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Chern-Simons theory on a general Seifert 3-manifold

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The path integral for the partition function of Chern-Simons gauge theory with a compact gauge group is evaluated on a general Seifert 3-manifold. This extends previous results and relies on abelianisation, a background field method and local application of the Kawasaki Index theorem.

1. Introduction

The main determination of the Reshetikhin-Turaev-Witten (RWT) [19, 20, 22] invariants of a 3-manifold has been through the use of the Reshetikin-Turaev construction or conformal field theory methods. A sampling of these approaches is [10, 11, 13, 15, 21]. There are also path integral evaluations such as semi-classical evaluations [13] as well as evaluations based on localisation [1, 2] and those based on supersymmetric localisation [14]. Though it must be said that the localisation approaches are not exact (so far) in case there is a moduli space of flat connections that is not made up of isolated points.

In a series of papers [4–8] two of us introduced the concept of diagonalisation as a gauge fixing condition in gauge theories. If one starts with a trivial G-bundle over a manifold M[1], with Lie algebra $\mathfrak{g}$, then in principle diagonalisation leaves one with a $\mathfrak{t}$ (some Cartan sub algebra) bundle and associated vector bundles. That procedure requires, however, that the 3-manifold be a principal bundle or fibration (over an orbifold) and that one make non-smooth gauge transformations to achieve the required gauge. The rationale for the first requirement is that, as explained in [5], this diagonalisation works "best" on 2-dimensional manifolds, since the resulting diagonalised gauge fields have precisely the singularity structure that allows them to be interpreted as non-singular connections on a non-trivial bundle. Generically in more than 2 dimensions the required gauge transformations

[1]We have indicated how non trivial gauge bundles can be incorporated in [3].

and resulting gauge fields are too singular to lend themselves to such an intepretation, and thus diagonalisation can only be applied if it is possible to reduce the calculations to 2 dimensions. In the case of 3-manifolds, this is possible in principle if the 3-manifold has the structure of a fibration over a 2-dimensional orbifold, to which the calculation can be "pushed down", and this singles out Seifert 3-manifolds among all possible 3-manifolds as those to which diagonalisation (at least as understood by us at present) can be applied.

A notable feature of this approach to the calculation of the Chern-Simons partition function of Seifert 3-manifolds [7] is that it completely bypasses the (possibly arduous) task of having to integrate over some moduli space of non-Abelian flat connections, as it essentially reduces the partition function to that of an Abelian gauge theory on a 2-dimensional orbifold.

The singular gauge transformations "Abelianise" the theory so that the fields are well defined but are now sections of non-trivial Abelian bundles. The obstructions [5] to using smooth gauge transformations to accomplish this are then reflected in the fact that one must sum over the Abelian bundles that are generated in this way. In all of the cases considered thus far the non trivial bundles that arise are always some power of a fixed line bundle $\mathcal{L}_M$, over the orbifold base, depending on the underlying 3-manifold M. Hence, there has only ever been the need to sum over one integer (the first Chern class of $\mathcal{L}_M$) in the path integral. The general class of Seifert three manifolds for which this is true we dubbed $\mathbb{Q}\mathrm{HS}[g]$ (genus g generalisations of rational homology spheres) in [7].

Our aim here is to extend the diagonalisation method to general Seifert 3-Manifolds. In order to diagonalise on Seifert 3-manifolds, which are not $\mathbb{Q}\mathrm{HS}[g]$, requires some new techniques. Firstly, we note that on a Riemann surface with N orbifold points Σ_V a general line V-bundle may be decomposed as

$$\mathcal{L} = \mathcal{L}_0^{n_0} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_N^{n_N} \tag{1.1}$$

with $0 \leq n_i < a_i$ where a_i is the order of the i'th orbifold point while $\mathcal{L}_0$ is a smooth line bundle and $n_0 \in \mathbb{Z}$. By Theorem 2.3 in [9] for M a smooth Seifert 3-manifold

$$\mathrm{H}^2\left(M, \mathbb{Z}\right) \simeq \mathrm{Pic}^{\mathrm{t}}\left(\Sigma_V\right) / \mathbb{Z}\left[\mathcal{L}_M\right] \oplus \mathbb{Z}^{2g} \tag{1.2}$$

where $\mathrm{Pic}^{\mathrm{t}}\left(\Sigma_V\right)$ is the topological Picard group of topological isomorphism classes of line V bundles over Σ_V. There is a more detailed statement namely

Proposition 5.3 in [17] which explains the relationship between bundles on M and those on Σ_V.

As all such bundles arise on diagonalisation we will need, for each line bundle, in the gauge bundle, to sum over the set of integers (n_0, n_i) (a $\mathrm{rk}\,(G)$'s set of such integers for structure group G). Consequently we will need to incorporate into the path integral that we are integrating over connections on such non-trivial bundles. To do this we introduce a background connection in anticipation that the connection is, infact, non-trivial.

To describe the background connections in detail we need to explain the orbifold construction on a Riemann surface and line V bundles in some detail, the principal bundle structure of Seifert 3-manifolds and the relationship between these. This is done in Section 2, Section 3 and in Section 3.4 respectively. One consequence of having an explicit background connection is that one does not need to introduce such a background implicitly in the evaluation of the determinants in Section 5.2 which is unlike the situation in the original evaluation of such determinants given in [4]. The reason for being so explicit is that one needs to keep to the fore the fact that on diagonalisation the smooth line bundles that are generated on the 3-manifold come from line V bundles below as essentially all the calculations are done on the orbifold.

The calculational part rests in Section 5.1. The original evaluation of the determinants in [4] shows that one is really dealing with densities on the underlying (V-) surface. The background fields localise the calculations to their support. Once one realises that the only changes that need to be made are to express the Kawasaki index theorem in a manner which takes into account local information then the calculations in this paper become essentially a commentary on [7] explaining where modifications need to be made, especially as we have alreay incorporated the background connection. The only point to be aware of is that we change our orientation and normalisation conventions in Section 5.1 to make it easier to use the results of [7].

2. 2-dimensional orbifolds and Seifert 3-manifolds

For us a compact closed 2-dimensional orbifold or V manifold Σ_V is a genus g Riemann surface with N discs D_i removed and replaced with the cones $U_i \simeq D_i/\mathbb{Z}_{a_i}$ for $i = 1, \ldots, N$. The apex of the cone is the orbifold point and we denote those points by x_i. The local model is, for $z \in D_i$,

$$z \simeq \zeta.z, \quad \zeta \in \mathbb{Z}_{a_i} \tag{2.1}$$

so that local holomorphic coordinates on the U_i are z^{a_i} and we think of ζ as a complex a_i'th root of unity.

Complex line V bundles $\mathcal{L} \to \Sigma_V$ are described in a similar fashion. Around an orbifold point the local description is

$$(z, w) \simeq (\zeta.z, \zeta^b.w), \quad w \in \mathbb{C} \tag{2.2}$$

where $0 < b < a$ and $\rho(\zeta) = \zeta^b$ is thought of as a representation of $\mathbb{Z}_a$. We note that the circle V bundle $S(\mathcal{L})$, with $|w| = 1$ in (2.2), is smooth as long as the $\gcd(a, b) = 1$ since there are no fixed points of the discrete action $\zeta^b.w$ in this case (otherwise with $a = cd$ and $b = ce$ where $c > 1$ one could take $\zeta = \exp(2\pi id/a)$ so that $\zeta^b = 1$).

