Branched holomorphic Cartan geometry on Sasakian manifolds

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We extend the notion of (branched) holomorphic Cartan geometry on a complex manifold to the context of Sasakian manifolds. Branched holomorphic Cartan geometries on Sasakian Calabi-Yau manifolds are investigated in particular.

1. Introduction

Contact manifolds are the odd dimensional counterparts of the symplectic manifolds. Just as the total space of the cotangent bundle of a C^∞ manifold M is a typical (local) model of a symplectic manifold, the total space of the projective bundle $P(T^*M)$ is a typical (local) model of a contact manifold. In a similar vein, Sasakian manifolds are the odd dimensional counterparts of the Kähler manifolds. A compact regular Sasakian manifold is the unit circle bundle inside a holomorphic Hermitian line bundle of positive curvature on a complex projective manifold. More generally, a compact quasi-regular Sasakian manifold is the unit circle bundle inside a holomorphic Hermitian line bundle of positive curvature on a Kähler orbifold. The global structure of compact irregular Sasakian manifolds does not admit any such explicit description. Sasakian manifolds were introduced by Sasaki [Sa], [SH], which explains the terminology. In last fifteen years, there has been a substantial increase of the interest in Sasakian manifolds (see [BG2] and references therein). It is evident from the references that a very large part of this recent investigations into Sasakian manifolds were carried out by C. Boyer and K. Galicki. Another aspect of this recent activities in Sasakian manifolds is the discovery of their relevance in string theory. This was initiated in the works of J. Maldacena [Mal]. For further developments in this direction see [GMSW], [MSY], [MS] and references therein.

Let G be a complex Lie group and $H \subset G$ a complex Lie subgroup. A holomorphic Cartan geometry of type (G, H) is a complex manifold equipped with an infinitesimal structure that mimics the infinitesimal structure of the quotient manifold G/H. In particular, a flat Cartan geometry

of type (G, H) is a complex manifold equipped with local charts modeled on open subsets of G/H such that all the transition functions are given by the elements of G acting on G/H as left–multiplication. See [Sh] and references therein for Cartan geometry. It should be mentioned that Cartan geometry has close ties with theoretical physics; see [RS], [Ha], [AFL], [Ho] and references therein. Being motivated by the works of Mandelbaum [Man1], [Man2], the notion of a holomorphic Cartan geometry was enhanced to branched holomorphic Cartan geometry; this was done in [BD].

Our aim here is to develop an analog of holomorphic Cartan geometries and of branched holomorphic Cartan geometries in the context of Sasakian manifolds. Most of our efforts were spent in building the foundations. In the last section we investigate holomorphic Cartan geometries on compact quasi-regular Sasakian manifolds that are Calabi-Yau. We hope in future to investigate further this topic of holomorphic Cartan geometries on Sasakian manifolds.

2. Holomorphic principal bundles on Sasakian manifolds

2.1. Sasakian manifolds

Let X be a connected oriented smooth real manifold of dimension $2m + 1$, where m is a positive integer, and let g be a C^∞ Riemannian metric on X. The Levi-Civita connection on the real tangent bundle TX associated to g will be denoted by ∇.

Definition 2.1 ([BG1, Definition-Theorem 10]). The above Riemannian manifold (X, g) is called a *Sasakian manifold* if any of the following three equivalent conditions hold:

(i) There is a Killing vector field ξ on X of length one such that the section

$$(2.1) \qquad \Phi \in C^\infty(X, TX \otimes (TX)^*)$$

defined by $\Phi(v) = -\nabla_v \xi$ satisfies the identity

$$(2.2) \qquad (\nabla_v \Phi)(w) = g(v, w)\xi - g(\xi, w)v$$

for all $v, w \in T_x X$ and all $x \in X$.

(ii) There is a Killing vector field ξ on X of unit length such that the Riemann curvature tensor R of (X, g) satisfies the identity

$$R(v, \xi)w = g(\xi, w)v - g(v, w)\xi$$

for all v and w as above.

(iii) The metric cone $(\mathbb{R}_+ \times X, dr^2 \oplus r^2 \cdot g)$ is Kähler.

Given a Killing vector field ξ of unit length satisfying condition (i) in Definition 2.1, the Kähler structure on $\mathbb{R}_+ \times X$ asserted in statement (iii) is constructed as follows. Let F be the distribution on X of rank $2m$ given by the orthogonal complement $\xi^\perp$ of ξ. The homomorphism Φ (defined in (2.1)) preserves F, and furthermore,

(2.3) $$(\Phi|_F)^2 = -\mathrm{Id}_F.$$

Let $\widetilde{J}$ be the almost complex structure on $\mathbb{R}_+ \times X$ defined as follows:

$$\widetilde{J}|_F = \Phi|_F, \quad \widetilde{J}\left(\frac{d}{dr}\right) = \xi, \quad \widetilde{J}(\xi) = -\frac{d}{dr}.$$

This almost complex structure is in fact integrable. The Riemannian metric $dr^2 \oplus r^2 \cdot g$ on $\mathbb{R}_+ \times X$ is Kähler with respect to this complex structure $\widetilde{J}$ [BG1], [BG2].

Conversely, if the metric cone $(\mathbb{R}_+ \times X, dr^2 \oplus r^2 \cdot g)$ is Kähler, then consider the vector field on $\mathbb{R}_+ \times X$ given by $J(\frac{d}{dr})$, where J is the almost complex structure on $\mathbb{R}_+ \times X$. The vector field ξ on X obtained by restricting this vector field to $X = \{1\} \times X \subset \mathbb{R}_+ \times X$ satisfies both condition (i) and (ii) in Definition 2.1, with respect to the induced Riemannian metric g.

As the three conditions in Definition 2.1 are equivalent, the vector field ξ and the Kähler structure on $\mathbb{R}_+ \times X$ will be considered as part of the definition of a Sasakian manifold.

A connected Sasakian manifold (X, g, ξ) with X a compact manifold is called *quasi-regular* if all the orbits of the unit vector field ξ are closed. It (X, g, ξ) is not quasi-regular, then it is called an *irregular* Sasakian manifold. A quasi-regular connected Sasakian manifold (X, g, ξ) is called *regular* if the vector field ξ integrates into a free and faithful action of $S^1 = \mathrm{U}(1)$ on X.

