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Two-dimensional supersymmetric gauge theories with exceptional gauge groups

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We apply the recent proposal for mirrors of nonabelian (2,2) supersymmetric two-dimensional gauge theories to make predictions for two-dimensional supersymmetric gauge theories with exceptional gauge groups G_2, F_4, E_6, E_7, and E_8. We compute the mirror Landau-Ginzburg models and predict excluded Coulomb loci and Coulomb branch relations (quantum cohomology). We also discuss the relationship between weight lattice normalizations and theta angle periodicities in the proposal, and explore different conventions for the mirrors. Finally, we discuss the behavior of pure gauge theories with exceptional gauge groups under RG flow, and describe evidence that any pure supersymmetric two-dimensional gauge theory with connected and simply-connected semisimple gauge group flows in the IR to a free theory of as many twisted chiral superfields as the rank of the gauge group, extending previous results for SU, SO, and Sp gauge theories.

1. Introduction

Mirror symmetry is a well-known duality of string theory, whose original form has been extended in a variety of ways. For two-dimensional abelian gauged linear sigma models, constructive proofs and various aspects thereof were described in [1, 2]. In particular, the paper [1] gave an explicit construction of a Landau-Ginzburg model mirror to many abelian gauged linear sigma models. However, one open problem for many years has been to find an analogous construction for two-dimensional supersymmetric nonabelian gauged linear sigma models.

Recently, a proposal was made in [3] for mirrors to two-dimensional supersymmetric nonabelian gauge theories. Specifically, it gave a construction of Landau-Ginzburg orbifolds for supersymmetric nonabelian gauge theories. That work checked the proposal against a wide variety of results for two-dimensional theories with classical gauge groups. To further develop the underlying machinery, in this paper we will apply the proposed mirror construction of [3] to two-dimensional supersymmetric nonabelian gauge theories with the exceptional gauge groups G_2, F_4, $E_{6,7,8}$, to make predictions for excluded loci and Coulomb branch relations (analogues of quantum cohomology relations).

Working through these computations will also allow us to explore some properties of those mirror superpotentials, which take the form

$$W = \sum_{a=1}^{r} \sigma_a \left(\sum_{i=1}^{N} \rho_i^a Y_i - \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) + \sum_{i=1}^{N} \exp\left(-Y_i\right) + \sum_{\tilde{\mu}=1}^{n-r} X_{\tilde{\mu}}. \tag{1.1}$$

In the expression above, ρ_i^a are components of weight vectors for matter representations of the original gauge theory, and $\alpha_{\tilde{\mu}}^a$ are root vectors (here taken to form a sublattice of the weight lattice). As described in [3], the σs encode theta angles in the Cartan subalgebra of the original gauge theory, and have periodicities reflecting the weight lattice, or at least the sublattice generated by the matter representations. However, the weight lattice need not be normalized in the same way as a charge lattice. It is always possible to find a basis for the weight lattice (in terms of fundamental weights) so that the coefficients in the σ terms are all integers, reflecting 2π theta angle periodicities and standard charge lattice conventions, but one can also consistently work in other bases as well. For the case of G_2 gauge theories, we will use a naive basis which results in nonstandard theta angle periodicities and charge lattices. For F_4, we explain in detail how to use instead a basis of fundamental weights, which results in standard theta angle periodicities and charge lattice normalizations, and we will use that convention for all of the other gauge theories discussed in this paper (except G_2, which we retain as an illustrative example).

In each case, we shall also study the mirror to the pure gauge theory, to follow up observations in [4]. In particular, [4] argued that two-dimensional pure (2,2) supersymmetric $SU(k)$ gauge theories flow in the IR to a free theory of $k-1$ twisted chiral multiplets, which [3] checked at the level of topological field theory computations and extended to $SO(n)$ theories with discrete theta angles and to $Sp(k)$ gauge theories. In each case, for one discrete theta angle, evidence in TFT computations was given that the theory flowed to a pure gauge theory of as many twisted chiral multiplets as the rank of the gauge group. We shall check the analogous claim for pure gauge theories with exceptional gauge groups in this paper, at the level of topological field theory computations, and will find evidence for the same result – that the pure gauge theories (for simply-connected gauge groups) flow in the IR to a theory of as many twisted chiral multiplets as the rank of the gauge group.

Combining the results of this paper with those in [3], a simple conjecture emerges: a pure two-dimensional (2,2) supersymmetric gauge theory with connected and simply-connected semisimple gauge group flows in the IR to a free theory of as many twisted chiral superfields as the rank of the gauge group. A check of this conjecture for Spin gauge theories can be derived from the results for SO gauge theories in [3]. Now, SO groups are not simply-connected; however, we can apply two-dimensional decomposition [5, 6] and the results for SO theories with various discrete theta angles to argue that a pure Spin gauge theory flows in the IR to a free theory of as many twisted chiral superfields as the rank. Combined with the results in this paper for[1] pure two-dimensional supersymmetric G_2, F_4, and $E_{6,7,8}$ gauge theories, we have the conjecture above.

We begin in Section 2 by reviewing the nonabelian mirror proposal of [3], which will be applied in this paper to theories with exceptional gauge groups. In Section 3 we compute the mirror Landau-Ginzburg orbifold of G_2

[1] G_2, F_4, and E_8 have no center, but E_6 has center $\mathbb{Z}_3$ and E_7 has center $\mathbb{Z}_2$, so for those groups we must specify the simply-connected cover. See [8, appendix A] for further details on centers.

gauge theories with matter in copies of the fundamental **7** dimensional representation. In Section 4 we compute the mirror Landau-Ginzburg orbifold of F_4 gauge theories with matter in copies of the fundamental **26** representation. In Section 5, we compute the mirror Landau-Ginzburg orbifold of E_6 with matter in copies of the **27** representation. In Sections 6, 7 we perform the same analysis for E_7 and E_8 with matter fields in copies of the **56** representation of E_7 and **248** of E_8.

In the published version of this paper, we have omitted a number of extremely lengthy expressions for superpotentials and ring relations from the analyses of $E_{6,7,8}$ gauge theories, which are straightforward to derive using the same methods as for G_2 and F_4 gauge theories. Those expressions can be found in the online version of this article, at [7].

2. Brief review of the nonabelian mirror proposal

The nonabelian mirror proposal of [3] is a generalization of the abelian duality described in [1] (see also [2]). It takes the following form. For an A-twisted two-dimensional (2,2) supersymmetric gauge theory with connected gauge group G, the mirror is a B-twisted Landau-Ginzburg orbifold, defined by (twisted) chiral multiplets

- Y_i, corresponding to the N matter fields of the original gauge theory,
- $X_{\tilde{\mu}}$, corresponding to nonzero roots $\tilde{\mu}$ of the Lie algebra $\mathfrak{g}$ of G, of dimension n,
- $\sigma_a = \overline{D}_+ D_- V_a$, as many as the rank r of G, corresponding to a choice of Cartan subalgebra of $\mathfrak{g}$, the Lie algebra of G,

with superpotential

$$W = \sum_{a=1}^{r} \sigma_a \left(\sum_{i=1}^{N} \rho_i^a Y_i - \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) + \sum_{i=1}^{N} \exp\left(-Y_i\right) + \sum_{\tilde{\mu}=1}^{n-r} X_{\tilde{\mu}} - \sum_i \tilde{m}_i Y_i, \tag{2.1}$$

In the expression above, the ρ_i^a are components of weight vectors for the matter representations appearing in the original gauge theory, and $\alpha_{\tilde{\mu}}^a$ are components of nonzero roots (here viewed as defining a sublattice of the weight lattice). (Also, sometimes one uses $Z = -\ln X$ for simplicity.) The t_a are constants, corresponding to Fayet-Iliopoulos parameters of the original gauge theory, and the $\tilde{m}_i$ are twisted masses in the original gauge theory. One then orbifolds by the Weyl group, which acts naturally on all the fields above, and leaves the superpotential invariant. The expression above was written for A-twisted gauge theories without a superpotential, but can be generalized to mirrors of gauge theories with superpotentials by assigning suitable R-charges and changing the fundamental fields accordingly, as explained in [3].

In the analysis of this theory, it was argued that some loci are dynamically excluded – specifically, loci where any $X_{\tilde{\mu}}$ vanishes. These loci turn out to reproduce excluded loci on Coulomb branches of the original gauge theories. Furthermore, critical loci of the superpotential above obey relations which correspond to relations in the OPE ring of the original A-twisted gauge theory. For gauge theories with $U(1)$ factors in G, one has continuous Fayet-Iliopoulos parameters, so one can speak of weak coupling limits, and those OPE relations are known as quantum cohomology relations. In cases in which G has no $U(1)$ factors, so that there are no continuous Fayet-Iliopoulos parameters, there is no weak coupling limit, and so referring to such relations as 'quantum cohomology' relations is somewhat misleading. In such cases, we refer to the relations as defining the Coulomb ring or Coulomb branch ring.

The work [3] checked the predictions of this proposal for excluded loci and Coulomb branch and quantum cohomology relations against known results for two-dimensional gauge theories in *e.g.* [9–12], and gave general arguments for why correlation functions in this B-twisted theory should match correlation functions in corresponding A-twisted gauge theories, such as in *e.g.* [13–15]. It also studied mirrors to pure gauge theories, to test and refine predictions for IR behavior described in [4]. In this paper, we will apply this mirror construction to make predictions for two-dimensional (2,2) supersymmetric gauge theories with exceptional gauge groups. To make all of these comparisons, the paper [3] utilized the following operator mirror map:

$$\exp(-Y_i) = -\tilde{m}_i + \sum_{a=1}^{r} \sigma_a \rho_i^a, \tag{2.2}$$

$$X_{\tilde{\mu}} = \sum_{a=1}^{r} \sigma_a \alpha_{\tilde{\mu}}^a, \tag{2.3}$$

which we shall also use in this paper.

3. G_2

In this section we will consider the mirror Landau-Ginzburg orbifold of G_2 gauge theory with matter fields in copies of the **7** representation, and then we compute quantum cohomology ring.

3.1. Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has fields

- $Y_{i\beta}$, $i \in \{1, \ldots, n\}$, $\beta \in \{0, \ldots, 6\}$, corresponding to the matter fields in n copies of the **7** of G_2,
- X_m, $\tilde{X}_m$, $m \in \{1, \ldots, 6\}$, corresponding to the short, respectively long roots of G_2,
- σ_a, $a \in \{1, 2\}$.

We associate the roots and weights to fields as listed in Table 1 and Figures 1, 2.

Field	Short root	Field	Long root	Field	Weight
X_1	$(1,0)$	$\tilde{X}_1$	$(-3/2, \sqrt{3}/2)$	Y_{i1}	$(1,0)$
X_2	$(-1,0)$	$\tilde{X}_2$	$(3/2, -\sqrt{3}/2)$	Y_{i2}	$(-1,0)$
X_3	$(1/2, \sqrt{3}/2)$	$\tilde{X}_3$	$(3/2, \sqrt{3}/2)$	Y_{i3}	$(1/2, \sqrt{3}/2)$
X_4	$(-1/2, -\sqrt{3}/2)$	$\tilde{X}_4$	$(-3/2, -\sqrt{3}/2)$	Y_{i4}	$(-1/2, -\sqrt{3}/2)$
X_5	$(-1/2, \sqrt{3}/2)$	$\tilde{X}_5$	$(0, \sqrt{3})$	Y_{i5}	$(-1/2, \sqrt{3}/2)$
X_6	$(1/2, -\sqrt{3}/2)$	$\tilde{X}_6$	$(0, -\sqrt{3})$	Y_{i6}	$(1/2, -\sqrt{3}/2)$
				Y_{i0}	$(0,0)$

Table 1: Roots and weights for G_2 and associated fields.

The mirror superpotential takes the form

$$W = \sigma_1 \Bigg(\sum_i (Y_{i1} - Y_{i2} + (1/2)Y_{i3} - (1/2)Y_{i4} - (1/2)Y_{i5} + (1/2)Y_{i6}) \\ + (Z_1 - Z_2 + (1/2)Z_3 - (1/2)Z_4 - (1/2)Z_5 + (1/2)Z_6) \\ + \left(-(3/2)\tilde{Z}_1 + (3/2)\tilde{Z}_2 + (3/2)\tilde{Z}_3 - (3/2)\tilde{Z}_4 \right) \Bigg) + \tag{3.1}$$

$$+ \sigma_2 \Bigg((\sqrt{3}/2) \sum_i (Y_{i3} - Y_{i4} + Y_{i5} - Y_{i6}) + (\sqrt{3}/2)(Z_3 - Z_4 + Z_5 - Z_6)$$
$$+ (\sqrt{3}/2)\left(\tilde{Z}_1 - \tilde{Z}_2 + \tilde{Z}_3 - \tilde{Z}_4 + 2\tilde{Z}_5 - 2\tilde{Z}_6\right)\Bigg)$$
$$+ \sum_i \sum_{\alpha=0}^{6} \exp(-Y_{i\alpha}) + \sum_{m=1}^{6} X_m + \sum_{m=1}^{6} \tilde{X}_m - \sum_{i,\alpha} \tilde{m}_i Y_{i\alpha},$$

where $X_m = \exp(-Z_m)$, $\tilde{X}_m = \exp(-\tilde{Z}_m)$, with X_m, $\tilde{X}_m$ the fundamental fields and $\tilde{m}_i$ are the twisted masses.

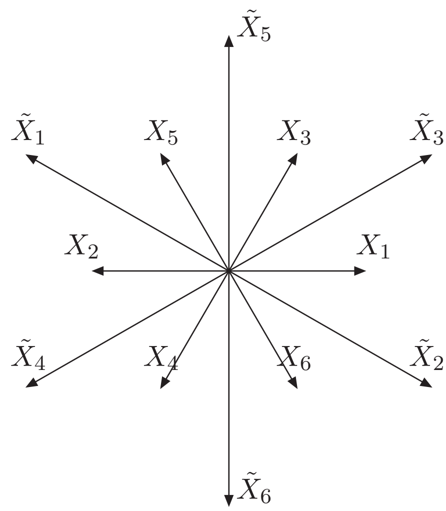

Figure 1: Roots of G_2.

The logic of the assignments above is that X_{odd}, $\tilde{X}_{\text{odd}}$ correspond to positive roots, X_{even}, $\tilde{X}_{\text{even}}$ correspond to their opposites, and the weight vectors are associated to matter fields similarly. We follow the conventions of [16, chapter 22]: short roots are given by

$$(\pm 1, 0), \quad \pm(+1/2, \sqrt{3}/2), \quad \pm(-1/2, \sqrt{3}/2),$$

long roots are given by

$$\pm(-3/2, \sqrt{3}/2), \quad \pm(+3/2, \sqrt{3}/2), \quad \pm(0, \sqrt{3}),$$

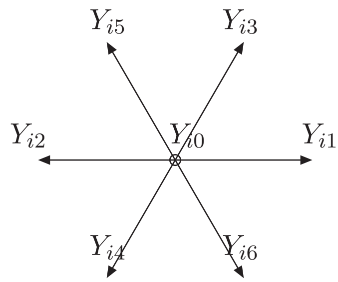

Figure 2: Weights of **7** of G_2.

and the weights of the **7** are given by

$$\pm(1,0),\quad \pm(1/2,\sqrt{3}/2),\pm(-1/2,\sqrt{3}/2),\quad (0,0).$$

Before moving on, there is an important subtlety in the expression for the mirror superpotential above, involving the theta angle periodicities. As described in [3], the factors multiplied by σs are not single-valued, reflecting the fact that the σ terms encode theta angles in the abelian subgroup determined by the choice of Cartan subgroup of the original gauge group. The periodicities[2] of these theta angles are determined by 2π times the weight lattice, or at least the sublattice generated by the matter representations. However, the weight lattice need not be normalized in the same way as a charge lattice. For example, in our conventions for the weight lattice of G_2 above, the σ_1 terms determine a theta angle periodicity of $2\pi/2 = \pi$ rather than 2π, and the σ_2 terms determine a theta angle periodicity of $(\sqrt{3}/2)(2\pi) = \sqrt{3}\pi$ rather than 2π.

Now, on the one hand, the normalization of the charge lattice is ultimately a convention, and so long as one is consistent, one can work with alternative conventions. On the other hand, it is also often helpful to work with standard conventions.

For the case of G_2, we shall use the normalization above, hence a nonstandard charge lattice normalization. However, it is always possible to rotate to a conventional charge lattice normalization by writing the weights

[2]On a noncompact worldsheet, the theta angles generate electric fields with periodicities determined by the matter representations – as theta increases, the electric field density eventually becomes strong enough to allow pair creation of matter fields.

in a basis of fundamental weights, for which any other weight is an integer linear combination. In terms of that mathematical basis, the theta angle periodicities determined by σs are all 2π, reflecting a standard charge lattice normalization. We will discuss this alternative basis in more detail for F_4, and in fact will use that alternative basis (and standard charge normalization) to study all the other gauge theories in this paper, after G_2. We study G_2 in nonstandard conventions for illustrative purposes.

3.2. Weyl group

Now, let us explicitly describe the action of the Weyl group on the fields of this theory and outline explicitly why the superpotential is invariant in this case. (General arguments appeared in [3], but as the Weyl group action is more complicated here than in the examples in that paper, a more detailed verification seems in order.)

For any root α, recall that the Weyl group reflection generated by α acts on a weight μ as follows:

$$\mu \mapsto \mu - \frac{2(\alpha \cdot \mu)}{\alpha^2}\alpha. \tag{3.2}$$

For example, for the Weyl reflection generated by $\alpha = (1, 0)$, it is straightforward to compute that the group action on fields corresponding to roots is given by

$$X_1 \leftrightarrow X_2, \quad X_3 \leftrightarrow X_5, \quad X_4 \leftrightarrow X_6, \tag{3.3}$$

$$\tilde{X}_1 \leftrightarrow \tilde{X}_3, \quad \tilde{X}_2 \leftrightarrow \tilde{X}_4, \tag{3.4}$$

and $\tilde{X}_{5,6}$ are invariant. The action on matter fields is

$$Y_{i1} \leftrightarrow Y_{i2}, \quad Y_{i3} \leftrightarrow Y_{i5}, \quad Y_{i4} \leftrightarrow Y_{i6}, \tag{3.5}$$

with Y_{i7} invariant. This is just a reflection about the y axis, which multiplies the first coordinate by -1 but leaves the second invariant. It is straightforward to check that the superpotential will be invariant under this reflection so long as

$$\sigma_1 \leftrightarrow -\sigma_1, \tag{3.6}$$

and σ_2 is invariant.