Of special interest to us are the building blocks of such bundles which we denote by $\mathcal{L}_i$. The $\mathcal{L}_i$ are trivial outside of the local neighbourhood U_i and have local data on $D_i \times \mathbb{C}$

$$(z, w) \simeq (\zeta_i.z, \zeta_i.w), \quad \zeta_i \in \mathbb{Z}_{a_i} \tag{2.3}$$

Such holomorphic 'point' V bundles can be described as follows [9]

$$\mathcal{L}_i = (\Sigma_i \times \mathbb{C}) \cup_\psi (D_i \times \mathbb{C}) / \mathbb{Z}_{a_i} \tag{2.4}$$

where Σ_i is the smooth Riemann surface Σ with x_i removed and the clutching map is defined, away from $z = 0$, by

$$\psi(z, w) = (z^{a_i}, z^{-1}.w) \tag{2.5}$$

and ψ can be thought of as a $\mathbb{Z}_{a_i}$ invariant map on $D/\{0\} \times \mathbb{C}$ which descends to $(D/\{0\} \times \mathbb{C})/\mathbb{Z}_{a_i}$. The n_i-th tensor power of this bundle, $\mathcal{L}_i^{\otimes n_i}$ has clutching map

$$\psi(z, w) = (z^{a_i}, z^{-n_i}.w) \tag{2.6}$$

A general holomorphic V bundle $\mathcal{L}$ over Σ_V is then obtained by performing this construction at each of the N orbifold points and at one regular point.

We are also interested in the unit disc V bundle $D(\mathcal{L})$ of $\mathcal{L}$ which is obtained by taking $|w| \leq 1$ in (2.2) and which is realated to the circle V bundle $S(\mathcal{L})$ by $\partial D(\mathcal{L}) = S(\mathcal{L})$.

A Seifert 3-manifold $M[\deg(\mathcal{L}_M), g, (a_1, b_1), \ldots, (a_N, b_N)]$ is a smooth circle V bundle $S(\mathcal{L}_M)$ over a genus g Riemann surface with N orbifold

points with Seifert data (a_i, b_i) such that

$$0 < b_i < a_i, \quad \gcd(a_i, b_i) = 1 \tag{2.7}$$

The first condition means that we are using normalised Seifert invariants while the second is the condition that, as we saw, the bundle is smooth.

Throughout we will have in mind a decomposition of the base space Σ_V into open sets U_i for $i = 0, 1, \ldots, N$ where $U_0 = \Sigma_0$ and the U_i $i = 1, \ldots, N$ are the cones $D/\mathbb{Z}_{a_i}$ about the orbifold points x_i, while Σ_0 is just Σ_V with the cones excised and the line V-bundles $\mathcal{L}_i$ will be 'point' bundles localised on the U_i.

2.1. Sections and connections on V bundles over Σ_V

There is a natural section of the $\mathcal{L}_i^{\otimes n_i}$ namely on D_i the section is

$$s_i(z) = z^{n_i} \tag{2.8}$$

which can be extended over the rest of Σ as the constant section 1 via the clutching map (2.6). The first Chern class is

$$c_1\left(\mathcal{L}_i^{\otimes n_i}\right) = \frac{n_i}{a_i} \tag{2.9}$$

A suitable local connection form on $D_i/\{0\}$ for $\mathcal{L}_i^{\otimes n_i}$ is

$$\alpha_i^{\otimes n_i} = g(z\overline{z})\, d\ln\left(z^{n_i}\right) + \overline{w} dw \tag{2.10}$$

providing that g is the identity much of the way into D_i (we always take the D_i to be unit discs). Having such a g is consistent with the clutching map (2.6). On $S\left(\mathcal{L}_i^{\otimes n_i}\right)$ this is, with $w = \exp(i\sigma)$, the connection

$$\alpha_i^{\otimes n_i} = g(z\overline{z})\, d\ln\left(z^{n_i}\right) + i d\sigma \tag{2.11}$$

so $d\alpha_i^{\otimes n_i}$ is horizontal and

$$d\alpha_i^{\otimes n_i} = n_i\, d\alpha_i \tag{2.12}$$

with holonomy

$$c_1\left(\mathcal{L}_i^{\otimes n_i}\right) = \frac{1}{2\pi i}\int_{U_i} d\alpha_i^{\otimes n_i} = \frac{1}{2\pi i}\frac{1}{a_i}\int_{\partial D_i} \alpha_i^{\otimes n_i} = \frac{n_i}{a_i} \tag{2.13}$$

as required. The α_i are also locally contact structures with

$$\frac{1}{(2\pi i)^2}\int_{S(\mathcal{L}_i^{\otimes n_i})}\alpha_i(\mathcal{L}_i^{\otimes n_i})\wedge d\alpha_i(\mathcal{L}_i^{\otimes n_i}) = \frac{n_i}{a_i} \tag{2.14}$$

$\mathcal{L}_i$ is then the holomorphic line V bundle with first Chern class $c_1(\mathcal{L}_i) = 1/a_i$ and with divisor at the i'th orbifold point with $i \in 1, \ldots, N$ and allow, for $i = 0$, $\mathcal{L}_0$ to be the line bundle at a smooth point with first Chern class $c_1(\mathcal{L}_0) = 1$. Then we have that any smooth holomorphic line V bundle $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{L}_0^{\otimes n_0}\otimes\mathcal{L}_1^{\otimes n_1}\otimes\cdots\otimes\mathcal{L}_N^{\otimes n_N} \tag{2.15}$$

with

$$n_0 \in \mathbb{Z}, \quad 0 < n_i < a_i, \quad i = 1, \ldots, N \tag{2.16}$$

3. Surgery, connections and Chern classes

This section is meant to connect the line bundle view point of the previous section with the direct construction of the Seifert 3-manifold. We begin with a topological description of the circle V-bundles that we considered in the previous section. This is followed by a surgery prescription on glueing boundaries along tori relevant to creating Seifert 3-manifolds.

3.1. Solid tori with S^1 action of (a, b) type

We fix an element of $SL(2,\mathbb{Z})$ in this section

$$\begin{pmatrix} a & b \\ -r & -s \end{pmatrix}, \quad br = 1 + as, \quad \gcd(a,b) = 1, \quad 0 < b < a, \quad 0 < r < a \tag{3.1}$$

Consider a solid torus $D^2 \times S^1$, where D^2 is a unit disc in $\mathbb{C}$ with center at the origin and with local coordinates $(\rho e^{i\phi}, e^{i\psi})$. The standard S^1 action of type (a, b) is

$$\left(\rho\,\mathrm{e}^{\,i\phi},\, \mathrm{e}^{\,i\psi}\right).\mathrm{e}^{\,i\theta} = \left(\rho\,\mathrm{e}^{\,(i\phi + ir\theta)},\, \mathrm{e}^{\,(i\psi + ia\theta)}\right) \tag{3.2}$$

We can quotient with this action (use θ to set $\psi = 0$ and we still have those transformations generated by $\zeta = \exp(i\theta)$ where $\theta = 2\pi/a$ as these do not change the value of $\exp(i\psi) = 1$) to be left with $D^2/\mathbb{Z}_a$. Denote

the solid torus with this action by $V_{(a,b)}$ then we have the S^1 V-bundle $V_{(a,b)} \longrightarrow D^2/\mathbb{Z}_a$.

The vector field corresponding to the generator of the $U(1)$ action on $V_{(a,b)}$ is

$$\xi = r\frac{\partial}{\partial\phi} + a\frac{\partial}{\partial\psi} \tag{3.3}$$

and the 'vertical'dual one-form is

$$d\theta = bd\phi - sd\psi \tag{3.4}$$

while the horizontal 1-forms, the space of which we quite generally denote by Ω^1_H, are spanned by

$$d\rho, \quad \text{and} \quad d\chi = ad\phi - rd\psi \tag{3.5}$$

3.2. Surgery to obtain Seifert 3-manifolds

The exposition here partially follows that of Jankins and Neumann [12] and of Orlik [16].

A solid torus is $D^2 \times S^1$ where D^2 is a unit disc in $\mathbb{C}$ with center at the origin. Let λ be a longitude, that is a simple non contractible curve on the T^2 boundary of $D^2 \times S^1$, and for definiteness, fix the point $\{1\} \in \partial D$ and take λ to be $\{1\} \times S^1$. We also set μ to be a meridian, that is a contractible loop in $D^2 \times S^1$ lying on the boundary of $D^2 \times S^1$ with unit intersection with λ, which we take to be $\partial D \times \{1\}$.