We refer the reader to [BG2] for Sasakian manifolds.

2.2. Smooth principal bundles

Let M be a C^∞ manifold equipped with a C^∞ distribution $S \subset TM$ of rank r. Let H be a Lie group and $p : E_H \longrightarrow M$ a C^∞ principal H–bundle on M. A *partial connection* on E_H in the direction of S is a C^∞ distribution

$$\widetilde{S} \subset TE_H$$

of rank r such that

1) $\widetilde{S}$ is preserved by the action of H on E_H, and

2) the differential $dp : TE_H \longrightarrow p^*TM$ of p restricts to an isomorphism between $\widetilde{S}$ and the subbundle $p^*S \subset p^*TM$.

A partial connection $\widetilde{S}$ is called *integrable* if the distribution $\widetilde{S}$ is integrable. Note that if $\widetilde{S}$ is integrable, then S must also be integrable.

If $S = TM$, then a partial connection is a usual connection.

In general, S can be a complex distribution, meaning a subbundle of the complex vector bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. Note that the Lie bracket operation on the locally defined vector fields on M extends to a Lie bracket operation on the locally defined smooth sections of $TM \otimes_{\mathbb{R}} \mathbb{C}$. The notion of a partial connection extends to complex distributions in an obvious way.

Let $\rho : H \longrightarrow Q$ be a homomorphism of Lie groups. Let

$$E_Q := E_H \times^\rho Q \longrightarrow M$$

be the principal Q–bundle over M obtained by extending the structure group of E_H using the above homomorphism ρ. A partial connection $\widetilde{S}$ on E_H produces a partial connection on E_Q for the same distribution S on M. To see this, we recall that E_Q is the quotient of $E_H \times Q$ where two points (e_1, q_1) and (e_2, q_2) of $E_H \times Q$ are identified if there is an element $h \in H$ such that $e_2 = e_1 h$ and $q_2 = \rho(h^{-1})q_1$. Let $\widetilde{S}'$ be the distribution of rank r on $E_H \times Q$ given by the distribution $\widetilde{S}$ on E_H. More precisely, for any $(e_1, q_1) \in E_H \times Q$, the subspace

$$\widetilde{S}'(e_1, q_1) \subset T_{(e_1, q_1)}E_H \times Q = T_{e_1}E_H \oplus T_{q_1}Q$$

is $\widetilde{S}(e_1) \subset T_{e_1}E_H$. This distribution $\widetilde{S}'$ descends to a distribution on the quotient space E_Q of $E_H \times Q$ by the quotient map. The resulting distribution on E_Q is in fact a partial connection on E_Q for the distribution S.

Let $\mathcal{X} := (X, g, \xi)$ be a connected Sasakian manifold. A principal H–bundle on $\mathcal{X}$ is defined to be a C^∞ principal H–bundle on X equipped with a partial connection $\widetilde{\xi}$ for the one dimensional distribution $\mathbb{R} \cdot \xi$ on X.

For a principal bundle on X with structure group a complex Lie group H, it is always assumed that the fibers of the principal bundle are complex manifolds and the action of H on the principal bundle preserves the complex structure of the fibers of the principal bundle. To explain this condition, let H be a complex Lie group. Let $p : E_H \longrightarrow X$ be a principal H–bundle. This implies that

- the subbundle $\mathrm{kernel}(dp) \subset TE_H$ is equipped with a C^∞ automorphism $J_E : \mathrm{kernel}(dp) \longrightarrow \mathrm{kernel}(dp)$ such that $J_E \circ J_E = -\mathrm{Id}$,

- for every $x \in X$, the almost complex structure $J_E|_{p^{-1}(x)}$ on the fiber $p^{-1}(x)$ is integrable, and

- the action of H on $p^{-1}(x)$ is holomorphic for every $x \in X$.

Consider the differential $dp : TE_H \longrightarrow p^*TX$ of the projection p. Using J_E we get a decomposition of the complex vector bundle

$$\mathbb{K} := \mathrm{kernel}(dp) \otimes_\mathbb{R} \mathbb{C} \longrightarrow X$$

as follows. Define

$$(2.4) \quad \mathbb{K}^{0,1} := \{v + \sqrt{-1} \cdot J_E(v) \mid v \in \mathrm{kernel}(dp)\} \subset \mathbb{K} := \mathrm{kernel}(dp) \otimes_\mathbb{R} \mathbb{C}$$

and

$$(2.5) \qquad \mathbb{K}^{1,0} := \{v - \sqrt{-1} \cdot J_E(v) \mid v \in \mathrm{kernel}(dp)\} \subset \mathbb{K},$$

so we have $\mathbb{K} = \mathbb{K}^{0,1} \oplus \mathbb{K}^{1,0}$ and $\mathbb{K}^{1,0} = \overline{\mathbb{K}^{0,1}}$.

Let $\mathcal{X} := (X, g, \xi)$ be a connected Sasakian manifold. Let (E_H, p, J_E) be a principal H–bundle on X, where H is a complex Lie group. Let $\widetilde{\xi} \subset TE_H$ be a partial connection on E_H for the one dimensional distribution $\mathbb{R} \cdot \xi$ on X. Consider the (unique) vector field $\widehat{\xi}$ on E_H such that

- $\widehat{\xi}(z) \subset \widetilde{\xi}_z$ for all $z \in E_H$, and also

- $dp(\widehat{\xi}(z)) = \xi(p(z))$.

Note that the flow on E_H associated to the above vector field $\widehat{\xi}$ takes a fiber of p to another fiber of p. Therefore, the Lie derivative $L_{\widetilde{\xi}} J_E$ is a C^∞

endomorphism of kernel(dp) that anti-commutes with J_E, meaning $J_E \circ (L_{\widetilde{\xi}} J_E) + (L_{\widetilde{\xi}} J_E) \circ J_E = 0$.