For another example, for Weyl reflections generated by $\alpha = (3/2, \sqrt{3}/2)$, it is straightforward to compute that the group action on fields corresponding to roots is given by

$$X_1 \leftrightarrow X_4, \quad X_2 \leftrightarrow X_3, \tag{3.7}$$

with $X_{5,6}$ invariant, and

$$\tilde{X}_1 \leftrightarrow \tilde{X}_5, \quad \tilde{X}_2 \leftrightarrow \tilde{X}_6, \quad \tilde{X}_3 \leftrightarrow \tilde{X}_4. \tag{3.8}$$

The action on matter fields is the same as on the mirrors to the short roots:

$$Y_{i1} \leftrightarrow Y_{i4}, \quad Y_{i2} \leftrightarrow Y_{i3}, \tag{3.9}$$

with Y_{i5}, Y_{i6} invariant. The σ_a fields are similarly rotated:

$$\begin{aligned} \sigma_1 &\mapsto -\frac{1}{2}\sigma_1 - \frac{\sqrt{3}}{2}\sigma_2, \\ \sigma_2 &\mapsto -\frac{\sqrt{3}}{2}\sigma_1 + \frac{1}{2}\sigma_2. \end{aligned}$$

(Note that if we describe the action above as mapping $\vec{\sigma} \mapsto A\vec{\sigma}$ for a 2×2 matrix A, then for the choice of A implicit above, it is straightforward to check $A = A^{-1}$.) It is straightforward to check that the superpotential is invariant under the action above. For example, the terms

$$\begin{aligned} &\sigma_1 \left(Z_1 - Z_2 + (1/2)Z_3 - (1/2)Z_4 - (1/2)Z_5 + (1/2)Z_6 \right) \\ &+ \sigma_2(\sqrt{3}/2) \left(Z_3 - Z_4 + Z_5 - Z_6 \right) \\ \mapsto &\left(-(1/2)\sigma_1 - (\sqrt{3}/2)\sigma_2 \right) \\ &\times \left(Z_4 - Z_3 + (1/2)Z_2 - (1/2)Z_1 - (1/2)Z_5 + (1/2)Z_6 \right) \\ &+ \left(-(\sqrt{3}/2)\sigma_1 + (1/2)\sigma_2 \right) (\sqrt{3}/2) \left(Z_2 - Z_1 + Z_5 - Z_6 \right), \end{aligned}$$

which is easily checked to be the same as the starting point,

$$\begin{aligned} &\sigma_1 \left(Z_1 - Z_2 + (1/2)Z_3 - (1/2)Z_4 - (1/2)Z_5 + (1/2)Z_6 \right) \\ &+ \sigma_2(\sqrt{3}/2) \left(Z_3 - Z_4 + Z_5 - Z_6 \right). \end{aligned}$$

Similar statements are true of other terms, and so the superpotential is preserved.

To be thorough, we will consider one more example of a Weyl group action, this time a reflection defined by a short root, specifically $\alpha = (1/2, \sqrt{3}/2)$. It is straightforward to compute that the group action on fields corresponding to roots is given by

$$X_1 \leftrightarrow X_6, \quad X_2 \leftrightarrow X_5, \quad X_3 \leftrightarrow X_4, \tag{3.10}$$

and

$$\tilde{X}_3 \leftrightarrow \tilde{X}_6, \quad \tilde{X}_4 \leftrightarrow \tilde{X}_5, \tag{3.11}$$

with $\tilde{X}_{1,2}$ invariant. The action on the matter fields is the same as on the mirrors to the short roots:

$$Y_{i1} \leftrightarrow Y_{i6}, \quad Y_{i2} \leftrightarrow Y_{i5}, \quad Y_{i3} \leftrightarrow Y_{i4}. \tag{3.12}$$

This is another reflection about the axis pass through X_5 and X_6. The σ_a fields are similarly rotated:

$$\begin{aligned}
\sigma_1 &\mapsto \frac{1}{2}\sigma_1 - \frac{\sqrt{3}}{2}\sigma_2, \\
\sigma_2 &\mapsto -\frac{\sqrt{3}}{2}\sigma_1 - \frac{1}{2}\sigma_2.
\end{aligned}$$

It is straightforward to check that the superpotential is invariant.

3.3. Coulomb ring relations

Integrating out the sigma fields in the superpotential (3.1), we obtain two constraints:

$$\begin{aligned}
&\sum_i (2Y_{i1} - 2Y_{i2} + Y_{i3} - Y_{i4} - Y_{i5} + Y_{i6}) \\
&\quad + (2Z_1 - 2Z_2 + Z_3 - Z_4 - Z_5 + Z_6) \\
&\quad + \left(-3\tilde{Z}_1 + 3\tilde{Z}_2 + 3\tilde{Z}_3 - 3\tilde{Z}_4\right) = 0, \\
&\sum_i (Y_{i3} - Y_{i4} + Y_{i5} - Y_{i6}) + (Z_3 - Z_4 + Z_5 - Z_6) \\
&\quad + \left(\tilde{Z}_1 - \tilde{Z}_2 + \tilde{Z}_3 - \tilde{Z}_4 + 2\tilde{Z}_5 - 2\tilde{Z}_6\right) = 0.
\end{aligned}$$

With the two constraints above, we are free to eliminant two fundamental fields, which we will take to be Y_{n3} and Y_{n6}:

$$
\begin{aligned}
-Y_{n3} &= \sum_{i=1}^{n} (Y_{i1} - Y_{i2} - Y_{i4}) + \sum_{i=1}^{n-1} Y_{i3} + (Z_1 - Z_2 + Z_3 - Z_4) \\
&\quad + \left(-\tilde{Z}_1 + \tilde{Z}_2 + 2\tilde{Z}_3 - 2\tilde{Z}_4 + \tilde{Z}_5 - \tilde{Z}_6\right), \\
-Y_{n3} &= \sum_{i=1}^{n} (Y_{i1} - Y_{i2} - Y_{i5}) + \sum_{i=1}^{n-1} Y_{i6} + (Z_1 - Z_2 - Z_5 + Z_6) \\
&\quad + \left(-2\tilde{Z}_1 + 2\tilde{Z}_2 + \tilde{Z}_3 - \tilde{Z}_4 - \tilde{Z}_5 + \tilde{Z}_6\right).
\end{aligned}
$$

For convenience, let's define:

$$
\begin{aligned}
\Pi_3 &\equiv \exp\left(-Y_{n3}\right), \\
&= \prod_{i=1}^{n} \exp\left(Y_{i1} - Y_{i2} - Y_{i4}\right) \prod_{i=1}^{n-1} \exp\left(Y_{i3}\right) \frac{X_2 X_4}{X_1 X_3} \frac{\tilde{X}_1 \tilde{X}_4^2 \tilde{X}_6}{\tilde{X}_2 \tilde{X}_3^2 \tilde{X}_5}, \qquad (3.13) \\
\Pi_6 &\equiv \exp\left(-Y_{n6}\right), \\
&= \prod_{i=1}^{n} \exp\left(Y_{i1} - Y_{i2} - Y_{i5}\right) \prod_{i=1}^{n-1} \exp\left(Y_{i6}\right) \frac{X_2 X_5}{X_1 X_6} \frac{\tilde{X}_1^2 \tilde{X}_4 \tilde{X}_5}{\tilde{X}_2^2 \tilde{X}_3 \tilde{X}_6}. \qquad (3.14)
\end{aligned}
$$

Then, the superpotential (3.1) reduces to

$$
\begin{aligned}
W = & \sum_{i=1}^{n} \left[\exp\left(-Y_{i0}\right) + \exp\left(-Y_{i1}\right) + \exp\left(-Y_{i2}\right) + \exp\left(-Y_{i4}\right) + \exp\left(-Y_{i5}\right) \right] \\
& + \sum_{i=1}^{n-1} \left[\exp\left(-Y_{i3}\right) + \exp\left(-Y_{i6}\right) \right] + \Pi_3 + \Pi_6 + \sum_{m=1}^{6} \left(X_m + \tilde{X}_m \right) \\
& - \sum_{i=1}^{n} \tilde{m}_i \left(Y_{i0} + Y_{i1} + Y_{i2} + Y_{i4} + Y_{i5}\right) \\
& - \sum_{i=1}^{n-1} \tilde{m}_i \left(Y_{i3} + Y_{i6}\right) + \tilde{m}_n (\ln \Pi_3 + \ln \Pi_6).
\end{aligned}
$$

Notice that the superpotential has poles at $X_1 \neq 0$, $X_3 \neq 0$, $X_6 \neq 0$, $\tilde{X}_2 \neq 0$, $\tilde{X}_3 \neq 0$, $\tilde{X}_5 \neq 0$ and $\tilde{X}_6 \neq 0$. With the mirror maps,

$$\begin{aligned} \exp(-Y_{i\beta}) &= -\tilde{m}_i + \sum_{a=1,2} \sigma_a \rho^a_{i\beta}, \\ X_m = \sum_{a=1,2} \sigma_a \alpha^a_m, &\quad \tilde{X}_m = \sum_{a=1,2} \sigma_a \tilde{\alpha}^a_m, \end{aligned} \tag{3.15}$$

one can get the excluded loci:

$$\sigma_1\sigma_2(\sigma_1^2 - 3\sigma_2^2)(3\sigma_1^2 - \sigma_2^2) \neq 0, \tag{3.16}$$

$$\begin{aligned} &\prod_i (-m_i + \sigma_1)(-m_i - \sigma_1)\left(-m_i + \frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2\right) \\ &\quad \times \left(-m_i - \frac{1}{2}\sigma_1 - \frac{\sqrt{3}}{2}\sigma_2\right)\left(-m_i - \frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2\right) \\ &\quad \times \left(-m_i + \frac{1}{2}\sigma_1 - \frac{\sqrt{3}}{2}\sigma_2\right) \neq 0. \end{aligned} \tag{3.17}$$

The critical locus is given by

$$\begin{aligned}
&\frac{\partial W}{\partial Y_{i0}} : \exp(-Y_{i0}) = -\tilde{m}_i, \quad \text{for} \quad i = 1, \ldots, n, \\
&\frac{\partial W}{\partial Y_{i1}} : \exp(-Y_{i1}) = \Pi_3 + \Pi_6 - \tilde{m}_i + 2\tilde{m}_n, \quad \text{for} \quad i = 1, \ldots, n, \\
&\frac{\partial W}{\partial Y_{i2}} : \exp(-Y_{i2}) = -\Pi_3 - \Pi_6 - \tilde{m}_i - 2\tilde{m}_n, \quad \text{for} \quad i = 1, \ldots, n, \\
&\frac{\partial W}{\partial Y_{i3}} : \exp(-Y_{i3}) = \Pi_3 - \tilde{m}_i + \tilde{m}_n, \quad \text{for} \quad i = 1, \ldots, n-1, \\
&\frac{\partial W}{\partial Y_{i4}} : \exp(-Y_{i4}) = -\Pi_3 - \tilde{m}_i - \tilde{m}_n, \quad \text{for} \quad i = 1, \ldots, n, \\
&\frac{\partial W}{\partial Y_{i5}} : \exp(-Y_{i5}) = -\Pi_6 - \tilde{m}_i - \tilde{m}_n, \quad \text{for} \quad i = 1, \ldots, n, \\
&\frac{\partial W}{\partial Y_{i6}} : \exp(-Y_{i6}) = \Pi_6 - \tilde{m}_i + \tilde{m}_n, \quad \text{for} \quad i = 1, \ldots, n-1, \\
&\frac{\partial W}{\partial X_1} : X_1 = \Pi_3 + \Pi_6 + 2\tilde{m}_n, \\
&\frac{\partial W}{\partial X_2} : X_2 = -\Pi_3 - \Pi_6 - 2\tilde{m}_n,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial W}{\partial X_3} &: X_3 = \Pi_3 + \tilde{m}_n, \\
\frac{\partial W}{\partial X_4} &: X_4 = -\Pi_3 - \tilde{m}_n, \\
\frac{\partial W}{\partial X_5} &: X_5 = -\Pi_6 - \tilde{m}_n, \\
\frac{\partial W}{\partial X_6} &: X_6 = \Pi_6 + \tilde{m}_n, \\
\frac{\partial W}{\partial \tilde{X}_1} &: \tilde{X}_1 = -\Pi_3 - 2\Pi_6 - 3\tilde{m}_n, \\
\frac{\partial W}{\partial \tilde{X}_2} &: \tilde{X}_2 = \Pi_3 + 2\Pi_6 + 3\tilde{m}_n, \\
\frac{\partial W}{\partial \tilde{X}_3} &: \tilde{X}_3 = 2\Pi_3 + \Pi_6 + 3\tilde{m}_n, \\
\frac{\partial W}{\partial \tilde{X}_4} &: \tilde{X}_4 = -2\Pi_3 - \Pi_6 - 3\tilde{m}_n, \\
\frac{\partial W}{\partial \tilde{X}_5} &: \tilde{X}_5 = \Pi_3 - \Pi_6, \\
\frac{\partial W}{\partial \tilde{X}_6} &: \tilde{X}_6 = -\Pi_3 + \Pi_6,
\end{aligned}$$

Plug the above equations back to (3.13), (3.14), one obtains the Coulomb branch relations:

$$\begin{aligned}
\Pi_3 = \prod_{i=1}^{n-1} (\Pi_3 - \tilde{m}_i + \tilde{m}_n)^{-1} \prod_{i=1}^{n} (\Pi_3 + \Pi_6 - \tilde{m}_i + 2\tilde{m}_n)^{-1} \\
\times (-\Pi_3 - \Pi_6 - \tilde{m}_i - 2\tilde{m}_n)(-\Pi_3 - \tilde{m}_i - \tilde{m}_n),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\Pi_6 = \prod_{i=1}^{n-1} (\Pi_6 - \tilde{m}_i + \tilde{m}_n)^{-1} \prod_{i=1}^{n} (\Pi_3 + \Pi_6 - \tilde{m}_i + 2\tilde{m}_n)^{-1} \\
\times (-\Pi_3 - \Pi_6 - \tilde{m}_i - 2\tilde{m}_n)(-\Pi_6 - \tilde{m}_i - \tilde{m}_n).
\end{aligned} \tag{3.19}$$

With the mirror map (3.15), on the critical locus relations, one finds

$$\Pi_3 = \frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2 - \tilde{m}_n, \quad \Pi_6 = \frac{1}{2}\sigma_1 - \frac{\sqrt{3}}{2}\sigma_2 - \tilde{m}_n.$$

Plugging them back in, one obtains the Coulomb (quantum cohomology) ring relations for G_2,

$$
\begin{aligned}
(3.20) \qquad & \prod_{i=1}^{n}(-\sigma_1-\tilde{m}_i)\left(-\frac{1}{2}\sigma_1-\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right) \\
&= \prod_{i=1}^{n}(\sigma_1-\tilde{m}_i)\left(\frac{1}{2}\sigma_1+\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right),\\
(3.21) \qquad & \prod_{i=1}^{n}(-\sigma_1-\tilde{m}_i)\left(-\frac{1}{2}\sigma_1+\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right) \\
&= \prod_{i=1}^{n}(\sigma_1-\tilde{m}_i)\left(\frac{1}{2}\sigma_1-\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right).
\end{aligned}
$$

Combining the above two relations, one gets

$$
\begin{aligned}
(3.22) \qquad & \prod_{i=1}^{n}(-\sigma_1-\tilde{m}_i)^2\left(-\frac{1}{2}\sigma_1-\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right)\left(-\frac{1}{2}\sigma_1+\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right) \\
&= \prod_{i=1}^{n}(\sigma_1-\tilde{m}_i)^2\left(\frac{1}{2}\sigma_1+\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right)\left(\frac{1}{2}\sigma_1-\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right),\\
(3.23) \qquad & \prod_{i=1}^{n}\left(\frac{1}{2}\sigma_1+\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right)\left(-\frac{1}{2}\sigma_1+\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right) \\
&= \prod_{i=1}^{n}\left(-\frac{1}{2}\sigma_1-\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right)\left(\frac{1}{2}\sigma_1-\frac{\sqrt{3}}{2}\sigma_2-\tilde{m}_i\right).
\end{aligned}
$$

3.4. Vacua

In this section, we will count the number of vacua in cases with small numbers n of fundamental fields. To solve the Coulomb branch (quantum cohomology) relations (3.20), (3.21) in general is not easy. However, since the superpotential is invariant under the Weyl group, the Coulomb ring relations (3.20), (3.21) will be covariant under the Weyl group action, which we check explicitly.

The Weyl group of G_2 is the dihedral group D_{12} of degree 6 and order 12, which can be described as [17, Section 7]

$$D_{12} = \left\{a^i x^j \mid a^6 = 1 = x^2, xax = a^{-1}\right\}.$$

(See *e.g.* [17, Section 47] for a discussion of representations of the dihedral groups.) Among the twelve elements of the Weyl group, there are six reflections, and below we list group elements and the field corresponding to the root about which the reflection takes place:

$$\begin{aligned} X_1 &\leftrightarrow a^3 x, \quad X_3 \leftrightarrow a^5 x, \quad X_5 \leftrightarrow ax, \\ \tilde{X}_1 &\leftrightarrow a^2 x, \quad \tilde{X}_3 \leftrightarrow a^4 x, \quad \tilde{X}_5 \leftrightarrow x. \end{aligned}$$

Notice that the reflections are also generated by the Weyl group reflection (3.2) and we denote the reflection matrices by the fields correspond to the positive simple roots. There are also five nontrivial rotations, corresponding to $\langle a \rangle \subset D_{12}$.

Now we can start to solve for the vacua (solutions of the Coulomb ring relations (3.20), (3.21)) begining with the case of small number of fundamental matter fields.

- $n = 1$, the only solution is $\sigma_1 = \sigma_2 = 0$ and it is excluded by the constraints (3.16), (3.17),
- $n = 2$, there are seven solutions but all of them are excluded by the constraints (3.16), (3.17),
- $n = 3$, there are ninteen solutions but all of them are excluded by the constraints (3.16), (3.17).

Starting with the case $n = 4$, we begin to obtain non-trivial solutions. First, let us analyze the case of $n = 4$ in detail. For simplicity, from now on, we will take $m_i = m_j = m$, $\forall i \neq j$ and will rescale the σ_i fields to $\sigma_i = \sigma_i / m$. There are thirty-seven solutions in total and twelve of them are true vacua (meaning, not on the excluded locus):

$$\begin{aligned} i &= 1, \ldots, 4, \quad s_i = \left\{ \sigma_1 = \pm i\sqrt{5}, \quad \sigma_2 = \pm i\sqrt{3} \right\}, \\ i &= 5, \ldots, 8, \quad s_i = \left\{ \sigma_1 = \pm i\sqrt{\frac{7}{2} - \frac{3\sqrt{5}}{2}}, \quad \sigma_2 = \pm i\sqrt{\frac{3}{2}(3 + \sqrt{5})} \right\}, \\ i &= 9, \ldots, 12, \quad s_i = \left\{ \sigma_1 = \pm i\sqrt{\frac{7}{2} + \frac{3\sqrt{5}}{2}}, \quad \sigma_2 = \pm i\sqrt{\frac{3}{2}(3 - \sqrt{5})} \right\}. \end{aligned}$$

	e	X_1	X_3	X_5	$\tilde{X}_1$	$\tilde{X}_3$	$\tilde{X}_5$	a_5	a_4	a_3	a_2	a_1
1	1	3	8	9	5	12	2	7	11	4	6	10
2	2	4	10	7	11	6	1	9	5	3	12	8
3	3	1	11	6	10	7	4	12	8	2	9	5
4	4	2	5	12	8	9	3	6	10	1	7	11
5	5	7	4	10	1	11	6	3	12	8	2	9
6	6	8	9	3	12	2	5	10	1	7	11	4
7	7	5	12	2	9	3	8	11	4	6	10	1
8	8	6	1	11	4	10	7	2	9	5	3	12
9	9	11	6	1	7	4	10	5	3	12	8	2
10	10	12	2	5	3	8	9	1	7	11	4	6
11	11	9	3	8	2	5	12	4	6	10	1	7
12	12	10	7	4	6	1	11	8	2	9	5	3

Table 2: Weyl group actions on the vacua of the case $n = 4$.