We wish to perform surgery on $\Sigma \times S^1$ where Σ is a compact closed Riemann surface. Let $\Sigma_0 = \Sigma/D_1^2 \cup \cdots \cup D_N^2$ be the surface with the interiors of N disjoint discs excised and, with obvious notation, $\partial\Sigma_0 = S_1^1 \cup \cdots \cup S_N^1$. We consider the manifold $\Sigma_0 \times S^1$. Denote the boundary curve in $\Sigma_0 \times S^1$ of the i'th excised disc in Σ_0 by $c_i = S_i^1 \times \{1\} \subset S_i^1 \times S^1$. Likewise, denote $h_i = \{1\} \times S^1 \subset S_i^1 \times S^1$.

Clearly we can regain $\Sigma \times S^1$ by glueing solid tori to all of the boundaries of $\Sigma_0 \times S^1$ where we simply identify the c_i with the meridian μ_i and h_i with the longitude λ_i of the i'th solid torus at the i'th boundary.

More generally we could glue in the N solid tori with the identification, a homeomorphism f,

$$f_* : \begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix} \longrightarrow \begin{pmatrix} a_i & b_i \\ -r_i & -s_i \end{pmatrix} . \begin{pmatrix} c_i \\ h_i \end{pmatrix}, \quad b_i r_i = 1 + a_i s_i \tag{3.6}$$

so that, in homology,

$$\begin{aligned} \mu_i &= a_i\, c_i + b_i\, h_i \\ \lambda_i &= -r_i\, c_i - s_i\, h_i \end{aligned} \tag{3.7}$$

which reads, μ_i wraps b_i times around h_i and a_i times about c_i while λ_i wraps s_i times around $-h_i$ and r_i times around $-c_i$. The image of $\{0\} \times S^1$ is called the singular fibre. Inverting the relationship (3.7) we have

$$\begin{aligned} c_i &= -s_i\, \mu_i - b_i\, \lambda_i \\ h_i &= r_i\, \mu_i + a_i\, \lambda_i \end{aligned} \tag{3.8}$$

The manifolds that have just been created, $M[g, (a_1, b_1), \ldots, (a_N, b_N)]$, are Seifert manifolds but with non-normalised Seifert invariants (so that b_i is not necessarily smaller than a_i).

The S^1 action (3.2) is designed to coincide with the wrapping of h_i on $\partial V_{(a_i,b_i)}$. To see this in detail let the coordinate on h_i be θ_i then by (3.8) the map $S^1 \to T^2$ with coordinates (ϕ_i, ψ_i) on T^2 sends θ_i to $(r\theta_i, a_i\theta_i)$ and the dual 1-form (3.4) pulls back to $d\theta_i$. Notice that this means that the solid tori $V_{(a_i,b_i)}$ come complete with their surgery data, that is one glues the solid torus to the rest of the manifold with the data (3.1) which is used in (3.7) and (3.8).

As an example take $M = S^2 \times S^1 = \left(D^2 \cup D^2\right) \times S^1$ and take out the right hand $D^2 \times S^1$ (leaving us with another solid torus namely the left $D^2 \times S^1$) now glue back according to (3.6). The 3-manifold obtained in this way is the Lens space $L(b, a)$ and in particular $S^3 = L(1, 0)$ is obtained with f given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.9}$$

The prescription (3.6) is not the one required when one takes out the tubular neighbourhood of a knot or link in S^3 and then glues back[2].

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[2]Rather, one uses instead a homeomorphism $\widehat{f}$, $\begin{pmatrix} b & -a \\ r & -s \end{pmatrix} = \begin{pmatrix} a & b \\ s & r \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which first undoes the first glueing to get S^3 from $S^2 \times S^1$. In this case we have $\widehat{f}_*(\mu) = b.c + a.h$.

3.3. Fractional monopole bundles, flat connections and surgery

Now we would like to provide connections for the principal bundle structure of the Seifert 3-Manifold as well as connections on bundles over M. In the first case we wish to provide smooth 1-forms on the Seifert 3-Manifold obeying the usual conditions.

A natural connection one form on $V_{(a_i,b_i)}$ is (there is no sum over a repeated index unless explicitely shown)

$$\begin{aligned} \sigma_i &= if(\rho_i)\ (b_i\, d\phi_i - s_i\, d\psi_i) + i\frac{1-f(\rho_i)}{a_i}\, d\psi_i \\ d\sigma_i &= i\frac{b_i}{a_i} df(\rho_i)\ \wedge\ (a_i d\phi_i - r_i d\psi_i) \end{aligned} \tag{3.10}$$

where $f(0) = 0$ and $f(1) = 1$. The σ_i satisfy

$$\iota_{\xi_i}.\,\sigma_i = i, \quad \text{and} \quad \iota_{\xi_i}.\,d\sigma_i = 0. \tag{3.11}$$

The first Chern class can be determined by integrating over the disc in $V_{(a_i,b_i)}$ defined by $\psi = 0$,

$$c_1 = \frac{1}{a_i}\frac{1}{2\pi i}\int_D ib_i\, df(\rho_i) \wedge d\phi_i = \frac{b_i}{a_i} \tag{3.12}$$

Note that

$$\int_{V_{(a_i,b_i)}} \sigma_i \wedge d\sigma_i = (2\pi i)^2\, \frac{b_i}{a_i} \tag{3.13}$$

If one adds $i\, n_i f(\rho_i)(a_i d\phi_i - r_i d\psi_i)$ then we have a connection with $c_1 = b_i/a_i + n$.

The holonomies for this connection along the meridian and longitude (that is at $\rho = 1$) are

$$\mathrm{hol}_{\sigma_i}(\mu_i) = \exp(2\pi i b_i) = 1, \quad \mathrm{hol}_{\sigma_i}(\lambda_i) = \exp(-2\pi i s_i) = 1 \tag{3.14}$$

so looking from the outside, as far as the boundary is concerned, one is dealing with a flat connection. Consequently, if we glue the i-th solid torus into Σ_0 with (3.8) and demand that the holonomy match we may extend the connection into Σ_0 as a flat connection. From our previous discussion the extension into $\Sigma_0 \times S^1$ is as $d\theta$. Consequently, we define a continuous

global 1-form κ which on $\Sigma_0 \times S^1$ is $d\theta$ and

$$\sigma|_{V(a_i,b_i)} = \sigma_i \tag{3.15}$$

With any suitable choice of $f(\rho_i)$, so that all its derivatives vanish at $\rho_i = 1$, $f^{(n)}(1) = 0$ when $n \geq 1$, we obtain a smooth connection one form. Indeed with such a choice the curvature 2-form, $d\sigma_i$, vanishes at $\rho_i = 1$. With these choices σ is a well defined smooth connection 1-form, such that

$$\int_M \sigma \wedge d\sigma = \sum_{i=0}^{N} \int_{V_{(a_i,b_i)}} \sigma_i \wedge d\sigma_i = (2\pi i)^2 \left(b_0 + \sum_{i=1}^{N} \frac{b_i}{a_i} \right) \tag{3.16}$$

Notice that, in this way, we have defined a 'global' principal bundle structure on M.

This bundle description can be made to be trivial away from the fibres over the orbifold points, since we may choose f to be one almost all the way into the center of the disc so that $d\sigma_i$ eventually has delta function support at the orbifold point. In particular we have, suggestively in that limit,

$$\frac{d\sigma}{2\pi i} = b_0\, \delta(x_0) + \sum_{i=1}^{N} \frac{b_i}{a_i}\, \delta(x_i), \tag{3.17}$$

with the $\delta(x_i)$ being 2-form de-Rham currents.

3.4. Holomorphic description and connections on $\mathcal{L}_i^{\otimes n_i}$

Now we wish to connect the surgery prescription with that of complex line V-bundles of the previous section.