A complex principal H–bundle on a Sasakian manifold $\mathcal{X} := (X,\, g,\, \xi)$ is a principal H–bundle $(E_H,\, p,\, J_E)$ on X, equipped with a partial connection $\widetilde{\xi}$ for the one dimensional distribution $\mathbb{R} \cdot \xi$ on X, such that

$$L_{\widetilde{\xi}} J_E = 0.$$

2.3. Holomorphic principal bundles

For a connected Sasakian manifold $\mathcal{X} := (X,\, g,\, \xi)$ of dimension $2m + 1$, consider F equipped with the almost complex structure $\Phi|_F$ (see (2.3)). Define

(2.6) $$F^{0,1} := \{v + \sqrt{-1} \cdot \Phi(v) \mid v \in F\} \subset F \otimes_{\mathbb{R}} \mathbb{C}$$

and

(2.7) $$F^{1,0} := \{v - \sqrt{-1} \cdot \Phi(v) \mid v \in F\} \subset F \otimes_{\mathbb{R}} \mathbb{C},$$

so $F \otimes_{\mathbb{R}} \mathbb{C} = F^{1,0} \oplus F^{0,1}$ and $F^{1,0} = \overline{F^{0,1}}$. We note that $F^{1,0}$ is a holomorphic Hermitian vector bundle over $\mathcal{X}$ (see [BS, p. 551, Definition 3.3] and [BS, p. 551, Definition 3.5] for definition of holomorphic Hermitian bundles over a Sasakian manifold). The partial connection on $F^{1,0}$ defining its holomorphic structure is given by the Lie bracket operation of vector fields; the Hermitian structure on $F^{1,0}$ is given by the Riemannian metric g on X.

The complex distribution on X

(2.8) $$\mathcal{F}^{0,1} := F^{0,1} \oplus \mathbb{C} \cdot \xi \subset TX \otimes_{\mathbb{R}} \mathbb{C}$$

of complex dimension $m + 1$ is integrable [BS, p. 550, Lemma 3.4].

Let H be a complex Lie group. Let $p : E_H \longrightarrow X$ be a principal H–bundle on X; as before, the almost complex structure on the fibers of E_H will be denoted by J_E. Let $\widetilde{\xi} \subset TE_H$ be a distribution such that $(E_H,\, p,\, J_E,\, \widetilde{\xi})$ is a complex principal H–bundle on $\mathcal{X}$. A *holomorphic structure* on $(E_H,\, p,\, J_E,\, \widetilde{\xi})$ is a complex distribution

$$\widetilde{\mathcal{F}} \subset TE_H \otimes_{\mathbb{R}} \mathbb{C}$$

of complex dimension $m + 1$ satisfying the following five conditions:

1) the distribution $\widetilde{\mathcal{F}}$ is integrable,

2) $\widetilde{\xi} \subset \widetilde{\mathcal{F}}$,

3) the complexified differential

$$dp \otimes_{\mathbb{R}} \mathbb{C} : TE_H \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow p^* TX \otimes_{\mathbb{R}} \mathbb{C}$$

of the projection p restricts to an isomorphism between $\widetilde{\mathcal{F}}$ and $p^* \mathcal{F}^{0,1}$ (defined in (2.8)),

4) the action of H on $TE_H \otimes_{\mathbb{R}} \mathbb{C}$, given by the action of H on E_H, preserves the subbundle $\widetilde{\mathcal{F}}$, and

5) the complex distribution $\widetilde{\mathcal{F}} \oplus \mathbb{K}^{0,1}$ (see (2.4)) on E_H is integrable.

In particular, $\widetilde{\mathcal{F}}$ is a partial connection on E_H for the complex distribution $\mathcal{F}^{0,1}$ on X.

A holomorphic principal H–bundle on $\mathcal{X}$ is a complex principal H–bundle on $\mathcal{X}$ equipped with a holomorphic structure.

Let

(2.9) $$\rho : H \longrightarrow Q$$

be a holomorphic homomorphism of complex Lie groups. Take a holomorphic principal H–bundle $(E_H, p, J_E, \widetilde{\xi}, \widetilde{\mathcal{F}})$ on $\mathcal{X}$. Let

$$E_Q := E_H \times^\rho Q \xrightarrow{p'} X$$

be the principal Q–bundle over X obtained by extending the structure group of E_H using the above homomorphism ρ. The complex structure J_E on the fibers of E_H and the complex structure of Q together produce a complex structure J_{E_Q} on the fibers of E_Q. As noted before, a partial connection on E_H produces a partial connection on E_Q. Therefore, the partial connections $\widetilde{\xi}$ and $\widetilde{\mathcal{F}}$ produce partial connections on E_Q for the distributions ξ and $\mathcal{F}^{0,1}$ respectively. Let $\widetilde{\xi}_Q$ and $\widetilde{\mathcal{F}}_Q$ denote these partial connections on E_Q for the distributions ξ and $\mathcal{F}^{0,1}$ respectively. Now it is straight-forward to check that

(2.10) $$(E_Q, p', J_{E_Q}, \widetilde{\xi}_Q, \widetilde{\mathcal{F}}_Q)$$

is a holomorphic principal Q–bundle on $\mathcal{X}$.

Set $H = \mathrm{GL}(r, \mathbb{C})$. Take a holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle $(E, p, J_E, \widetilde{\xi}, \widetilde{\mathcal{F}})$ on $\mathcal{X}$. Then the vector bundle $E \times^{\mathrm{GL}(r,\mathbb{C})} \mathbb{C}^r$ associated to E for the standard representation of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$ is a holomorphic vector bundle

over $\mathcal{X}$ (see [BS] for holomorphic vector bundles on Sasakian manifolds). Conversely, if V is a holomorphic vector bundle on $\mathcal{X}$ of rank r, then the frame bundle for V is a holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle over $\mathcal{X}$.

For a general complex Lie group H, assume that Q in (2.9) is $\mathrm{GL}(r, \mathbb{C})$. Consequently, $(E_Q, p, J_{E_Q}, \widetilde{\xi}_Q, \widetilde{\mathcal{F}}_Q)$ in (2.10) produces a holomorphic vector bundle on $\mathcal{X}$ of rank r.

