11)\rangle+|(12,21,22),(11),(12,21,22),(11)\rangle \\
&+|(11),(12,21,22),(11),(12,21,22)\rangle+|(11),(11),(12,21,22),(12,21,22)\rangle \\
&-|(11),(12),(11,21,22),(12,21,22)\rangle-|(12,21,22),(11,21,22),(12),(11)\rangle \\
&-|(12,21,22),(12),(11,21,22),(11)\rangle-|(12),(12,21,22),(11),(11,21,22)\rangle \\
&-i|(11,21,22),(12,21,22),(11),(11)\rangle-i|(11,21,22),(11),(12,21,22),(11)\rangle \\
&+i|(12),(12,21,22),(11),(12,21,22)\rangle+i|(12),(11),(12,21,22),(12,21,22)\rangle \\
&-i|(12,21,22),(11,21,22),(11),(11)\rangle+i|(12,21,22),(12),(12,21,22),(11)\rangle \\
&-i|(11),(11,21,22),(11),(12,21,22)\rangle+i|(11),(11),(12,21,22),(11,21,22)\rangle
\end{aligned}
$$

$$\diamondsuit \qquad \diamondsuit \qquad \diamondsuit$$

(10.11)

$$
\begin{aligned}
&-\frac{1}{4}S_{23}|(12,21,22),(11),(12,21,22),(11)\rangle \\
&=|(12,21,22),(12,21,22),(11),(11)\rangle+|(12,21,22),(11),(12,21,22),(11)\rangle \\
&+|(11),(12,21,22),(11),(12,21,22)\rangle+|(11),(11),(12,21,22),(12,21,22)\rangle \\
&+|(11),(12),(11,21,22),(12,21,22)\rangle+|(12,21,22),(11,21,22),(12),(11)\rangle \\
&+|(12,21,22),(12),(11,21,22),(11)\rangle+|(12),(12,21,22),(11),(11,21,22)\rangle \\
&-i|(11,21,22),(12,21,22),(11),(11)\rangle-i|(11,21,22),(11),(12,21,22),(11)\rangle \\
&+i|(12),(12,21,22),(11),(12,21,22)\rangle+i|(12),(11),(12,21,22),(12,21,22)\rangle \\
&+i|(12,21,22),(11,21,22),(11),(11)\rangle-i|(12,21,22),(12),(12,21,22),(11)\rangle \\
&+i|(11),(11,21,22),(11),(12,21,22)\rangle-i|(11),(11),(12,21,22),(11,21,22)\rangle
\end{aligned}
$$

$$\diamondsuit \qquad \diamondsuit \qquad \diamondsuit$$

(10.12)

$$
\begin{aligned}
&-\frac{1}{4}S_{23}|(11),(11),(12,21,22),(12,21,22)\rangle \\
&=|(12,21,22),(12,21,22),(11),(11)\rangle+|(12,21,22),(11),(12,21,22),(11)\rangle \\
&+|(11),(12,21,22),(11),(12,21,22)\rangle+|(11),(11),(12,21,22),(12,21,22)\rangle \\
&-|(11),(12),(11,21,22),(12,21,22)\rangle-|(12,21,22),(11,21,22),(12),(11)\rangle \\
&-|(12,21,22),(12),(11,21,22),(11)\rangle-|(12),(12,21,22),(11),(11,21,22)\rangle \\
&+i|(11,21,22),(12,21,22),(11),(11)\rangle+i|(11,21,22),(11),(12,21,22),(11)\rangle \\
&-i|(12),(12,21,22),(11),(12,21,22)\rangle-i|(12),(11),(12,21,22),(12,21,22)\rangle \\
&+i|(12,21,22),(11,21,22),(11),(11)\rangle-i|(12,21,22),(12),(12,21,22),(11)\rangle \\
&+i|(11),(11,21,22),(11),(12,21,22)\rangle-i|(11),(11),(12,21,22),(11,21,22)\rangle
\end{aligned}
$$