Signs are assigned in each group of four solutions in the order $\{-,-\}$, $\{-,+\}$, $\{+,-\}$, $\{+,+\}$. For example,

$$X_1 = \{\sigma_1 = -i\sqrt{5}, \quad \sigma_2 = -i\sqrt{3}\}, \quad X_2 = \{\sigma_1 = -i\sqrt{5}, \quad \sigma_2 = +i\sqrt{3}\},$$
$$X_3 = \{\sigma_1 = +i\sqrt{5}, \quad \sigma_2 = -i\sqrt{3}\}, \quad X_4 = \{\sigma_1 = +i\sqrt{5}, \quad \sigma_2 = +i\sqrt{3}\}.$$

Under the Weyl group actions, the solutions transform as in Table 2.[3] One can see that the twelve vacua are covariant and form one Weyl orbit under the Weyl group action.

When there are five fundamental matter multiplets, there are sixty-one solutions and twenty-four of them are non-trivial. Following the same conventions, those non-trivial vacua are

$$i = 1, \ldots, 4, \quad s_i = \left\{\sigma_1 = \pm i\sqrt{5 - \frac{6}{\sqrt{5}}}, \quad \sigma_2 = \pm i\sqrt{3 - \frac{6}{\sqrt{5}}}\right\},$$
$$i = 5, \ldots, 8, \quad s_i = \left\{\sigma_1 = \pm i\sqrt{5 + \frac{6}{\sqrt{5}}}, \quad \sigma_2 = \pm i\sqrt{3 + \frac{6}{\sqrt{5}}}\right\},$$

[3]Note that the Weyl group acts on σs by the inverse of the group elements. In the table, we denote the action by the original group elements instead of the inverse of the elements. Table 3 adopts the same notation.

$$
\begin{aligned}
i = 9, \ldots, 12, \quad s_i &= \Big\{ \sigma_1 = \pm i \sqrt{(1/10)(35 + 12\sqrt{5} + 3(185 + 80\sqrt{5})^{1/2}}, \\
&\qquad \sigma_2 = \pm i \sqrt{(3/10)(15 + 4\sqrt{5} - (185 + 80\sqrt{5})^{1/2}} \Big\}, \\
i = 13, \ldots, 16, \quad s_i &= \Big\{ \sigma_1 = \pm i \sqrt{(1/10)(35 + 12\sqrt{5} - 3(185 + 80\sqrt{5})^{1/2}}, \\
&\qquad \sigma_2 = \pm i \sqrt{(3/10)(15 + 4\sqrt{5} + (185 + 80\sqrt{5})^{1/2}} \Big\}, \\
i = 17, \ldots, 20, \quad s_i &= \Big\{ \sigma_1 = \pm i \sqrt{(1/10)(35 - 12\sqrt{5} + 3(185 + 80\sqrt{5})^{1/2}}, \\
&\qquad \sigma_2 = \pm i \sqrt{(3/10)(15 - 4\sqrt{5} - (185 + 80\sqrt{5})^{1/2}} \Big\}, \\
i = 21, \ldots, 24, \quad s_i &= \Big\{ \sigma_1 = \pm i \sqrt{(1/10)(-35 + 12\sqrt{5} + 3(185 + 80\sqrt{5})^{1/2}}, \\
&\qquad \sigma_2 = \pm i \sqrt{(3/10)(15 - 4\sqrt{5} + (185 + 80\sqrt{5})^{1/2}} \Big\}.
\end{aligned}
$$

The vacua form two Weyl orbits, each of which contains twelve elements. The first orbit consists of the first through fourth solutions, the seventeenth through twentieth solutions, and the twenty-first through twenty-fourth solutions. The rest of the solutions form the second Weyl orbit. We summarize the results for the Weyl group actions in Table 3.

We checked one more case, $n = 6$. In this case, there are ninety-one solutions in total, of which forty-eight solutions are not on the excluded locus. As expected, these vacua form four Weyl orbits under the Weyl group action and each orbit contain twelve vacua. The solutions in this case are much more complicated, and so we do not list them explicitly.

3.5. Pure gauge theory

In this section, we will check (at the level of topological field theory computations) that the pure G_2 theory flows in the IR to a free theory of two

	e	X_1	X_3	X_5	$\tilde{X}_1$	$\tilde{X}_3$	$\tilde{X}_5$	a_5	a_4	a_3	a_2	a_1
1	1	3	22	17	23	20	2	21	19	4	24	18
2	2	4	18	21	19	24	1	17	23	3	20	22
3	3	1	19	24	18	21	4	20	22	2	17	23
4	4	2	23	20	22	17	3	24	18	1	21	19
17	17	19	24	1	21	4	18	23	3	20	22	2
18	18	20	2	23	3	22	17	1	21	19	4	24
19	19	17	3	22	2	23	20	4	24	18	1	21
20	20	18	21	4	24	1	19	22	2	17	23	3
21	21	23	20	2	17	3	22	19	4	24	18	1
22	22	24	1	19	4	18	21	2	17	23	3	20
23	23	21	4	18	1	19	24	3	20	22	2	17
24	24	22	17	3	20	2	23	18	1	21	19	4
5	5	7	16	9	13	12	6	15	11	8	14	10
6	6	8	10	15	11	14	5	9	13	7	12	16
7	7	5	11	14	10	15	8	12	16	6	9	13
8	8	6	13	12	16	9	7	14	10	5	15	11
9	9	11	14	5	15	8	10	13	7	12	16	6
10	10	12	6	13	7	16	9	5	15	11	8	14
11	11	9	7	16	6	13	12	8	14	10	5	15
12	12	10	15	8	14	5	11	16	6	9	13	7
13	13	15	8	10	5	11	14	7	12	16	6	9
14	14	16	9	7	12	6	13	10	5	15	11	8
15	15	13	12	6	9	7	16	11	8	14	10	5
16	16	14	5	11	8	10	15	6	9	13	7	12

Table 3: Weyl group actions on the vacua of five fundamental matter multiplets.

chiral multiplets. The superpotential of the pure gauge theory is

$$
\begin{aligned}
W &= \sum_{a=1}^{2} \sigma_a \left(\alpha^a_m Z_m + \tilde{\alpha}^a_m \tilde{Z}_m \right) + \sum_m (X_m + \tilde{X}_m), \\
&= \sigma_1 \Bigg[\Big(Z_1 - Z_2 + (1/2) Z_3 - (1/2) Z_4 - (1/2) Z_5 + (1/2) Z_6 \Big) \\
&\qquad + \Big(-(3/2)\tilde{Z}_1 + (3/2)\tilde{Z}_2 + (3/2)\tilde{Z}_3 - (3/2)\tilde{Z}_4 \Big) \Bigg] \\
&\quad + \sigma_2 (\sqrt{3}/2) \Big(Z_3 - Z_4 + Z_5 - Z_6 + \tilde{Z}_1 - \tilde{Z}_2 + \tilde{Z}_3 - \tilde{Z}_4 + 2\tilde{Z}_5 - 2\tilde{Z}_6 \Big) \\
&\quad + \sum_{m=1}^{6} (X_m + \tilde{X}_m).
\end{aligned}
$$

Integrating out X_m and $\tilde{X}_m$, one obtains the constraints,

$$X_m = \sum_a \sigma_a \alpha_m^a, \qquad \tilde{X}_m = \sum_a \sigma \tilde{\alpha}_m^a.$$

The point is that all the X_m fields and $\tilde{X}_m$ fields correspond to the nonzero roots of G_2 which come in pairs, positive roots and their negatives. As a result, pluging the constraints above back into the superpoential, one gets $W = 0$. Therefore, the pure gauge theory indeed flows to a free theory of two twisted chiral multiplies in the IR limit.

On the other hand, integrating out σ_1 and σ_2, one obtains the constraints,

$$\begin{aligned}
&-\ln X_1 + \ln X_2 - (1/2)\ln X_3 + (1/2)\ln X_4 + (1/2)\ln X_5 - (1/2)\ln X_6 \\
&\qquad + (3/2)\ln \tilde{X}_1 - (3/2)\ln \tilde{X}_2 - (3/2)\tilde{X}_4 + (3/2)\tilde{X}_4 = 0, \\
&-\frac{\sqrt{3}}{2}(\ln X_3 - \ln X_4 + \ln X_5 - \ln X_6) \\
&\qquad - \frac{\sqrt{3}}{2}(\tilde{X}_1 - \tilde{X}_2 + \tilde{X}_3 - \tilde{X}_4 + 2\tilde{X}_5 - 2\tilde{X}_6) = 0
\end{aligned}$$

With those two constraints, one can eliminate two fields in the superpotential,

$$X_4 = a\frac{X_1 X_3 \tilde{X}_2 \tilde{X}_3^2 \tilde{X}_5}{X_2 \tilde{X}_1 \tilde{X}_4^2 \tilde{X}_6}, \qquad X_5 = b\frac{X_1 X_6 \tilde{X}_2^2 \tilde{X}_3 \tilde{X}_6}{X_2 \tilde{X}_1^2 \tilde{X}_4 \tilde{X}_5},$$

with $a = \pm 1$ and $b = \pm 1$. Plugging this back into the superpotential, we get

$$\begin{aligned}
W = {} & X_1 + X_2 + X_3 + X_6 + \tilde{X}_1 + \tilde{X}_2 + \tilde{X}_3 + \tilde{X}_4 + \tilde{X}_5 + \tilde{X}_6 \\
& + a\frac{X_1 X_3 \tilde{X}_2 \tilde{X}_3^2 \tilde{X}_5}{X_2 \tilde{X}_1 \tilde{X}_4^2 \tilde{X}_6} + b\frac{X_1 X_6 \tilde{X}_2^2 \tilde{X}_3 \tilde{X}_6}{X_2 \tilde{X}_1^2 \tilde{X}_4 \tilde{X}_5}.
\end{aligned} \tag{3.24}$$

The critical loci are given by

$$\begin{aligned}
& a = b = 1, \\
& X_1 = -X_2 = X_3 + X_6, \\
& \tilde{X}_1 = -\tilde{X}_2 = -X_3 - 2X_6, \\
& \tilde{X}_3 = -\tilde{X}_4 = 2X_3 + X_6, \\
& \tilde{X}_5 = -\tilde{X}_6 = X_3 - X_6.
\end{aligned}$$

One can easily see that, on the critical locus, the above superpotential (3.24) vanishes with two free fields X_3 and X_6. Therefore, the pure gauge theory again flows to free theories of two chiral multiplies in the IR.

3.6. Comparison with A model results

In this section, we will discuss the A-twisted gauge theory with gauge group G_2 and n chiral superfields in the **7**, and compare it to results from our proposed mirror, as a check of our methods. In principle, this should necessarily work, for reasons discussed in [3, Section 3]; however, we will check for the special case of G_2 that indeed everything works as it should, which will also give us the opportunity to discuss the role of theta angle periodicities and charge lattice normalizations.

The one-loop effective twisted superpotential $\tilde{W}_{\text{eff}}$ of the A-twisted gauge theory is given by [13, equ'ns (2.17), (2.19)], [14, equ'ns (3.16), (3.17)]

$$\begin{aligned}\tilde{W}_{\text{eff}} = &-\sum_{i,\alpha}(\sigma_1\rho^{'1}_{i,\alpha} + \sigma_2\rho^{'2}_{i,\alpha} - \tilde{m}_i)\left(\ln(\sigma_1\rho^{'1}_{i,\alpha} + \sigma_2\rho^{'2}_{i,\alpha} - \tilde{m}_i) - 1\right)\\ &+\sum_{m}(\sigma_1\alpha^{'1}_{m} + \sigma_2\alpha^{'2}_{m})\left(\ln(\sigma_1\alpha^{'1}_{m} + \sigma_2\alpha^{'2}_{m}) - 1\right)\\ &+\sum_{m}(\sigma_1\tilde{\alpha}^{'1}_{m} + \sigma_2\tilde{\alpha}^{'2}_{m})\left(\ln(\sigma_1\tilde{\alpha}^{'1}_{m} + \sigma_2\tilde{\alpha}^{'2}_{m}) - 1\right).\end{aligned}$$

Since the logarithm branch cuts in the expressions above are supposed to reflect (standard) theta angle periodicities of 2π, we have rescaled ρ and α to ρ' and α'. Specifically, we have rescaled all the charges under σ_1 by a factor of 2 and all the charges under σ_2 by a factor of $2/\sqrt{3}$.

Since the roots and weights of G_2 come in positive/negative pairs, we can further simplify the effective superpotential:

$$\begin{aligned}\tilde{W}_{\text{eff}} = &-\sum_{i,\alpha}(\sigma_1\rho^{'1}_{i,\alpha} + \sigma_2\rho^{'2}_{i,\alpha} - \tilde{m}_i)\ln(\sigma_1\rho^{'1}_{i,\alpha} + \sigma_2\rho^{'2}_{i,\alpha} - \tilde{m}_i)\\ &-7\sum_{i}\tilde{m}_i + 6\pi i(\sigma_1 + \sigma_2).\end{aligned}$$

The vacua are given by

$$\prod_{i,\alpha}(\sigma_1\rho_{i,\alpha}^{'1} + \sigma_2\rho_{i,\alpha}^{'2} - \tilde{m}_i)^{\rho_{i,\alpha}^{'1}} = 1,$$
$$\prod_{i,\alpha}(\sigma_1\rho_{i,\alpha}^{'1} + \sigma_2\rho_{i,\alpha}^{'2} - \tilde{m}_i)^{\rho_{i,\alpha}^{'2}} = 1.$$

Plugging in the charges, we get

$$\begin{aligned}
&\prod_i(2\sigma_1 - \tilde{m}_i)^2(\sigma_1 + \sigma_2 - \tilde{m}_i)(\sigma_1 - \sigma_2 - \tilde{m}_i) \\
&= \prod_i(-2\sigma_1 - \tilde{m}_i)^2(-\sigma_1 - \sigma_2 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \tilde{m}_i), \\
&\prod_i(\sigma_1 + \sigma_2 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \tilde{m}_i) \\
&= \prod_i(-\sigma_1 - \sigma_2 - \tilde{m}_i)(\sigma_1 - \sigma_2 - \tilde{m}_i).
\end{aligned}$$

The relations above are the same as the Coulomb ring relations (3.22), (3.23) we derived from the B model with a suitable rescaling of the σ fields,

$$\sigma_1 \to \frac{1}{2}\sigma_1, \qquad \sigma_2 \to \frac{\sqrt{3}}{2}\sigma_2.$$

Thus, we see that A model results match those of the B model mirror, as expected, after correctly taking into account subtleties in theta angle periodicities.

3.7. Comparison to other bases for weight lattice

So far in this section, we have used a particular basis for the weight lattice for G_2. In principle, other bases are related by field redefinitions. To make this more explicit, in this section we will outline corresponding results in a different basis for the weight lattice, specifically a basis of fundamental weights. We will describe this basis in greater detail in the section on F_4, as it will be used for the rest of the exceptional gauge groups in this paper, but for the moment, will content ourselves to briefly outline results.

In terms of that basis of fundamental weights, it can be shown that the roots and pertinent weights of G_2 are expanded as given in Table 4.

Field	Short root	Field	Long root	Field	Weight
X_1	$(1,0)$	$\tilde{X}_1$	$(0,1)$	Y_{i1}	$(1,0)$
X_2	$(-1,1)$	$\tilde{X}_2$	$(3,-1)$	Y_{i2}	$(-1,1)$
X_3	$(2,-1)$	$\tilde{X}_3$	$(-3,2)$	Y_{i3}	$(2,-1)$
X_4	$(-1,0)$	$\tilde{X}_4$	$(0,-1)$	Y_{i4}	$(-1,0)$
X_5	$(1,-1)$	$\tilde{X}_5$	$(-3,1)$	Y_{i5}	$(1,-1)$
X_6	$(-2,1)$	$\tilde{X}_6$	$(3,-2)$	Y_{i6}	$(-2,1)$
				Y_{i0}	$(0,0)$

Table 4: Roots and weights for G_2 and associated fields.

Repeating the same mirror analysis as described earlier in this section, we derive the Coulomb branch relations

$$\prod_{i=1}^{n}(\sigma_1' - \tilde{m}_i)(2\sigma_1' - \sigma_2' - \tilde{m}_i) = \prod_{i=1}^{n}(-\sigma_1' - \tilde{m}_i)(-2\sigma_1' + \sigma_2' - \tilde{m}_i), \tag{3.25}$$

$$\prod_{i=1}^{n}(\sigma_1' - \sigma_2' - \tilde{m}_i)(2\sigma_1' - \sigma_2' - \tilde{m}_i) = \prod_{i=1}^{n}(-\sigma_1' + \sigma_2' - \tilde{m}_i)(-2\sigma_1' + \sigma_2' - \tilde{m}_i), \tag{3.26}$$

and excluded loci

$$\sigma_1'\sigma_2'(\sigma_1' - \sigma_2')(2\sigma_1' - \sigma_2')(3\sigma_1' - \sigma_2')(3\sigma_1' - 2\sigma_2') \neq 0, \tag{3.27}$$

$$\prod_{i=1}^{n}(\sigma_1' - \tilde{m}_i)(-\sigma_1' + \sigma_2' - \tilde{m}_i)(2\sigma_1' - \sigma_2' - \tilde{m}_i) \times (-\sigma_1' - \tilde{m}_i)(\sigma_1' - \sigma_2' - \tilde{m}_i)(-2\sigma_1' + \sigma_2' - \tilde{m}_i) \neq 0. \tag{3.28}$$

Comparing with earlier reuslts for the critical locus (3.20), (3.21) and excluded loci (3.16), (3.17), computed in the earlier basis, we find that the results above are related by the following linear field redefinitions

$$\sigma_1' = \frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2, \tag{3.29}$$

$$\sigma_2' = \sqrt{3}\sigma_2, \tag{3.30}$$

or equivalently

$$\sigma_1 = 2\sigma_1' - \sigma_2', \tag{3.31}$$

$$\sigma_2 = \frac{1}{\sqrt{3}}\sigma_2'. \tag{3.32}$$

4. F_4

In this section we will consider the mirror Landau-Ginzburg orbifold of an F_4 gauge theory with matter fields in the **26** fundamental representation, and then compute Coulomb branch relations. We also consider the pure gauge theory without matter fields.

4.1. Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has fields

- $Y_{i,\beta}$, $i \in \{1, \ldots, n\}$, $\beta \in \{1, \ldots, 26\}$, corresponding to the matter fields in n copies of the fundamental **26** dimensional representation of F_4,
- X_m, $m \in \{1, \ldots, 48\}$, corresponding to the roots of F_4,
- σ_a, $a \in \{1, 2, 3, 4\}$.

We associate the roots, α_m^a, to X_m fields and the weights, $\rho_{i,\beta}^a$, of the fundamental **26** representation to $Y_{i,\beta}$.

Now, previously for G_2, we worked with a basis in which the θ-angle periodicities were unusual: $\theta_1 \sim \theta_1 + \pi i$, $\theta_2 \sim \theta_2 + 4\pi i/\sqrt{3}$. This essentially just corresponded to a nonstandard charge lattice normalization. This was convenient for relating to Lie algebras, but, is rather unusual for physics.

Here, for F_4 and all the later examples we will discuss in this paper, we would like instead to work with a basis for the roots and weights that corresponds to an integer charge lattice, so that the θ-angle periodicities take a more nearly standard form. In particular, the superpotential is invariant under such basis changes, since its terms are tensor contractions such as

$$\sum_a \sigma_a \rho_i^a Y^i.$$

We can pick any basis we like, so long as we consistently change coordinates in the tensors above. In particular, the superpotential (for this B-twisted Landau-Ginzburg model) does not depend explicitly upon *e.g.* the Cartan matrix, so the metric on the Lie algebra is not directly relevant in the presentation above. Thus, we have the flexibility to pick a basis such that the weights have integer coordinates, which yields standard θ-angle periodicities.