Let $U(1)$ act on $\mathbb{C}$ by the character $e^{i\theta} \mapsto e^{-in\theta}$. There are associated complex line V-bundles $\mathcal{L}_i^{\otimes nb_i}$ over the orbifold $D^2/\mathbb{Z}_a$

$$\mathcal{L}_i^{\otimes nb_i} = V_{(a_i,b_i)} \times_n \mathbb{C} \tag{3.18}$$

meaning that $\mathcal{L}_i^{\otimes b_i}$ is the quotient of $V_{(a_i,b_i)} \times \mathbb{C}$ according to $(p, w).e^{i\theta} = (p.e^{i\theta}, e^{im\theta}.w)$. As before we can use the S^1 action to set $\psi = 0$ but we are left with a $\mathbb{Z}_a$ action

$$(z, w) \simeq (\zeta^r.z, \zeta^n.w) \tag{3.19}$$

and $z = \rho\, .e^{i\phi}$. On using ζ^{b_i} as the generator rather than ζ we get $(z, w) \simeq (\zeta.z, \zeta^{nb_i}.w)$ which agrees with (2.2). The associated bundle construction

(3.18) allows us to identify a holomorphic connection on the line bundle given the connection (3.10). We are tasked to make the identification

$$(z,t,w) \simeq (z.\mathrm{e}^{ir_i\theta}, t.\mathrm{e}^{ia_i\theta}, \mathrm{e}^{-in\theta}.w) \tag{3.20}$$

which we do by taking $\mathrm{e}^{i\theta} = t^{-1/a_i}$ (and we still have to make the identification under $\mathbb{Z}_{a_i}$). This provides us with a map from $V_{(a_i,b_i)}$ to $\mathcal{L}_i^{\otimes n_i b_i}$ given by

$$\begin{aligned}(\rho_i \exp i\phi_i, \exp i\psi_i) \xrightarrow{\tau} (\rho_i.\exp(i\phi_i - ir_i\psi_i/a_i), \exp(in_i\psi_i/a_i)) \\ = (z_i, w_i)\end{aligned} \tag{3.21}$$

The connection (2.10) on $\mathcal{L}_i^{\otimes n_i b_i}$ pulls back as

$$\begin{aligned}\tau^*\left(\alpha_i^{\otimes n_i b_i}\right) = in_i b_i g(\rho_i^2)(d\phi_i - r_i d\psi_i/a_i) \\ + in_i d\psi_i/a_i + n_i b_i g(\rho_i^2) d\ln\rho_i\end{aligned} \tag{3.22}$$

If we set $g(\rho_i^2) = f(\rho_i)$, which we do, then we have the equality

$$\tau^*\left(\alpha_i^{\otimes n_i b_i}\right) = n_i\sigma_i + d\Lambda(\rho_i) \tag{3.23}$$

We are really interested in the 'classes' that these forms represent and so we simply substitute $\alpha_i^{\otimes n_i b_i}$ with $n_i\sigma_i$. In particular we have that the curvature 2-forms agree,

$$\tau^*\left(d\alpha_i^{\otimes n_i b_i}\right) = n_i d\sigma_i \tag{3.24}$$

4. Chern-Simons theory on a Seifert 3-manifold

The Chern-Simons action is

$$S = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(\mathbb{A}d\mathbb{A} + \frac{2}{3}\mathbb{A}^3\right) \tag{4.1}$$

where $\mathbb{A}$ is a connection on a trivializable (and trivialized) G-bundle over M.

We consider the class of 3-manifolds M as described in the previous section, namely circle bundles $S(\mathcal{L})$ of holomorphic line V bundles $\mathcal{L}$ over an orbifold Σ_V. As in [7], we take the gauge group G to be a compact, semi-simple, connected and simply connected Lie group.

Given the principal bundle structure κ one can decompose fields in a Fourier series along the fibre direction as done previously [6] and [7]. One

cannot completely follow the derivation in those papers directly for reasons that we explained in the Introduction though we will try to follow it as closely as possible.

Our conventions in [7] had the vector field generating the $U(1)$ action denoted by ξ and the real dual 1-form κ satisfying

$$\iota_\xi \kappa = 1, \quad \iota_\xi d\kappa = 0 \tag{4.2}$$

with

$$d\kappa = -c_1(\mathcal{L}_M)\,\omega, \quad \int_M \kappa \wedge d\kappa = -c_1(\mathcal{L}_M), \quad \text{where} \quad \int_{\Sigma_V} \omega = 1 \tag{4.3}$$

We can achieve this be setting

$$\kappa = \frac{1}{2\pi i}\sigma, \quad \kappa|_{V(a_i,b_i)} = \kappa_i = \frac{1}{2\pi i}\sigma_i \tag{4.4}$$

and by understanding that the fibre has length 1 (rather than 2π). The minus sign in (4.3) implies that we are using the opposite orientation for the Seifert 3-manifold M to that in previous sections.

Now we decompose fields as

$$\mathbb{A} = A + \kappa\phi \tag{4.5}$$

with $\iota_\xi A = 0$ so that A is a horizontal field with respect to this fibration and ϕ is the component that lies along the fibre. Note that both A and ϕ are anti-Hermitian.

With this decomposition the Chern-Simons action becomes,

$$S_{CS}[\mathbb{A}] = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(A \wedge \kappa \wedge L_\phi A + 2\phi\,\kappa \wedge d\,A + \phi^2\,\kappa \wedge d\,\kappa\right) \quad . \tag{4.6}$$

The Lie derivative is denoted by $L_\xi = \{\iota_\xi, d\}$ and for the covariant Lie derivative we set $L_\phi = L_\xi + [\phi,$

4.1. Background gauge fields and patches

We know from the outset that once we try to impose the condition that ϕ only takes values in the Cartan subalgebra that we will have to sum over all possible non-trivial Abelian bundles that are 'liberated' in this procedure. In

anticipation of this we will work directly with a connection plus a background Abelian connection

$$\mathbb{A} \to \mathbb{A} + \mathbb{A}_B \tag{4.7}$$

where $\mathbb{A}$ is a Lie algebra valued 1-form and $\mathbb{A}_B$ will be specified in the next section. To explain where these background fields come from, recall that firstly we set $\iota_\xi d\phi = 0$ with a well defined gauge transformation and then we follow this by diagonalising ϕ with a 'time' independent gauge transformation $\iota_\xi dg = 0$ which is, necessarily, singular. All of this is now happening on Σ_V and so the singular gauge transformations give rise to non-trivial bundles on Σ_V so that one should be dealing with the connections we called α in Section 2.1. However, we need to pull those back to our 3-manifold M as in Section 3.4 and modulo a couple of caveats that pull back (a sum of multiples of the κ_i) is our background connection.

Given that all the non-trivial bundles are encoded in the $\mathbb{A}_B$ we demand that $\mathbb{A}$ is a smooth globally defined form (actually section). As the background is fixed gauge transformations act as follows

$$\mathbb{A}^g = g^{-1}\mathbb{A}g + g^{-1}dg + g^{-1}\mathbb{A}_B g - \mathbb{A}_B \tag{4.8}$$

Next we impose the gauge condition that ϕ is constant along the fibre $\iota_\xi d\phi = 0$. The variation of this condition involves the operator

$$L_{\phi+\phi_B} \tag{4.9}$$

where $\phi_B = \iota_\xi \mathbb{A}_B$ and it is this operator that appears in the ghost determinant.

4.2. Abelianization on a Seifert manifold

As we still have gauge invariance under those gauge transformations g that satisfy $\iota_\xi dg = 0$ we would like to Abelianize the field ϕ, that is set $\phi^{\mathfrak{k}} = 0$ where we have decomposed the Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ into a Cartan subalgebra and root spaces. If we do so then we must follow this by summing over all available line V bundles on the orbifold Σ_V. In previous works on Abelianization in Chern-Simons theory this amounted to a sum over one integer. The reason for that is that we had previously considered Seifert 3-manifolds M (the $\mathbb{Q}$HS[g]-manifolds of [7]) on which every line V bundle on the base orbifold could be given as a tensor power of some unique line

V bundle $\mathcal{L}_M$. We are certainly far away from that situation in the present context where we will need to sum over all possible line V bundles.