Now take ρ to be the adjoint representation of H in its Lie algebra $\mathrm{Lie}(H)$. Then the corresponding holomorphic vector bundle $(E_Q, p', J_{E_Q}, \widetilde{\xi}_Q, \widetilde{\mathcal{F}}_Q)$ on $\mathcal{X}$ will be called the adjoint bundle for E_H, and it will be denoted by $\mathrm{ad}(E_H)$. We note that

$$(2.11) \qquad\qquad \mathrm{ad}(E_H) = \mathbb{K}^{1,0}/H,$$

where $\mathbb{K}^{1,0}$ is constructed in (2.5); the isomorphism in (2.11) is obtained from the action of H on E_H that identifies $\mathbb{K}^{1,0}$ with the trivial vector bundle on E_H with fiber $\mathrm{Lie}(H)$.

Given holomorphic vector bundles W_1 and W_2 on $\mathcal{X}$, holomorphic homomorphisms from W_1 to W_2 are defined in [BS, p. 551, Definition 3.4]. We recall that a holomorphic homomorphism is a fiber-wise $\mathbb{C}$–linear C^∞ map

$$(2.12) \qquad\qquad \Psi : W_1 \longrightarrow W_2$$

such that Ψ intertwines the partial connections on W_1 and W_2 in the direction of the distribution $\mathcal{F}^{0,1}$ defined in (2.8).

3. Holomorphic connections on a principal bundle

As before, $\mathcal{X} := (X, g, \xi)$ is a connected Sasakian manifold of dimension $2m + 1$. Consider the complex vector bundle $F^{1,0} \longrightarrow X$ in (2.7); the complex structure on the fibers of it is given by $\Phi|_F$ in (2.3). As noted before, $F^{1,0}$ is a holomorphic vector bundle on $\mathcal{X}$ of complex rank m; the partial connections are given by Lie derivative.

Let $(E_H, p, J_E, \widetilde{\xi}, \widetilde{\mathcal{F}})$ be a holomorphic principal H–bundle on $\mathcal{X}$. Consider the subbundle $F^{0,1} \subset \mathcal{F}^{0,1}$ in (2.8). Recall that the complexified differential $dp \otimes_{\mathbb{R}} \mathbb{C} : TE_H \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow p^*TX \otimes_{\mathbb{R}} \mathbb{C}$ of the projection p restricts to an isomorphism between $\widetilde{\mathcal{F}}$ and $p^*\mathcal{F}^{0,1}$. Let

$$\mathbb{F} \subset \widetilde{\mathcal{F}}$$

be the subbundle corresponding to $p^*F^{0,1}$ under this isomorphism between $\widetilde{\mathcal{F}}$ and $p^*\mathcal{F}^{0,1}$ given by $dp \otimes_{\mathbb{R}} \mathbb{C}$. Now define the complex distribution

$$(3.1) \qquad\qquad \mathbb{T} := \mathbb{K}^{1,0} \oplus \overline{\mathbb{F}} \subset TE_H \otimes_{\mathbb{R}} \mathbb{C}$$

on E_H, where $\mathbb{K}^{1,0}$ is defined in (2.5). Since the complexified differential $dp \otimes_{\mathbb{R}} \mathbb{C}$ gives an isomorphism between $\mathbb{F}$ and $p^*F^{0,1}$, it also gives an isomorphism between $\overline{\mathbb{F}}$ and $p^*F^{1,0}$ (defined in (2.7)). Note that the action of H on $TE_H \otimes_{\mathbb{R}} \mathbb{C}$, given by the action of H on E_H, preserves this subbundle $\mathbb{T}$ in (3.1). The quotient

$$(3.2) \qquad\qquad \mathrm{At}(E_H) := \mathbb{T}/H \longrightarrow E_H/H = X$$

is a complex vector bundle. Although the complex distribution $\mathbb{T}$ is not integrable, it is straightforward to check that the complex distribution $\mathbb{T} \oplus \mathbb{C} \cdot \xi$ on E_H is integrable. Using this it follows that $\mathrm{At}(E_H)$ is a holomorphic vector bundle over the Sasakian manifold $\mathcal{X}$. This $\mathrm{At}(E_H)$ will be called the *Atiyah bundle* for E_H. In view of (3.1), comparing (2.11) and (3.2) it follows that

$$(3.3) \qquad\qquad \mathrm{ad}(E_H) \subset \mathrm{At}(E_H);$$

this inclusion of $\mathrm{ad}(E_H)$ in $\mathrm{At}(E_H)$ is holomorphic. We shall now investigate the quotient vector bundle $\mathrm{At}(E_H)/\mathrm{ad}(E_H)$.

The isomorphism $(dp \otimes_{\mathbb{R}} \mathbb{C})|_{\overline{\mathbb{F}}}$ between $\overline{\mathbb{F}}$ and $p^*F^{1,0}$ and the zero homomorphism $\mathbb{K}^{1,0} \longrightarrow p^*F^{1,0}$ together produce a homomorphism

$$0 \oplus (dp \otimes_{\mathbb{R}} \mathbb{C})|_{\overline{\mathbb{F}}} : \mathbb{T} := \mathbb{K}^{1,0} \oplus \overline{\mathbb{F}} \longrightarrow p^*F^{1,0}.$$

The above homomorphism $0 \oplus (dp \otimes_{\mathbb{R}} \mathbb{C})|_{\overline{\mathbb{F}}}$ is H–equivariant, and hence it descends to a homomorphism

$$\mathrm{At}(E_H) := \mathbb{T}/H \xrightarrow{d'p} (p^*F^{1,0})/H = F^{1,0}.$$

This descended homomorphism $d'p$ is evidently surjective, and for the holomorphic subbundle $\mathrm{ad}(E_H)$ in (3.3) we have

$$d'p(\mathrm{ad}(E_H)) = 0.$$

We have the following short exact sequence of holomorphic vector bundles on $\mathcal{X}$:

$$(3.4) \qquad\qquad 0 \longrightarrow \mathrm{ad}(E_H) \longrightarrow \mathrm{At}(E_H) \xrightarrow{d'p} F^{1,0} \longrightarrow 0;$$

holomorphic homomorphisms of holomorphic vector bundles on $\mathcal{X}$ are defined in (2.12). The short exact sequence in (3.4) will be called the *Atiyah exact sequence* for E_H.