♢ ♢ ♢

(10.13)

$$
\begin{aligned}
&-\frac{1}{4}S_{23}|(11),(12,21,22),(11),(12,21,22)\rangle \\
&=|(12,21,22),(12,21,22),(11),(11)\rangle + |(12,21,22),(11),(12,21,22),(11)\rangle \\
&+|(11),(12,21,22),(11),(12,21,22)\rangle + |(11),(11),(12,21,22),(12,21,22)\rangle \\
&+|(11),(12),(11,21,22),(12,21,22)\rangle + |(12,21,22),(11,21,22),(12),(11)\rangle \\
&+|(12,21,22),(12),(11,21,22),(11)\rangle + |(12),(12,21,22),(11),(11,21,22)\rangle \\
&+i|(11,21,22),(12,21,22),(11),(11)\rangle + i|(11,21,22),(11),(12,21,22),(11)\rangle \\
&-i|(12),(12,21,22),(11),(12,21,22)\rangle - i|(12),(11),(12,21,22),(12,21,22)\rangle \\
&-i|(12,21,22),(11,21,22),(11),(11)\rangle + i|(12,21,22),(12),(12,21,22),(11)\rangle \\
&-i|(11),(11,21,22),(11),(12,21,22)\rangle + i|(11),(11),(12,21,22),(11,21,22)\rangle
\end{aligned}
$$

Using these expressions, it is straightforward to show that under the action of $S_{23}S_A$, $S_{23}S_{03}$ or $S_{23}S_{12}$, the ground state transforms into itself.

The rest of the discrete symmetry operators are not independent and can be constructed using the operators we have considered so far. So, the information we have is sufficient to show that the ground state is unique under all the symmetries we have identified.

We now explain the degeneracy of the eigenvalue $+4\sqrt{3}$. This eigenvalue appears in the sets 1,15 and 16. The $+4\sqrt{3}$ eigenstates in the 15 and 16 sets transform into each other under the action of the operator $S_{12}S_A$. We can further show that under the action of $S_{23}S_A$, the $+4\sqrt{3}$ eigenstates of set 1 and set 16 transform into each other. The action of other operators does not lead to any other new states. That is, we find that the degeneracy of the eigenvalue $+4\sqrt{3}$ is three.

In a similar way, we can explain the degeneracies of all the eigenvalues except 0 and $\pm 2\sqrt{2}$. For these exceptions, using our symmetry operators, we can explain the degeneracies partially. By that we mean that there are states having same[16] eigenvalue which are not related via any of the symmetries that we have identified. Thus, we need to find some other symmetries to explain all the degeneracies. One of the drawbacks of our symmetry operators is that the singlets of $(p_{2,4}, p_{2,4})$ do not mix with the singlets of the other groups under any of our symmetry operators. Also, the singlets (21)-(36) of

[16]As an example, consider the first states of sets 1, 2 and 11. All of them have zero eigenvalue and are not related by symmetries. It is good to keep this example in mind for the rest of the discussion.

(p_3, p_3) group transform among themselves under the action of our symmetries. So, the extra symmetry operators we need to identify should overcome these problems. In defining the symmetry operators, we have treated the fermions of all the colors on an equal footing and there is no reason to do that. We believe that the new symmetry operator(s) that can explain all the degeneracies should indeed treat the colors in a different way.

11. Chaos in the gauged $n = 2$ Gurau-Witten model

In the previous sections, we have identified the gauge spectrum of $n = 2$ Gurau-Witten model explicitly. Now, we investigate whether there are any signs of chaos in the gauged sector. Even though the number of distinct eigenvalues is small, we find that the spectral form factor has a dip-ramp-plateau structure indicating the signs of chaos.

Before investigating chaos, let us first understand the eigenvalue spectrum of the gauged model. There are 11 distinct eigenvalues in the spectrum. There is a large degeneracy at zero energy and the spectrum has spectral mirror symmetry as is obvious from the plot of density of eigenvalues in Figure 1. Note that all the eigenvalues in the Table 2 are present in the numerical diagonalization as well and this provides a non-trivial check of our results.