To be specific, we will write the weights and roots in terms of a basis of fundamental weights. Recall the fundamental weights are defined as follows. First, let $\{\alpha_\mu\}$ be a basis of simple roots, normalized so that the Cartan matrix $C_{\mu\nu}$ is given as

$$C_{\mu\nu} = 2\frac{\alpha_\mu \cdot \alpha_\nu}{\alpha_\nu^2}. \tag{4.1}$$

The fundamental weights $\{\omega_\mu\}$ are then defined by the property that [18, Section 13.1]

$$2\frac{\alpha_\mu \cdot \omega_\nu}{\alpha_\mu^2} = \delta_{\mu\nu}. \tag{4.2}$$

Furthermore, the fundamental weights form an integer basis for the weight lattice – every element of the weight lattice is a linear combination of fundamental weights with integer coefficients [18, Section 13.1]. This is perfect for our purposes, as this basis yields standard θ-angle periodicities, and we will use this basis for all computations in this and later sections. To compute root and weight vectors as linear combinations of the fundamental weights, as displayed in tables in this and later sections, we used the Mathematica package LieART [19].

The long roots and associated fields are listed in Table 5. The short roots and associated fields are listed in Table 6. The weights of the **26** and associated fields are listed in Table 7.

Now, plugging the information above into the mirror superpotential with twisted masses

$$\begin{aligned} W = &\sum_{a=1}^{4} \sigma_a \left(\sum_{i=1}^{n} \sum_{\beta=1}^{26} \rho_{i,\beta}^a Y_{i,\beta} + \sum_{m=1}^{48} \alpha_m^a Z_m \right) \\ &- \sum_{i=1}^{n} \tilde{m}_i \sum_{\beta=1}^{26} Y_{i,\beta} + \sum_{i=1}^{n} \sum_{\beta=1}^{26} \exp(-Y_{i,\beta}) + \sum_{m=1}^{48} X_m, \end{aligned} \tag{4.3}$$

where $X_m = \exp(-Z_m)$ and X_m are the fundamental fields, we get

$$W = \sum_{a=1}^{4} \sigma_a \mathcal{C}^a - \sum_{i=1}^{n} \tilde{m}_i \sum_{\beta=1}^{26} Y_{i,\beta} + \sum_{i=1}^{n} \sum_{\beta=1}^{26} \exp(-Y_{i,\beta}) + \sum_{m=1}^{48} X_m. \tag{4.4}$$

Field	Positive root	Field	Negative root
X_1	$(1,0,0,0)$	X_{25}	$(-1,0,0,0)$
X_2	$(-1,1,0,0)$	X_{26}	$(1,-1,0,0)$
X_3	$(0,-1,2,0)$	X_{27}	$(0,1,-2,0)$
X_4	$(0,1,-2,2)$	X_{28}	$(0,-1,2,-2)$
X_5	$(1,-1,0,2)$	X_{29}	$(-1,1,0,-2)$
X_6	$(-1,0,0,2)$	X_{30}	$(1,0,0,-2)$
X_7	$(0,1,0,-2)$	X_{31}	$(0,-1,0,2)$
X_8	$(1,-1,2,-2)$	X_{32}	$(-1,1,-2,2)$
X_9	$(-1,0,2,-2)$	X_{33}	$(1,0,-2,2)$
X_{10}	$(1,1,-2,0)$	X_{34}	$(-1,-1,2,0)$
X_{11}	$(-1,2,-2,0)$	X_{35}	$(1,-2,2,0)$
X_{12}	$(2,-1,0,0)$	X_{36}	$(-2,1,0,0)$

Table 5: Long roots of F_4 and associated fields.

Field	Positive root	Field	Negative root
X_{13}	$(0,0,0,1)$	X_{37}	$(0,0,0,-1)$
X_{14}	$(0,0,1,-1)$	X_{38}	$(0,0,-1,1)$
X_{15}	$(0,1,-1,0)$	X_{39}	$(0,-1,1,0)$
X_{16}	$(1,-1,1,0)$	X_{40}	$(-1,1,-1,0)$
X_{17}	$(-1,0,1,0)$	X_{41}	$(1,0,-1,0)$
X_{18}	$(1,0,-1,1)$	X_{42}	$(-1,0,1,-1)$
X_{19}	$(-1,1,-1,1)$	X_{43}	$(1,-1,1,-1)$
X_{20}	$(1,0,0,-1)$	X_{44}	$(-1,0,0,1)$
X_{21}	$(-1,1,0,-1)$	X_{45}	$(1,-1,0,1)$
X_{22}	$(0,-1,1,1)$	X_{46}	$(0,1,-1,-1)$
X_{23}	$(0,-1,2,-1)$	X_{47}	$(0,1,-2,1)$
X_{24}	$(0,0,-1,2)$	X_{48}	$(0,0,1,-2)$

Table 6: Short roots of F_4 and associated fields.

where the $\mathcal{C}^a$ are given as follows:

$$\begin{aligned}\mathcal{C}^1 = \sum_{i=1}^{n} \big(& Y_{i,4} - Y_{i,5} + Y_{i,6} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,19} \\ & - Y_{i,20} + Y_{i,21} - Y_{i,16} + Y_{i,17} - Y_{i,18} \big) \\ & + Z_1 - Z_2 + Z_5 - Z_6 + Z_{16} - Z_{17} + Z_8 + Z_{18} - Z_9 - Z_{19} + Z_{20} - Z_{21} \\ & + Z_{10} - Z_{11} + 2Z_{12} - Z_{25} + Z_{26} - Z_{29} + Z_{30} - Z_{40} + Z_{41} - Z_{32} - Z_{42} \\ & + Z_{33} + Z_{43} - Z_{44} + Z_{45} - Z_{34} + Z_{35} - 2Z_{36},\end{aligned}$$

Field	Weight	Field	Weight
$Y_{i,1}$	$(0,0,0,1)$	$Y_{i,13}$	$(0,0,0,-1)$
$Y_{i,2}$	$(0,0,1,-1)$	$Y_{i,14}$	$(0,0,-1,1)$
$Y_{i,3}$	$(0,1,-1,0)$	$Y_{i,15}$	$(0,-1,1,0)$
$Y_{i,4}$	$(1,-1,1,0)$	$Y_{i,16}$	$(-1,1,-1,0)$
$Y_{i,5}$	$(-1,0,1,0)$	$Y_{i,17}$	$(1,0,-1,0)$
$Y_{i,6}$	$(1,0,-1,1)$	$Y_{i,18}$	$(-1,0,1,-1)$
$Y_{i,7}$	$(-1,1,-1,1)$	$Y_{i,19}$	$(1,-1,1,-1)$
$Y_{i,8}$	$(1,0,0,-1)$	$Y_{i,20}$	$(-1,0,0,1)$
$Y_{i,9}$	$(-1,1,0,-1)$	$Y_{i,21}$	$(1,-1,0,1)$
$Y_{i,10}$	$(0,-1,1,1)$	$Y_{i,22}$	$(0,1,-1,-1)$
$Y_{i,11}$	$(0,-1,2,-1)$	$Y_{i,23}$	$(0,1,-2,1)$
$Y_{i,12}$	$(0,0,-1,2)$	$Y_{i,24}$	$(0,0,1,-2)$
$Y_{i,25}$	$(0,0,0,0)$	$Y_{i,26}$	$(0,0,0,0)$

Table 7: Weights of **26** of F_4 and associated fields.

$$
\begin{aligned}
\mathcal{C}^2 = \sum_{i=1}^{n} \big(&Y_{i,3} - Y_{i,4} + Y_{i,7} + Y_{i,9} - Y_{i,10} - Y_{i,11} - Y_{i,19} \\
&- Y_{i,21} + Y_{i,22} + Y_{i,23} - Y_{i,15} + Y_{i,16}\big) \\
&+ Z_2 - Z_3 + Z_4 + Z_{15} - Z_5 + Z_7 - Z_{16} - Z_8 + Z_{19} + Z_{21} - Z_{22} + Z_{10} \\
&+ 2Z_{11} - Z_{23} - Z_{12} - Z_{26} + Z_{27} - Z_{28} - Z_{39} + Z_{29} - Z_{31} + Z_{40} + Z_{32} \\
&- Z_{43} - Z_{45} + Z_{46} - Z_{34} - 2Z_{35} + Z_{47} + Z_{36},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}^3 = \sum_{i=1}^{n} \big(&Y_{i,2} - Y_{i,3} + Y_{i,4} + Y_{i,5} - Y_{i,6} - Y_{i,7} + Y_{i,10} + 2Y_{i,11} - Y_{i,12} + Y_{i,19} \\
&- Y_{i,22} - 2Y_{i,23} + Y_{i,24} - Y_{i,14} + Y_{i,15} - Y_{i,16} - Y_{i,17} + Y_{i,18}\big) \\
&+ 2Z_3 + Z_{14} - 2Z_4 - Z_{15} + Z_{16} + Z_{17} + 2Z_8 - Z_{18} + 2Z_9 - Z_{19} + Z_{22} \\
&- 2Z_{10} - 2Z_{11} + 2Z_{23} - Z_{24} - 2Z_{27} - Z_{38} + 2Z_{28} + Z_{39} - Z_{40} - Z_{41} \\
&- 2Z_{32} + Z_{42} - 2Z_{33} + Z_{43} - Z_{46} + 2Z_{34} + 2Z_{35} - 2Z_{47} + Z_{48},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}^4 = \sum_{i=1}^{n} \big(&Y_{i,1} - Y_{i,2} + Y_{i,6} + Y_{i,7} - Y_{i,8} - Y_{i,9} + Y_{i,10} - Y_{i,11} + 2Y_{i,12} \\
&- Y_{i,19} + Y_{i,20} + Y_{i,21} - Y_{i,22} + Y_{i,23} - 2Y_{i,24} - Y_{i,13} + Y_{i,14} - Y_{i,18}\big) \\
&+ Z_{13} - Z_{14} + 2Z_4 + 2Z_5 + 2Z_6 - 2Z_7 - 2Z_8 + Z_{18} - 2Z_9 + Z_{19} - Z_{20} \\
&- Z_{21} + Z_{22} - Z_{23} + 2Z_{24} - Z_{37} + Z_{38} - 2Z_{28} - 2Z_{29} - 2Z_{30} + 2Z_{31} \\
&+ 2Z_{32} - Z_{42} + 2Z_{33} - Z_{43} + Z_{44} + Z_{45} - Z_{46} + Z_{47} - 2Z_{48}.
\end{aligned}
$$

Integrating out σ_a fields, we get four constraints $\mathcal{C}^a = 0$. So we are free to eliminate four fundamental fields. Our choice here will be $Y_{n,1}$, $Y_{n,2}$, $Y_{n,3}$ and $Y_{n,4}$.

$$\begin{aligned}
-Y_{n,1} = \sum_{i=1}^{n-1} Y_{i,1} + \sum_{i=1}^{n} \big(&Y_{i,5} + Y_{i,7} - Y_{i,8} + Y_{i,10} + Y_{i,12} - Y_{i,19} \\
&+ Y_{i,20} - Y_{i,22} - Y_{i,24} - Y_{i,13} - Y_{i,17}\big) \\
+ Z_2 + Z_3 &+ Z_{13} + Z_4 + Z_5 + 2Z_6 - Z_7 + Z_{17} - Z_8 + Z_{19} - Z_{20} \\
+ Z_{22} - Z_{10} &+ Z_{24} - Z_{12} - Z_{26} - Z_{27} - Z_{37} - Z_{28} - Z_{29} - 2Z_{30} \\
+ Z_{31} - Z_{41} &+ Z_{32} - Z_{43} + Z_{44} - Z_{46} + Z_{34} - Z_{48} + Z_{36},
\end{aligned}$$

$$\begin{aligned}
-Y_{n,2} = \sum_{i=1}^{n-1} Y_{i,2} + \sum_{i=1}^{n} \big(&Y_{i,5} - Y_{i,6} + Y_{i,9} + Y_{i,11} - Y_{i,12} - Y_{i,21} \\
&- Y_{i,23} + Y_{i,24} - Y_{i,14} - Y_{i,17} + Y_{i,18}\big) \\
+ Z_2 + Z_3 &+ Z_{14} - Z_4 - Z_5 + Z_7 + Z_{17} + Z_8 - Z_{18} + 2Z_9 + Z_{21} \\
- Z_{10} + Z_{23} &- Z_{24} - Z_{12} - Z_{26} - Z_{27} - Z_{38} + Z_{28} + Z_{29} - Z_{31} \\
- Z_{41} - Z_{32} &+ Z_{42} - 2Z_{33} - Z_{45} + Z_{34} - Z_{47} + Z_{48} + Z_{36},
\end{aligned}$$

$$\begin{aligned}
-Y_{n,3} = \sum_{i=1}^{n-1} + \sum_{i=1}^{n} \big(&- Y_{i,5} + Y_{i,6} + Y_{i,8} - Y_{i,10} - Y_{i,11} - Y_{i,20} \\
&+ Y_{i,22} + Y_{i,23} - Y_{i,15} + Y_{i,17} - Y_{i,18}\big) \\
+ Z_1 - Z_3 &+ Z_4 + Z_{15} - Z_6 + Z_7 - Z_{17} + Z_{18} - Z_9 + Z_{20} - Z_{22} \\
+ 2Z_{10} + Z_{11} &- Z_{23} + Z_{12} - Z_{25} + Z_{27} - Z_{28} - Z_{39} + Z_{30} - Z_{31} \\
+ Z_{41} - Z_{42} &+ Z_{33} - Z_{44} + Z_{46} - 2Z_{34} - Z_{35} + Z_{47} - Z_{36},
\end{aligned}$$

$$\begin{aligned}
-Y_{n,4} = \sum_{i=1}^{n-1} Y_{i,4} + \sum_{i=1}^{n} \big(&- Y_{i,5} + Y_{i,6} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,19} \\
&- Y_{i,20} + Y_{i,21} - Y_{i,16} + Y_{i,17} - Y_{i,18}\big) \\
+ Z_1 - Z_2 &+ Z_5 - Z_6 + Z_{16} - Z_{17} + Z_8 + Z_{18} - Z_9 - Z_{19} + Z_{20} \\
- Z_{21} + Z_{10} &- Z_{11} + 2Z_{12} - Z_{25} + Z_{26} - Z_{29} + Z_{30} - Z_{40} + Z_{41} \\
- Z_{32} - Z_{42} &+ Z_{33} + Z_{43} - Z_{44} + Z_{45} - Z_{34} + Z_{35} - 2Z_{36}.
\end{aligned}$$

For convenience, we define:

(4.5)

$$\begin{aligned}
&\Pi_1 \equiv \exp(-Y_{n,1}),\\
&= \prod_{i=1}^{n} \exp(Y_{i,5} + Y_{i,7} - Y_{i,8} + Y_{i,10} + Y_{i,12} - Y_{i,19} + Y_{i,20}\\
&\qquad - Y_{i,22} - Y_{i,24} - Y_{i,13} - Y_{i,17})\\
&\times \prod_{i=1}^{n-1} \exp(Y_{i,1}) \cdot \frac{X_7 X_8 X_{20} X_{10} X_{12} X_{26} X_{27} X_{37} X_{28} X_{29} X_{30}^2 X_{41} X_{43} X_{46} X_{48}}{X_2 X_3 X_{13} X_4 X_5 X_6^2 X_{17} X_{19} X_{22} X_{24} X_{31} X_{32} X_{44} X_{34} X_{36}},
\end{aligned}$$

(4.6)

$$\begin{aligned}
&\Pi_2 \equiv \exp(-Y_{n,2}),\\
&= \prod_{i=1}^{n} \exp(Y_{i,5} - Y_{i,6} + Y_{i,9} + Y_{i,11} - Y_{i,12} - Y_{i,21} - Y_{i,23}\\
&\qquad + Y_{i,24} - Y_{i,14} - Y_{i,17} + Y_{i,18})\\
&\times \prod_{i=1}^{n-1} \exp(Y_{i,2}) \cdot \frac{X_4 X_5 X_{18} X_{10} X_{24} X_{12} X_{26} X_{27} X_{38} X_{31} X_{41} X_{32} X_{33}^2 X_{45} X_{47}}{X_2 X_3 X_{14} X_7 X_{17} X_8 X_9^2 X_{21} X_{23} X_{28} X_{29} X_{42} X_{34} X_{48} X_{36}},
\end{aligned}$$

(4.7)

$$\begin{aligned}
&\Pi_3 \equiv \exp(-Y_{n,3}),\\
&= \prod_{i=1}^{n} \exp(-Y_{i,5} + Y_{i,6} + Y_{i,8} - Y_{i,10} - Y_{i,11} - Y_{i,20} + Y_{i,22}\\
&\qquad + Y_{i,23} - Y_{i,15} + Y_{i,17} - Y_{i,18})\\
&\times \prod_{i=1}^{n-1} \exp(Y_{i,3}) \cdot \frac{X_3 X_6 X_{17} X_9 X_{22} X_{23} X_{25} X_{28} X_{39} X_{31} X_{42} X_{44} X_{34}^2 X_{35} X_{36}}{X_1 X_4 X_{15} X_7 X_{18} X_{20} X_{10}^2 X_{11} X_{12} X_{27} X_{30} X_{41} X_{33} X_{46} X_{47}},
\end{aligned}$$

(4.8)

$$\begin{aligned}
&\Pi_4 \equiv \exp(-Y_{n,4}),\\
&= \prod_{i=1}^{n} \exp(-Y_{i,5} + Y_{i,6} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,19} - Y_{i,20}\\
&\qquad + Y_{i,21} - Y_{i,16} + Y_{i,17} - Y_{i,18})\\
&\times \prod_{i=1}^{n-1} \exp(Y_{i,4}) \cdot \frac{X_2 X_6 X_{17} X_9 X_{19} X_{21} X_{11} X_{25} X_{29} X_{40} X_{32} X_{42} X_{44} X_{34} X_{36}^2}{X_1 X_5 X_{16} X_8 X_{18} X_{20} X_{10} X_{12}^2 X_{26} X_{30} X_{41} X_{33} X_{43} X_{45} X_{35}}.
\end{aligned}$$

Integrating out the sigma fields and eliminating the fields, $Y_{n,1}$, $Y_{n,2}$, $Y_{n,3}$ and $Y_{n,4}$, the superpotential reduces to

$$
\begin{aligned}
W = & \sum_{i=1}^{n-1}\sum_{b=1}^{26}\big(\exp(-Y_{i,b}) - \tilde{m}_i Y_{i,b}\big) + \big(\Pi_1 + \tilde{m}_n \ln \Pi_1\big) + \big(\Pi_2 + \tilde{m}_n \ln \Pi_2\big) \\
& + \big(\Pi_3 + \tilde{m}_n \ln \Pi_3\big) + \big(\Pi_4 + \tilde{m}_n \ln \Pi_4\big) \\
& + \sum_{a=5}^{26}\big(\exp(-Y_{n,a}) - \tilde{m}_n Y_{n,a}\big) + \sum_{m=1}^{48} X_m
\end{aligned}
$$

The superpotential is only well defined when the X_m fields in the denominator of Π_a's are non-zero.