A *holomorphic connection* on E_H is a holomorphic homomorphism of vector bundles on $\mathcal{X}$

$$D : F^{1,0} \longrightarrow \mathrm{At}(E_H)$$

such that $(d'p) \circ D = \mathrm{Id}_{F^{1,0}}$, where $d'p$ is the projection in (3.4).

The above definition of a holomorphic connection on E_H is modeled on the definition of a holomorphic connection on a holomorphic principal bundle over a complex manifold (see [At]).

Using the orthogonal splitting of the real tangent bundle

$$TX = \mathbb{R} \cdot \xi \oplus \xi^{\perp} = \mathbb{R} \cdot \xi \oplus F,$$

we consider F^* as a subbundle of the real cotangent bundle of X. So $(F^{1,0})^*$ is a subbundle of $(TX)^* \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma 3.1. *Let D be a holomorphic connection on E_H. Then D defines a usual connection $\widetilde{D}$ on the C^{∞} principal H–bundle E_H. The curvature of $\widetilde{D}$ is a C^{∞} section of the vector bundle $\mathrm{ad}(E_H) \otimes \bigwedge^2 (F^{1,0})^*$.*

Proof. The homomorphism D gives a partial connection on E_H in the direction of $F = \xi^{\perp}$. So D and the given partial connection on E_H in the direction of ξ together produce a usual connection on E_H. The curvature of this connection is evidently a section of the vector bundle $\mathrm{ad}(E_H) \otimes \bigwedge^2 (F^{1,0})^*$. $\square$

The Reeb vector field ξ on X defines a flow on X while the vector field $\widetilde{\xi}$ on E_H defines a flow on E_H. The projection $p : E_H \longrightarrow X$ intertwines these two flows. The curvature of the connection $\widetilde{D}$ in Lemma 3.1 is actually preserved by this flow.

Definition 3.2. A holomorphic connection D on a holomorphic principal H–bundle E_H over the Sasakian manifold $\mathcal{X}$ will be called *flat* if the curvature of the corresponding usual connection $\widetilde{D}$ in Lemma 3.1 vanishes identically.

Let $\rho : H \longrightarrow Q$ be a holomorphic homomorphism of complex Lie groups. Consider the holomorphic principal Q–bundle $(E_Q, p', J_{E_Q}, \widetilde{\xi}_Q, \widetilde{\mathcal{F}}_Q)$ on $\mathcal{X}$

constructed in (2.10) from the holomorphic principal H–bundle E_H. Let

$$\rho' : \mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_Q)$$

be the homomorphism of holomorphic vector bundles given by the homomorphism of Lie algebras corresponding to the above homomorphism ρ of Lie groups. Consider the injective homomorphism of holomorphic vector bundles

$$\mathrm{ad}(E_H) \longrightarrow \mathrm{At}(E_H) \oplus \mathrm{ad}(E_Q), \quad v \longmapsto (-v, \rho'(v));$$

the above inclusion map $\mathrm{ad}(E_H) \hookrightarrow \mathrm{At}(E_H)$ is the one in (3.4). Then we have

$$\mathrm{At}(E_Q) = (\mathrm{At}(E_H) \oplus \mathrm{ad}(E_Q))/\mathrm{ad}(E_H).$$

If $D : F^{1,0} \longrightarrow \mathrm{At}(E_H)$ is a holomorphic connection on E_H, then the homomorphism

$$F^{1,0} \longrightarrow \mathrm{At}(E_H) \oplus \mathrm{ad}(E_Q), \quad v \longmapsto (D(v), 0)$$

descends to a homomorphism $F^{1,0} \longrightarrow \mathrm{At}(E_Q)$ that defines a holomorphic connection on the principal Q–bundle E_Q over $\mathcal{X}$; it is called the holomorphic connection on E_Q induced by D.

From Lemma 3.1 we know that the above holomorphic connection on E_Q induced by D defines a usual connection on E_Q. This (usual) connection on E_Q clearly coincides with the connection on E_Q induced by the connection $\widetilde{D}$ in Lemma 3.1 given by D.

4. Branched holomorphic Cartan geometry

4.1. Definitions

Let G be a connected complex Lie group and $H \subset G$ a complex Lie subgroup. The Lie algebras of G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Take a holomorphic principal H–bundle

$$\mathcal{E}_H = (E_H, \, p, \, J_E, \, \widetilde{\xi}, \, \widetilde{\mathcal{F}})$$

on $\mathcal{X}$. Let

$$\mathcal{E}_G = (E_G, \, p', \, J'_E, \, \widetilde{\xi}', \, \widetilde{\mathcal{F}}')$$

be the holomorphic principal G–bundle on $\mathcal{X}$ obtained by extending the structure group of $\mathcal{E}_H$ using the inclusion of H in G. The inclusion of $\mathfrak{h}$ in $\mathfrak{g}$,

being H–equivariant, produces an inclusion of $\mathrm{ad}(\mathcal{E}_H)$ in $\mathrm{ad}(\mathcal{E}_G)$; this inclusion map is holomorphic. We have the following two short exact sequences of holomorphic vector bundles on $\mathcal{X}$ with a common first term:

(4.1) $$0 \longrightarrow \mathrm{ad}(\mathcal{E}_H) \xrightarrow{\iota_1} \mathrm{ad}(\mathcal{E}_G) \longrightarrow \mathrm{ad}(\mathcal{E}_G)/\mathrm{ad}(\mathcal{E}_H) \longrightarrow 0$$

and

(4.2) $$0 \longrightarrow \mathrm{ad}(\mathcal{E}_H) \xrightarrow{\iota_2} \mathrm{At}(\mathcal{E}_H) \xrightarrow{d'p} F^{1,0} \longrightarrow 0$$

(see (3.4)).