The tool we use to investigate chaos is the spectral form factor (SFF). It is defined as:

$$F(\beta, t) = \left| \frac{Z(\beta, t)}{Z(\beta, 0)} \right|^2 ; \quad Z(\beta, t) = \mathrm{Tr}\left(e^{-(\beta + it)H} \right) \tag{11.1}$$

For chaotic systems, SFF initially decays up to a certain time called the dip-time (t_d). After that, it starts raising until the plateau time (t_p) and then finally stabilizes to a value called the plateau height. That is, the SFF of chaotic systems have a dip-ramp-plateau structure. We compute the SFF for the singlet spectrum and report it in the Figure 2 after a sliding time average with different sliding intervals Δt. Even though we have only 11 distinct eigenvalues, the SFF qualitatively has a dip-ramp-plateau structure which can be understood as a primitive sign of chaos.

Exact solutions of strongly coupled (fermionic) gauge theories should be useful in extending the understanding of eigenstate thermalization [19], entanglement/entanglement entropy [20] and evolution of complexity [21] in the context of holography.

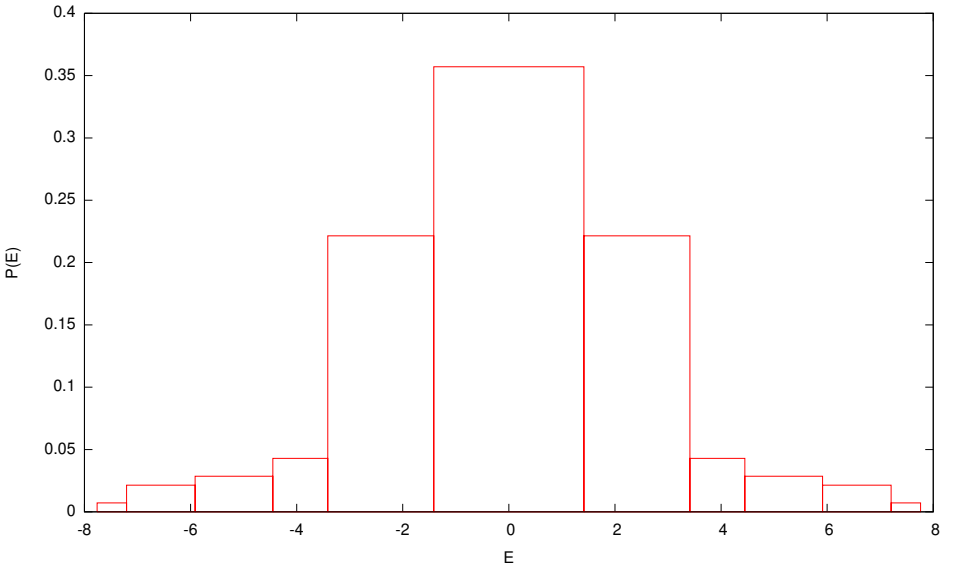

Figure 1: Density of states for the singlet spectrum of $n = 2$ Gurau-Witten model.