The critical locus is given as follows:

For $i < n$:

$$\frac{\partial W}{\partial Y_{i,1}} : \exp\left(-Y_{i,1}\right) = \Pi_1 + \tilde{m}_n - \tilde{m}_i, \tag{4.9}$$

$$\frac{\partial W}{\partial Y_{i,2}} : \exp\left(-Y_{i,2}\right) = \Pi_2 + \tilde{m}_n - \tilde{m}_i, \tag{4.10}$$

$$\frac{\partial W}{\partial Y_{i,3}} : \exp\left(-Y_{i,3}\right) = \Pi_3 + \tilde{m}_n - \tilde{m}_i, \tag{4.11}$$

$$\frac{\partial W}{\partial Y_{i,4}} : \exp\left(-Y_{i,4}\right) = \Pi_4 + \tilde{m}_n - \tilde{m}_i, \tag{4.12}$$

For $i \leq n$:

$$\frac{\partial W}{\partial Y_{i,5}} : \exp\left(-Y_{i,5}\right) = \Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i, \tag{4.13}$$

$$\frac{\partial W}{\partial Y_{i,6}} : \exp\left(-Y_{i,6}\right) = -\Pi_2 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i, \tag{4.14}$$

$$\frac{\partial W}{\partial Y_{i,7}} : \exp\left(-Y_{i,7}\right) = \Pi_1 - \Pi_4 - \tilde{m}_i, \tag{4.15}$$

$$\frac{\partial W}{\partial Y_{i,8}} : \exp\left(-Y_{i,8}\right) = -\Pi_1 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i, \tag{4.16}$$

$$\frac{\partial W}{\partial Y_{i,9}} : \exp\left(-Y_{i,9}\right) = \Pi_2 - \Pi_4 - \tilde{m}_i, \tag{4.17}$$

$$\frac{\partial W}{\partial Y_{i,10}} : \exp(-Y_{i,10}) = \Pi_1 - \Pi_3 - \tilde{m}_i, \tag{4.18}$$

$$\frac{\partial W}{\partial Y_{i,11}} : \exp(-Y_{i,11}) = \Pi_2 - \Pi_3 - \tilde{m}_i, \tag{4.19}$$

$$\frac{\partial W}{\partial Y_{i,12}} : \exp(-Y_{i,12}) = \Pi_1 - \Pi_2 - \tilde{m}_i, \tag{4.20}$$

$$\frac{\partial W}{\partial Y_{i,13}} : \exp(-Y_{i,13}) = -\Pi_1 - \tilde{m}_n - \tilde{m}_i, \tag{4.21}$$

$$\frac{\partial W}{\partial Y_{i,14}} : \exp(-Y_{i,14}) = -\Pi_2 - \tilde{m}_n - \tilde{m}_i, \tag{4.22}$$

$$\frac{\partial W}{\partial Y_{i,15}} : \exp(-Y_{i,15}) = -\Pi_3 - \tilde{m}_n - \tilde{m}_i, \tag{4.23}$$

$$\frac{\partial W}{\partial Y_{i,16}} : \exp(-Y_{i,16}) = -\Pi_4 - \tilde{m}_n - \tilde{m}_i, \tag{4.24}$$

$$\frac{\partial W}{\partial Y_{i,17}} : \exp(-Y_{i,17}) = -\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i, \tag{4.25}$$

$$\frac{\partial W}{\partial Y_{i,18}} : \exp(-Y_{i,18}) = \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i, \tag{4.26}$$

$$\frac{\partial W}{\partial Y_{i,19}} : \exp(-Y_{i,19}) = -\Pi_1 + \Pi_4 - \tilde{m}_i, \tag{4.27}$$

$$\frac{\partial W}{\partial Y_{i,20}} : \exp(-Y_{i,20}) = \Pi_1 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i, \tag{4.28}$$

$$\frac{\partial W}{\partial Y_{i,21}} : \exp(-Y_{i,21}) = -\Pi_2 + \Pi_4 - \tilde{m}_i, \tag{4.29}$$

$$\frac{\partial W}{\partial Y_{i,22}} : \exp(-Y_{i,22}) = -\Pi_1 + \Pi_3 - \tilde{m}_i, \tag{4.30}$$

$$\frac{\partial W}{\partial Y_{i,23}} : \exp(-Y_{i,23}) = -\Pi_2 + \Pi_3 - \tilde{m}_i, \tag{4.31}$$

$$\frac{\partial W}{\partial Y_{i,24}} : \exp(-Y_{i,24}) = -\Pi_1 + \Pi_2 - \tilde{m}_i, \tag{4.32}$$

$$\frac{\partial W}{\partial Y_{i,25}} : \exp(-Y_{i,25}) = -\tilde{m}_i, \tag{4.33}$$

$$\frac{\partial W}{\partial Y_{i,26}} : \exp(-Y_{i,26}) = -\tilde{m}_i, \tag{4.34}$$

In the same way, $\partial W/\partial X_m$ gives:

$$X_1 = \Pi_3 + \Pi_4, \qquad X_{25} = -\Pi_3 - \Pi_4, \tag{4.35}$$
$$X_2 = \Pi_1 + \Pi_2 - \Pi_4, \qquad X_{26} = -\Pi_1 - \Pi_2 + \Pi_4, \tag{4.36}$$
$$X_3 = \Pi_1 + \Pi_2 - \Pi_3, \qquad X_{27} = -\Pi_1 - \Pi_2 + \Pi_3, \tag{4.37}$$
$$X_4 = \Pi_1 + \Pi_3, \qquad X_{28} = -\Pi_1 - \Pi_3, \tag{4.38}$$
$$X_5 = \Pi_1 - \Pi_2 + \Pi_4, \qquad X_{29} = -\Pi_1 + \Pi_2 - \Pi_4, \tag{4.39}$$
$$X_6 = 2\Pi_1 - \Pi_4, \qquad X_{30} = -2\Pi_1 + \Pi_4, \tag{4.40}$$
$$X_7 = -\Pi_1 + \Pi_2 + \Pi_3, \qquad X_{31} = \Pi_1 - \Pi_2 - \Pi_3, \tag{4.41}$$
$$X_8 = -\Pi_1 + \Pi_2 + \Pi_4, \qquad X_{32} = \Pi_1 - \Pi_2 - \Pi_4, \tag{4.42}$$
$$X_9 = 2\Pi_2 - \Pi_3 - \Pi_4, \qquad X_{33} = -2\Pi_2 + \Pi_3 + \Pi_4, \tag{4.43}$$
$$X_{10} = -\Pi_1 - \Pi_2 + 2\Pi_3 + \Pi_4, \qquad X_{34} = \Pi_1 + \Pi_2 - 2\Pi_3 - \Pi_4, \tag{4.44}$$
$$X_{11} = \Pi_3 - \Pi_4, \qquad X_{35} = -\Pi_3 + \Pi_4, \tag{4.45}$$
$$X_{12} = -\Pi_1 - \Pi_2 + \Pi_3 + 2\Pi_4, \qquad X_{36} = \Pi_1 + \Pi_2 - \Pi_3 - 2\Pi_4, \tag{4.46}$$
$$X_{13} = \Pi_1, \qquad X_{37} = -\Pi_1, \tag{4.47}$$
$$X_{14} = \Pi_2, \qquad X_{38} = -\Pi_2, \tag{4.48}$$
$$X_{15} = \Pi_3, \qquad X_{39} = -\Pi_3, \tag{4.49}$$
$$X_{16} = \Pi_4, \qquad X_{40} = -\Pi_4, \tag{4.50}$$
$$X_{17} = \Pi_1 + \Pi_2 - \Pi_3 - \Pi_4, \qquad X_{41} = -\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4, \tag{4.51}$$
$$X_{18} = -\Pi_2 + \Pi_3 + \Pi_4, \qquad X_{42} = \Pi_2 - \Pi_3 - \Pi_4, \tag{4.52}$$
$$X_{19} = \Pi_1 - \Pi_4, \qquad X_{43} = -\Pi_1 + \Pi_4, \tag{4.53}$$
$$X_{20} = -\Pi_1 + \Pi_3 + \Pi_4, \qquad X_{44} = \Pi_1 - \Pi_3 - \Pi_4, \tag{4.54}$$
$$X_{21} = \Pi_2 - \Pi_4, \qquad X_{45} = -\Pi_2 + \Pi_4, \tag{4.55}$$
$$X_{22} = \Pi_1 - \Pi_3, \qquad X_{46} = -\Pi_1 + \Pi_3, \tag{4.56}$$
$$X_{23} = \Pi_2 - \Pi_3, \qquad X_{47} = -\Pi_2 + \Pi_3, \tag{4.57}$$
$$X_{24} = \Pi_1 - \Pi_2, \qquad X_{48} = -\Pi_1 + \Pi_2. \tag{4.58}$$

Now, plug these constraints back into (4.5)i–(4.8) to get:

$$\begin{aligned}
\Pi_1 = \prod_{i=1}^{n-1} (\Pi_1 + \tilde{m}_n - \tilde{m}_i)^{-1} \prod_{i=1}^{n} &(\Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_4 - \tilde{m}_i)^{-1} \\
&\times (-\Pi_1 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i)(\Pi_1 - \Pi_3 - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_2 - \tilde{m}_i)^{-1} \\
&\times (-\Pi_1 + \Pi_2 - \tilde{m}_i)(\Pi_1 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i)^{-1} (-\Pi_1 + \Pi_3 - \tilde{m}_i) \\
&\times (-\Pi_1 + \Pi_2 - \tilde{m}_i)(-\Pi_1 - \tilde{m}_n - \tilde{m}_i)(-\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i),
\end{aligned}$$

$$\begin{aligned}\Pi_2 = \prod_{i=1}^{n-1}(\Pi_2+\tilde{m}_n-\tilde{m}_i)^{-1}\prod_{i=1}^{n}&(\Pi_1+\Pi_2-\Pi_3-\Pi_4-\tilde{m}_i)^{-1}\\ &\times(-\Pi_2+\Pi_3+\Pi_4+\tilde{m}_n-\tilde{m}_i)(\Pi_2-\Pi_4-\tilde{m}_i)^{-1}\\ &\times(\Pi_2-\Pi_3-\tilde{m}_i)^{-1}(\Pi_1-\Pi_2-\tilde{m}_i)(-\Pi_2+\Pi_4-\tilde{m}_i)\\ &\times(-\Pi_2+\Pi_3-\tilde{m}_i)(-\Pi_1+\Pi_2-\tilde{m}_i)^{-1}(-\Pi_2-\tilde{m}_n-\tilde{m}_i)\\ &\times(-\Pi_1-\Pi_2+\Pi_3+\Pi_4-\tilde{m}_i)(\Pi_2-\Pi_3-\Pi_4-\tilde{m}_n-\tilde{m}_i)^{-1},\end{aligned}$$

$$\begin{aligned}\Pi_3 = \prod_{i=1}^{n-1}(\Pi_3+\tilde{m}_n-\tilde{m}_i)^{-1}\prod_{i=1}^{n}&(\Pi_1+\Pi_2-\Pi_3-\Pi_4-\tilde{m}_i)\\ &\times(-\Pi_2+\Pi_3+\Pi_4+\tilde{m}_n-\tilde{m}_i)^{-1}(-\Pi_1+\Pi_3+\Pi_4+\tilde{m}_n-\tilde{m}_i)^{-1}\\ &\times(\Pi_1-\Pi_3-\tilde{m}_i)(\Pi_2-\Pi_3-\tilde{m}_i)(\Pi_1-\Pi_3-\Pi_4-\tilde{m}_n-\tilde{m}_i)\\ &\times(-\Pi_1+\Pi_3-\tilde{m}_i)^{-1}(-\Pi_2+\Pi_3-\tilde{m}_i)^{-1}(-\Pi_3-\tilde{m}_n-\tilde{m}_i)\\ &\times(-\Pi_1-\Pi_2+\Pi_3+\Pi_4-\tilde{m}_i)^{-1}(\Pi_2-\Pi_3-\Pi_4-\tilde{m}_n-\tilde{m}_i),\end{aligned}$$

$$\begin{aligned}\Pi_4 = \prod_{i=1}^{n-1}(\Pi_4+\tilde{m}_n-\tilde{m}_i)^{-1}\prod_{1}^{n}&(\Pi_1+\Pi_2-\Pi_3-\Pi_4-\tilde{m}_i)\\ &\times(-\Pi_2+\Pi_3+\Pi_4+\tilde{m}_n-\tilde{m}_i)^{-1}(\Pi_1-\Pi_4-\tilde{m}_i)\\ &\times(-\Pi_1+\Pi_3+\Pi_4+\tilde{m}_n-\tilde{m}_i)^{-1}(\Pi_2-\Pi_4-\tilde{m}_i)\\ &\times(-\Pi_1+\Pi_2-\tilde{m}_i)^{-1}(\Pi_1-\Pi_3-\Pi_4-\tilde{m}_n-\tilde{m}_i)\\ &\times(-\Pi_2+\Pi_4-\tilde{m}_i)^{-1}(-\Pi_4-\tilde{m}_n-\tilde{m}_i)\\ &\times(-\Pi_1-\Pi_2+\Pi_3+\Pi_4-\tilde{m}_i)^{-1}(\Pi_2-\Pi_3-\Pi_4-\tilde{m}_n-\tilde{m}_i).\end{aligned}$$

The mirror maps are given by,

$$\exp(-Y_{i,\beta}) \mapsto -\tilde{m}_i + \sum_{a=1}^{4}\sigma_a\rho^a_{i,\beta}, \quad X_m \mapsto \sum_{a=1}^{4}\sigma_a\alpha^a_m.$$

on the critical locus relations, one finds

$$\begin{aligned}\Pi_1 &= \exp(-Y_{n,1}) = \sigma_4-\tilde{m}_n, & \Pi_2 &= \exp(-Y_{n,2}) = \sigma_3-\sigma_4-\tilde{m}_n,\\ \Pi_3 &= \exp(-Y_{n,3}) = \sigma_2-\sigma_3-\tilde{m}_n, & \Pi_4 &= \exp(-Y_{n,4}) = \sigma_1-\sigma_2+\sigma_3-\tilde{m}_n.\end{aligned}$$

Plugging them back in, one obtains the Coulomb branch (quantum cohomology) ring relations for F_4:

(4.59)
$$\begin{aligned}
&\prod_{i=1}^{n}(-\sigma_1+\sigma_3-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_3+\sigma_4-\tilde{m}_i)(-\sigma_2+\sigma_3+\sigma_4-\tilde{m}_i)\\
&\times(-\sigma_3+2\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_4-\tilde{m}_i)(\sigma_4-\tilde{m}_i)\\
&=\prod_{i=1}^{n}(\sigma_1-\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_2+\sigma_3-\sigma_4-\tilde{m}_i)(\sigma_2-\sigma_3-\sigma_4-\tilde{m}_i)\\
&\times(\sigma_3-2\sigma_4-\tilde{m}_i)(-\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_3-\tilde{m}_i),
\end{aligned}$$

(4.60)
$$\begin{aligned}
&\prod_{i=1}^{n}(-\sigma_1+\sigma_3-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_4-\tilde{m}_i)(-\sigma_2+2\sigma_3-\sigma_4-\tilde{m}_i)\\
&\times(\sigma_3-2\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_3-\sigma_4-\tilde{m}_i)(\sigma_3-\sigma_4-\tilde{m}_i)\\
&=\prod_{i=1}^{n}(\sigma_1-\sigma_3+\sigma_4-\tilde{m}_i)(-\sigma_3+2\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_2+\sigma_4-\tilde{m}_i)\\
&\times(\sigma_2-2\sigma_3+\sigma_4-\tilde{m}_i)(-\sigma_3+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_3-\tilde{m}_i),
\end{aligned}$$

(4.61)
$$\begin{aligned}
&\prod_{i=1}^{n}(\sigma_1-\sigma_3+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_4-\tilde{m}_i)(\sigma_2-\sigma_3-\sigma_4-\tilde{m}_i)\\
&\times(\sigma_2-2\sigma_3+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_3-\tilde{m}_i)(\sigma_2-\sigma_3-\tilde{m}_i)\\
&=\prod_{i=1}^{n}(-\sigma_1+\sigma_3-\tilde{m}_i)(-\sigma_2+\sigma_3+\sigma_4-\tilde{m}_i)(-\sigma_2+2\sigma_3-\sigma_4-\tilde{m}_i)\\
&\times(-\sigma_1+\sigma_4-\tilde{m}_i)(-\sigma_2+\sigma_3-\tilde{m}_i)(-\sigma_1+\sigma_3-\sigma_4-\tilde{m}_i),
\end{aligned}$$

(4.62)
$$\begin{aligned}
&\prod_{i=1}^{n}(\sigma_1-\sigma_3+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_2+\sigma_3-\sigma_4-\tilde{m}_i)\\
&\times(\sigma_1-\sigma_2+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_3-\tilde{m}_i)(\sigma_1-\sigma_2+\sigma_3-\tilde{m}_i)\\
&=\prod_{i=1}^{n}(-\sigma_1+\sigma_3-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_3+\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_4-\tilde{m}_i)\\
&\times(-\sigma_1+\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_3-\tilde{m}_i)(-\sigma_1+\sigma_3-\sigma_4-\tilde{m}_i).
\end{aligned}$$

Next, let us described the excluded locus on the Coulomb branch. As discussed previously and in [3], part of the excluded locus is defined by the condition $X_m \neq 0$ for all m. This gives

$$
\begin{aligned}
(4.63)\quad & \sigma_1(2\sigma_1-\sigma_2)(-\sigma_1+\sigma_2)(\sigma_1+\sigma_2-2\sigma_3)(-\sigma_1+2\sigma_2-2\sigma_3)\\
&\times(\sigma_2-\sigma_3)(-\sigma_1+\sigma_3)(\sigma_1-\sigma_2+\sigma_3)(-\sigma_2+2\sigma_3)(\sigma_2-2\sigma_4)\\
&\times(-\sigma_1+2\sigma_3-2\sigma_4)(\sigma_1-\sigma_2+2\sigma_3-2\sigma_4)(\sigma_1-\sigma_4)(-\sigma_1+\sigma_2-\sigma_4)\\
&\times(\sigma_3-\sigma_4)(-\sigma_2+\sigma_3-\sigma_4)\sigma_4(\sigma_1-\sigma_3+\sigma_4)(-\sigma_1+2\sigma_4)\\
&\times(-\sigma_1+\sigma_2-\sigma_3+\sigma_4)(-\sigma_2+\sigma_3+\sigma_4)(\sigma_1-\sigma_2+2\sigma_4)\\
&\times(\sigma_2-2\sigma_3+2\sigma_4)(-\sigma_3+2\sigma_4)\neq 0.
\end{aligned}
$$

The second part of the excluded locus is determined by the condition that $\exp(-Y)\neq 0$. From the mirror map

$$\exp(-Y_{i,\beta}) = -\tilde{m}_i + \sum_{a=1}^{4}\sigma_a\rho^a_{i,\beta},$$

the excluded locus constraint becomes

$$-\tilde{m}_i + \sum_{a=1}^{4}\sigma_a\rho^a_{i,\beta}\neq 0$$

which is encoded in the expression below:

$$
\begin{aligned}
(4.64)\quad & \prod_{i=1}^{n}(\sigma_1-\sigma_3-\tilde{m}_i)(\sigma_2-\sigma_3-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_3-\tilde{m}_i)\\
&\times(-\sigma_1+\sigma_3-\tilde{m}_i)(-\sigma_2+\sigma_3-\tilde{m}_i)(\sigma_1-\sigma_2+\sigma_3-\tilde{m}_i)\\
&\times(\sigma_3-2\sigma_4-\tilde{m}_i)(-\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_4-\tilde{m}_i)\\
&\times(\sigma_2-\sigma_3-\sigma_4-\tilde{m}_i)(\sigma_3-\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_3-\sigma_4-\tilde{m}_i)\\
&\times(\sigma_1-\sigma_2+\sigma_3-\sigma_4-\tilde{m}_i)(-\sigma_2+2\sigma_3-\sigma_4-\tilde{m}_i)(\sigma_4-\tilde{m}_i)\\
&\times(-\sigma_1+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_2+\sigma_4-\tilde{m}_i)(-\sigma_3+2\sigma_4-\tilde{m}_i)\\
&\times(-\sigma_3+\sigma_4-\tilde{m}_i)(\sigma_1-\sigma_3+\sigma_4-\tilde{m}_i)(-\sigma_1+\sigma_2-\sigma_3+\sigma_4-\tilde{m}_i)\\
&\times(-\sigma_2+\sigma_3+\sigma_4-\tilde{m}_i)(\sigma_2-2\sigma_3+\sigma_4-\tilde{m}_i)\neq 0.
\end{aligned}
$$

4.2. Transformation under the Weyl group of F_4

The Weyl group of F_4 has $1152 = 2^7 \cdot 3^2$ elements[4] [20, Table 2.2], so explicitly listing orbits of vacua, for example, is not feasible, unlike the case of G_2. (Similarly [20, Table 2.2], the order of the Weyl group of E_6 is $2^7 \cdot 3^4 \cdot 5$, the order of the Weyl group of E_7 is $2^{10} \cdot 3^4 \cdot 5 \cdot 7$, and the order of the Weyl group of E_8 is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, so we will not be tracking orbits of vacua under the Weyl group in those cases either.) In this section, we will instead merely check that the critical locus equations transform into one another under Weyl reflections, a nontrivial check of our computations.