To sum over all of these possibilities we add to the connection $\mathbb{A}$ an Abelian background connection $\mathbb{A}_B$. The Chern-Simons action goes over to,

$$S_{CS}[\mathbb{A}+\mathbb{A}_B] = S_{CS}[\mathbb{A}] + \frac{1}{4\pi}\int_M \mathrm{Tr}\left(\mathbb{A}_B d\mathbb{A}_B + 2\mathbb{A}\wedge F_B + 2\mathbb{A}^2\mathbb{A}_B\right) \tag{4.10}$$

The last term only involves the charged components of the connection $\mathbb{A}$ so that, in particular, it does not involve the gauge fixed ϕ. One may wonder why it is that $2\mathbb{A}\wedge F_B$ appears in the action rather than $\mathbb{A}\wedge d\mathbb{A}_B + d\mathbb{A}\wedge\mathbb{A}_B$ as, even though these two only differ by an exact term $2\mathbb{A}\wedge F_B = \mathbb{A}\wedge d\mathbb{A}_B + d\mathbb{A}\wedge\mathbb{A}_B - d(\mathbb{A}\wedge\mathbb{A}_B)$, for singular forms a naive application of Stokes theorem is not correct. Actually the Chern Simons Lagrangian is not invariant under a gauge transformation

$$CS(\mathbb{A}^g) = CS(\mathbb{A}) + d\,\mathrm{Tr}\left(\mathbb{A}^g\wedge g^{-1}dg\right) - \frac{1}{3}\,\mathrm{Tr}\left(g^{-1}dg\right)^3$$

so, ignoring the winding number, we really should use

$$CS(\mathbb{A}) = CS(\mathbb{A}^g) - d\,\mathrm{Tr}\left(\mathbb{A}^g\wedge g^{-1}dg\right)$$

the $\mathbb{A}^g$ are our new 'quantum' fields and $g^{-1}dg$ is essentially $\mathbb{A}_B$.

The background bundles which are available to us are all of those that can appear in (2.15)

$$\mathcal{L} = \mathcal{L}_0^{\otimes n_0}\otimes\mathcal{L}_1^{\otimes n_1}\otimes\cdots\otimes\mathcal{L}_N^{\otimes n_N} \tag{4.11}$$

In the i-th patch the connection form σ_i is that for $V_{(a_i,b_i)}$ (or equivalently $\mathcal{L}_i^{\otimes b_i}$). We would like to represent connections of $\mathcal{L}_i^{\otimes n_i}$ but it seems that the best that we can do is have connections for $\mathcal{L}_i^{\otimes n_i b_i}$. In order to deal with this situation we use (4.11) as follows

$$\mathcal{L}_i^{\otimes n_i r_i b_i} = \mathcal{L}_i^{\otimes n_i}\otimes\mathcal{L}_0^{\otimes s_i n_i} \tag{4.12}$$

from which we deduce that the connection that we require on $\mathcal{L}_i^{\otimes n_i}$ pulls back to $n_i r_i\sigma_i - n_i s_i\sigma_0$ or, somewhat more correctly, the curvature 2-forms

satisfy

$$n_i[d\sigma_i] = n_i r_i[d\sigma_i] - s_i n_i[d\sigma_0] \tag{4.13}$$

The line bundles that appear live inside the gauge bundle that we are considering, so that the background connection is taken to be

$$\begin{aligned} \mathbb{A}_B &= \mathbf{n}_0\sigma_0 + \sum_{i=1}^{N} \mathbf{n}_i \left(r_i\sigma_i - s_i\sigma_0\right) \\ &= 2\pi i \left(\mathbf{n}_0\kappa_0 + \sum_{i=1}^{N} \mathbf{n}_i \left(r_i\kappa_i - s_i\kappa_0\right)\right) \end{aligned} \tag{4.14}$$

where the $\mathbf{n}$ are Hermitian. As the components of $\mathbf{n}_0$ range over the integers, and as we will sum over these, we can shift to absorb the $s_i\mathbf{n}_i$. With this understood, and with an abuse of notation, we write the background as

$$\kappa\,\phi_B = 2\pi i \left(\mathbf{n}_0\kappa_0 + \sum_{i=1}^{N} \mathbf{n}_i r_i \kappa_i\right) \tag{4.15}$$

It is usually appropriate for an Abelian theory to write

$$\frac{1}{4\pi}\int_M \mathrm{Tr}\left(\mathbb{A}_B d\mathbb{A}_B\right) = \frac{1}{4\pi}\int_X \mathrm{Tr}\left(F_B \wedge F_B\right) \tag{4.16}$$

where X is a 4-manifold that bounds M. Recall that M is itself the unit circle V bundle, $S(\mathcal{L}_M)$ of some line V bundle $\mathcal{L}_M$. We are fortunate in that there is a natural X available to us, namely we take X to be the unit disc bundle $D(\mathcal{L}_M)$ whose boundary is $S(\mathcal{L}_M) \equiv M$. Even though though the disc bundle is itself singular one could follow this through [18], however, we are in the even happier situation that we are able to determine the left hand side directly, which we now proceed to do.

We can evaluate the second Chern-Simons contribution in (4.10) as follows (with $r_0 = 1$)

$$\begin{aligned} \frac{1}{4\pi}\int_M \mathrm{Tr}\left(\mathbb{A}_B d\mathbb{A}_B\right) &= -\pi \sum_{i=0}^{N} r_i^2 \int_{V_{(a_i,b_i)}} \mathrm{Tr}\,\mathbf{n}_i^2\, \kappa_i \wedge d\kappa_i \\ &= \pi \sum_{i=0}^{N} \frac{r_i^2 b_i}{a_i}\,\mathrm{Tr}\,\mathbf{n}_i^2 \end{aligned} \tag{4.17}$$

For a simply connected group G, $\pi \operatorname{Tr} \mathbf{n}_i^2$ is an element of $2\pi\mathbb{Z}$. The order of the normal point is 1 ($a_0 = 1$, $b_0 = 1$) so that the exponential of that term gives unity and so may be neglected. Furthermore, on replacing $r_i b_i = 1 + a_i s_i$ in (4.17), the only terms that will contribute in the exponential are

$$\pi \sum_{i=1}^{N} \frac{r_i}{a_i} \operatorname{Tr} \mathbf{n}_i^2 \tag{4.18}$$

For a non-simply connected group one will also have to take into account signs that depend on the length of each of the $\mathbf{n}_i$.

In the second last term of (4.10) only the ϕ component of $\mathbb{A}$ is present as F_B is horizontal,

$$\frac{1}{2\pi} \int_{(D_i \times S^1)/\mathbb{Z}_{a_i}} \operatorname{Tr} (\kappa\phi \wedge F_B) = -i \frac{b_i r_i}{a_i} \operatorname{Tr} \phi \mathbf{n}_i \tag{4.19}$$

and we have made use of the fact that when we integrate over Cartan valued A we have a delta function constraint on ϕ which implies it is constant and

$$\kappa|_{D_i} = d\theta + \beta_i \tag{4.20}$$

and the integral on the fibre for $d\theta$ is one.

The last piece of the puzzle is the

$$\frac{1}{2\pi} \int_M \operatorname{Tr} \mathbb{A}^2 \mathbb{A}_B = \frac{1}{4\pi} \int_M \operatorname{Tr} A \wedge \kappa \wedge [\phi_B, A] \tag{4.21}$$

term where $\phi_B = \iota_\xi \mathbb{A}_B$. This piece appears in the determinants that we have still to evaluate.

4.3. Collecting terms in the action

The total action becomes

$$\begin{aligned} ikS_{CS} = {} & \frac{ik}{4\pi} \int_M \operatorname{Tr} \left(A \wedge \kappa \wedge L_{(\phi+\phi_B)} A + 2\phi\, \kappa \wedge d\, A + \phi^2\, \kappa \wedge d\, \kappa \right) \\ & + ik \sum_{i=1}^{N} \frac{r_i}{a_i} \operatorname{Tr} \left(-ib_i \phi \mathbf{n}_i + \pi \mathbf{n}_i^2 \right) + k \operatorname{Tr} (\phi \mathbf{n}_0) \end{aligned} \tag{4.22}$$

Clearly integrating over $A^{\mathfrak{t}}$ gives us the condition that $d(\kappa\phi) = 0$ which together with the gauge condition on ϕ implies that ϕ is constant, $d\phi = 0$.