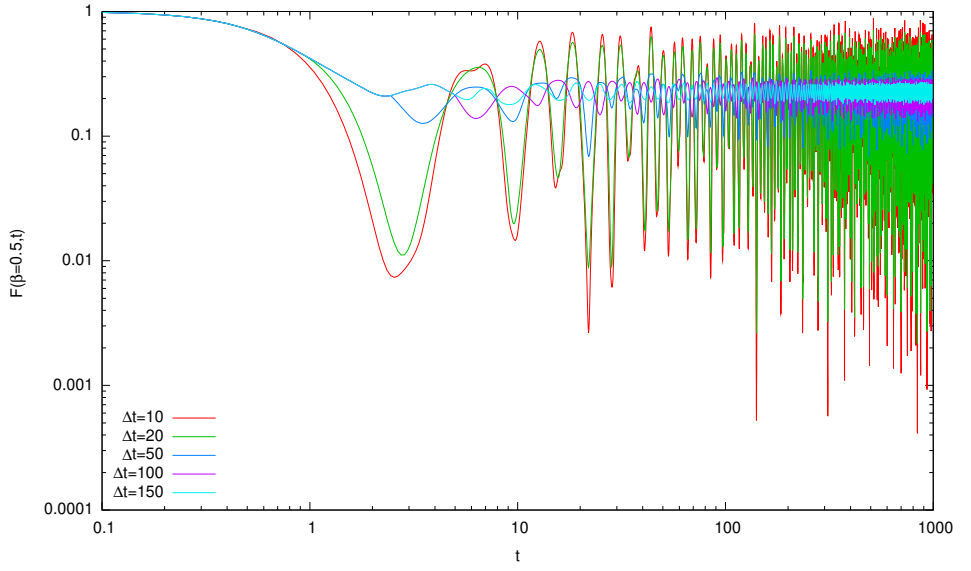

Figure 2: SFF for the singlet spectrum of $n = 2$ Gurau-Witten model for $\beta = 0.5$.

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Appendix A. $n = 2$ uncolored model

In this appendix, we give relevant details of the $n = 2$ uncolored model. These are useful especially for finding singlets via method-II. For more details on the $n = 2$ uncolored model, see [16].

The Noether charges Of $n = 2$ uncolored model are given by:

$$\begin{aligned} Q_1^{12} &= i\left(\psi^{111^+}\psi^{211^-} + \psi^{111^-}\psi^{211^+} + \psi^{121^+}\psi^{221^-} + \psi^{121^-}\psi^{221^+}\right) \\ Q_2^{12} &= i\left(\psi^{111^+}\psi^{121^-} + \psi^{111^-}\psi^{121^+} + \psi^{211^+}\psi^{221^-} + \psi^{211^-}\psi^{221^+}\right) \\ Q_3^{12} &= 2 - \psi^{111^+}\psi^{111^-} - \psi^{121^+}\psi^{121^-} - \psi^{211^+}\psi^{211^-} - \psi^{221^+}\psi^{221^-} \end{aligned} \tag{A.1}$$

The action of the first two charges on level 0 and level 4 states is given by:

$$\begin{aligned} Q_{1,2}^{12}|\,\rangle &= 0 \\ Q_{1,2}^{12}\left(\psi^{111^+}\psi^{121^+}\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= 0 \end{aligned} \tag{A.2}$$

At level 1, we have:

$$\begin{aligned} Q_1^{12}\ \psi^{111^+}|\,\rangle &= -\psi^{211^+} \\ Q_1^{12}\ \psi^{211^+}|\,\rangle &= +\psi^{111^+} \\ Q_1^{12}\ \psi^{121^+}|\,\rangle &= -\psi^{221^+} \\ Q_1^{12}\ \psi^{221^+}|\,\rangle &= +\psi^{121^+} \end{aligned} \tag{A.3}$$

At level 3, we have:

$$\begin{aligned} Q_1^{12}\left(\psi^{111^+}\psi^{121^+}\psi^{211^+}\right)|\,\rangle &= +\psi^{111^+}\psi^{211^+}\psi^{221^+}|\,\rangle \\ Q_1^{12}\left(\psi^{111^+}\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= -\psi^{111^+}\psi^{121^+}\psi^{211^+}|\,\rangle \\ Q_1^{12}\left(\psi^{111^+}\psi^{121^+}\psi^{221^+}\right)|\,\rangle &= +\psi^{121^+}\psi^{211^+}\psi^{221^+}|\,\rangle \\ Q_1^{12}\left(\psi^{121^+}\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= -\psi^{111^+}\psi^{121^+}\psi^{221^+}|\,\rangle \end{aligned} \tag{A.4}$$

From these relations, we can see that:

$$\left((Q_1)^2+1\right)|\text{Level 1/3 state}\rangle = 0 \tag{A.5}$$

At level 2:

(A.6)

$$\begin{aligned}
Q_1^{12}\left(\psi^{111^+}\psi^{211^+}\right)|\,\rangle &= 0\\
Q_1^{12}\left(\psi^{121^+}\psi^{221^+}\right)|\,\rangle &= 0\\
Q_1^{12}\left(\psi^{111^+}\psi^{221^+}+\psi^{121^+}\psi^{211^+}\right)|\,\rangle &= 0\\
Q_1^{12}\left(\psi^{111^+}\psi^{121^+}+\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= 0\\
Q_1^{12}\left(\psi^{111^+}\psi^{121^+}-\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= 2\left(\psi^{121^+}\psi^{211^+}-\psi^{111^+}\psi^{221^+}\right)|\,\rangle\\
Q_1^{12}\left(\psi^{111^+}\psi^{221^+}-\psi^{121^+}\psi^{211^+}\right)|\,\rangle &= 2\left(\psi^{111^+}\psi^{121^+}-\psi^{211^+}\psi^{221^+}\right)|\,\rangle
\end{aligned}$$

Now, we consider the action of Q_2 charge. At level 1, we have the following relations:

$$\begin{aligned}
Q_2^{12}\ \psi^{111^+}|\,\rangle &= -\psi^{121^+}\\
Q_2^{12}\ \psi^{121^+}|\,\rangle &= +\psi^{111^+}\\
Q_2^{12}\ \psi^{211^+}|\,\rangle &= -\psi^{221^+}\\
Q_2^{12}\ \psi^{221^+}|\,\rangle &= +\psi^{211^+}
\end{aligned} \tag{A.7}$$

At level 3:

$$\begin{aligned}
Q_2^{12}\left(\psi^{111^+}\psi^{121^+}\psi^{211^+}\right)|\,\rangle &= -\psi^{111^+}\psi^{121^+}\psi^{221^+}|\,\rangle\\
Q_2^{12}\left(\psi^{111^+}\psi^{121^+}\psi^{221^+}\right)|\,\rangle &= +\psi^{111^+}\psi^{121^+}\psi^{211^+}|\,\rangle\\
Q_2^{12}\left(\psi^{111^+}\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= -\psi^{121^+}\psi^{211^+}\psi^{221^+}|\,\rangle\\
Q_2^{12}\left(\psi^{121^+}\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= +\psi^{111^+}\psi^{211^+}\psi^{221^+}|\,\rangle
\end{aligned} \tag{A.8}$$

As in the case of Q_1, we have:

$$\left((Q_2)^2+1\right)|\text{Level 1/3 state}\rangle = 0 \tag{A.9}$$

At level 2, we have:
(A.10)

$$
\begin{aligned}
Q_2^{12}\left(\psi^{111^+}\psi^{121^+}\right)|\,\rangle &= 0\\
Q_2^{12}\left(\psi^{211^+}\psi^{221^+}\right)|\,\rangle &= 0\\
Q_2^{12}\left(\psi^{111^+}\psi^{221^+}-\psi^{121^+}\psi^{211^+}\right)|\,\rangle &= 0\\
Q_2^{12}\left(\psi^{111^+}\psi^{211^+}+\psi^{121^+}\psi^{221^+}\right)|\,\rangle &= 0\\
Q_2^{12}\left(\psi^{111^+}\psi^{211^+}-\psi^{121^+}\psi^{221^+}\right)|\,\rangle &= -2\left(\psi^{121^+}\psi^{211^+}+\psi^{111^+}\psi^{221^+}\right)|\,\rangle\\
Q_2^{12}\left(\psi^{111^+}\psi^{221^+}+\psi^{121^+}\psi^{211^+}\right)|\,\rangle &= 2\left(\psi^{111^+}\psi^{211^+}-\psi^{121^+}\psi^{221^+}\right)|\,\rangle
\end{aligned}
$$

We conclude by pointing out that the charges Q_1 and Q_2 commute and this fact is useful in determining the singlets of $(p_{2,4}, p_{2,4})$ and also in uniquely fixing such singlets to be of the form (7.9) .

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Center for High Energy Physics
Indian Institute of Science
Bangalore 560012, India
E-mail address: chethan.krishnan@gmail.com
E-mail address: kumar.pavan56@gmail.com