As reviewed earlier, the Weyl transformation acts on vectors, roots and weights:

$$S_\alpha(v^a) = v^a - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha^a. \tag{4.65}$$

The Euclidean inner product takes the following metric matrix in this coordinate

$$[g_{ab}] = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & 3/2 \\ 1 & 2 & 3/2 & 1 \end{pmatrix}. \tag{4.66}$$

The σ_a transform as covectors under the same Weyl transformation. F_4 has four simple roots, which can be taken to be

$$\begin{aligned} A &= (2, -1, 0, 0), & B &= (-1, 2, -2, 0), \\ C &= (0, -1, 2, -1), & D &= (0, 0, -1, 2). \end{aligned}$$

[4]For the curious, information on the representation theory of the Weyl group of F_4 can be found in [21].

so the Weyl group of F_4 has four distinguished elements, $S_A, \ldots, S_D$, whose actions are given by

$$\begin{aligned} S_A(v^1, v^2, v^3, v^4) &= (-v^1, v^1 + v^2, v^3, v^4), \\ S_A(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_2 - \sigma_1, \sigma_2, \sigma_3, \sigma_4), \end{aligned} \tag{4.67}$$

$$\begin{aligned} S_B(v^1, v^2, v^3, v^4) &= (v^1 + v^2, -v^2, 2v^2 + v^3, v^4), \\ S_B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_1, \sigma_1 - \sigma_2 + 2\sigma_3, \sigma_3, \sigma_4), \end{aligned} \tag{4.68}$$

$$\begin{aligned} S_C(v^1, v^2, v^3, v^4) &= (v^1, v^2 + v^3, -v^3, v^3 + v^4), \\ S_C(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_1, \sigma_2, \sigma_2 - \sigma_3 + \sigma_4, \sigma_4), \end{aligned} \tag{4.69}$$

$$\begin{aligned} S_D(v^1, v^2, v^3, v^4) &= (v^1, v^2, v^3 + v^4, -v^4), \\ S_D(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_1, \sigma_2, \sigma_3, \sigma_3 - \sigma_4). \end{aligned} \tag{4.70}$$

The superpotential is, by construction, invariant under Weyl reflections, hence to be consistent, the critical locus equations should transform into one another under these reflections. We check this below. For example, under the action of S_D, equation (4.59)

$$\begin{aligned} &\prod_{i=1}^{n} (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_2 + \sigma_3 + \sigma_4 - \tilde{m}_i) \\ &\times (-\sigma_3 + 2\sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_4 - \tilde{m}_i)(\sigma_4 - \tilde{m}_i) \\ = &\prod_{i=1}^{n} (\sigma_1 - \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_2 - \sigma_3 - \sigma_4 - \tilde{m}_i) \\ &\times (\sigma_3 - 2\sigma_4 - \tilde{m}_i)(-\sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i), \end{aligned}$$

transforms into

$$\begin{aligned} &\prod_{i=1}^{n} (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_4 - \tilde{m}_i)(-\sigma_2 + 2\sigma_3 - \sigma_4 - \tilde{m}_i) \\ &\times (\sigma_3 - 2\sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_3 - \sigma_4 - \tilde{m}_i) \\ = &\prod_{i=1}^{n} (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_4 - \tilde{m}_i)(\sigma_2 - 2\sigma_3 + \sigma_4 - \tilde{m}_i) \\ &\times (-\sigma_3 + 2\sigma_4 - \tilde{m}_i)(-\sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i), \end{aligned}$$

which is equation (4.60).

Table 8 schematically describes how other critical locus equations transform under these Weyl reflections.

	Initial equ'n	Final equ'n
A	(4.59)	(4.59) $\times$ (4.62)
A	(4.60)	(4.60) $\times$ (4.62)
A	(4.61)	(4.61) $\times$ (4.62)
A	(4.62)	(4.62)
B	(4.59)	(4.59)
B	(4.60)	(4.60)
B	(4.61)	(4.62)
B	(4.62)	(4.61)
C	(4.59)	(4.59)
C	(4.60)	(4.61) $\times$ (4.62)
C	(4.61)	(4.60) $\times$ (4.62)
C	(4.62)	(4.62)
D	(4.59)	(4.60)
D	(4.60)	(4.59)
D	(4.61)	(4.61)
D	(4.62)	(4.62)

Table 8: Transformation of critical locus equations under four Weyl reflections.

The fact that the critical locus equations are closed under Weyl reflections associated to a set of simple roots provides a nontrivial consistency check on our results.

4.3. Pure gauge theory

In this section, we will consider the mirror to the pure supersymmetric F_4 gauge theory. The mirror superpotential is

(4.71)

$$W = \sigma_1 \Big(Z_1 - Z_2 + Z_5 - Z_6 + Z_{16} - Z_{17} + Z_8 + Z_{18} - Z_9 - Z_{19} + Z_{20} - Z_{21} + Z_{10} - Z_{11} + 2Z_{12} - Z_{25} + Z_{26} - Z_{29} + Z_{30} - Z_{40} + Z_{41} - Z_{32} - Z_{42} + Z_{33} + Z_{43} - Z_{44} + Z_{45} - Z_{34} + Z_{35} - 2Z_{36} \Big) +$$

$$
\begin{aligned}
&+\sigma_2\Big(Z_2 - Z_3 + Z_4 + Z_{15} - Z_5 + Z_7 - Z_{16} - Z_8 + Z_{19} + Z_{21} - Z_{22} + Z_{10} \\
&\qquad + 2Z_{11} - Z_{23} - Z_{12} - Z_{26} + Z_{27} - Z_{28} - Z_{39} + Z_{29} - Z_{31} + Z_{40} \\
&\qquad + Z_{32} - Z_{43} - Z_{45} + Z_{46} - Z_{34} - 2Z_{35} + Z_{47} + Z_{36} \Big) \\
&+\sigma_3\Big(2Z_3 + Z_{14} - 2Z_4 - Z_{15} + Z_{16} + Z_{17} + 2Z_8 - Z_{18} + 2Z_9 - Z_{19} + Z_{22} \\
&\qquad - 2Z_{10} - 2Z_{11} + 2Z_{23} - Z_{24} - 2Z_{27} - Z_{38} + 2Z_{28} + Z_{39} - Z_{40} - Z_{41} \\
&\qquad - 2Z_{32} + Z_{42} - 2Z_{33} + Z_{43} - Z_{46} + 2Z_{34} + 2Z_{35} - 2Z_{47} + Z_{48} \Big) \\
&+\sigma_4\Big(Z_{13} - Z_{14} + 2Z_4 + 2Z_5 + 2Z_6 - 2Z_7 - 2Z_8 + Z_{18} - 2Z_9 + Z_{19} - Z_{20} \\
&\qquad - Z_{21} + Z_{22} - Z_{23} + 2Z_{24} - Z_{37} + Z_{38} - 2Z_{28} - 2Z_{29} - 2Z_{30} + 2Z_{31} \\
&\qquad + 2Z_{32} - Z_{42} + 2Z_{33} - Z_{43} + Z_{44} + Z_{45} - Z_{46} + Z_{47} - 2Z_{48} \Big) \\
&+ \sum_{m=1}^{48} X_m.
\end{aligned}
$$

Now, let us consider the critical locus of the superpotential above. For each root μ, the fields X_μ and $X_{-\mu}$ appear paired with opoosite signs coupling to each σ. Therefore, one impliciation of the derivatives

$$\frac{\partial W}{\partial X_\mu} = 0$$

is that, on the critical locus,

$$X_\mu = -X_{-\mu}. \tag{4.72}$$

(Furthermore, on the critical locus, each X_μ is determined by σs.) Next, each derivative

$$\frac{\partial W}{\partial \sigma_a}$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check in the superpotential above that each σ_a is multiplied by an even number of such ratios (*i.e.* the number of Z's is a multiple of four). For example, the sum of the absolute values of the coefficients of the Z's multiplying σ_1 and σ_2 is $32 = 4 \cdot 8$, and the sum of the absolute values of the coefficients of the Z's multiplying σ_3 and σ_4 is $44 = 4 \cdot 11$. Thus, the constraint implied by the σ's is automatically satisfied.

As a result, following the same analysis in [3], we see in this case, that the critical locus is nonempty, and in fact is determined by four σs. In other words, at the level of these topological field theory computations, we have evidence that the pure supersymmetric F_4 gauge theory in two dimensions flows in the IR to a theory of four free twisted chiral superfields.

5. E_6

In this section we will consider the mirror Landau-Ginzburg orbifold superpotential of E_6 gauge theory when matter fields are in **27** fundamental representation of it and then we compute quantum cohomology ring of it. Also we will consider the pure theory without matter field.

5.1. Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has fields

- $Y_{i,\beta}$, $i \in \{1, \ldots, n\}$, $\beta \in \{1, \ldots, 27\}$, corresponding to the matter fields in n copies of the **27** representation **27**,
- X_m, $m \in \{1, \ldots, 72\}$, corresponding to the roots of E_6,
- σ_a, $a \in \{1, 2, 3, 4, 5, 6\}$.

We associate the roots, α_m^a, to X_m fields and the wights, $\rho_{i,\beta}^a$, of fundamental **27** representation of E_6 to $Y_{i,\beta}$.

The roots of E_6 and associated fields are listed in Tables 9, 10. The weights associated to the **27** of E_6 and their associated fields are listed in Table 11.

The weights in the tables in this section are written as linear combinations of the fundamental weights, computed with LieART [19], as discussed earlier, so as to get conventional θ-angle periodicities.

Field	Positive root	Field	Negative root
X_1	$(0,0,0,0,0,1)$	X_{37}	$(0,0,0,0,0,-1)$
X_2	$(0,0,1,0,0,-1)$	X_{38}	$(0,0,-1,0,0,1)$
X_3	$(0,1,-1,1,0,0)$	X_{39}	$(0,-1,1,-1,0,0)$
X_4	$(0,1,0,-1,1,0)$	X_{40}	$(0,-1,0,1,-1,0)$
X_5	$(1,-1,0,1,0,0)$	X_{41}	$(-1,1,0,-1,0,0)$
X_6	$(-1,0,0,1,0,0)$	X_{42}	$(1,0,0,-1,0,0)$
X_7	$(0,1,0,0,-1,0)$	X_{43}	$(0,-1,0,0,1,0)$
X_8	$(1,-1,1,-1,1,0)$	X_{44}	$(-1,1,-1,1,-1,0)$
X_9	$(-1,0,1,-1,1,0)$	X_{45}	$(1,0,-1,1,-1,0)$
X_{10}	$(1,-1,1,0,-1,0)$	X_{46}	$(-1,1,-1,0,1,0)$
X_{11}	$(1,0,-1,0,1,1)$	X_{47}	$(-1,0,1,0,-1,-1)$
X_{12}	$(-1,0,1,0,-1,0)$	X_{48}	$(1,0,-1,0,1,0)$
X_{13}	$(-1,1,-1,0,1,1)$	X_{49}	$(1,-1,1,0,-1,-1)$
X_{14}	$(1,0,-1,1,-1,1)$	X_{50}	$(-1,0,1,-1,1,-1)$
X_{15}	$(1,0,0,0,1,-1)$	X_{51}	$(-1,0,0,0,-1,1)$
X_{16}	$(-1,1,-1,1,-1,1)$	X_{52}	$(1,-1,1,-1,1,-1)$
X_{17}	$(-1,1,0,0,1,-1)$	X_{53}	$(1,-1,0,0,-1,1)$
X_{18}	$(0,-1,0,0,1,1)$	X_{54}	$(0,1,0,0,-1,-1)$

Table 9: First set of roots of E_6 and associated fields.

5.2. Superpotential

In this section, we describe the superpotential of the mirror Landau-Ginzburg orbifold. It is given by

$$W = \sum_{a=1}^{6} \sigma_a \left(\sum_{i=1}^{n} \sum_{\beta=1}^{27} \rho^a_{i,\beta} Y_{i,\beta} + \sum_{m=1}^{72} \alpha^a_m Z_m \right) - \sum_{i=1}^{n} \tilde{m}_i \sum_{b=1}^{27} Y_{i,\beta} + \sum_{i=1}^{n} \sum_{\beta=1}^{27} \exp(-Y_{i,\beta}) + \sum_{m=1}^{72} X_m.$$

where $X_m = \exp(-Z_m)$ and X_m are the fundamental fields. Using the results of the previous section we get

$$W = \sum_{a=1}^{6} \sigma_a \mathcal{C}^a - \sum_{i=1}^{n} \tilde{m}_i \sum_{\beta=1}^{27} Y_{i,\beta} + \sum_{i=1}^{n} \sum_{\beta=1}^{27} \exp(-Y_{i,\beta}) + \sum_{m=1}^{72} X_m. \tag{5.1}$$

Field	Positive root	Field	Negative root
X_{19}	$(1,0,0,-1,0,1)$	X_{55}	$(-1,0,0,1,0,-1)$
X_{20}	$(1,0,0,1,-1,-1)$	X_{56}	$(-1,0,0,-1,1,1)$
X_{21}	$(-1,1,0,-1,0,1)$	X_{57}	$(1,-1,0,1,0,-1)$
X_{22}	$(-1,1,0,1,-1,-1)$	X_{58}	$(1,-1,0,-1,1,1)$
X_{23}	$(0,-1,0,1,-1,1)$	X_{59}	$(0,1,0,-1,1,-1)$
X_{24}	$(0,-1,1,0,1,-1)$	X_{60}	$(0,1,-1,0,-1,1)$
X_{25}	$(1,0,1,-1,0,-1)$	X_{61}	$(-1,0,-1,1,0,1)$
X_{26}	$(-1,1,1,-1,0,-1)$	X_{62}	$(1,-1,-1,1,0,1)$
X_{27}	$(0,-1,1,-1,0,1)$	X_{63}	$(0,1,-1,1,0,-1)$
X_{28}	$(0,-1,1,1,-1,-1)$	X_{64}	$(0,1,-1,-1,1,1)$
X_{29}	$(0,0,-1,1,1,0)$	X_{65}	$(0,0,1,-1,-1,0)$
X_{30}	$(1,1,-1,0,0,0)$	X_{66}	$(-1,-1,1,0,0,0)$
X_{31}	$(-1,2,-1,0,0,0)$	X_{67}	$(1,-2,1,0,0,0)$
X_{32}	$(0,-1,2,-1,0,-1)$	X_{68}	$(0,1,-2,1,0,1)$
X_{33}	$(0,0,-1,0,0,2)$	X_{69}	$(0,0,1,0,0,-2)$
X_{34}	$(0,0,-1,2,-1,0)$	X_{70}	$(0,0,1,-2,1,0)$
X_{35}	$(0,0,0,-1,2,0)$	X_{71}	$(0,0,0,1,-2,0)$
X_{36}	$(2,-1,0,0,0,0)$	X_{72}	$(-2,1,0,0,0,0)$

Table 10: Second set of roots of E_6 and associated fields.

Field	Weight	Field	Weight	Field	Weight
$Y_{i,1}$	$(1,0,0,0,0,0)$	$Y_{i,2}$	$(-1,1,0,0,0,0)$	$Y_{i,3}$	$(0,-1,1,0,0,0)$
$Y_{i,4}$	$(0,0,-1,1,0,1)$	$Y_{i,5}$	$(0,0,0,-1,1,1)$	$Y_{i,6}$	$(0,0,0,1,0,-1)$
$Y_{i,7}$	$(0,0,0,0,-1,1)$	$Y_{i,8}$	$(0,0,1,-1,1,-1)$	$Y_{i,9}$	$(0,0,1,0,-1,-1)$
$Y_{i,10}$	$(0,1,-1,0,1,0)$	$Y_{i,11}$	$(0,1,-1,1,-1,0)$	$Y_{i,12}$	$(1,-1,0,0,1,0)$
$Y_{i,13}$	$(-1,0,0,0,1,0)$	$Y_{i,14}$	$(0,1,0,-1,0,0)$	$Y_{i,15}$	$(1,-1,0,1,-1,0)$
$Y_{i,16}$	$(-1,0,0,1,-1,0)$	$Y_{i,17}$	$(1,-1,1,-1,0,0)$	$Y_{i,18}$	$(-1,0,1,-1,0,0)$
$Y_{i,19}$	$(1,0,-1,0,0,1)$	$Y_{i,20}$	$(-1,1,-1,0,0,1)$	$Y_{i,21}$	$(1,0,0,0,0,-1)$
$Y_{i,22}$	$(-1,1,0,0,0,-1)$	$Y_{i,23}$	$(0,-1,0,0,0,1)$	$Y_{i,24}$	$(0,-1,1,0,0,-1)$
$Y_{i,25}$	$(0,0,-1,1,0,0)$	$Y_{i,26}$	$(0,0,0,-1,1,0)$	$Y_{i,27}$	$(0,0,0,0,-1,0)$

Table 11: Weights of **27** of E_6 and associated fields.

where $\mathcal{C}^a$ are listed in [7].

5.3. Coulomb ring relations

Integrating out the σ_a fields, we obtain six constraints $\mathcal{C}^a = 0$. Exponentiating these constraints, we obtain a series of equations from which the Coulomb ring relations will be derived. For reasons of notational sanity, we will also slightly simplify these expressions, as follows. To make predictions for the A model, we will evaluate the ring relations on the critical locus, where

$$\frac{X_m}{X_{m+63}} = -1.$$

It is straightforward to see that each of the constraints $\mathcal{C}^a$ contains 22 differences of corresponding Z's, so that the exponential of the constraints contains 22 factors of the form X_m/X_{m+63} – an even number of factors of -1, which will cancel out. Therefore, since on the critical locus those factors will cancel out, we will omit them, and solely relate the exponentiated constraints in terms of Ys.

The exponentiated constraints are listed in [7].

The mirror maps are given by

$$\exp(-Y_{i,\beta}) \mapsto -\tilde{m}_i + \sum_{a=1}^{6} \sigma_a \rho^a_{i,\beta}, \quad X_m \mapsto \sum_{a=1}^{6} \sigma_a \alpha^a_m.$$

Applying the operator mirror maps, the Coulomb ring relations are straightforward to derive and are listed in [7].