Now that ϕ is constant and noting that

$$\int_M \kappa \wedge d\kappa = -c_1(\mathcal{L}_M) \tag{4.23}$$

we may write

$$\frac{ik}{4\pi}\int_M \kappa \wedge d\kappa \,\mathrm{Tr}\left(\phi^2\right) = -\frac{ik}{4\pi}c_1(\mathcal{L}_M)\,\mathrm{Tr}\left(\phi^2\right) \tag{4.24}$$

Consequently the partition function becomes

$$\begin{aligned} Z_{CS} = &\sum_{\mathbf{n}_0\in\mathbb{Z}}\left(\prod_{i=1}^{N}\sum_{\mathbf{n}_i=1}^{a_i-1}\right) \\ &\times \int_{\mathfrak{t}} d\phi\, \frac{\mathrm{Det}_{\Omega^0(M,\mathfrak{k})}\left(iL_{\phi+\phi_B}\right)}{\sqrt{\mathrm{Det}_{\Omega^1_H(M,\mathfrak{k})}\left(*\kappa\,\wedge\, L_{\phi+\phi_B}\right)}}\,.\,\exp\left(ikI(\phi,\mathbf{n})\right) \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} I(\phi,\mathbf{n}) = &\sum_{i=1}^{N}\frac{r_i}{a_i}\,\mathrm{Tr}\left(-ib_i\phi\mathbf{n}_i + \pi\mathbf{n}_i^2\right) \\ &+ \mathrm{Tr}\left(\phi\mathbf{n}_0\right) - \frac{1}{4\pi}c_1(\mathcal{L}_M)\,\mathrm{Tr}\left(\phi^2\right) \end{aligned} \tag{4.26}$$

5. One loop effects and the Kawasaki index theorem

We borrow heavily from the calculations in [7]. In order to make contact with that work we will need to explain, along the way, how working locally mimics the global calculations there. Furthermore, we need to take into account that ϕ_B unlike ϕ is not constant on Σ_V. Lastly, one needs to note that the Kawasaki index theorem tells us that the number of holomorphic sections of the line V bundle only depends on the desingularisation $|\mathcal{L}|$ over the smooth manifold $\Sigma \equiv |\Sigma_V|$ of the holomorphic line V bundle $\mathcal{L}$ over Σ_V,

$$\chi(\Sigma_V,\mathcal{L}) = 1 - g + \deg\left(\mathcal{L}\right) \tag{5.1}$$

where $\deg\left(\mathcal{L}\right) = c_1(|\mathcal{L}|)$.

Firstly we write the ratio of determinants in terms of Fourier modes as sections of powers, $\mathcal{L}_M^{\otimes n}$ of the line V bundle that defines M,

$$\frac{\mathrm{Det}_{\Omega^0(M,\mathfrak{k})}\left(L_{\phi+\phi_B}\right)}{\sqrt{\mathrm{Det}_{\Omega^1_H(M,\mathfrak{k})}\left(*\kappa\wedge L_{\phi+\phi_B}\right)}} \tag{5.2}$$

We regularise both the absolute value and the phase of the ratio of determinants as follows

$$\sqrt{\mathrm{Det}\,Q}=\sqrt{|\mathrm{Det}\,Q|}\,\exp\left(\frac{+i\pi}{2}\,\eta(Q)\right) \tag{5.3}$$

where $\eta(Q)=\frac{1}{2}\sum_{\lambda\in\mathrm{spec}(Q)}\mathrm{sign}(\lambda)$

$$|\mathrm{Det}\,Q|\,(s)=\exp\sum_{\lambda\in\mathrm{spec}(Q)}e^{s\Delta}\ln|\lambda| \tag{5.4}$$

$$\eta(Q,\,s)=\frac{1}{2}\sum_{\lambda\in\mathrm{spec}(Q)}\frac{\mathrm{sign}(\lambda)}{|\lambda|^s}\exp\left(s\Delta\right) \tag{5.5}$$

for Δ the Laplacian of the twisted Dolbeault operator.

5.1. The absolute value of the determinants and Ray-Singer torsion

Had ϕ_B been constant then the regularisation would have led us to consider [1]

$$\chi\left(\Sigma,\mathcal{L}_M^{\otimes n}\right)+\chi\left(\Sigma,\mathcal{L}_M^{\otimes -n}\right)=2-2g-N+\sum_{i=1}\phi_{a_i}\left(n\right) \tag{5.6}$$

What $\phi_{a_i}(n)$ measures are the number of 'honest' line bundles in the tensor power $\mathcal{L}_i^{\otimes nb_i(\mathcal{L}_M)}$, that survive in the sum $\deg\mathcal{L}_M^{\otimes n}+\deg\mathcal{L}_M^{\otimes -n}$, and these line bundles are, by construction, at the i'th orbifold point. Note that we always have $\gcd\left(a_i,\,b_i(\mathcal{L}_M)\right)=1$ so that an honest bundle only arises when $a_i|n$.

However, ϕ_B is not constant. As explained between (6.16) and (6.19) in [4] when dealing with a non constant ϕ the ratio of determinants takes the form of an integral of the density representing the characteristic classes of the Dolbeault operator and the, log of, the operator itself. Applying that in

the orbifold case leads us to objects of the form

$$\int_{\Sigma_V} \mathcal{I}\,(R,\, F_0, \ldots, F_N) \ln M(\phi + \phi_B) \tag{5.7}$$

in the effective action. Here $\mathcal{I}\,(R,\, F_0, \ldots, F_N)$ is the local density function of the characteristic classes and $M(\phi + \phi_B)$ is essentially $\mathrm{Det}_{\mathfrak{k}}\,(L_{\phi+\phi_B})|_{S^1}$ (which varies over Σ_V).

We use the local decomposition (2.15) for line V bundles whose support is about the specified points $x_0,\, x_1, \ldots, x_N$ and we recall that contributions to the index theorem are local to express (5.7) as

$$\int_{\Sigma_0} \mathcal{I}\,(R,\, F_0) \ln M(\phi + \phi_B) + \sum_{i=1}^{N} \int_{D_i/\mathbb{Z}_{a_i}} \mathcal{I}\,(R,\, F_i) \ln M(\phi + \phi_B) \tag{5.8}$$

where Σ_0 is Σ_V with the N discs about the orbifold points removed. As the Kawasaki index comes from the holomorphic Lefshetz fixed point formula there are contributions coming from the orbifold points which we have implicitly incorporated in the integrals over the $D_i/\mathbb{Z}_{a_i}$. Indeed as we saw previously by appropriate choice of f in (3.10), we can have delta function support for the curvature 2-forms and thus 'localise' the contribution to the fixed points.

On each region ϕ_B is constant

$$\phi_B|_{U_i} = 2\pi i r_i \mathbf{n}_i \tag{5.9}$$

Over Σ_0 there are only smooth line bundles which cancel out in the sum of degrees which leaves us with the Euler characteristic which is $2 - 2g - N$ and following the discussion in Section 5.1 of [7] this leads us to a factor of

$$T_{S^1}\,(\phi + 2\pi i \mathbf{n}_0)^{1-g-N/2} = T_{S^1}\,(\phi)^{1-g-N/2} \tag{5.10}$$

where $T_{S^1}(\varphi)$ is the Ray-Singer torsion on S^1 of a constant connection $\varphi\, d\theta$ (and all connections are gauge equivalent to such a connection). So, on Σ_0, $M = T_{S^1}$ and

$$T_{S^1}(\varphi) = \det{}_{\mathfrak{k}}\left(1 - \mathrm{Ad}\,\mathrm{e}^{\varphi}\right) \tag{5.11}$$

where the right hand side is a determinant on the $\mathfrak{k}$ part of the Lie algebra $\mathfrak{g}$.