Definition 4.1. A *branched holomorphic Cartan geometry* on $\mathcal{X}$ of type (G, H) is a holomorphic principal H–bundle $\mathcal{E}_H$ on $\mathcal{X}$ together with a holomorphic homomorphism of

$$\varphi : \mathrm{At}(\mathcal{E}_H) \longrightarrow \mathrm{ad}(\mathcal{E}_G)$$

satisfying the following two conditions:

   1) φ is an isomorphism over a nonempty subset of X, and

   2) for any $v \in \mathrm{ad}(\mathcal{E}_H)$, the equality

$$\varphi(\iota_2(v)) = \iota_1(v)$$

   holds, where ι_1 and ι_2 are the homomorphisms in (4.1) and (4.2) respectively.

If φ is an isomorphism over X, then the pair $(\mathcal{E}_H, \varphi)$ is called a *holomorphic Cartan geometry*.

   The *branching locus* of a branched holomorphic Cartan geometry $(\mathcal{E}_H, \varphi)$ is the subset of X where φ fails to be an isomorphism.

   From Definition 4.1 it follows immediately that a branched holomorphic Cartan geometry $(\mathcal{E}_H, \varphi)$ produces the following commutative diagram of holomorphic homomorphisms of vector bundles on $\mathcal{X}$

(4.3)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(\mathcal{E}_H) & \longrightarrow & \mathrm{At}(\mathcal{E}_H) & \xrightarrow{d'p} & F^{1,0} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\varphi} & & \downarrow{\phi} & & \\
0 & \longrightarrow & \mathrm{ad}(\mathcal{E}_H) & \longrightarrow & \mathrm{ad}(\mathcal{E}_G) & \longrightarrow & \mathrm{ad}(\mathcal{E}_G)/\mathrm{ad}(\mathcal{E}_H) & \longrightarrow & 0
\end{array}
$$

where ϕ is induced by φ. We note that ϕ is an isomorphism over a point $x \in X$ if and only if φ is an isomorphism over x. Therefore, ϕ is an isomorphism over a nonempty subset of X. This nonempty subset is evidently open and dense. If $(\mathcal{E}_H, \varphi)$ is a holomorphic Cartan geometry, then ϕ is an isomorphism over X.

4.2. Standard examples

Take a connected complex Lie group G and a complex Lie subgroup $H \subset G$. Assume that G/H is equipped with a Kähler form ω satisfying the following condition: there is a holomorphic line bundle L over G/H equipped with a Hermitian structure h_L such that the curvature of the Chern connection on L for h_L coincides with $\sqrt{-1} \cdot \omega$. Consider the real hypersurface

$$X_0 := \{v \in L \mid h_L(v) = 1\} \subset L$$

in the total space of L. Let

(4.4) $$q : X_0 \longrightarrow G/H$$

be the natural projection; note that q makes X_0 a principal S^1–bundle over G/H. Then ω and the Chern connection for h_L together produce a Sasakian structure on X_0. The Chern connection for h_L decomposes the real tangent bundle TX_0 of X_0 as

$$TX_0 = q^*T(G/H) \oplus ((G/H) \times \mathbb{R}),$$

where $T(G/H)$ is the real tangent bundle of G/H; more precisely, $q^*T(G/H)$ is the horizontal tangent space and $(G/H) \times \mathbb{R}$ is the vertical tangent space for the connection on the S^1–bundle X_0 given by the Chern connection on L for h_L. The vector field on X_0 given by the action of S^1 on X_0 is the Reeb vector field ξ. So X_0 is a regular Sasakian manifold. We will denote by $\mathcal{X}_0$ this manifold X_0 equipped with the Sasakian structure.

The quotient map $G \longrightarrow G/H$ defines a holomorphic principal H–bundle over G/H. This holomorphic principal H–bundle over G/H will be denoted by E_H^1. Let

$$E_G^1 := E_H^1 \times^H G \longrightarrow G/H$$

be the holomorphic principal G–bundle over G/H obtained by extending the structure group of E_H^1 using the inclusion of H in G. We note that E_G^1 is identified with the trivial principal G–bundle $(G/H) \times G \longrightarrow G/H$.

Indeed, E_G^1 is the quotient of $G \times G$ where two elements (g_1, g_1') and (g_2, g_2') are identified if there is an element $h \in H$ such that $g_2 = g_1 h$ and $g_2' = h^{-1} g_1'$. So the self-map of $G \times G$ defined by $(g_1, g_1') \longmapsto (g_1, g_1 g_1')$ identifies E_G^1 with the trivial principal G–bundle $(G/H) \times G$ over G/H. Identify the holomorphic tangent bundle TG with the trivial holomorphic bundle $G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, using right-invariant vector fields on G. This trivialization produces an isomorphism

$$\mu : \mathrm{At}(E_H^1) \longrightarrow \mathrm{ad}(E_G^1)$$

using the isomorphisms

$$\mathrm{At}(E_H^1) := (TG)/H = (G \times \mathfrak{g})/H = (G/H) \times \mathfrak{g} = \mathrm{ad}(E_G^1);$$

recall that E_G^1 is the trivial principal G–bundle $(G/H) \times G \longrightarrow G/H$, so $\mathrm{ad}(E_G^1)$ is the trivial vector bundle $(G/H) \times \mathfrak{g}$ over G/H. Hence E_H^1 and the above isomorphism μ together produce a tautological holomorphic Cartan geometry on G/H of type (G, H) [Sh].

Consider the pull back $(q^* E_H^1, q^* \mu)$, where q is the projection in (4.4). This pair defines a holomorphic Cartan geometry of type (G, H) on the Sasakian manifold $\mathcal{X}_0$ constructed above.

Let $\mathcal{X} := (X, g, \xi)$ be a Sasakian manifold, and let

(4.5) $f : \mathcal{X} \longrightarrow G/H$

be a holomorphic map. We recall that the holomorphicity of f means that the differential

$$df : TX \longrightarrow f^* T(G/H)$$

(here $T(G/H)$ is the real tangent bundle of G/H) satisfies the following two conditions:

- $df(\xi) = 0$, and

- df intertwines the automorphism $\Phi|_F$ in (2.3) and the automorphism of $f^* T(G/H)$ given by the almost complex structure on the complex manifold G/H.