Part of the excluded locus is defined by the vanishing locus of the X_m, and is given by

(5.2)

$$\begin{aligned}
&(2\sigma_1 - \sigma_2)(\sigma_1 + \sigma_2 - \sigma_3)(-\sigma_1 + 2\sigma_2 - \sigma_3)(\sigma_4 - \sigma_1)(\sigma_1 - \sigma_2 + \sigma_4)\\
&\times (\sigma_2 - \sigma_3 + \sigma_4)(\sigma_2 - \sigma_5)(-\sigma_1 + \sigma_3 - \sigma_5)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_5)\\
&\times (-\sigma_3 + 2\sigma_4 - \sigma_5)(\sigma_2 - \sigma_4 + \sigma_5)(-\sigma_1 + \sigma_3 - \sigma_4 + \sigma_5)\\
&\times (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \sigma_5)(-\sigma_3 + \sigma_4 + \sigma_5)(2\sigma_5 - \sigma_4)(\sigma_3 - \sigma_6)\\
&\times (\sigma_1 + \sigma_3 - \sigma_4 - \sigma_6)(-\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4 - \sigma_6)(-\sigma_2 + 2\sigma_3 - \sigma_4 - \sigma_6)\\
&\times (\sigma_1 + \sigma_4 - \sigma_5 - \sigma_6)(-\sigma_1 + \sigma_2 + \sigma_4 - \sigma_5 - \sigma_6)(-\sigma_2 + \sigma_3 + \sigma_4 - \sigma_5 - \sigma_6)\\
&\times (\sigma_1 + \sigma_5 - \sigma_6)(-\sigma_1 + \sigma_2 + \sigma_5 - \sigma_6)(-\sigma_2 + \sigma_3 + \sigma_5 - \sigma_6)\,\sigma_6\,(\sigma_1 - \sigma_4 + \sigma_6)\\
&\times (-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_6)(-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_6)(-\sigma_2 + \sigma_4 - \sigma_5 + \sigma_6)\\
&\times (\sigma_1 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_6)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_6)(-\sigma_2 + \sigma_5 + \sigma_6)\\
&\times (\sigma_1 - \sigma_3 + \sigma_5 + \sigma_6)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_5 + \sigma_6)(2\sigma_6 - \sigma_3) \neq 0.
\end{aligned}$$

Similarly,

$$\exp(-Y_{i,\beta}) = -\tilde{m}_i + \sum_{a=1}^{6} \sigma_a \rho^a_{i,\beta}$$

on the critical locus, so

$$-\tilde{m}_i + \sum_{a=1}^{6} \sigma_a \rho^a_{i,\beta} \neq 0$$

which determines the remainder of the excluded locus:

(5.3)
$$\begin{aligned}
&(\sigma_1 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \tilde{m}_i)(-\sigma_2 + \sigma_3 - \tilde{m}_i)(\sigma_2 - \sigma_4 - \tilde{m}_i)\\
&\times(-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i)(-\sigma_3 + \sigma_4 - \tilde{m}_i)\\
&\times(-\sigma_5 - \tilde{m}_i)(-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i)\\
&\times(\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i)(-\sigma_1 + \sigma_5 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_5 - \tilde{m}_i)\\
&\times(\sigma_2 - \sigma_3 + \sigma_5 - \tilde{m}_i)(-\sigma_4 + \sigma_5 - \tilde{m}_i)(\sigma_1 - \sigma_6 - \tilde{m}_i)\\
&\times(-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i)(-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i)(\sigma_4 - \sigma_6 - \tilde{m}_i)\\
&\times(\sigma_3 - \sigma_5 - \sigma_6 - \tilde{m}_i)(\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \tilde{m}_i)(-\sigma_2 + \sigma_6 - \tilde{m}_i)\\
&\times(\sigma_1 - \sigma_3 + \sigma_6 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i)(-\sigma_3 + \sigma_4 + \sigma_6 - \tilde{m}_i)\\
&\times(-\sigma_5 + \sigma_6 - \tilde{m}_i)(-\sigma_4 + \sigma_5 + \sigma_6 - \tilde{m}_i) \neq 0.
\end{aligned}$$

5.4. Pure gauge theory

In this part we will consider the mirror to the pure supersymmetric E_6 gauge theory. The mirror Landau-Ginzburg superpotential is given in [7].

We can analyze this mirror in the same way as previous pure gauge theory mirrors. As discussed previously, for each root μ, the fields X_μ and $X_{-\mu}$ appear paired with opoosite signs coupling to each σ. Therefore, one impliciation of the derivatives

$$\frac{\partial W}{\partial X_\mu} = 0$$

is that, on the critical locus,

$$X_\mu = -X_{-\mu}. \tag{5.4}$$

(Furthermore, on the critical locus, each X_μ is determined by σs.) Next, each derivative

$$\frac{\partial W}{\partial \sigma_a}$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check that, just as in the previous examples, in the superpotential above each σ multiplies a number of Zs that is divisible by four, *i.e.* an even number of ratios $X_\mu / X_{-\mu}$. Specifically, the sum of the absolute values of the coefficients of the Z's multiplying each σ is $44 = 4 \cdot 11$. Thus, the constraint implied by integrating out the σ's is automatically satisfied.

As a result, following the same analysis as earlier and [3], the critical locus is nonempty, and is determined by the six σs. Thus, at the level of these topological field theory computations, we have evidence that the pure supersymmetric E_6 gauge theory in two dimensions flows in the IR to a theory of six free twisted chiral superfields.

6. E_7

In this section we will consider the mirror Landau-Ginzburg orbifold to an E_7 gauge theory with matter fields in the **56** fundamental representation. As before, we will compute Coulomb branch (quantum cohomology) ring relations and excluded loci. We will also study the pure E_7 gauge theory without matter.

6.1. Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has superfields

- $Y_{i,\beta}$, $i \in \{1, \ldots, n\}$, $\beta \in \{1, \ldots, 56\}$, corresponding to the matter fields in n copies of the fundamental **56** representation of E_7,
- X_m, $m \in \{1, \ldots, 126\}$, corresponding to the nonzero roots of E_7,
- σ_a, $a \in \{1, \ldots, 7\}$.

We associate the roots, α_m^a, to X_m fields and the weights, $\rho_{i,\beta}^a$ to the $Y_{i,\beta}$.

The nonzero roots of E_7 are listed in Tables 12, 13, and 14. The weights of the **56** of E_7 are listed in Table 15. All weights are given as linear combinations of fundamental weights, as discussed earlier, and computed with LieART [19], so as to have conventional θ-angle periodicites.

Field	Positive root	Field	Negative root
X_1	$(1,0,0,0,0,0,0)$	X_{64}	$(-1,0,0,0,0,0,0)$
X_2	$(-1,1,0,0,0,0,0)$	X_{65}	$(1,-1,0,0,0,0,0)$
X_3	$(0,-1,1,0,0,0,0)$	X_{66}	$(0,1,-1,0,0,0,0)$
X_4	$(0,0,-1,1,0,0,1)$	X_{67}	$(0,0,1,-1,0,0,-1)$
X_5	$(0,0,0,-1,1,0,1)$	X_{68}	$(0,0,0,1,-1,0,-1)$
X_6	$(0,0,0,1,0,0,-1)$	X_{69}	$(0,0,0,-1,0,0,1)$
X_7	$(0,0,0,0,-1,1,1)$	X_{70}	$(0,0,0,0,1,-1,-1)$
X_8	$(0,0,1,-1,1,0,-1)$	X_{71}	$(0,0,-1,1,-1,0,1)$
X_9	$(0,0,0,0,0,-1,1)$	X_{72}	$(0,0,0,0,0,1,-1)$
X_{10}	$(0,0,1,0,-1,1,-1)$	X_{73}	$(0,0,-1,0,1,-1,1)$
X_{11}	$(0,1,-1,0,1,0,0)$	X_{74}	$(0,-1,1,0,-1,0,0)$
X_{12}	$(0,0,1,0,0,-1,-1)$	X_{75}	$(0,0,-1,0,0,1,1)$
X_{13}	$(0,1,-1,1,-1,1,0)$	X_{76}	$(0,-1,1,-1,1,-1,0)$
X_{14}	$(1,-1,0,0,1,0,0)$	X_{77}	$(-1,1,0,0,-1,0,0)$
X_{15}	$(-1,0,0,0,1,0,0)$	X_{78}	$(1,0,0,0,-1,0,0)$
X_{16}	$(0,1,-1,1,0,-1,0)$	X_{79}	$(0,-1,1,-1,0,1,0)$
X_{17}	$(0,1,0,-1,0,1,0)$	X_{80}	$(0,-1,0,1,0,-1,0)$
X_{18}	$(1,-1,0,1,-1,1,0)$	X_{81}	$(-1,1,0,-1,1,-1,0)$
X_{19}	$(-1,0,0,1,-1,1,0)$	X_{82}	$(1,0,0,-1,1,-1,0)$
X_{20}	$(0,1,0,-1,1,-1,0)$	X_{83}	$(0,-1,0,1,-1,1,0)$
X_{21}	$(1,-1,0,1,0,-1,0)$	X_{84}	$(-1,1,0,-1,0,1,0)$
X_{22}	$(1,-1,1,-1,0,1,0)$	X_{85}	$(-1,1,-1,1,0,-1,0)$
X_{23}	$(-1,0,0,1,0,-1,0)$	X_{86}	$(1,0,0,-1,0,1,0)$
X_{24}	$(-1,0,1,-1,0,1,0)$	X_{87}	$(1,0,-1,1,0,-1,0)$
X_{25}	$(0,1,0,0,-1,0,0)$	X_{88}	$(0,-1,0,0,1,0,0)$
X_{26}	$(1,-1,1,-1,1,-1,0)$	X_{89}	$(-1,1,-1,1,-1,1,0)$

Table 12: First set of roots of E_7 and assocaited fields.

Field	Positive root	Field	Negative root
X_{27}	$(1,0,-1,0,0,1,1)$	X_{90}	$(-1,0,1,0,0,-1,-1)$
X_{28}	$(-1,0,1,-1,1,-1,0)$	X_{91}	$(1,0,-1,1,-1,1,0)$
X_{29}	$(-1,1,-1,0,0,1,1)$	X_{92}	$(1,-1,1,0,0,-1,-1)$
X_{30}	$(1,-1,1,0,-1,0,0)$	X_{93}	$(-1,1,-1,0,1,0,0)$
X_{31}	$(1,0,-1,0,1,-1,1)$	X_{94}	$(-1,0,1,0,-1,1,-1)$
X_{32}	$(1,0,0,0,0,1,-1)$	X_{95}	$(-1,0,0,0,0,-1,1)$
X_{33}	$(-1,0,1,0,-1,0,0)$	X_{96}	$(1,0,-1,0,1,0,0)$
X_{34}	$(-1,1,-1,0,1,-1,1)$	X_{97}	$(1,-1,1,0,-1,1,-1)$
X_{35}	$(-1,1,0,0,0,1,-1)$	X_{98}	$(1,-1,0,0,0,-1,1)$
X_{36}	$(0,-1,0,0,0,1,1)$	X_{99}	$(0,1,0,0,0,-1,-1)$
X_{37}	$(1,0,-1,1,-1,0,1)$	X_{100}	$(-1,0,1,-1,1,0,-1)$
X_{38}	$(1,0,0,0,1,-1,-1)$	X_{101}	$(-1,0,0,0,-1,1,1)$
X_{39}	$(-1,1,-1,1,-1,0,1)$	X_{102}	$(1,-1,1,-1,1,0,-1)$
X_{40}	$(-1,1,0,0,1,-1,-1)$	X_{103}	$(1,-1,0,0,-1,1,1)$
X_{41}	$(0,-1,0,0,1,-1,1)$	X_{104}	$(0,1,0,0,-1,1,-1)$
X_{42}	$(0,-1,1,0,0,1,-1)$	X_{105}	$(0,1,-1,0,0,-1,1)$
X_{43}	$(1,0,0,-1,0,0,1)$	X_{106}	$(-1,0,0,1,0,0,-1)$
X_{44}	$(1,0,0,1,-1,0,-1)$	X_{107}	$(-1,0,0,-1,1,0,1)$
X_{45}	$(-1,1,0,-1,0,0,1)$	X_{108}	$(1,-1,0,1,0,0,-1)$
X_{46}	$(-1,1,0,1,-1,0,-1)$	X_{109}	$(1,-1,0,-1,1,0,1)$
X_{47}	$(0,-1,0,1,-1,0,1)$	X_{110}	$(0,1,0,-1,1,0,-1)$
X_{48}	$(0,-1,1,0,1,-1,-1)$	X_{111}	$(0,1,-1,0,-1,1,1)$
X_{49}	$(0,0,-1,1,0,1,0)$	X_{112}	$(0,0,1,-1,0,-1,0)$
X_{50}	$(1,0,1,-1,0,0,-1)$	X_{113}	$(-1,0,-1,1,0,0,1)$
X_{51}	$(-1,1,1,-1,0,0,-1)$	X_{114}	$(1,-1,-1,1,0,0,1)$
X_{52}	$(0,-1,1,-1,0,0,1)$	X_{115}	$(0,1,-1,1,0,0,-1)$

Table 13: Second set of roots of E_7 and associated fields.

Field	Positive root	Field	Negative root
X_{53}	$(0,-1,1,1,-1,0,-1)$	X_{116}	$(0,1,-1,-1,1,0,1)$
X_{54}	$(0,0,-1,1,1,-1,0)$	X_{117}	$(0,0,1,-1,-1,1,0)$
X_{55}	$(0,0,0,-1,1,1,0)$	X_{118}	$(0,0,0,1,-1,-1,0)$
X_{56}	$(1,1,-1,0,0,0,0)$	X_{119}	$(-1,-1,1,0,0,0,0)$
X_{57}	$(-1,2,-1,0,0,0,0)$	X_{120}	$(1,-2,1,0,0,0,0)$
X_{58}	$(0,-1,2,-1,0,0,-1)$	X_{121}	$(0,1,-2,1,0,0,1)$
X_{59}	$(0,0,-1,0,0,0,2)$	X_{122}	$(0,0,1,0,0,0,-2)$
X_{60}	$(0,0,-1,2,-1,0,0)$	X_{123}	$(0,0,1,-2,1,0,0)$
X_{61}	$(0,0,0,-1,2,-1,0)$	X_{124}	$(0,0,0,1,-2,1,0)$
X_{62}	$(0,0,0,0,-1,2,0)$	X_{125}	$(0,0,0,0,1,-2,0)$
X_{63}	$(2,-1,0,0,0,0,0)$	X_{126}	$(-2,1,0,0,0,0,0)$

Table 14: Third set of roots of E_7 and associated fields.

6.2. Superpotential

Plugging into the general expression for the mirror superpotential, we find for this case that the mirror superpotential is given by

$$\begin{aligned} W = {} & \sum_{a=1}^{7} \sigma_a \left(\sum_{i=1}^{n} \sum_{\beta=1}^{56} \rho_{i,\beta}^a Y_{i,\beta} + \sum_{m=1}^{126} \alpha_m^a Z_m \right) \\ & - \sum_{i=1}^{n} \tilde{m}_i \sum_{\beta=1}^{56} Y_{i,\beta} + \sum_{i=1}^{n} \sum_{\beta=1}^{56} \exp(-Y_{i,\beta}) + \sum_{m=1}^{126} X_m. \end{aligned}$$

where $X_m = \exp(-Z_m)$ and X_m are the fundamental fields, we get:

$$W = \sum_{a=1}^{7} \sigma_a \mathcal{C}^a - \sum_{i=1}^{n} \tilde{m}_i \sum_{\beta=1}^{56} Y_{i,\beta} + \sum_{i=1}^{n} \sum_{\beta=1}^{56} \exp(-Y_{i,\beta}) + \sum_{m=1}^{126} X_m. \tag{6.1}$$

where $\mathcal{C}^a$ are given in [7].

6.3. Coulomb ring relations

Integrating out the σ_a fields, we obtain seven constraints $\mathcal{C}^a = 0$. Exponentiating these constraints, we obtain a series of equations from which the Coulomb ring relations will be derived. For reasons of notational sanity, we

Field	Weight	Field	Weight
$Y_{i,1}$	$(0,0,0,0,0,1,0)$	$Y_{i,29}$	$(0,0,0,0,0,-1,0)$
$Y_{i,2}$	$(0,0,0,0,1,-1,0)$	$Y_{i,30}$	$(0,0,0,0,-1,1,0)$
$Y_{i,3}$	$(0,0,0,1,-1,0,0)$	$Y_{i,31}$	$(0,0,0,-1,1,0,0)$
$Y_{i,4}$	$(0,0,1,-1,0,0,0)$	$Y_{i,32}$	$(0,0,-1,1,0,0,0)$
$Y_{i,5}$	$(0,1,-1,0,0,0,1)$	$Y_{i,33}$	$(0,-1,1,0,0,0,-1)$
$Y_{i,6}$	$(0,1,0,0,0,0,-1)$	$Y_{i,34}$	$(0,-1,0,0,0,0,1)$
$Y_{i,7}$	$(1,-1,0,0,0,0,1)$	$Y_{i,35}$	$(-1,1,0,0,0,0,-1)$
$Y_{i,8}$	$(-1,0,0,0,0,0,1)$	$Y_{i,36}$	$(1,0,0,0,0,0,-1)$
$Y_{i,9}$	$(1,-1,1,0,0,0,-1)$	$Y_{i,37}$	$(-1,1,-1,0,0,0,1)$
$Y_{i,10}$	$(-1,0,1,0,0,0,-1)$	$Y_{i,38}$	$(1,0,-1,0,0,0,1)$
$Y_{i,11}$	$(1,0,-1,1,0,0,0)$	$Y_{i,39}$	$(-1,0,1,-1,0,0,0)$
$Y_{i,12}$	$(-1,1,-1,1,0,0,0)$	$Y_{i,40}$	$(1,-1,1,-1,0,0,0)$
$Y_{i,13}$	$(1,0,0,-1,1,0,0)$	$Y_{i,41}$	$(-1,0,0,1,-1,0,0)$
$Y_{i,14}$	$(-1,1,0,-1,1,0,0)$	$Y_{i,42}$	$(1,-1,0,1,-1,0,0)$
$Y_{i,15}$	$(0,-1,0,1,0,0,0)$	$Y_{i,43}$	$(0,1,0,-1,0,0,0)$
$Y_{i,16}$	$(1,0,0,0,-1,1,0)$	$Y_{i,44}$	$(-1,0,0,0,1,-1,0)$
$Y_{i,17}$	$(-1,1,0,0,-1,1,0)$	$Y_{i,45}$	$(1,-1,0,0,1,-1,0)$
$Y_{i,18}$	$(0,-1,1,-1,1,0,0)$	$Y_{i,46}$	$(0,1,-1,1,-1,0,0)$
$Y_{i,19}$	$(1,0,0,0,0,-1,0)$	$Y_{i,47}$	$(-1,0,0,0,0,1,0)$
$Y_{i,20}$	$(-1,1,0,0,0,-1,0)$	$Y_{i,48}$	$(1,-1,0,0,0,1,0)$
$Y_{i,21}$	$(0,-1,1,0,-1,1,0)$	$Y_{i,49}$	$(0,1,-1,0,1,-1,0)$
$Y_{i,22}$	$(0,0,-1,0,1,0,1)$	$Y_{i,50}$	$(0,0,1,0,-1,0,-1)$
$Y_{i,23}$	$(0,-1,1,0,0,-1,0)$	$Y_{i,51}$	$(0,1,-1,0,0,1,0)$
$Y_{i,24}$	$(0,0,-1,1,-1,1,1)$	$Y_{i,52}$	$(0,0,1,-1,1,-1,-1)$
$Y_{i,25}$	$(0,0,0,0,1,0,-1)$	$Y_{i,53}$	$(0,0,0,0,-1,0,1)$
$Y_{i,26}$	$(0,0,-1,1,0,-1,1)$	$Y_{i,54}$	$(0,0,1,-1,0,1,-1)$
$Y_{i,27}$	$(0,0,0,-1,0,1,1)$	$Y_{i,55}$	$(0,0,0,1,0,-1,-1)$
$Y_{i,28}$	$(0,0,0,1,-1,1,-1)$	$Y_{i,56}$	$(0,0,0,-1,1,-1,1)$

Table 15: Weights of **56** of E_7 and associated fields.

will also slightly simplify these expressions, as follows. To make predictions for the A model, we will evaluate the ring relations on the critical locus, where

$$\frac{X_m}{X_{m+63}} = -1.$$

It is straightforward to see that each of the constraints $\mathcal{C}^a$ contains 34 differences of corresponding Z's, so that the exponential of the constraints contains 34 factors of the form X_m/X_{m+63} – an even number of factors of -1, which will cancel out. Therefore, since on the critical locus those factors will cancel out, we will omit them, and solely relate the exponentiated constraints in terms of Ys.