As we saw before there can also be contributions of honest line bundles at the orbifold points (though we will have to consider the orbifold points

to be 'smoothed out'). Recall that the ϕ_{a_i} count those line bundles over D_i which cancel in the sum of Euler characteristics. Indeed $\phi_{a_i}(n)$ arises as

$$\phi_{a_i}(n) = 1 - \frac{1}{a_i}\left(b_i(\mathcal{L}_i^{\otimes nb_i}) + b_i(\mathcal{L}_i^{\otimes -nb_i})\right) \tag{5.12}$$

where the 1 is the Euler characteristic of the disc and the second term is 0 if $a_i|n$ and one otherwise. When $a_i|n$ then line V bundles are line bundles and the second term vanishes (honest line bundles drop out in the sum).

In any case, once more following the discussion in Section 5.1 of [7] gives us the factor

$$T_{S^1}\left((\phi + 2\pi i r_i \mathbf{n}_i)/a_i\right)^{1/2} \tag{5.13}$$

All together then we have that the absolute value is

$$T_M(\phi;\, \mathbf{n}_i) = T_{S^1}(\phi)^{1-g-N/2} \cdot \prod_{i=1}^{N} T_{S^1}\left((\phi + 2\pi i r_i \mathbf{n}_i)/a_i\right)^{1/2} \tag{5.14}$$

5.2. The phase of the ratio of determinants and η invariants

We recall the regularised formulae for the phase with ϕ_B constant and then take into account the fact that it is not so.

In Section 5 of [7] the phase is split into two pieces one depending on the charges of the fields but not on the line V bundles defining M while the second has dependence on $\mathcal{L}_M$ but not on the smooth line bundles V_α,

$$\eta(L_{\phi+\phi_B}, s) = \sigma(L_{\phi+\phi_B}, V_{\mathfrak{k}}, s) + \gamma(L_{\phi+\phi_B}, \mathcal{L}_M, s)$$

where

$$\begin{aligned}\sigma(L_{\phi+\phi_B}, V_{\mathfrak{k}}, s) = &-2\sum_{\alpha>0} \deg(V_\alpha)|i\alpha(\phi+\phi_B)|^{-s} \\ &- 2\sum_{\alpha>0}\deg(V_\alpha)\sum_{n\geq 1}(2\pi n + i\alpha(\phi+\phi_B))^{-s} \\ &+ 2\sum_{\alpha>0}\deg(V_\alpha)\sum_{n\geq 1}(2\pi n - i\alpha(\phi+\phi_B))^{-s}\end{aligned} \tag{5.15}$$

and

$$\gamma(L_{\phi+\phi_B}, \mathcal{L}_M, s) = -\sum_{n\geq 1}\sum_{\alpha>0} \left[\deg\left(\mathcal{L}_M^{\otimes n}\right) - \deg\left(\mathcal{L}_M^{\otimes -n}\right)\right] \times \left[(2\pi n + i\alpha(\phi+\phi_B))^{-s} + (2\pi n - i\alpha(\phi+\phi_B))^{-s}\right] \tag{5.16}$$

However, here it is not the case that the bundle dependence neatly seperates. Both terms depend on the background gauge field and we are in danger of overcounting. It is straightforward to see that one generates equivalent terms in $\sigma(L_{\phi+\phi_B}, V_{\mathfrak{k}}, s)$ and $\gamma(L_{\phi+\phi_B}, \mathcal{L}_M, s)$ if one allows both to have the background field dependence. As we have extracted the background gauge fields we understand the field strength associated with the V_α in the above formula to be dA, and that the background field dependence should be turned off. With this understood the contribution to (5.15) is

$$\begin{aligned}\sigma(L_\phi, V_{\mathfrak{k}}, s) = &-2\sum_{\alpha>0}\deg\left(V_\alpha\right)|i\alpha(\phi)|^{-s} \\ &- 2\sum_{\alpha>0}\deg\left(V_\alpha\right)\sum_{n\geq 1}(2\pi n + i\alpha(\phi))^{-s} \\ &+ 2\sum_{\alpha>0}\deg\left(V_\alpha\right)\sum_{n\geq 1}(2\pi n - i\alpha(\phi))^{-s}\end{aligned} \tag{5.17}$$

In the limit as $s \to 0$

$$\sigma(L_{\phi+\phi_B}, V_{\mathfrak{k}}, s) = -2\sum_{\alpha>0}\deg\left(V_\alpha\right)\left(1 + \frac{1}{\pi}i\alpha(\phi)\right) + \mathcal{O}(s) \tag{5.18}$$

The term $\sum_{\alpha>0}\deg\left(V_\alpha\right)$ does not contribute to the phase, given our assumption that the group is simply-connected, so we are left with

$$\sigma(L_{\phi+\phi_B}, V_{\mathfrak{k}}, 0) \to -\frac{2i}{\pi}\sum_{\alpha>0} c_1\left(V_\alpha\right)\alpha(\phi) \tag{5.19}$$

In order to define and then evaluate (5.16) when ϕ_B is not constant we must define what we mean by the right hand side of

$$\deg\left(\mathcal{L}_M^{\otimes n}\right) - \deg\left(\mathcal{L}_M^{\otimes -n}\right) = 2n.c_1(\mathcal{L}_M) - 2\sum_{i=1}^{N}\left(\left(\frac{nb_i(\mathcal{L}_M)}{a_i}\right)\right) \tag{5.20}$$

The first Chern character $c_1(\mathcal{L}_M) = c_1(\mathcal{L}_0^{\otimes b_0}) \oplus_{i=1}^{N} c_1(\mathcal{L}_i^{\otimes b_i})$ and each summand is supported on the corresponding open set. The terms involving

the double bracket symbol $((nb_i/a_i))$ come from densities that have support on U_i.

In Section 5, Equation (5.26) of [7] the phase proportional to $c_1(\mathcal{L}_M)$ is determined to be

$$c_1(\mathcal{L}_M)\sum_{\alpha>0}\left(\frac{1}{3}+\frac{1}{2\pi^2}\alpha(\phi)^2\right)+\mathcal{O}(s) \tag{5.21}$$

which now goes over to

$$\begin{aligned}
&\sum_{i=0}^{N} c_1(\mathcal{L}_i^{\otimes b_i})\sum_{\alpha>0}\left(\frac{1}{3}+\frac{1}{2\pi^2}\alpha(\phi+2\pi i r_i\mathbf{n}_i)^2\right)+\mathcal{O}(s)\\
&\longrightarrow \sum_{\alpha>0}\left(\frac{c_1(\mathcal{L}_M)}{2\pi^2}\alpha(\phi)^2+\frac{2i}{\pi}\alpha(\phi)\alpha(\mathbf{n}_0)\right.\\
&\qquad\left.+\frac{2}{\pi}\sum_{i=1}^{N}\frac{b_i r_i}{a_i}\left[i\alpha(\phi)\alpha(\mathbf{n}_i)-\pi r_i\alpha(\mathbf{n}_i)^2\right]\right)\\
&\qquad+\dim(G/T)\,\frac{c_1(\mathcal{L}_M)}{6}
\end{aligned} \tag{5.22}$$

The determination of the phase coming from the double bracket symbol is presented between (5.26) and (5.27) in [7]. As one can see there that calculation is done for each line V bundle $\mathcal{L}_i$ independently and does not depend on ϕ consequently (5.27) there immediately goes over to

$$\begin{aligned}
&2\sum_{\alpha>0}\sum_{n\geq 0}\sum_{\pm}\left(\left(\frac{nb_i}{a_i}\right)\right)\cdot\frac{1}{(2\pi n\pm i\alpha(\phi+\phi_B))^s}\\
&=-2\dim(G/T)\,s(b_i,a_i)+\mathcal{O}(s)
\end{aligned}$$

without change.