Assume that the restriction $(df)|_F$ is an isomorphism over some point of X; this implies that $(df)|_F$ is an isomorphism over an open dense subset of X. Then $(f^* E_H^1, f^* \mu)$ is a holomorphic branched Cartan geometry on $\mathcal{X}$ of type (G, H). The branching locus for this holomorphic branched Cartan

geometry $(f^*E_H^1,\ f^*\mu)$ is the closed subset of X where $(df)|_F$ fails to be an isomorphism.

4.3. A connection defined by branched Cartan geometry

Let $\mathcal{X} := (X,\ g,\ \xi)$ be a connected Sasakian manifold. Take a pair $(\mathcal{E}_H,\ \varphi)$, where $\mathcal{E}_H := (E_H,\ p,\ J_E,\ \widetilde{\xi},\ \widetilde{\mathcal{F}})$ is a holomorphic principal H–bundle on $\mathcal{X}$, that defines a branched holomorphic Cartan geometry on $\mathcal{X}$ of type $(G,\ H)$. As before, let $\mathcal{E}_G = (E_G,\ p',\ J'_E,\ \widetilde{\xi}',\ \widetilde{\mathcal{F}}')$ be the holomorphic principal G–bundle on $\mathcal{X}$ obtained by extending the structure group of $\mathcal{E}_H$ using the inclusion of H in G.

Proposition 4.2. *The above holomorphic principal G–bundle $\mathcal{E}_G$ has a natural holomorphic connection given by φ.*

Proof. We will first describe $\mathrm{At}(\mathcal{E}_G)$. Consider the homomorphism

$$\iota_3 : \mathrm{ad}(\mathcal{E}_H) \longrightarrow \mathrm{ad}(\mathcal{E}_G) \oplus \mathrm{At}(\mathcal{E}_H), \quad v \longmapsto (\iota_1(v),\ -\iota_2(v)),$$

where ι_1 and ι_2 are the homomorphisms in (4.1) and (4.2) respectively. Then we have

$$(\mathrm{ad}(\mathcal{E}_G) \oplus \mathrm{At}(\mathcal{E}_H))/\iota_3(\mathrm{ad}(\mathcal{E}_H)) = \mathrm{At}(\mathcal{E}_G).$$

Indeed, this follows immediately from the construction of the Atiyah bundle (see (3.2)). Let

$$\alpha : \mathrm{ad}(\mathcal{E}_G) \oplus \mathrm{At}(\mathcal{E}_H) \longrightarrow (\mathrm{ad}(\mathcal{E}_G) \oplus \mathrm{At}(\mathcal{E}_H))/\iota_3(\mathrm{ad}(\mathcal{E}_H)) = \mathrm{At}(\mathcal{E}_G)$$

be the quotient map. Let

(4.6) $$0 \longrightarrow \mathrm{ad}(\mathcal{E}_G) \xrightarrow{\iota'} \mathrm{At}(\mathcal{E}_G) \xrightarrow{d'p'} F^{1,0} \longrightarrow 0$$

be the Atiyah exact sequence for $\mathcal{E}_G$ (see (3.4)). The injective homomorphism ι' in (4.6) coincides $\alpha \circ j_1$, where j_1 is the inclusion of $\mathrm{ad}(\mathcal{E}_G)$ in $\mathrm{ad}(\mathcal{E}_G) \oplus \mathrm{At}(\mathcal{E}_H)$ and α is the above quotient map. Now consider the homomorphism

$$\beta : \mathrm{ad}(\mathcal{E}_G) \oplus \mathrm{At}(\mathcal{E}_H) \longrightarrow \mathrm{ad}(\mathcal{E}_G), \quad (v,\ w) \longmapsto v + \varphi(w),$$

where φ is the homomorphism in the statement of the proposition. Since $\beta \circ \iota_3 = 0$, it follows that β descends to a homomorphism

$$\beta' : \mathrm{At}(\mathcal{E}_G) \longrightarrow \mathrm{ad}(\mathcal{E}_G).$$

Now it is straightforward to check that $\beta' \circ \iota' = \mathrm{Id}_{\mathrm{ad}(\mathcal{E}_G)}$, where ι' is the homomorphism in (4.6). Consequently, β' produces a holomorphic splitting of the short exact sequence in (4.6). Hence β' gives a holomorphic connection on $\mathcal{E}_G$. $\qquad\square$

Definition 4.3. A branched holomorphic Cartan geometry $(\mathcal{E}_H, \varphi)$ of type (G, H) on $\mathcal{X}$ is called *flat* if the holomorphic connection on $\mathcal{E}_G$ in Proposition 4.2 is flat (see Definition 3.2).

The branched holomorphic Cartan geometries in Section 4.2 are flat.

4.4. Developing map for flat Cartan geometries

Let $(\mathcal{E}_H, \varphi)$ be a flat branched holomorphic Cartan geometry of type (G, H) on $\mathcal{X}$. Consider the holomorphic connection on $\mathcal{E}_G$ in Proposition 4.2. Let $\widehat{D}$ be the flat connection on E_G given by it in Lemma 3.1.

Now assume that the manifold X is simply connected. Fix a point $x_0 \in X$. Using the flat connection $\widehat{D}$ on E_G, the principal G–bundle E_G gets identified with the trivial principal G–bundle $X \times (E_G)_{x_0}$, where $(E_G)_{x_0}$ is the fiber of E_G over the base point x_0. This identification between E_G and the trivial principal G–bundle $X \times (E_G)_{x_0}$ is clearly holomorphic. So we have

$$E_H \subset E_G = E_G \times (E_G)_{x_0}.$$

Let

$$(4.7) \qquad\qquad \gamma : X \longrightarrow (E_G)_{x_0}/H$$

be the map that sends any $x \in X$ to the H–orbit $(E_H)_x \subset (E_G)_{x_0}$, where $(E_H)_x$ is the fiber of E_H over the point x. Since the identification between E_G and the trivial principal G–bundle $X \times (E_G)_{x_0}$ is holomorphic with respect to the holomorphic structure on E_G given by the holomorphic structure on E_H, it follows immediately that the map γ in (4.7) is holomorphic. In particular, γ is constant on the orbits of the flow on X given by the Reeb vector field ξ.