The exponentiated constraints are listed in [7].

The mirror map is given by,

$$\exp(-Y_{i,\beta}) \mapsto -\tilde{m}_i + \sum_{a=1}^{7} \sigma_a \rho^a_{i,\beta}, \quad X_m \mapsto \sum_{a=1}^{7} \sigma_a \alpha^a_m.$$

After applying the mirror map, the constraints adopt the form listed in [7].

Part of the excluded locus is defined by the condition that the $X_m \neq 0$. This part of the excluded locus is encoded by a relation listed in [7]. The other part of the excluded locus is determined by the fact that $\exp(-Y) \neq 0$. Since on the critical locus,

$$\exp(-Y_{i,\beta}) = -\tilde{m}_i + \sum_{a=1}^{7} \sigma_a \rho^a_{i,\beta},$$

so

$$-\tilde{m}_i + \sum_{a=1}^{7} \sigma_a \rho^a_{i,\beta} \neq 0,$$

which is given more explicitly in [7].

6.4. Pure gauge theory

In this part we will consider the mirror to the pure E_7 gauge theory. The mirror superpotential is given in [7].

Now, we can proceed as in previous sections. For the reasons discussed there, since each σ is multiplied by both Z_μ and $Z_{-\mu}$ with opposite signs, the critical locus equations

$$\frac{\partial W}{\partial X_\mu} = 0$$

imply that on the critical locus,

$$X_\mu = -X_{-\mu}. \tag{6.2}$$

(Furthermore, on the critical locus, each X_μ is determined by σs.) In addition, each derivative

$$\frac{\partial W}{\partial \sigma_a}$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check in the superpotential above that each σ_a is multiplied by an even number of such ratios (*i.e.* the number of Z's is a multiple of four). Specifically, the sum of the absolute values of the Z's multiplying each σ is $68 = 4 \cdot 17$. Thus, the constraint implied by the σ's is automatically satisfied.

As a result, following the same analysis in [3], we see in this case, that the critical locus is nonempty, and in fact is determined by the seven σs. In other words, at the level of these topological field theory computations, we have evidence that the pure supersymmetric E_7 gauge theory in two dimensions flows in the IR to a theory of seven free twisted chiral superfields.

7. E_8

In this section, we will discuss the mirror theory to a two-dimensional $_8$ gauge theory. The group E_8 and its algebra are the largest and most complicated exceptional groups, we shall only list results.

7.1. Mirror Landau-Ginzburg orbifold

We will consider an E_8 gauge theory with n matter fields in the **248**, the lowest-dimensional representation, which also happens to be the adjoint representation. The mirror Landau-Ginzburg model has fields

- $Y_{i\alpha}$, $i \in \{1, \ldots, n\}$, $\alpha \in \{1, \ldots, 248\}$
- X_m, $m \in \{1, 2, \ldots, 120\}$, correponding to positive roots, and X_{120+m}, associated with the negative roots of those associated to X_m,
- σ_a, $a \in \{1, 2, \ldots, 8\}$.

As before, we work with an integer-lattice-basis for the roots and weights, corresponding to standard theta angle periodicities. We associate the roots and weights to fields as listed in the Tables 16, 17, and 18.

For the rest of the fields, the roots and weights are given by

$$X_{a+120} = -X_a, \quad a = 1, \ldots, 120,$$
$$Y_{i,a+120} = -Y_{i,120}, \quad a = 1, \ldots, 120.$$

Field	Positive root/weight	Field	Positive root/weight
X_1, $Y_{i,1}$,	$(0,0,0,0,0,0,1,0)$	X_2, $Y_{i,2}$	$(0,0,0,0,0,1,-1,0)$
X_3, $Y_{i,3}$	$(0,0,0,0,1,-1,0,0)$	X_4, $Y_{i,4}$	$(0,0,0,1,-1,0,0,0)$
X_5, $Y_{i,5}$	$(0,0,1,-1,0,0,0,0)$	X_6, $Y_{i,6}$	$(0,1,-1,0,0,0,0,1)$
X_7, $Y_{i,7}$	$(0,1,0,0,0,0,0,-1)$	X_8, $Y_{i,8}$	$(1,-1,0,0,0,0,0,1)$
X_9, $Y_{i,9}$	$(-1,0,0,0,0,0,0,1)$	X_{10}, $Y_{i,10}$	$(1,-1,1,0,0,0,0,-1)$
X_{11}, $Y_{i,11}$	$(-1,0,1,0,0,0,0,-1)$	X_{12}, $Y_{i,12}$	$(1,0,-1,1,0,0,0,0)$
X_{13}, $Y_{i,13}$	$(-1,1,-1,1,0,0,0,0)$	X_{14}, $Y_{i,14}$	$(1,0,0,-1,1,0,0,0)$
X_{15}, $Y_{i,15}$	$(-1,1,0,-1,1,0,0,0)$	X_{16}, $Y_{i,16}$	$(0,-1,0,1,0,0,0,0)$
X_{17}, $Y_{i,17}$	$(1,0,0,0,-1,1,0,0)$	X_{18}, $Y_{i,18}$	$(-1,1,0,0,-1,1,0,0)$,
X_{19}, $Y_{i,19}$	$(0,-1,1,-1,1,0,0,0)$	X_{20}, $Y_{i,20}$	$(1,0,0,0,0,-1,1,0)$
X_{21}, $Y_{i,21}$	$(-1,1,0,0,0,-1,1,0)$	X_{22}, $Y_{i,22}$	$(0,-1,1,0,-1,1,0,0)$
X_{23}, $Y_{i,23}$	$(0,0,-1,0,1,0,0,1)$	X_{24}, $Y_{i,24}$	$(1,0,0,0,0,0,-1,0)$
X_{25}, $Y_{i,25}$	$(-1,1,0,0,0,0,-1,0)$	X_{26}, $Y_{i,26}$	$(0,-1,1,0,0,-1,1,0)$
X_{27}, $Y_{i,27}$	$(0,0,-1,1,-1,1,0,1)$	X_{28}, $Y_{i,28}$	$(0,0,0,0,1,0,0,-1)$
X_{29}, $Y_{i,29}$	$(0,-1,1,0,0,0,-1,0)$	X_{30}, $Y_{i,30}$	$(0,0,-1,1,0,-1,1,1)$
X_{31}, $Y_{i,31}$	$(0,0,0,-1,0,1,0,1)$	X_{32}, $Y_{i,32}$	$(0,0,0,1,-1,1,0,-1)$
X_{33}, $Y_{i,33}$	$(0,0,-1,1,0,0,-1,1)$	X_{34}, $Y_{i,34}$	$(0,0,0,-1,1,-1,1,1)$
X_{35}, $Y_{i,35}$	$(0,0,0,1,0,-1,1,-1)$	X_{36}, $Y_{i,36}$	$(0,0,1,-1,0,1,0,-1)$
X_{37}, $Y_{i,37}$	$(0,0,0,-1,1,0,-1,1)$	X_{38}, $Y_{i,38}$	$(0,0,0,0,-1,0,1,1)$
X_{39}, $Y_{i,39}$	$(0,0,0,1,0,0,-1,-1)$	X_{40}, $Y_{i,40}$	$(0,0,1,-1,1,-1,1,-1)$
X_{41}, $Y_{i,41}$	$(0,1,-1,0,0,1,0,0)$	X_{42}, $Y_{i,42}$	$(0,0,0,0,-1,1,-1,1)$
X_{43}, $Y_{i,43}$	$(0,0,1,-1,1,0,-1,-1)$	X_{44}, $Y_{i,44}$	$(0,0,1,0,-1,0,1,-1)$
X_{45}, $Y_{i,45}$	$(0,1,-1,0,1,-1,1,0)$	X_{46}, $Y_{i,46}$	$(1,-1,0,0,0,1,0,0)$
X_{47}, $Y_{i,47}$	$(-1,0,0,0,0,1,0,0)$	X_{48}, $Y_{i,48}$	$(0,0,0,0,0,-1,0,1)$

Table 16: First set of roots of E_8 and associated fields.

Field	Positive root/weight	Field	Positive root/weight
X_{49}, $Y_{i,49}$	$(0,0,1,0,-1,1,-1,-1)$	X_{50}, $Y_{i,50}$	$(0,1,-1,0,1,0,-1,0)$
X_{51}, $Y_{i,51}$	$(0,1,-1,1,-1,0,1,0)$	X_{52}, $Y_{i,52}$	$(1,-1,0,0,1,-1,1,0)$
X_{53}, $Y_{i,53}$	$(-1,0,0,0,1,-1,1,0)$	X_{54}, $Y_{i,54}$	$(0,0,1,0,0,-1,0,-1)$
X_{55}, $Y_{i,55}$	$(0,1,-1,1,-1,1,-1,0)$	X_{56}, $Y_{i,56}$	$(0,1,0,-1,0,0,1,0)$
X_{57}, $Y_{i,57}$	$(1,-1,0,0,1,0,-1,0)$	X_{58}, $Y_{i,58}$	$(1,-1,0,1,-1,0,1,0)$
X_{59}, $Y_{i,59}$	$(-1,0,0,0,1,0,-1,0)$	X_{60}, $Y_{i,60}$	$(-1,0,0,1,-1,0,1,0)$
X_{61}, $Y_{i,61}$	$(0,1,-1,1,0,-1,0,0)$	X_{62}, $Y_{i,62}$	$(0,1,0,-1,0,1,-1,0)$
X_{63}, $Y_{i,63}$	$(1,-1,0,1,-1,1,-1,0)$	X_{64}, $Y_{i,64}$	$(1,-1,1,-1,0,0,1,0)$
X_{65}, $Y_{i,65}$	$(-1,0,0,1,-1,1,-1,0)$	X_{66}, $Y_{i,66}$	$(-1,0,1,-1,0,0,1,0)$
X_{67}, $Y_{i,67}$	$(0,1,0,-1,1,-1,0,0)$	X_{68}, $Y_{i,68}$	$(1,-1,0,1,0,-1,0,0)$
X_{69}, $Y_{i,69}$	$(1,-1,1,-1,0,1,-1,0)$	X_{70}, $Y_{i,70}$	$(1,0,-1,0,0,0,1,1)$
X_{71}, $Y_{i,71}$	$(-1,0,0,1,0,-1,0,0)$	X_{72}, $Y_{i,72}$	$(-1,0,1,-1,0,1,-1,0)$
X_{73}, $Y_{i,73}$	$(-1,1,-1,0,0,0,1,1)$	X_{74}, $Y_{i,74}$	$(0,1,0,0,-1,0,0,0)$
X_{75}, $Y_{i,75}$	$(1,-1,1,-1,1,-1,0,0)$	X_{76}, $Y_{i,76}$	$(1,0,-1,0,0,1,-1,1)$
X_{77}, $Y_{i,77}$	$(1,0,0,0,0,0,1,-1)$	X_{78}, $Y_{i,78}$	$(-1,0,1,-1,1,-1,0,0)$
X_{79}, $Y_{i,79}$	$(-1,1,-1,0,0,1,-1,1)$	X_{80}, $Y_{i,80}$	$(-1,1,0,0,0,0,1,-1)$
X_{81}, $Y_{i,81}$	$(0,-1,0,0,0,0,1,1)$	X_{82}, $Y_{i,82}$	$(1,-1,1,0,-1,0,0,0)$
X_{83}, $Y_{i,83}$	$(1,0,-1,0,1,-1,0,1)$	X_{84}, $Y_{i,84}$	$(1,0,0,0,0,1,-1,-1)$
X_{85}, $Y_{i,85}$	$(-1,0,1,0,-1,0,0,0)$	X_{86}, $Y_{i,86}$	$(-1,1,-1,0,1,-1,0,1)$
X_{87}, $Y_{i,87}$	$(-1,1,0,0,0,1,-1,-1)$	X_{88}, $Y_{i,88}$	$(0,-1,0,0,0,1,-1,1)$
X_{89}, $Y_{i,89}$	$(0,-1,1,0,0,0,1,-1)$	X_{90}, $Y_{i,90}$	$(1,0,-1,1,-1,0,0,1)$
X_{91}, $Y_{i,91}$	$(1,0,0,0,1,-1,0,-1)$	X_{92}, $Y_{i,92}$	$(-1,1,-1,1,-1,0,0,1)$
X_{93}, $Y_{i,93}$	$(-1,1,0,0,1,-1,0,-1)$	X_{94}, $Y_{i,94}$	$(0,-1,0,0,1,-1,0,1)$

Table 17: Second set of roots of E_8 and associated fields.

7.2. Superpotential

In this section, we give the superpotential for the Landau-Ginzburg orbifold mirror to the theory above.

$$W = \sum_{a=1}^{8} \sigma_a \mathcal{C}^a - \sum_{i=1}^{n} \tilde{m}_i \sum_{\alpha=1}^{248} Y_{i,\alpha} + \sum_{i=1}^{n} \sum_{\alpha=1}^{248} \exp(-Y_{i,\alpha}) + \sum_{m=1}^{240} X_m, \tag{7.1}$$

where the $\mathcal{C}^a$ are given in [7].

Field	Positive root/weight	Field	Positive root/weight
X_{95}, $Y_{i,95}$	$(0,-1,1,0,0,1,-1,-1)$	X_{96}, $Y_{i,96}$	$(0,0,-1,1,0,0,1,0)$
X_{97}, $Y_{i,97}$	$(1,0,0,-1,0,0,0,1)$	X_{98}, $Y_{i,98}$	$(1,0,0,1,-1,0,0,-1)$
X_{99}, $Y_{i,99}$	$(-1,1,0,-1,0,0,0,1)$	X_{100}, $Y_{i,100}$	$(-1,1,0,1,-1,0,0,-1)$
X_{101}, $Y_{i,101}$	$(0,-1,0,1,-1,0,0,1)$	X_{102}, $Y_{i,102}$	$(0,-1,1,0,1,-1,0,-1)$
X_{103}, $Y_{i,103}$	$(0,0,-1,1,0,1,-1,0)$	X_{104}, $Y_{i,104}$	$(0,0,0,-1,1,0,1,0)$
X_{105}, $Y_{i,105}$	$(1,0,1,-1,0,0,0,-1)$	X_{106}, $Y_{i,106}$	$(-1,1,1,-1,0,0,0,-1)$
X_{107}, $Y_{i,107}$	$(0,-1,1,-1,0,0,0,1)$	X_{108}, $Y_{i,108}$	$(0,-1,1,1,-1,0,0,-1)$
X_{109}, $Y_{i,109}$	$(0,0,-1,1,1,-1,0,0)$	X_{110}, $Y_{i,110}$	$(0,0,0,-1,1,1,-1,0)$
X_{111}, $Y_{i,111}$	$(0,0,0,0,-1,1,1,0)$	X_{112}, $Y_{i,112}$	$(1,1,-1,0,0,0,0,0)$
X_{113}, $Y_{i,113}$	$(-1,2,-1,0,0,0,0,0)$	X_{114}, $Y_{i,114}$	$(0,-1,2,-1,0,0,0,-1)$
X_{115}, $Y_{i,115}$	$(0,0,-1,0,0,0,0,2)$	X_{116}, $Y_{i,116}$	$(0,0,-1,2,-1,0,0,0)$
X_{117}, $Y_{i,117}$	$(0,0,0,-1,2,-1,0,0)$	X_{118}, $Y_{i,118}$	$(0,0,0,0,-1,2,-1,0)$
X_{119}, $Y_{i,119}$	$(0,0,0,0,0,-1,2,0)$	X_{120}, $Y_{i,120}$	$(2,-1,0,0,0,0,0,0)$
$Y_{i,241}$,$Y_{i,242}$	$(0,0,0,0,0,0,0,0)$	$Y_{i,243}$,$Y_{i,244}$	$(0,0,0,0,0,0,0,0)$
$Y_{i,245}$,$Y_{i,246}$	$(0,0,0,0,0,0,0,0)$	$Y_{i,247}$,$Y_{i,248}$	$(0,0,0,0,0,0,0,0)$

Table 18: Third set of roots of E_8 and associated fields.

7.3. Coulomb ring relations

For the group E_8, we obtain eight Coulomb ring relations, using the same methods as before. The results for these ring relations are listed in [7].

7.4. Pure gauge theory

In this part we will consider the mirror to the pure E_8 gauge theory. For brevity, we will not rewrite the superpotential here, explicitly omitting Y fields, but instead merely refer to the expression (7.1) given earlier, leaving the reader to omit Y fields.

Now, let us consider the critical locus of the superpotential above. For each root μ, the fields X_μ and $X_{-\mu}$ appear paired with opoosite signs coupling to each σ. Therefore, one impliciation of the derivatives

$$\frac{\partial W}{\partial X_\mu} = 0$$

is that, on the critical locus,

$$X_\mu = -X_{-\mu}. \tag{7.2}$$

(Furthermore, on the critical locus, each X_μ is determined by σs.) Next, each derivative

$$\frac{\partial W}{\partial \sigma_a}$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check that in the superpotential above that each σ_a is multiplied by an even number of such ratios (*i.e.* the number of Z's is a multiple of four). Specifically, for each σ, the sum of the absolute values of the coefficients of the Z's multiplying it is $116 = 4 \cdot 29$. Thus, the constraint implied by the σ's is automatically satisfied.

As a result, following the same analysis as in [3] and previous sections, we see that the critical locus is nonempty, and in fact is determined by eight σs. In other words, at the level of these topological field theory computations, we have evidence that the pure supersymmetric E_8 gauge theory in two dimensions flows in the IR to a theory of eight free twisted chiral superfields.

8. Conclusions

In this paper we applied the recent nonabelian mirrors proposal [3] to examples of two-dimensional A-twisted gauge theories with exceptional gauge groups $G_2, F_4, E_{6,7,8}$. In each case, we explicitly compute the proposed mirror Landau-Ginzburg orbifold and derived the Coulomb ring relations (the analogue of quantum cohomology ring relations). In the cases of the G_2 and F_4 gauge theories, we studied the action of the Weyl group on the critical locus equations, which allowed us to perform consistency checks on the results here, and in the case of G_2, performed a detailed analysis of Weyl group orbits of the critical locis (vacua).

We also studied pure gauge theories with each gauge group, and provided evidence (at the level of these topological-field-theory-type computations) that each pure gauge theory (with simply-connected gauge group) flows in the IR to a free theory of as many twisted chiral multiplets as the rank of the gauge group.

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