Collecting all the contributions including one from the T valued fields we have (mod $4\mathbb{Z}$)

$$\begin{aligned}
\eta(0)=&\sum_{\alpha>0}\left(\frac{c_1(\mathcal{L}_M)}{2\pi^2}\alpha(\phi)^2-\frac{2c_1(V_\alpha)}{\pi}i\alpha(\phi)+\frac{2i}{\pi}\alpha(\phi)\alpha(\mathbf{n}_0)\right)\\
&+\sum_{\alpha>0}\frac{2}{\pi}\sum_{i=1}^{N}\frac{b_i r_i}{a_i}\left(i\alpha(\phi)\alpha(\mathbf{n}_i)-\pi r_i\alpha(\mathbf{n}_i)^2\right)\\
&+\dim G\left(\frac{c_1(\mathcal{L}_M)}{6}-2\sum_{i=1}^{N}s(b_i,a_i)\right)
\end{aligned}$$

so that we have finally determined the phase to be

$$-\frac{i\pi}{2}\eta(0) = 4\pi i\Phi(\mathcal{L}_M) - \frac{ic_{\mathfrak{g}}}{4\pi}c_1(\mathcal{L}_M)\,\mathrm{Tr}\,(\phi^2) + \frac{ic_{\mathfrak{g}}}{2\pi}\int_\Sigma \mathrm{Tr}\,\phi F_A$$
$$+ c_{\mathfrak{g}}\,\mathrm{Tr}\,(\phi\mathbf{n}_0) + ic_{\mathfrak{g}}\sum_{i=1}^{N}\frac{b_i r_i}{a_i}\,\mathrm{Tr}\left(-i\phi\mathbf{n}_i + \pi r_i\mathbf{n}_i^2\right) \tag{5.23}$$

The term $\int_\Sigma \mathrm{Tr}\,\phi F_A$ should not really be considered as we have already taken ϕ constant and all the non-trivial bundle structure resides in the background fields. This is unlike previous works on abelianisation where $\int_\Sigma F_A \neq 0$. However, thinking of the gauge field A in (4.22) as a background field, for the purposes of this calculation, then we indeed get the appropriate shift in k for this term too.

5.3. The partion function

The net effect of the phase is to give us the famous shift $k \to k_{\mathfrak{g}} = k + c_{\mathfrak{g}}$ as well as the framing term

$$\Phi(\mathcal{L}_M) = -\frac{\dim G}{48}\left(c_1(\mathcal{L}_M) - 12\sum_{i=1}^{N} s(b_i, a_i)\right) \tag{5.24}$$

Consequently the partition function becomes

$$Z_{CS} = \sum_{\mathbf{n}_0\in\mathbb{Z}}\left(\prod_{i=1}^{N}\sum_{\mathbf{n}_i=1}^{a_i-1}\right)\int_{\mathfrak{t}} d\phi\ \ \sqrt{T_M(\phi;\,\mathbf{n}_i)}\,.\exp\left(4\pi i\Phi(\mathcal{L}_M) + ik_{\mathfrak{g}}I(\phi,\mathbf{n})\right) \tag{5.25}$$

There is still a large symmetry available to us. The first Chern class is a rational number so we set[3] $c_1(\mathcal{L}_M) = d/P$ where $P = a_1 \ldots a_N$

$$\phi \to \phi - 2\pi P\mathbf{s}, \quad \mathbf{n}_0 \to \mathbf{n}_0 + d\mathbf{s} \tag{5.26}$$

Just as in [7] (2.4) one may use this symmetry to write the partition function in various forms. To write the partition function completely as a sum one only needs to note that using the symmetry we may constrain the ϕ integrals

[3]We are not claiming that there exists an $\mathcal{L}_0$ for which $\mathcal{L}_M$ is the d'th tensor power.

to lie between zero and $2\pi P$ while performing the sum over $\mathbf{n}_0$ sets

$$\phi = 2\pi \mathbf{n}/k_{\mathfrak{g}} \tag{5.27}$$

Note that if $c_1(\mathcal{L}_M) = 0$, there is still the symmetry one must set $d = 0$.

On setting ϕ to be as in (5.27) then every occurrance of the product $b_i r_i$ in the exponential in (5.25) can be taken to be unity thanks to an argument we have used a number of times. With this substitution understood then these formulae agree well (up to an overall factor, which can be determined) with [10, 11, 21].

6. Odds and ends

It might seem that the introduction of the background fields changes the fibre Wilson loop observables that we are able to evaluate. However, this is not the case. Depending over which open set U_i we are the observable becomes, on abelianisation and noting that ϕ is gauge fixed to be constant on the fibre,

$$\begin{aligned} \mathrm{Tr}_R \left(P \exp \left(\oint \kappa(\phi + \phi_B) \right) \right) &= \mathrm{Tr}_R \left(\exp \left(\phi + 2\pi b_i^* \mathbf{n}_i \right) \right) \\ &= \mathrm{Tr}_R \left(\exp \left(\phi \right) \right) \end{aligned} \tag{6.1}$$

This shows us that for such loops it is not important which smooth point on the base they go through in the fibration $S^1 \to M \to \Sigma_V$. To evaluate the expectation value of products of such knots one may simply insert the appropriate operators with representations in (5.25).

As explained in [2] and developed in detail for complex Chern Simons theory in [8] one can use different Seifert representations of the same manifold to obtain the invariants of different knots.

The same arguments that we have given apply to other theories such as BF theory and Chern Simons theory with a complex gauge group.

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References

[1] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, J. Diff. Geom. **70**, 183–323.

[2] C. Beasley, *Localization for Wilson loops in Chern-Simons theory*, Adv. Theor. Math. Phys. **17** (2013), 1–240.

[3] M. Blau, F. Hussain, and G. Thompson, *Grassmannian topological Kazama-Suzuki models and cohomology*, Nucl. Phys. **B488** (1997), 599–652.

[4] M. Blau and G. Thompson, *Derivation of the Verlinde formula from Chern-Simons theory and the G/G model*, Nucl. Phys. **B408** (1993), 345–390.

[5] M. Blau and G. Thompson, *On diagonalization in Map(M,G)*, Commun. Math. Phys. **171** (1995), 639–660.

[6] M. Blau and G. Thompson, *Chern-Simons theory on S^1-bundles: Abelianisation and q-deformed Yang-Mills theory*, JHEP **0605** (2006) 003.

[7] M. Blau and G. Thompson, *Chern-Simons theory on Seifert 3-manifolds*, JHEP **1309** (2013) 033.

[8] M. Blau and G. Thompson, *Chern-Simons theory with complex gauge group on Seifert-Fibred 3-manifolds*, `arXiv:1603.01149 [hep-th]`.

[9] M. Furuta and B. Steer, *Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann sirfaces with marked points*, Adnaces in Math. **96** (1992), 38–102.

[10] S. Hansen, *Reshetikhin-Turaev invariants of Seifert 3-manifolds and a rational surgery formula*, Alg. Geom. Topology **1** (2001), 627–686.

[11] S. Hansen and T. Takata, *Quantum invariants of Seifert 3-manifolds and their asymptotic expansions*, in: Geormetry and Topology Monographs, Vol. **4**, Invariants of Knots and 3-Manifolds Kyoto (2001), pp. 69–87.

[12] M. Jankin and W. Neumann, Lectures on Seifert Manifolds, Brandeis Lecture Notes (1981).

[13] L. C. Jeffrey *Chern-Simons-Witten invariants of lens spaces and torus bundles and the semi-classical approximation*, Commun. Math. Phys. **147** (1992), 563–604.

[14] J. Källén, *Cohomological localisation of Chern-Simons theory*, JHEP **08** (2011) 008.

[15] R. Lawrence and L. Rozansky, *Witten-Reshetikhin-Turaev invariants of Seifert manifolds*, Commun. Math. Phys. **205** (1999), 287–334.

[16] P. Orlik, *Seifert manifolds*, Lect. Notes in Math. **291** (1972).

[17] T. Mrowka, P. Ozsváth, and B. Yu, *Seiberg-Witten monopoles on Seifert fibered spaces*, Comm. Annal. Geom. **5** (1997), 685–791.

[18] M. Ouyang, *Geometric invariants for Seifert* 3*-manifolds*, Trans. Am. Math. Soc. **346** (1994), 641–659.

[19] N. Reshetikhin and V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.

[20] N. Reshetikhin and V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

[21] L. Rozansky, *Residue formulas for the large k asymptotics of Witten's invariants of Seifert manifolds. The SU*(2) *case*, Commun. Math. Phys. **178** (1996), 27–60.

[22] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–398.

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