Fixing a point $y_0 \in (E_G)_{x_0}$, we may identify G with $(E_G)_{x_0}$ by the map $g \longmapsto y_0 g$. In that case, γ is a holomorphic map from X to G/H. If we set f in (4.5) to be the map γ in (4.7), then the pulled back holomorphic branched Cartan geometry $(f^* E_H^1, f^* \mu)$ in Section 4.2 is identified with the holomorphic branched Cartan geometry $(\mathcal{E}_H, \varphi)$ that we started with.

The map γ in (4.7) will be called the *developing map* for $(\mathcal{E}_H, \varphi)$.

Lemma 4.4. *Let $\mathcal{X} := (X, g, \xi)$ be a connected Sasakian manifold such that X is compact and simply connected. Then there is no flat branched holomorphic Cartan geometry on $\mathcal{X}$ of type (G, H) if G/H is noncompact.*

Proof. Since X is compact and connected while G/H is noncompact, there is no holomorphic map from $\mathcal{X}$ to G/H satisfying the condition that its restriction to some nonempty open subset of X is a submersion. Hence there is no developing map γ as in (4.7). $\qquad\square$

5. Quasi-regular Sasakians that are Calabi-Yau

Let $\mathcal{X} := (X, g, \xi)$ be a connected compact quasi-regular Sasakian manifold of dimension $2m + 1$ satisfying the following condition: the holomorphic line bundle $\bigwedge^m F^{1,0}$ on $\mathcal{X}$ (see (2.7)) admits a holomorphic connection.

Since $\mathcal{X}$ is compact and quasi-regular, the space of orbits for the Reeb flow ξ has the structure of a smooth compact Kähler orbifold of complex dimension m [BG2]. We will denote by $\mathbb{X}$ this compact Kähler orbifold of complex dimension m. The above condition that the holomorphic line bundle $\bigwedge^m F^{1,0}$ on $\mathcal{X}$ in (2.7) admits a holomorphic connection is equivalent to the condition that $c_1(\mathbb{X}) = 0$. This condition implies that $\mathbb{X}$ admits a Ricci flat Kähler metric; this was conjectured by Calabi and it was proved by Yau in [Ya], and for orbifolds it was proved in [Ca].

Giving a branched holomorphic Cartan geometry on $\mathcal{X}$ of type (G, H) is equivalent to giving a branched holomorphic Cartan geometry of type (G, H) on the orbifold $\mathbb{X}$. See [BD] for branched holomorphic Cartan geometries on a complex manifold; the definition in [BD] extends to smooth orbifolds in a straightforward way.

Theorem 5.1. *Assume that the orbifold fundamental group of $\mathbb{X}$ is trivial. Let E be a holomorphic vector bundle over the orbifold $\mathbb{X}$ admitting a holomorphic connection D. Then*

  1) *the holomorphic vector bundle E is holomorphically trivial, and*

  2) *D is the trivial connection on the trivial holomorphic vector bundle E.*

Proof. This was proved in [BD] for compact simply connected Kähler manifolds M with $c_1(M) = 0$ (see [BD, Theorem 6.2]). The proof given in [BD] extends to the case of smooth compact orbifolds with vanishing c_1 once some straightforward modifications are incorporated. The main point to note is that the results from [Si1] used in [BD, Theorem 6.2] remain valid for compact quasi-regular Sasakian manifolds [BM]. (See also [Si2].) $\qquad\square$

Proposition 5.2. *Assume that the orbifold fundamental group of* $\mathbb{X}$ *is trivial. Let* $(\mathcal{E}_H, \varphi)$ *be a branched holomorphic Cartan geometry of type* (G, H), *with* G *a complex affine Lie group, on* $\mathcal{X}$. *Then there is a holomorphic map* $\gamma : \mathcal{X} \longrightarrow G/H$ *such that* $(\mathcal{E}_H, \varphi)$ *is the pullback, by* γ, *of the standard Cartan geometry on* G/H *of type* (G, H). *Also,* G/H *is compact.*

Recall that G is a complex affine Lie group if there exist a positive integer r and holomorphic homomorphism of complex Lie groups $\rho : G \longrightarrow \mathrm{GL}(r, \mathbb{C})$, with discrete kernel. The corresponding Lie algebra representation ρ' is an injective Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(\mathfrak{r}, \mathbb{C})$. Notice that complex simply connected Lie groups and complex semi-simple Lie groups are complex affine. Indeed, for G complex simply connected, holomorphic representations with discrete kernel do exist by Ado's theorem. For G complex semi-simple, holomorphic representations with discrete kernel are also known to exist (see Theorem 3.2, chapter XVII in [Hoc]).

Proof. By Proposition 4.2 the holomorphic principal G–bundle $\mathcal{E}_G$ constructed by extension of the structure group has a natural holomorphic connection given by φ.

Let us consider a holomorphic homomorphism of complex Lie groups $\rho : G \longrightarrow \mathrm{GL}(r, \mathbb{C})$, with discrete kernel. Then the associated holomorphic vector bundle of rank r inherits a holomorphic connection, which must be flat by Theorem 5.1. Since the Lie algebra homomorphism $\rho' : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{r}, \mathbb{C})$ is injective, the curvature of the holomorphic connection of the holomorphic principal bundle $\mathcal{E}_G$ also vanishes.

It follows that the branched holomorphic Cartan geometry $(\mathcal{E}_H, \varphi)$ is flat (see Definition 4.3).

Consider the developing map

$$\gamma : \mathcal{X} \longrightarrow G/H$$

constructed in (4.7). As observed in Section 4.4, the branched holomorphic Cartan geometry $(\mathcal{E}_H, \varphi)$ on $\mathcal{X}$ is the pullback of the standard Cartan geometry on G/H of type (G, H) by the map γ.

From Lemma 4.4 we know that G/H is compact. $\square$

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